Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 31–52

Applications of Jacobi and Riccati Equations along Flows to Riemannian Geometry

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Abstract.

In the present paper we show a model for geodesic flows on the unit tangent bundles of complete Riemannian manifolds. By treating it as in the study of manifolds without conjugate points we have two theorems of the same type as E. Hopf and L. Green proved. One is for spaces of constant curvature instead of flat manifolds. The other is for differentiable flows without conjugate points, and in particular, gradient flows. In addition, we give the formula of the same type as R. Ossermann and P. Sarnak did. As its application, we get the simpler proof of the extension due to W. Ballmann and W. P. Wojtkowski.

§1. Introduction

Let N be a manifold with volume form ω and let $f^t : N \longrightarrow N$ be a (complete) flow preserving ω. Let $\pi : E \longrightarrow N$ be a vector bundle over N with inner product $\langle \cdot, \cdot \rangle$. Let $L(E) = \{D | D(p) : E_p \longrightarrow E_p$ is a linear map for each $p \in N\}$ and $S(E) = \{A | A(p) : E_p \longrightarrow E_p$ is a symmetric linear map for each $p \in N\}$. We assume that there is a connection ∇ along the flow f^t such that

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for any sections Y, Z on N into E where X is the vector field on N generating the flow f^t. For $A \in S(E)$ we consider the E-valued differential equation of Jacobi type along the flow.

$$(J_A) \qquad \nabla_X \nabla_X Y + AY = 0.$$

And also, the $L(E)$-valued differential equation of Riccati type.

$$(R_A) \qquad \nabla_X U + U^2 + A = 0.$$

Received December 7, 1990.
Revised March 7, 1991.

We say that (J_A) is *disconjugate on* N if any nontrivial E-valued solution of (J_A) has at most one zero on each trajectory $\{f^t p| -\infty < t < \infty\}$, or equivalently (R_A) has a symmetric solution defined on each trajectory $\{f^t p| -\infty < t < \infty\}$.

To state the Theorems we need some definitions. We say that a point $p \in N$ is *nonwandering* if there exist sequences $\{p_n\} \subset N$ and $\{t_n\} \subset \mathbf{R}$ such that $t_n \longrightarrow \infty$, $p_n \longrightarrow p$ and $f^{t_n} p_n \longrightarrow p$ as $n \longrightarrow \infty$. We denote by Ω_0 the set of all nonwandering points such that their trajectories are not bounded in N with respect to a complete Riemannian metric on N and by Ω the set of all nonwandering points. A function $F : N \longrightarrow \mathbf{R}$ is by definition *summable* if $|F|$ is integrable over N.

Theorem A. *Suppose $A \in S(E)$ and the equation (J_A) is disconjugate on N. Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If the trace of A is summable on N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0$$

and equality holds only if $A = 0$ identically.

Theorem A can be proven by almost the same way as in [In4]. The examples and applications are seen in Subsections 1.1 and 1.2.

In the following theorem let U denote a symmetric solution of (R_A) defined on N, i.e., $U(f^t p)$ satisfies the equation (R_A) along each trajectory $\{f^t p| -\infty < t < \infty\}$.

Theorem B. *Suppose $A \in S(E)$ and $A(p) \leq 0$ for all $p \in N$. If N is compact, then*

$$-\int_N \operatorname{tr} U \, d\omega \geq \int_N \operatorname{tr} \sqrt{-A} \, d\omega$$

and equality holds only if A is parallel along the flow f^t, i.e., $\nabla_X A = 0$ identically.

The application of Theorem B can be seen in Subsection 1.3.

1.1. Geodesic flows

Let M be a complete Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle^0$ and let $N = SM$ be the unit tangent bundle with volume form $\omega = \sigma \wedge \theta$ where σ is the volume form induced from the Riemannian metric of M and θ is the canonical volume form of the unit sphere S^{n-1}, $n = \dim M$. Let $f^t : N \longrightarrow N$ be the geodesic flow, namely $f^t v = \dot{\gamma}_v(t)$

for any $t \in \mathbf{R}$ where $v \in N$ and $\gamma_v : \mathbf{R} \longrightarrow M$ is the geodesic with $\dot{\gamma}_v(0) = v$. Then, f^t preserves ω. Let $E = \cup_{v\in N} v^\perp$ where $v^\perp$ is the subspace of $T_{\pi(v)}M$ orthogonal to v with respect to the Riemannian metric of M. The inner product $\langle \cdot, \cdot \rangle$ on E and connection ∇ along the geodesic flow is defined as follows:

Let $Y, Z \in E_v = v^\perp, v \in N$. Then, $\langle Y, Z\rangle = \langle \bar{Y}, \bar{Z}\rangle^0(\pi(v))$ where we think as $\bar{Y} = Y, \bar{Z} = Z \in v^\perp \subset T_{\pi(v)}M$. Let X be the vector field on N generating the geodesic flow f^t, i.e., $X(v) = \left.\frac{df^t v}{dt}\right|_{t=0}$ for any $v \in N$. Define ∇ by $\nabla_{X(v)}Y := \nabla_v \bar{Y}$ for any E-valued vector field Y along $f^t v$. Then, ∇ satisfies the condition of compatibility.

Let R be the Riemannian curvature tensor given by

$$R(Y,Z)W = \nabla_Y \nabla_Z W - \nabla_Z \nabla_Y W - \nabla_{[Y,Z]} W$$

for any vector fields Y, Z, W on M. We put $A(v) = R(\cdot, v)v$ for any $v \in N$. Then, we have $A \in S(E)$. In this situation, E. Hopf ([Ho]) and L. Green ([Gr]) proved that if M is compact and without conjugate points, then the integral of $\operatorname{tr} A$ on N is nonpositive, and it vanishes only if M is flat. Here we note that the integral of $\operatorname{tr} A$ over N is that of the scalar curvature S of M over M with constant multiple. This theorem is extended by N. Innami ([In4]). However, Theorem A shows that this integral inequality is not Riemannian. Combined with Schur's lemma we have the following.

Theorem C. *Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If there is a function F on M such that $(J_{A-(F\circ\pi)I})$ is disconjugate on N and* $\operatorname{tr}(A - (F \circ \pi)I)$ *is summable over N, then*

$$\int_M (S - n(n-1)F)\, d\sigma \leq 0.$$

Equality holds only if $A = (F \circ \pi)I$, and in particular, M is a space of constant curvature if $\dim M \geq 3$.

It should be noted that a space of positive constant curvature is compact ([CE]). In Section 4 we discuss the assumption that Ω_0 has finite measure. There we have the following.

(1) M has finite volume. In this case $\operatorname{vol}(\Omega) = \operatorname{vol}(N)$.
(2) M is simply connected and without conjugate points. In this case $\Omega = \phi$.
(3) M has nonnegative sectional curvature outside a compact set. In this case there exists an exhaustion Lipschitz continuous convex

function B on M, and $\mathrm{vol}(\Omega)$ is less than or equal to the measure of the unit tangent bundle of the minimum level set of B.

(4) There exists an increasing sequence $\{C_i\}$ of compact totally convex sets such that $\cup_{i=1}^{\infty} C_i = M$ and any sequence $\{p_i\}, p_i \in C_{m_{i+1}} - C_{m_i}$, has no accumulation point. In this case $\Omega_0 = \phi$. We find such surfaces in Section 4.

Here we say that $C \subset M$ is *totally convex* in M if any geodesic segment with endpoints of both sides in C is contained in C ([CG]). It is noted that if we want the goal in the equality case of Theorem C to be symmetric spaces of rank one, we have only to use a symmetric linear map that comes from a suitable curvature tensor satisfying the second Bianchi identity as A does, instead of using the identity map in $A - (F \circ \pi)I$ (see [In6], Lemma).

1.2. Gradient flows

Let N be a manifold and let $f^t : N \longrightarrow N$ be a flow which is generated by the vector field X on N. Assume that f^t preserves a volume form ω on N. Let $\pi : E = TN \longrightarrow N$ be the tangent bundle over N. Let $\langle \cdot, \cdot \rangle$ be a complete Riemannian metric on N and $P(t,p)$ the parallel translation from p to $f^t p$ along the curve $c_p : [0,t] \longrightarrow N$ with $c_p(s) = f^s p$ by Riemannian connection ∇. Define $D(t,p) = df_p^t \circ P(t,p)^{-1}$ for all $t \in \mathbf{R}$ and $p \in N$, and $D(t,p)$ is a $(1,1)$-tensor along c_p for each $p \in N$. Then, $D(t,p)$ are linear isomorphisms of $T_{f^t p}N = E_{f^t p}$ for all $t \in \mathbf{R}$ and $p \in N$. We define $(1,1)$-tensors U and A on N by

$$U(p) = \nabla_X D(0,p) \left(= \frac{d}{dt} D(t,p) \Big|_{t=0} \right)$$
$$A(p) = -\nabla_X \nabla_X D(0,p)$$

for each $p \in N$. The $D(t,p)$ satisfies the following differential equations:

$$(L_U) \qquad \nabla_X D(t,p) = U(f^t p) D(t,p),$$

$$(J_A) \qquad \nabla_X \nabla_X D(t,p) + A(f^t p) D(t,p) = 0,$$

$$(R_A) \qquad \nabla_X U(f^t p) + U(f^t p)^2 + A(f^t p) = 0,$$

If U is symmetric on N, then there exists a function $F : \widetilde{N} \longrightarrow \mathbf{R}$ such that $\mathrm{grad} F = \widetilde{X}$ where $\widetilde{N}$ is the universal covering space of N and $\widetilde{X}$ is the lift of X to $\widetilde{N}$ (see Section 5). If $\langle X, X \rangle$ is constant on N in addition to symmetry of U, then the trajectories of the flow f^t are geodesics and A is therefore the curvature tensor of N (Section 5). We say that the flow f^t is *disconjugate on* N if the differential equation (J_A) on all trajectories $\{f^tp| -\infty < t < \infty\}$ are disconjugate. It is known that f^t is disconjugate on N if U is symmetric on N ([Ha], Theorem 10.2 and [In5]). As an application of Theorem A we have the following.

Theorem D. *Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If U is symmetric on N and the trace of A is summable over N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0.$$

Equality holds only if either f^t are the identity map of N for all $t \in \mathbf{R}$, or N and f^t are such that

(1) *the universal covering space $\widetilde{N}$ of N is isometric to the Riemannian product $M \times \mathbf{R}$,*
(2) *there exists a constant a such that the lift $\tilde{f}^t$ of f^t to $\widetilde{N}$ is given by $\tilde{f}^t(p,s) = (p, s + at)$ for any $(p,s) \in M \times \mathbf{R}$ and $t \in \mathbf{R}$,*
(3) *if $N = \widetilde{N}/\Gamma$ where Γ is an isometry group of $\widetilde{N}$, then each element of Γ splits, namely for any $\widetilde{\varphi} \in \Gamma$ there exists an isometry φ of M and a constant b such that $\widetilde{\varphi}(p,s) = (\varphi(p), s + b)$ for any $(p,s) \in M \times \mathbf{R}$.*

In particular, if $\langle X, X \rangle$ is in addition constant on N, the integral inequality is written as

$$\int_N \operatorname{Ric}(X) \, d\omega \leq 0$$

where $\operatorname{Ric}(X)$ *is the Ricci curvature of X.*

It should be noted that if f^t is disconjugate and A is symmetric on N, then the integral inequality holds without symmetric hypothesis of U, but we cannot determine what the flow is isometrically as seen in the example of Section 5. The most interesting cases having symmetric A are geodesic flows on the unit tangent bundles. However, in this case, the disconjugacy of the geodesic flow on the unit tangent bundle is equivalent to that of the underlying manifold ([In5]). The manifolds without conjugate points have already been studied in [In4].

Corollary E. *If there exists a function $F : N \longrightarrow \mathbf{R}$ such that $X = \operatorname{grad} F$, and if* $\operatorname{tr} A$ *is summable over N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0$$

and equality holds only if either F is constant on N, or F is a non-trivial affine function on N, i.e., for any geodesic $\alpha : \mathbf{R} \longrightarrow N$ there exists constants a and b such that $F \circ \alpha(t) = at + b$. In the latter case N is isometric to the Riemannian product $F^{-1}(0) \times \mathbf{R}$.

In Section 5 we discuss what the symmetry of U implies and what relation there is between A and the Riemannian curvature tensor. We give the proofs of Theorem D and Corollary E in Section 5.

1.3. Measure theoretic entropy of geodesic flows

We use the same notation here as in Subsection 1.1. R. Ossermann and P. Sarnak ([OS]) give the lower estimate of measure entropy for geodesic flows on compact negatively curved manifold in terms of the curvature invariant as an application of Pesin's formula ([Pe]). And, W. Ballmann and M. P. Wojtkowski ([BW]) extend it to the case of nonpositive curvature. The main part of their proofs is to show that the integral inequality as in Theorem B. We write their estimate of measure entropy here:

Theorem F. *Suppose M is compact and has nonpositive curvature. Then,*

$$h_\omega(f^1) \geq \int_{SM} \operatorname{tr} \sqrt{-A}\, d\omega,$$

where $h_\omega(f^1)$ is the measure theoretic entropy of the geodesic flow f^t on the unit tangent bundle SM. Equality holds if and only if M is a locally symmetric space.

W. Ballmann ([Ba]) give other information concerning this result. We give the proof of Theorem F in Section 3. The proof of ours will be achived by approximation of the Riemannian curvature tensor A from below, which does not mean that of the Riemannian metric on M.

§2. Proofs of Theorems A and C

The proof of Theorem A is the same as in [In4] which is the version of the geodesic flows on the unit tangent bundles of Riemannian manifolds without conjugate points. We start discussing the assumption of decomposition of Ω_0.

2.1. The decomposition of Ω_0 and Ω

We can prove the following.

Lemma 2.1. *If Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure, then so is Ω.*

Proof. Let $\{D_i\}$ be an increasing sequence of compact sets D_i such that $\cup_{i=1}^{\infty} D_i = N$ and any sequence $\{p_i\}$, $p_i \in D_{m_i+1} - D_{m_i}$, has no accumulation point. Such a sequence always exists. Let $\Omega'_i = \{p \in \Omega | f^t p \in D_i \text{ for any } t \in (-\infty, \infty)\}$, and let $\Omega_i = \Omega'_i - \Omega'_{i-1}$ for each $i \geq 2$ and $\Omega_1 = \Omega'_1$. Then, Ω_i is an f^t-invariant measurable set with finite measure for each i. Obviously, $\cup_{i=1}^{\infty} \Omega_i = \Omega - \Omega_0$. Therefore, we can get a decomposition of at most countably many f^t-invariant sets of Ω with finite measure. Q.E.D.

The condition $\mathrm{vol}_\omega(\Omega_0) = 0$ will be discussed in Section 4. For the proof of Theorem A we need some preliminalies.

2.2. The trajectories of the flow

We introduce an equivalence relation $\sim$ in $N - \Omega$ in such a way that $p \sim q$ if $p = f^t q$ for some $t \in (-\infty, \infty)$, where $p, q \in N - \Omega$. Let M be the set of all equivalence classes $[p]$, $p \in N - \Omega$. Since $N - \Omega$ is open and f^t-invariant, for any $p \in N$ there exists locally a hypersurface H in $N - \Omega$ containing p and diffeomorphic to an open subset in $\mathbf{R}^{n-1}$ such that $[q] \cap H = \{q\}$ and H intersects $[q]$ transversely for any $q \in H$. The collection of such hypersurfaces H yields a differentiable structure of M with dimension $n - 1$. We define the volume form η on M such that $\eta_{[p]} \wedge dt = \omega_p$ for any $[p] \in M$. Then we have, for any summable function F on $N - \Omega$,

$$\int_{N-\Omega} F\, d\omega = \int_{[p]\in M} d\eta \int_{-\infty}^{\infty} F_{[p]}(f^t p)\, dt, \tag{2.1}$$

where $F_{[p]} : [p] \longrightarrow \mathbf{R}$ is given by $F_{[p]}(q) = F(q)$ for any $q \in [p]$.

2.3. The Birkhoff ergodic theorem

Let D be a f^t-invariant subset of N with finite measure. The Birkhoff ergodic theorem (see [AA]) says that for any summable function F on D

1) $$F_*(p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(f^t p)\, dt$$

exist and are f^t-invariant for almost all $p \in D$,

2) $$\int_B F_*\, d\omega = \int_B F\, d\omega$$

for any f^t-invariant measurable subset $B \subset D$.

We say that a $p \in D$ is *uniformly recurrent* if for any neighborhood U of p, we have

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \chi_U(f^t p)\,dt > 0,$$

where $\chi_U : D \longrightarrow \mathbf{R}$ is the characteristic function of U. We denote by $W(D)$ the set of all uniformly recurrent points in D. It follows from the Birkhoff ergodic theorem that $W(D)$ has full measure in D (see [BBE]).

2.4. The integral of $\operatorname{tr} A$ on $N - \Omega$

We will prove the following.

Lemma 2.2.

$$\int_{N-\Omega} \operatorname{tr} A\, d\omega \le 0$$

and equality holds only if $A = 0$ identically on $N - \Omega$.

Proof. Since $\operatorname{tr} A$ is summable and by the formula (2.1), the integral of the absolute of $\operatorname{tr} A$ is finite along the trajectory $\{f^t p; -\infty < t < \infty\}$ for almost all $p \in N - \Omega$. It follows (cf. [Ha], [In4]) that

$$\int_{-\infty}^{\infty} \operatorname{tr} A(f^t p)\,dt \le 0$$

for almost all $p \in N - \Omega$. Integrating it on M as in Subsection 2.2, we obtain

$$\int_{N-\Omega} \operatorname{tr} A\, d\omega = \int_{[p]\in M} d\eta \int_{-\infty}^{\infty} \operatorname{tr} A(f^t p)\,dt \le 0.$$

Equality means (cf. [In4]) that $A(p) = 0$ for almost all $p \in N - \Omega$. Since $A(p)$ depend continuously on the points $p \in N$, we see that A is identically zero on $N - \Omega$. Q.E.D.

2.5. The integral of $\operatorname{tr} A$ on Ω

Let Ω_1 be an f^t-invariant subset of Ω with finite measure. We work in Ω_1 and prove the following.

Lemma 2.3.

$$\int_{\Omega_1} \operatorname{tr} A\, d\omega \le 0,$$

and equality holds only if $A(p) = 0$ for any $p \in \Omega_1$.

Proof. Let $X(\Omega_1)$ be the set of all points p such that $(\operatorname{tr} A)_*(p)$ exists as in 1) of Subsection 2.3. Then, $X(\Omega_1) \cap W(\Omega_1)$ has full measure in Ω_1. Let a point $p \in X(\Omega_1) \cap W(\Omega_1)$ and let K be a compact neighborhood of p in Ω_1. It follows (cf. [In4], [Go]) that there exists a constant $C(K) > 0$ such that $\|U(q)\| < C(K)$ for any $q \in K$, where $U(f^t q)$ is the minimal symmetric solution of (R_A) along the trajectory $\{f^t q; -\infty < t < \infty\}$. Since p is uniformly recurrent, there exists a sequence $\{T_n\} \subset \mathbf{R}$ such that $T_n \longrightarrow \infty$, $f^{T_n} p \longrightarrow p$ as $n \longrightarrow \infty$ and $f^{T_n} p \in K$ for all n. Since $U(f^t p)$ satisfies the equation (R_A), we have

$$\frac{1}{T_n}(\operatorname{tr} U(f^{T_n} p) - \operatorname{tr} U(p)) + \frac{1}{T_n} \int_0^{T_n} \operatorname{tr} U(f^t p)^2 \, dt \\ + \frac{1}{T_n} \int_0^{T_n} \operatorname{tr} A(f^t p) \, dt = 0.$$

Taking $n \longrightarrow \infty$ we obtain

$$(\operatorname{tr} A)_*(p) = - \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \operatorname{tr} U(f^t p)^2 \, dt \leq 0.$$

Hence, by the Birkhoff ergodic theorem, i.e., 2) of Subsection 2.3, we get

$$\int_{\Omega_1} \operatorname{tr} A d\omega = \int_{\Omega_1} (\operatorname{tr} A)_* d\omega \leq 0.$$

Suppose equality holds. Then, $X_0(\Omega_1) = \{p \in \Omega | (\operatorname{tr} A)_*(p) = 0\}$ has full measure in Ω_1, and, hence, $X_0(\Omega_1) \cap W(\Omega_1)$ has full measure in Ω_1. We will prove that $A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$. The idea of the proof is seen in [In3]. Let a $p \in X_0(\Omega_1) \cap W(\Omega_1)$. Suppose $X(p) \neq 0$. Let $\gamma(t) = f^t p$ for any $t \in (-\infty, \infty)$. We put $U(t) = U(f^t v)$ and $\operatorname{tr} A(t) = \operatorname{tr} A(f^t v)$ for all $t \in (-\infty, \infty)$. Let B be the neighborhood of p in N such that $B = \{f^t q | -\ell \leq t \leq \ell,\, q \in H\}$ for some $\ell > 0$ and some hypersurface H containing p diffeomorphic to the closed disk in $\mathbf{R}^{n-1}$ and B is differmorphic to $H \times [-\ell, \ell]$. Since $(\operatorname{tr} A)_*(p) = 0$ and $p \in W(\Omega_1)$, it follows from the argument above that

$$\lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \operatorname{tr} U(t)^2 \, dt = 0,$$

if a sequence $\{T_n\} \subset \mathbf{R}$ is such that $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\gamma(T_n)$ lie in the boundary of B for all n.

Assertion. *There exists a sequence $\{t_n\} \subset [0, \infty)$ such that*

(1) *$t_n \longrightarrow \infty$ as $n \longrightarrow \infty$,*

(2) *if* $U_n(t)$ *is the matrix given by* $U_n(t) = U(t_n + t)$ *for any* $t \in [-\ell, \ell]$, *then*

$$\int_{-\ell}^{\ell} \operatorname{tr} U_n(t)^2\, dt \longrightarrow 0 \qquad as \qquad n \longrightarrow \infty,$$

and $\operatorname{tr} U_n(t) \longrightarrow 0$ *for almost all* $t \in [-\ell, \ell]$ *as* $n \longrightarrow \infty$,

(3) *if* $\gamma_n : [-\ell, \ell] \longrightarrow N$ *is given by* $\gamma_n(t) = f^{t_n + t}p$ *for any* $t \in [-\ell, \ell]$, *then* γ_n *converges to the curve* $\gamma_0 : [-\ell, \ell] \longrightarrow N$ *with* $\gamma_0(t) = f^t p$ *for any* $t \in [-\ell, \ell]$ *as* $n \longrightarrow \infty$.

Proof. We denote the set $\gamma^{-1}(H)$ by $\{t_i\}$. We assume that if $i < j$ then $t_i < t_j$ and $t_1 = 0$. Put $a_i = t_i - \ell$ and $b_i = t_i + \ell$ for each $i = 1, 2, \cdots$. Then, $\gamma([a_i, b_i]) \subset B$. Suppose

$$\liminf_{i \to \infty} \int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt > \alpha > 0.$$

For any n, we have

$$\frac{1}{T_n} \int_0^{T_n} \operatorname{tr} U(t)^2\, dt \geq \frac{1}{T_n} \sum_{i=1}^{m_n} \int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt$$

$$\geq \frac{1}{T_n} \sum_{i=1}^{m} \int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt + \frac{\alpha}{2\ell T_n} \sum_{i=m+1}^{m_n} (b_i - a_i)$$

$$\geq \frac{\alpha}{2\ell T_n} \sum_{i=m+1}^{m_n} (b_i - a_i)$$

$$\geq \frac{\alpha}{2\ell T_n} \int_0^{T_n} \chi_{B \cap B(\ell/k)}(\gamma(t))\, dt - \frac{\alpha}{2\ell T_n} \sum_{i=1}^{m} (b_i - a_i),$$

where m_n and m are chosen so that

$$b_{m_n} < T_n < a_{m_n + 1} \qquad \text{and} \qquad \inf_{i \geq m} \int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt > \alpha,$$

and $B(\ell/k)$ is the ball with center p radius ℓ/k for a Riemannian metric on N. This implies that

$$0 = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \operatorname{tr} U(t)^2\, dt$$

$$\geq \frac{\alpha}{2\ell} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \chi_{B\cap B(\ell/k)}(f^t p)dt > 0,$$

a contradiction. Thus we can find an integer $i(k) \geq k$ such that

$$\gamma(t_{i(k)}) \in H \cap B(\ell/k) \qquad \text{and} \qquad \int_{a_{i(k)}}^{b_{i(k)}} \operatorname{tr} U(t)^2\, dt \leq \frac{1}{k},$$

If we change the notation by $t_k = t_{i(k)}$, then the sequence $\{t_k\}$ satisfies the condition (1) and the first part of (2). For the second part of (2) and (3) we have only to choose a suitable subsequence $\{t_n\}$ of $\{t_k\}$ if necessary. This completes the proof of Assertion. Q.E.D.

We return to the proof of $\operatorname{tr} A(p) = 0$. Rewriting the equation (R_A) for $U(f^t p)$ in terms of (2), we get for each n

$$\operatorname{tr} U_n'(t) + \operatorname{tr} U_n(t)^2 + \operatorname{tr} A_n(t) = 0 \tag{2.2}$$

for any $t \in [-\ell, \ell]$, where $\operatorname{tr} A_n(t) = \operatorname{tr} A(t_n + t)$. It should be noted that $\operatorname{tr} A_n(t)$ converges to $\operatorname{tr} A(t)$ uniformly in $t \in [-\ell, \ell]$ as $n \longrightarrow \infty$. Suppose $\operatorname{tr} A(0) = \operatorname{tr} A(p) \neq 0$, say $\operatorname{tr} A(p) > 0$. Then, there exist a and $b \in [-\ell, \ell]$, $a < 0 < b$ such that $\operatorname{tr} A(t) > 0$ for any $t \in [a, b]$ and $\operatorname{tr} U_n(a)$, $\operatorname{tr} U_n(b) \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, by integrating (2.2) on the interval $[a, b]$ and taking n to infinity, we have

$$\int_a^b \operatorname{tr} A(t)\, dt = 0,$$

a contradiction. Therefore, $\operatorname{tr} A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$ with $X(p) \neq 0$, namely p is not a fixed point of the flow. Next we assume that $X(p) = 0$, namely $f^t p = p$ for any $t \in (-\infty, \infty)$. Then, we have that $\operatorname{tr} A(p) = (\operatorname{tr} A)_*(p) = 0$. Thus, it follows ([In4]) that $A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$. Since $A(p)$ depends continuously on the points $p \in N$, we see that A is identically zero on Ω_1. Lemma 2.3 is proved. Q.E.D.

Now we can prove Theorem A.

Proof of Theorem A. Let Ω_1, Ω_2, $\cdots$ be the decomposition of f^t-invariant sets of Ω with finite measure. Then,

$$\int_N \operatorname{tr} A\, d\omega = \int_{N-\Omega} \operatorname{tr} A\, d\omega + \sum_{i=1}^{\infty} \int_{\Omega_i} \operatorname{tr} A\, d\omega \leq 0.$$

Equality holds only if $\int_{N-\Omega} \operatorname{tr} A \, d\omega = 0$ and $\int_{\Omega_i} \operatorname{tr} A \, d\omega = 0$ for all $i = 1, 2, \cdots$. Lemmas 2.2 and 2.3 state that $A = 0$ identically. This completes the proof of Theorem A.

Proof of Theorem C. We use the notation in Subsection 1.1. Theorem A states that

$$0 \geq \int_{SM} \operatorname{tr}(A - F \circ \pi I) \, d\omega = \int_{SM} \operatorname{Ric} \, d\omega - (n-1) F \circ \pi \, d\omega$$

$$= \frac{\theta_{n-1}}{n} \int_M S - n(n-1) F \, d\sigma,$$

where θ_{n-1} is the volume of the unit sphere in the n-dimensional Euclidean space, and therefore

$$\int_M S - n(n-1) F \, d\sigma \leq 0.$$

Equality holds only if $A(v) = F(\pi(v))I$ for any $v \in SM$. In that case Schur's lemma (cf. [Ch], [Sc]) shows that M is a space of constant curvature. This completes the proof of Theorem C.

§3. Proofs of Theorems B and F

We shall use the following fact without proof, since J.-H. Eschenburg stated it in [Es]. In the statement $S(n)$ denotes the set of all (n, n) symmetric matrices.

Lemma 3.1. *Let $A_0 : [a, b] \longrightarrow S(n)$ be such that (J_{A_0}) is disconjugate on $[a, b]$. Suppose a sequence of $A_i : [a, b] \longrightarrow S(n)$ satisfies that $A_i(t) \leq A_0(t)$ for any $t \in [a, b]$ and each $i = 1, 2, \cdots$. If the sequence $\{A_i\}$ converges to A_0, then the sequence of minimal symmetric solutions $U_{is} : [a, s) \longrightarrow S(n)$ of (R_{A_i}) converges to that of (R_{A_0}), $U_{0s} : [a, s) \longrightarrow S(n)$, where $s \in [a, b]$.*

We need the following to prove Theorem B according to R. Ossermann and P. Sarnak ([OS]). Hereafter we assume that N is compact.

Lemma 3.2. *If (J_A) is disconjugate on N and $U(p)$ is invertible for all $p \in N$, then*

$$-\int_N \operatorname{tr} U \, d\omega = \int_N \operatorname{tr} AU^{-1} \, d\omega.$$

Proof. Since $U'U^{-1} + U + AU^{-1} = 0$ on N, we have

$$(\log|\det U(f^t p)|)' + \operatorname{tr} U(f^t p) + \operatorname{tr} A(f^t p) U^{-1}(f^t p) = 0.$$

Integrating it over $[0,1] \times N$ we get

$$\int_N \operatorname{tr} U \, d\omega + \int_N \operatorname{tr} AU^{-1} \, d\omega = 0.$$

Q.E.D.

Now we prove Theorem B. Put $A_\varepsilon = A - \varepsilon I$ for $\varepsilon > 0$. Since $A_\varepsilon \leq -\varepsilon I$, the minimal symmetric solution U_ε of (R_{A_ε}) is invertible and $U_\varepsilon \longrightarrow U$ as $\varepsilon \longrightarrow 0$ as seen in Lemma 3.1. Since

$$0 \leq \operatorname{tr}\left(\sqrt{-U_\varepsilon} - \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}\right)\left(\sqrt{-U_\varepsilon} - \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}\right)^*$$
$$= -\operatorname{tr} U_\varepsilon - 2\operatorname{tr}\sqrt{-A_\varepsilon} + \operatorname{tr} A_\varepsilon U_\varepsilon^{-1},$$

(This formula is due to W. Ballmann ([Ba]).) and Lemma 3.2 is true, we have

$$0 \leq -2\int_N \operatorname{tr} U_\varepsilon \, d\omega - 2\int_N \operatorname{tr}\sqrt{-A_\varepsilon} \, d\omega.$$

Taking ε to zero we have that

$$0 \leq -2\int_N \operatorname{tr} U \, d\omega - 2\int_N \operatorname{tr}\sqrt{-A} \, d\omega.$$

If equality holds, then the convergence of $\sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}$ to $\sqrt{-U}$ is in L^2. Hence, if $X_\varepsilon = \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}$, then $X_\varepsilon\sqrt{-U_\varepsilon} = \sqrt{-A_\varepsilon} \longrightarrow \left(\sqrt{-U}\right)^2$ as $\varepsilon \longrightarrow 0$ in L^2. Therefore, $\sqrt{-A(p)} = -U(p)$ i.e., $U(p)^2 + A(p) = 0$ for almost all $p \in N$. Continuity of U on each trajectory of the flow implies that $U'(f^t p) = 0$ for almost all p and all $t \in (-\infty, \infty)$. It follows from this that $A'(f^t p) = -U'(f^t p)U(f^t p) - U(f^t p)U'(f^t p) = 0$ for almost all $p \in N$. Since A is continuous on N, we conclude that A is parallel along the flow. This completes the proof of Theorem B.

Proof of Theorem F. Ja. Pesin proved in [Pe] that the measure theoretic entropy of the geodesic flow of the unit tangent bundle SM of a compact manifold M without conjugate points is $-\int_{SM} \operatorname{tr} U \, d\omega$ where U is the minimal symmetric solution of (R_A), A is the Riemannian curvature tensor of M. If we use the Riemannian curvature tensor of M as A in Theorem A, we immediately have Theorem F.

§4. Decomposition of the set of nonwandering points

In this section we study the condition which decomposes the set of all nonwandering points into at most countably many f^t-invariant sets with finite measure.

Let N be a manifold and let $f^t : N \longrightarrow N$ be a flow. We say that a subset $C \subset N$ is *a pencil of* f^t*–segments* if $f^s p \in C$ and $f^u p \in C$ implies that $f^t p \in C$ for any t, $s < t < u$. The condition shows that $f^t p \notin C$ for all $t > s$ (or $t < s$) if $p \in C$ and $f^s p \notin C$ for some $s > 0$ (or $s < 0$, resp.).

Example. Let M be a complete Riemannian manifold and let $N = SM$ be the unit tangent bundle. We denote by f^t the geodesic flow. A subset $C_0 \subset M$ is called *totally convex* if $\gamma([a,b]) \subset C_0$ for any geodesic $\gamma : [a,b] \longrightarrow M$ with $\gamma(a) \in C_0$ and $\gamma(b) \in C_0$. Let $C_0 \subset M$ be a totally convex set in M and let $C = \{v \in SM \ ; \ \pi(v) \in C_0\}$ where π is the canonical projection of SM to M. Then, C is a pencil of f^t-segments.

We denote by Ω the set of all non-wandering points.

Lemma 4.1. *Suppose there exists an increasing sequence* $\{D_i\}$ *of compact pencils of* f^t*-segments such that* $\cup_{i=1}^{\infty} D_i = N$ *and any sequence* $\{p_i\}, p_i \in D_{m_{i+1}} - D_{m_i}$*, has no accumulation point. Then,* Ω *is decomposed into at most countably many* f^t*-invariant sets with finite measure. More precisely, there exists no nonwandering point* p *such that the trajectory through* p *is not contained in any compact set.*

Proof. Let $\Omega'_i = \{p \in \Omega | f^t p \in D_i$ for all $t \in (-\infty, \infty)\}$ for each i and let $\Omega_i = \Omega'_i - \Omega'_{i-1}$ for each $i \geq 2$ and $\Omega_1 = \Omega'_1$. It is clear that $\Omega_i \cap \Omega_j = \phi$ for any $i \neq j$ and Ω_i is f^t-invariant for each i. Since Ω'_i is a closed set in a compact set D_i we see that Ω'_i is compact and measurable, and hence, Ω_i has finite measure. We must prove that $\cup_{i=1}^{\infty}\Omega_i = \Omega$. Obviousely, $\cup_{i=1}^{\infty}\Omega_i \subset \Omega$. Let $p \in \Omega$. Then, there exists an i_0 with $p \in \mathrm{Int} D_{i_0}$ from the property of $\{D_i\}$ that $\{p_i\}$, $p_i \in D_{m_{i+1}} - D_{m_i}$, has no accumulation point. If $f^t p \in D_{i_0}$ for all $t \in (-\infty, \infty)$, then $p \in \Omega'_{i_0}$, and hence, $p \in \cup_{i=1}^{i_0}\Omega_i$. Suppose $f^s p \notin D_{i_0}$ for some s. We may assume that $s > 0$ because the same argument is valid in the case $s < 0$. Since D_{i_0} is compact, we have that $f^s p \in \mathrm{Int}(N - D_{i_0}) = N - D_{i_0}$. By definition of nonwandering point, there exist a sequence $p_n \longrightarrow p$ and a sequence $t_n \longrightarrow \infty$ such that $f^{t_n} p_n \longrightarrow p$ as $n \longrightarrow \infty$. Hence, we can find by continuity of $f^t : N \longrightarrow N$ an m such that $t_m > s$, $p_m \in D_{i_0}$, $f^{t_m} p_m \in D_{i_0}$ and $f^s p_m \notin D_{i_0}$, contradicting that D_{i_0} is a pencil of f^t-segments. Thus, $f^t p \in D_{i_0}$ for all $t \in (-\infty, \infty)$. This completes the proof. Q.E.D.

We shall describe the assumption in Lemma 4.1 by using a nealy f^t-peakless function. We say that a continuous function $F : [a, b] \longrightarrow \mathbf{R}$ is *nearly peakless* if $a \leq t_1 < t_2 < t_3 \leq b$ implies that $F(t_2) \leq \max\{F(t_1), F(t_3)\}$. The function was defined by H. Busemann and B. B. Phadke ([BP]) as convexities of functions degenerate. We say that a continuous function $F : N \longrightarrow \mathbf{R}$ is *nearly f^t-peakless* if for each $p \in N$ the function $F(f^t p)$ in $t \in (-\infty, \infty)$ is nearly peakless. We denote a sublevel set of F by $[F \leq b] = \{p \in N | F(p) \leq b\}$. If $F : N \longrightarrow \mathbf{R}$ is nearly f^t-peakless, then all sublevel sets are pencils of f^t-segments. A function $F : N \longrightarrow \mathbf{R}$ is said to be *exhaustive* if the sublevel sets $[F \leq b]$ are compact for all $b < \sup F(N)$. In these words we rewrite Lemma 4.1.

Proposition 4.2. *Suppose there exists a nearly f^t-peakless exhaustion function $F : N \longrightarrow \mathbf{R}$. Then, Ω is decomposed into at most countably many f^t-invariant sets with finite measure.*

We shall discuss some examples.

4.1. Example

Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature outside some compact set C. We construct an f^t-convex function on SM as follows: Let $\gamma : [0, \infty) \longrightarrow M$ be a ray with $\gamma(0) = o \in C$, namely a minimizing geodesic from o. We denote the Busemann function of γ by $B_\gamma : M \longrightarrow \mathbf{R}$ ([Bu]), $B_\gamma(p) = \lim_{t\to\infty}\{t - d(p, \gamma(t))\}$, where $d(\cdot, \cdot)$ is the distance function induced from the Riemannian metric. This function satisfies that $|B_\gamma(p) - B_\gamma(q)| \leq d(p, q)$. H. Wu ([Wu]) shows that B_γ is convex in the case $C = \phi$, namely $B_\gamma \circ \alpha(t)$ is a convex function in t for any geodesic $\alpha : (a, b) \longrightarrow M$. The theorem is still true in the case $C \neq \phi$ with slight modification. We say that a function $F : M \longrightarrow \mathbf{R}$ is *convex outside a set D* if $F \circ \alpha(t)$ is convex in t for any geodesic $\alpha : (a, b) \longrightarrow M - D$. In this words Wu's theorem is stated such as B_γ is convex outside a set $D = [B_\gamma \leq \sup B_\gamma(C)]$. Define a function $B : N \longrightarrow \mathbf{R}$ by $B(p) = \sup\{B_\gamma(p) | \gamma \text{ is a ray with } \gamma(0) = o\}$. This function satisfies that $|B(p) - B(q)| \leq d(p, q)$ for any p and $q \in M$, and therefore B is continuous on M. Furthermore, B is a convex exhaustion function outside some compact set. The convexity of the function is proved as follows: It should be noted that the convexity of functions is a local property. Let $b = \sup B(C)$ and let $p \notin [B \leq b]$. For any $\varepsilon > 0$ with $b < B(p) - \varepsilon$ there exists a ray $\gamma : [0, \infty) \longrightarrow M$ with $\gamma(0) = o$ such that $B(p) - \varepsilon \leq B_\gamma(p)$. Let $\alpha : (-a, a) \longrightarrow M$ be a geodesic with $\alpha(0) = p$ and $\alpha((-a, a)) \cap [B_\gamma \leq \sup B_\gamma(C)] = \phi$. By the above theorem of Wu we have that

$$(B_\gamma \circ \alpha)(\lambda t_1 + (1-\lambda)t_2) \le \lambda(B_\gamma \circ \alpha)(t_1) + (1-\lambda)(B_\gamma \circ \alpha)(t_2)$$

for any t_1, $t_2 \in (-a, a)$ and $0 < \lambda < 1$. Therefore, we get

$$(B \circ \alpha)(\lambda t_1 + (1-\lambda)t_2) \le \lambda(B \circ \alpha)(t_1) + (1-\lambda)(B \circ \alpha)(t_2)$$

for any t_1, $t_2 \in (-a, a)$ and $0 < \lambda < 1$. This proves that B is convex outside $[B \le b]$, and therefore any sublevel set $[B \le c]$ is totally convex for any $c > b$. Next we prove that B is an exhaustion function. Suppose for indirect proof that there exists a $c > b$ such that $[B \le c]$ is noncompact. Since $[B \le c]$ is totally convex, we can find a ray $\gamma : [0, \infty) \longrightarrow M$ with $\gamma(0) = o$ and $\gamma([0, \infty)) \subset [B \le c]$. Since $B_\gamma(\gamma(t)) = t \le B(\gamma(t))$, we have a contradiction.

Define a function $B_1 : M \longrightarrow \mathbf{R}$ as follows:

$$B_1(p) = \begin{cases} B(p) & \text{if} \quad p \notin [B \le b] \\ b & \text{if} \quad p \in [B \le b] \end{cases}$$

Then, B_1 is a convex function on M. Let $\widetilde{B} : SM \longrightarrow \mathbf{R}$ be a function given by $\widetilde{B} = B_1 \circ \pi$. We finally get an f^t-convex exhaustion function $\widetilde{B}$ on SM. The weaker version of the existence of a filtration of M by compact totally convex sets has been proved by G. Thorbergsson ([Th]) by an analogous way as in Cheeger and Gromoll's paper ([CG]).

4.2. Example

We shall get a surface such that the set of all nonwandering points of the geodesic flow is decomposed into countably many f^t-invariant sets with finite measure.

Let $\mathbf{R}^2$ be the affine plane and let $F : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be a positive function which depends only on the x-coordinate. We define a metric on $\mathbf{R}^2$ by $ds^2 = dx^2 + F(x)^2 dy^2$. Let $\theta_\alpha(t)$ is the angle of the velocity vector $\dot\alpha(t)$ of a curve $\alpha(t) = (x(t), y(t))$ and the curve $c(s) = (x(t), s)$, $-\infty < s < \infty$. Clairaut's theorem states that if $\alpha : (-\infty, \infty) \longrightarrow (\mathbf{R}^2, ds^2)$ is a geodesic with $\alpha(t) = (x(t), y(t))$, then $F(x(t)) \cos\theta_\alpha(t) = \text{const.}$ for any $t \in (-\infty, \infty)$ (cf. [In2]).

Lemma. *Let (x_0, y_0) be a point in $(\mathbf{R}^2, ds^2)$ such that $F(x_0) = \min\{F(x) | x \ge x_0\}$ and let $\alpha : [0, \infty) \longrightarrow (\mathbf{R}^2, ds^2)$ be a geodesic emanating from (x_0, y_0) with $\alpha(t) = (x(t), y(t))$ and $\dot x(0) \ge 0$. Then, $x(t)$ is either constant or strictly increasing in $t \in [0, \infty)$.*

Proof. Suppose that $\dot x(0) > 0$. If this lemma is not true, then there exists an $s > 0$ such that $\theta_\alpha(s) = 0$. By Clairaut's theorem we have that

$F(x_0)\cos\theta_\alpha(0) = F(x(s))$. However, since $\dot{x}(0) > 0$, namely $\theta_\alpha(0) \neq 0$, and by the choice of x_0, this is a contradiction. Suppose that $\dot{x}(0) = 0$. By taking the limit of a sequence of geodesics $\alpha_n = (x_n, y_n)$ with $\dot{x}_n(0) > 0$ we see that $\alpha([0,\infty)) \subset \{(x,y)|x \geq x_0, -\infty < y < \infty\}$. Assume in addition that $x(t)$ is not constant in $t \in [0,\infty)$. Let $t_0 > 0$ such that $x(t_0) > x_0$. If $x(t)$ is not strictly increasing in $t \geq t_0$, then there exists a $t_1 \geq t_0$ such that $\theta_\alpha(t_1) = 0$. By Clairaut's theorem again we have that $F(x(t_1)) = F(x_0)$. Hence, $F(x(t_1))$ is a minimum in a neighborhood of $x(t_1)$, so we have that $F'(x(t_1)) = 0$. Then, $\beta(t) = (x(t_1), \frac{t}{F(x(t_1))})$ is a geodesic other than α, contradicting the uniqueness of the geodesic with initial point and velocity. Let $t_2 = \inf\{t > 0|x(t) > x_0\}$. If $t_2 = 0$, then we have nothing to prove. Suppose $t_2 > 0$. Then, $\alpha : [0,t_2] \longrightarrow (\mathbf{R}^2, ds^2)$ is a geodesic contained in $\{(x_0,y)| -\infty < y < \infty\}$. Since the family of maps $\varphi_\lambda : (\mathbf{R}^2, ds^2) \longrightarrow (\mathbf{R}^2, ds^2)$ given by $\varphi_\lambda(x,y) = (x, y+\lambda)$ for all $(x,y) \in \mathbf{R}^2$ is a one-parameter group of isometries on $(\mathbf{R}^2, ds^2)$, this implies that $\beta(t) = (x_0, \frac{t}{F(x_0)} + y_0)$ is the geodesic with $\beta(t) = \alpha(t)$ for any $t \in [0,t_2]$, contradicting that $x(t)$ is not constant in $t \in [0,\infty)$. This completes the proof of Lemma. Q.E.D.

Let $\varphi_\lambda : (\mathbf{R}^2, ds^2) \longrightarrow (\mathbf{R}^2, ds^2)$ be given by $\varphi(x,y) = (x, y+\lambda)$. We consider the tube $T = \{(x,y)|x \geq x_0, -\infty < y < \infty\}/\{\varphi_\lambda^n\}_{n\in\mathbf{Z}}$, and the condition:

(C) *There exists a sequence* $\{x_i\}$, $x_i < x_{i+1}$, *such that* $x_i \longrightarrow \infty$ *as* $i \longrightarrow \infty$ *and* $F(x_i) = \min\{F(x)|x \geq x_i\}$ *for each* i.

Let M be a surface on which there exists a compact set K such that each connected component of $M - K$ is one of the following types:

(1) It is a tube constructed above and satisfying the condition (C).
(2) It has finite volume.

We denote by $T_1, \cdots, T_m$ the tubes of type (1) and by $\{x_{j,i}\}_{i=1,2,\cdots}$ a sequence in the condition (C) for the tube T_j, $j = 1, 2, \cdots, m$. Let $T_{j,i} = \{(x,y) \in T_j|x \geq x_{j,i}\}$. Then, it follows from Lemma that $D'_i = M - (\cup_{j=1}^{\infty} T_{j,i})$ is a totally convex set with finite measure for each i. Put $D_i = \{v \in SM|\pi(v) \in D'_i\}$. Although D_i are not compact, by using the sequence $\{D_i\}$ in the same way as Lemma 4.1, the set of all nonwandering points can be decomposed into countably many f^t-invariant sets with finite measure.

§5. Jacobi and Riccati equations from flows

In this section we use the same notation as in Subsection 1.2. We begin with the study of symmetry of U.

5.1. Flows with symmetric U

First of all we note that if $y \in T_pN$ and $Y(t) = P(t,p)y$, then $(DY)(t) := D(t,p)Y(t) = df_p^t y$, and hence

$$\begin{aligned}\nabla_X(DY)(t) &= (\nabla_X D)Y(t) = ((\nabla_X D)D^{-1})(DY)(t)\\ &= U(f^tp)(DY)(t) = U(f^tp)(df_p^t y).\end{aligned}$$

Let $\widetilde{N}$ be the universal covering space of N and $\widetilde{X}$ the lift of X to $\widetilde{N}$. Let $\widetilde{U}$ be the lift of U to $\widetilde{N}$.

Lemma 5.1. *If $\widetilde{U}$ is symmetric, then there exists a function $\widetilde{F} : \widetilde{N} \longrightarrow \mathbf{R}$ with $\widetilde{X} = \operatorname{grad} \widetilde{F}$.*

Proof. Define a 1-form η on $\widetilde{N}$ by $\eta(\widetilde{Y}) = \langle \widetilde{X}, \widetilde{Y}\rangle$ for any vector field $\widetilde{Y}$ on $\widetilde{N}$. Since $\widetilde{N}$ is simply connected, the Poincaré Lemma states Lemma 5.1 if $d\eta = 0$ on $\widetilde{N}$ (cf. [Wa]). Let y, $z \in T_{\tilde{p}}\widetilde{N}$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow \widetilde{N}$ be a variation such that

(1) $\varphi(0,0,0) = \tilde{p}$,
(2) $\varphi(t,s,u) = f^t\varphi(0,s,u)$,
(3) $d\varphi_{(0,0,0)}\left(\frac{\partial}{\partial s}\right) = y$, $d\varphi_{(0,0,0)}\left(\frac{\partial}{\partial u}\right) = z$.

If we put

$$\widetilde{Y}(t,s,u) := d\varphi_{(t,s,u)}\left(\frac{\partial}{\partial s}\right) = df^t_{\varphi(0,s,u)} d\varphi_{(0,s,u)}\left(\frac{\partial}{\partial s}\right),$$

and

$$\widetilde{Z}(t,s,u) := d\varphi_{(t,s,u)}\left(\frac{\partial}{\partial u}\right) = df^t_{\varphi(0,s,u)} d\varphi_{(0,s,u)}\left(\frac{\partial}{\partial u}\right),$$

then we have

$$\begin{aligned} d\eta(\widetilde{Y}, \widetilde{Z}) &= \widetilde{Y}\eta(\widetilde{Z}) - \widetilde{Z}\eta(\widetilde{Y}) - \eta([\widetilde{Y}, \widetilde{Z}])\\ &= \widetilde{Y}\langle \widetilde{X}, \widetilde{Z}\rangle - \widetilde{Z}\langle \widetilde{X}, \widetilde{Y}\rangle - \langle \widetilde{X}, [\widetilde{Y}, \widetilde{Z}]\rangle\\ &= \langle \nabla_{\widetilde{Y}}\widetilde{X}, \widetilde{Z}\rangle - \langle \nabla_{\widetilde{Z}}\widetilde{X}, \widetilde{Y}\rangle\\ &= \langle \nabla_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\rangle - \langle \nabla_{\widetilde{X}}\widetilde{Z}, \widetilde{Y}\rangle\\ &= \langle \widetilde{U}\widetilde{Y}, \widetilde{Z}\rangle - \langle \widetilde{U}\widetilde{Z}, \widetilde{Y}\rangle = 0,\end{aligned}$$

since $\widetilde{X}(\varphi(t,s,u)) = df^t_{\varphi(t,s,u)}\widetilde{X}(\varphi(0,s,u))$. Q.E.D.

Conversely we have the following.

Lemma 5.2. *If* $X = \operatorname{grad} F$ *for some function* $F : N \longrightarrow \mathbf{R}$*, then* U *is symmetric.*

Proof. Let y, $z \in T_pN$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow N$ be a variation as in the proof of Lemma 5.1. Then

$$\begin{aligned}&\langle UY, Z\rangle = \langle \nabla_X Y, Z\rangle = \langle \nabla_Y X, Z\rangle\\ =&Y\langle X, Z\rangle - \langle X, \nabla_Y Z\rangle = Y(ZF) - (\nabla_Y Z)F\\ =&Z(YF) - (\nabla_Z Y)F = \langle UZ, Y\rangle.\end{aligned}$$

This implies that $\langle Uy, z\rangle = \langle y, Uz\rangle$ at p. Q.E.D.

5.2. The relation between A and the curvature tensor

Let R be the curvature tensor of N. Let $p \in N$ and $y \in T_pN$. Put $Y(t) = P(t,p)y$ for any t. Then,

$$\begin{aligned}\nabla_X\nabla_X DY - \nabla_{DY}\nabla_X X &= \nabla_X \nabla_{DY} X - \nabla_{DY}\nabla_X X\\ &= R(X, DY)X.\end{aligned}$$

Hence, we have

$$A(y) = R(y, X(p))X(p) - \nabla_y \nabla_X X.$$

We prove the following.

Lemma 5.3. *If* $\langle X, X\rangle$ *is constant on* N *and* U *is symmetric on* N*, then the trajectories of* f^t *are geodesics in* N *and therefore* $A(y) = R(y, X(p))X(p)$ *for any* $p \in N$ *and* $y \in T_pN$*.*

Proof. To prove Lemma 5.3 it suffices that the trajectories of f^t are geodesics in N. Since $\langle X, X\rangle$ is constant on N, we see that

$$\begin{aligned}0 &= \langle \nabla_y X, X(p)\rangle = \langle \nabla_{X(p)} DY, X(p)\rangle\\ &= \langle U(p)y, X(p)\rangle = \langle y, U(p)X(p)\rangle = \langle y, \nabla_{X(p)} X\rangle.\end{aligned}$$

Therefore, $\nabla_X X$ is identically zero on N. Q.E.D.

5.3. Proofs of Theorem D and Corollary E

Since the integral inequality immediately follows from the disconjugacy of the flow which comes from the symmetric property of U, we have only to show what N is when $U = 0$ identically on N. The lift $\widetilde{U}$ of U to $\widetilde{N}$ is identically zero also. Let $\widetilde{F} : \widetilde{N} \longrightarrow \mathbf{R}$ be a function as in

Lemma 5.1. Let $y, z \in T_{\tilde{p}}\widetilde{N}$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow \widetilde{N}$ be a variation as in the proof of Lemma 5.1. We have

$$\langle \nabla_{\widetilde{Y}} \widetilde{X}, \widetilde{Z} \rangle = \langle \nabla_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \rangle = \langle \widetilde{U} \widetilde{Y}, \widetilde{Z} \rangle = 0,$$

and, in particular, $\widetilde{X}$ is parallel on $\widetilde{N}$. If $\alpha : (-\infty, \infty) \longrightarrow \widetilde{N}$ be a geodesic in $\widetilde{N}$, then

$$(\widetilde{F} \circ \alpha)"(t) = \dot{\alpha}(t) \langle \dot{\alpha}(t), \widetilde{X} \rangle = 0$$

for all $t \in (-\infty, \infty)$. Therefore, $\widetilde{F}$ is an affine function on $\widetilde{N}$. Theorem D follows from Main Theorem in [In1]. It should be noted that the nonwandering points p of gradient flows are fixed ones, and therefore $\Omega_0 = \phi$. Since the assumption of Main Theorem in [In1] does not require that the manifold is simply connected, Corollary E is true also by Lemma 5.2.

5.4. Example

We give an example such that $A = 0$ but U is not symmetric. Let $\widetilde{X}$ be the vector field on $\mathbf{E}^3$ given by

$$\widetilde{X}(x, y, z) = (\cos 2\pi z, \sin 2\pi z, 0),$$

where (x, y, z) are canonical coordinates on $\mathbf{E}^3$. Let

$$\Gamma = \left\{ \varphi_{\ell,m,n} | \varphi_{\ell,m,n}(x, y, z) = (x + \ell, y + m, z + n), \ell, m, n \in \mathbf{Z} \right\}$$

and let $T^3 = \mathbf{E}^3/\Gamma$. Then, there exists the vector field X on T^3 whose lift is $\widetilde{X}$. The flow generated from X preserves canonical volume form of T^3 and its trajectories are geodesics in the flat torus T^3. Thus we have a desired example.

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