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The Maass Zeta Functions Attached to Positive Definite Quadratic Forms

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§0. Introduction

Let m, n be positive integers with $m > n$. Put $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ and take a lattice L in the space $\mathbf{M}(m, n; \mathbb{R})$ of m by n real matrices stable under the action of Γ from the right. Denote by L' the set of matrices of rank n in L. In a series of papers ([M1], [M2], [M5]) Maass made precise investigations of the following zeta functions:

$$\zeta(Q, \phi, L; s) = \sum_{x \in L'/\Gamma} \frac{Q(x)\phi({}^t xx)}{\det({}^t xx)^{s+d/2n}},$$

where $Q(x)$ is a homogeneous polynomial of even degree d on $\mathbf{V} = \mathbf{M}(m, n)$ invariant under the action of $\mathbf{SL}(n)$ from the right and $\phi(Y)$ is an automorphic form of homogeneous degree 0 on the space of positive definite symmetric matrices of size n with respect to the arithmetic subgroup Γ. According to his results, the zeta functions can be extended to meromorphic functions in the whole complex plane and satisfy a functional equation under $s \longmapsto m/2 - s$; however Maass' functional equation involves derivatives of the form

$$L(D_x)\left(\phi({}^t xx)\det({}^t xx)^{-s}\right),$$

where $L(D_x)$ is a certain differential operator obtained from Q, and the derivatives have not been calculated explicitly unless $Q(x)$ is harmonic.

The aim of this paper is to present an approach to the Maass zeta functions based on the theory of prehomogeneous vector spaces and to obtain their explicit functional equation. In the present paper we restrict our attention to the case where the automorphic form ϕ is a constant function. The general case will be treated in a subsequent paper [S6] (see also [S5]).

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Put $\mathbf{G} = \mathbf{SO}(m) \times \mathbf{GL}(n)$. The group $\mathbf{G}$ acts linearly on the space $\mathbf{V} = \mathbf{M}(m,n)$ of m by n matrices via

$$x \longmapsto kxg^{-1} \qquad (k \in \mathbf{SO}(n), g \in \mathbf{GL}(n), x \in \mathbf{V}).$$

Then $(\mathbf{G}, \mathbf{V})$ is a prehomogeneous vector space with singular set

$$\mathbf{S} = \left\{ x \in \mathbf{V} \mid \det({}^t x x) = 0 \right\}$$

and the Maass zeta functions can be viewed as zeta functions associated with this prehomogeneous vector space.

Let $R = \mathbb{C}[\mathbf{M}(m,n)]^{\mathbf{SL}(n)}$ be the ring of $\mathbf{SL}(n)$-invariant polynomial functions on $\mathbf{V} = \mathbf{M}(m,n)$. To get an explicit functional equation of the Maass zeta function, it is necessary to decompose the ring R into direct sum of simple $\mathbf{G}$-modules. Put

$$n_0 = \min\{n, m-n\} \quad \text{and} \quad \kappa = \left[\frac{m}{2}\right].$$

Then simple components of R are parametrized by elements in the set

$$\Lambda = \left\{ (\lambda_0, \lambda) \in \mathbb{Z} \times \mathbb{Z}^\kappa \;\middle|\; \begin{array}{c} \lambda_0 \equiv \lambda_1 \equiv \cdots \equiv \lambda_{n_0} \pmod 2 \\ \lambda_0 \geq \cdots \geq \lambda_{n_0} \geq \lambda_{n_0+1} = \cdots = \lambda_\kappa = 0 \end{array} \right\}$$

if $m \neq 2n$, and

$$\Lambda = \left\{ (\lambda_0, \lambda) \in \mathbb{Z} \times \mathbb{Z}^n \;\middle|\; \begin{array}{c} \lambda_0 \equiv \lambda_1 \equiv \cdots \equiv \lambda_n \pmod 2 \\ \lambda_0 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \end{array} \right\}$$

if $m = 2n$ and hence $\kappa = n$. We denote by $\mathcal{R}_{\lambda_0,\lambda}$ the simple component corresponding to $(\lambda_0, \lambda) \in \Lambda$. Then our main theorem is the following:

Theorem. *Let $Q(x)$ be a polynomial in $\mathcal{R}_{\lambda_0,\lambda}$ $((\lambda_0, \lambda) \in \Lambda)$ and $\phi_0(Y)$ be a constant function. Then*

(i) *$\zeta(Q, L; s) = \zeta(Q, \phi_0, L; s)$ has an analytic continuation to a meromorphic function of s in $\mathbb{C}$ and the function*

$$\prod_{i=n_0+1}^{n} \left(s - \frac{\lambda_1 + i + 1}{2}\right)\left(s - \frac{\lambda_1 + m - i + 1}{2}\right)$$
$$\times \prod_{i=1}^{n_0} \left(s + \frac{\lambda_i - i - 1}{2}\right)\left(s - \frac{\lambda_i + m - i + 1}{2}\right) \zeta(Q, L; s)$$

is an entire function.

(ii) *Put*

$$\xi(Q,L;s) = v(L)^{1/2}\,\pi^{-ns} \times \prod_{i=n_0+1}^{n} \Gamma\left(s - \frac{i-\sigma-1}{2}\right) \prod_{i=1}^{n_0} \Gamma\left(s + \frac{\lambda_i - i + 1}{2}\right) \zeta(Q,L;s),$$

where $v(L) = \int_{\mathbf{V}_{\mathbb{R}}/L} dx$ *and* $\sigma = 0$ *or 1 according as* λ_1 *is even or odd. Then the following functional equation holds:*

$$\xi(Q,L^*;m/2-s) = \exp\left(\frac{\pi\sqrt{-1}}{2}\left(\sum_{i=1}^{n_0} \lambda_i + \sigma(n-n_0)\right)\right) \xi(Q,L;s).$$

As mentioned above, the Maass zeta function can be viewed as a zeta function associated with the prehomogeneous vector space $(\mathbf{SO}(m) \times \mathbf{GL}(n), \mathbf{M}(m,n))$. However to control $\mathbf{SL}(n)$-invariant functions appearing as coefficients of the zeta function, we need precise information on the prehomogeneous vector space $(\mathbf{B}(m) \times \mathbf{GL}(n), \mathbf{M}(m,n))$, where $\mathbf{B}(m)$ is the Borel subgroup of the special orthogonal group $\mathbf{SO}(m)$. In Section 1 the structure of $(\mathbf{B}(m) \times \mathbf{GL}(n), \mathbf{M}(m,n))$ is examined. The decomposition theorem of the ring R is due to Hoppe [H]. We give in this section a simple proof of Hoppe's decomposition theorem and make a correction to Hoppe's result in the case $m = 2n$.

Recall that the following are the facts lying behind the validity of functional equations of zeta functions associated with prehomogeneous vector spaces (cf. [SS], [S1]):

(1) local functional equations satisfied by complex powers of relative invariants,
(2) integral representation of zeta functions as a kind of Mellin transform of Theta series,
(3) the existence of the b-functions (the Bernstein-Sato polynomials of relative invariants).

Since the general theory of zeta functions associated with prehomogeneous vector spaces developed in [SS] and [S1] can be applied to the Maass zeta functions only when both $Q(x)$ and $\phi(Y)$ are constant functions, it is necessary for our purpose to generalize these three facts to the Maass zeta function $\zeta(Q,L;s)$. In Section 2 we give an integral representation of the Maass zeta function. We prove in Section 3 a generalization of local functional equations (Theorem 3.3) and give a proof of the main theorem (Theorem 3.1), assuming a formula for generalized b-functions (Theorem 3.4). The determination of generalized

b-functions corresponds to explicit calculation of $L(D_x)\left((\det {}^t\!xx)^{-s}\right)$. Section 4 is devoted to a calculation of the local functional equation and the b-functions of $\left(\mathbf{B}(m)\times\mathbf{GL}(n), \mathbf{M}(m,n)\right)$, which plays a key role in determining the explicit form of the functional equation of the Maass zeta function.

In the study of the Maass zeta functions, we are often forced to distinguish the cases $m \geq 2n$ and $n < m < 2n$. The use of the castling transform of prehomogeneous vector spaces allows us to reduce the case $m < 2n$ to the case $m \geq 2n$. In fact our argument in the present paper relies heavily upon results on the castling transform in [SO].

When $n = 1$, the Maass zeta function is nothing but the Epstein zeta function with spherical functions (cf. [E], [Si]). This special case has been precisely examined in [S2] by the same method.

A general theory of zeta functions with polynomial coefficients, which can be applied to the Maass zeta function $\zeta(Q, L; s)$, is developed in [S4]. However, for the sake of selfcontainedness, we do not assume the results in [S4], but only some basic knowledge on prehomogeneous vector spaces, for which we refer to [SK, §4], [SS] and [S1].

Notation. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of real numbers and the field of complex numbers, respectively. For an affine algebraic variety $\mathbf{X}$ defined over a field k, $\mathbf{X}_k$ stands for the set of k-rational points of $\mathbf{X}$. The space of rapidly decreasing functions on a real vector space V is denoted by $\mathcal{S}(V)$. The space of compactly supported C^∞-functions on a C^∞-manifold M is denoted by $C_0^\infty(M)$. We denote by 1_n and $0^{(m,n)}$ the identity matrix of size n and the m by n zero matrix, respectively. We often denote the zero matrix simply by 0, if its size is obvious from the context. The superscript (m,n) of a matrix $A = A^{(m,n)}$ indicates that the matrix A is of m rows and n columns. We write simply $A^{(m)}$ for $A^{(m,m)}$. For a real number a, we put $\operatorname{sgn}(a) = a/|a|$. The real part of a complex number z is denoted by $\Re(z)$.

§1. Structure of certain prehomogeneous vector spaces

Let m, n be positive integers with $m > n$. We put $n_0 = \min\{n, m-n\}$, $\kappa = [m/2]$ and $\delta = 0$ or 1 according as m is even or odd. Then $m = 2\kappa + \delta$. For a nondegenerate symmetric matrix $Y^{(m)}$, let $\mathbf{G} = \mathbf{SO}(Y) \times \mathbf{GL}(n)$ and $\mathbf{V} = \mathbf{M}(m,n)$. We consider the representation ρ of $\mathbf{G}$ on $\mathbf{V}$ defined by

$$\rho(h,g)x = hxg^{-1} \qquad \left(h \in \mathbf{SO}(Y), g \in \mathbf{GL}(n), x \in \mathbf{M}(m,n)\right).$$

Proposition 1.1 ([SK; §5, Proposition 23]). *The triple* $(\mathbf{G}, \rho, \mathbf{V})$ *is a regular prehomogeneous vector space with singular set*

$$\mathbf{S} = \left\{ x \in \mathbf{V} \;\middle|\; \det({}^t x Y x) = 0 \right\}.$$

The prehomogeneous vector space $(\mathbf{G}, \rho, \mathbf{V})$ is defined over the field $\mathbb{Q}(y_{ij}; 1 \le i \le j \le m)$ generated by the entries of Y. In this section we consider $(\mathbf{G}, \rho, \mathbf{V})$ as a prehomogeneous vector space defined over $\mathbb{C}$ and it is convenient to take the matrix

$$J = \begin{pmatrix} & & 1_\kappa \\ & 1_\delta & \\ 1_\kappa & & \end{pmatrix}$$

as Y. Real structures of $(\mathbf{G}, \rho, \mathbf{V})$ enter into the picture in Sections 2, 3 and 4. In Sections 2 and 3 we consider the real form of compact type, for which $Y = 1_m$ and in Section 4 the real form of split type, for which $Y = J$.

Put

$$\mathbf{B}(m) = \left\{ \begin{pmatrix} A & * & * \\ 0 & 1_\delta & * \\ 0 & 0 & {}^t A^{-1} \end{pmatrix} \in \mathbf{SO}(J) \;\middle|\; A \in \mathbf{Trig}(\kappa) \right\},$$

where $\mathbf{Trig}(\kappa)$ is the group of nondegenerate upper triangular matrices of size κ. Then the group $\mathbf{B}(m)$ is a Borel subgroup of $\mathbf{SO}(J)$. Every element b of $\mathbf{B}(m)$ can be written as

$$\begin{aligned} b =& b_1(A) b_2(v) b_3(B), \\ & b_1(A) = \begin{pmatrix} A & & \\ & 1_\delta & \\ & & {}^t A^{-1} \end{pmatrix} \qquad \big(A \in \mathbf{Trig}(\kappa)\big), \\ & b_2(v) = \begin{pmatrix} 1_\kappa & v & -2^{-1} v {}^t v \\ & 1_\delta & -{}^t v \\ & & 1_\kappa \end{pmatrix} \qquad \big(v \in \mathbb{C}^\kappa\big), \\ & b_3(B) = \begin{pmatrix} 1_\kappa & 0 & B \\ & 1_\delta & 0 \\ & & 1_\kappa \end{pmatrix} \qquad \big(B \in \mathbf{M}(\kappa), {}^t B = -B\big). \end{aligned}$$

Also put $\mathbf{P} = \mathbf{B}(m) \times \mathbf{GL}(n)$. We denote the representation of $\mathbf{P}$ on $\mathbf{V}$ obtained from ρ by restricting it to $\mathbf{P}$ by the same symbol ρ. If it is necessary to indicate the dependence on n, we use the superscript (n) such as $\mathbf{P}^{(n)}$, $\rho^{(n)}$, $\mathbf{V}^{(n)}$, etc.

Proposition 1.2. *The triple* $(\mathbf{P}, \rho, \mathbf{V})$ *is a regular prehomogeneous vector space.*

Proof. The triple $(\mathbf{P}^{(m-n)}, \rho^{(m-n)}, \mathbf{V}^{(m-n)})$ is equivalent to the castling transform of $(\mathbf{P}^{(n)}, \rho^{(n)}, \mathbf{V}^{(n)})$. Hence, by [SK, §2, Proposition 7] and [SO, Lemma 1.5], it is enough to prove the proposition in the case $m \geq 2n$. Consider the point

$$x_0 = \begin{pmatrix} 1_n \\ 0^{(\kappa-n+\delta,n)} \\ 1_n \\ 0^{(\kappa-n,n)} \end{pmatrix} \in \mathbf{V}.$$

Then, by an elementary calculation, it is easy to see that every element of the isotropy subgroup $\mathbf{P}_{x_0}$ of $\mathbf{P}$ at x_0 is of the form

$$\left(b_1 \begin{pmatrix} U^{(n)} & 0 \\ 0 & A^{(\kappa-n)} \end{pmatrix} b_2 \begin{pmatrix} 0^{(n,1)} \\ v^{(\kappa-n,1)} \end{pmatrix} b_3 \begin{pmatrix} 0^{(n)} & 0 \\ 0 & B^{(\kappa-n)} \end{pmatrix}, U\right), \tag{1.1}$$

where $U = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \in \mathbf{M}(n)$, $A \in \mathbf{Trig}(\kappa - n)$, $B \in \mathbf{M}(\kappa - n)$, ${}^tB = -B$ and $v \in \mathbb{C}^{\kappa-n}$. Hence

$$\mathbf{P}_{x_0} \cong \{\pm 1\}^n \times \mathbf{B}(2(\kappa - n) + \delta)$$

and

$$\begin{aligned} \dim \mathbf{P} - \dim \mathbf{P}_{x_0} &= (\kappa^2 + \kappa\delta + n^2) - \{(\kappa - n)^2 + \delta(\kappa - n)\} \\ &= (2\kappa + \delta)n = mn = \dim \mathbf{V}. \end{aligned}$$

This shows that the triple $(\mathbf{P}, \rho, \mathbf{V})$ is a prehomogeneous vector space (cf. [SK; §2, Proposition 2]). The regularity of $(\mathbf{P}, \rho, \mathbf{V})$ follows directly from the regularity of $(\mathbf{G}, \rho, \mathbf{V})$. Q.E.D.

Now we shall determine the singular set and relative invariants of $(\mathbf{P}, \rho, \mathbf{V})$.

For a symmetric matrix T we denote by $d_i(T)$ the i-th principal minor, namely the determinant of the upper left i by i block of T. Using the block decomposition

$$x = \begin{pmatrix} x_1^{(\kappa,n)} \\ y^{(\delta,n)} \\ x_2^{(\kappa,n)} \end{pmatrix} \in \mathbf{V},$$

we define rational functions $P_0(x), \ldots, P_{n_0}(x)$ by

$$P_0(x) = \det({}^t x J x),$$

$$P_i(x) = P_0(x) \cdot d_i(x_2({}^t x J x)^{-1}\,{}^t x_2) \qquad (1 \le i \le n_0 - 2),$$

$$P_{n_0-1}(x) = \begin{cases} P_0(x) \cdot d_{n_0-1}(x_2({}^t x J x)^{-1}\,{}^t x_2) & \text{if } m \ne 2n, \\ \dfrac{P_0(x)}{P_{n_0}(x)} \cdot d_{n_0-1}(x_2({}^t x J x)^{-1}\,{}^t x_2) & \text{if } m = 2n, \end{cases}$$

$$P_{n_0}(x) = \left(P_0(x) \cdot d_{n_0}(x_2({}^t x J x)^{-1}\,{}^t x_2)\right)^{1/2}.$$

Note that $P_{n_0}(x)$ is equal to the determinant of the matrix formed by the first (resp. last) n rows of x_2 (resp. x), if $m \ge 2n$ (resp. $m < 2n$). Then it is easy to check the first part of the following proposition:

Proposition 1.3. (i) *The functions $P_0(x), \ldots, P_{n_0}(x)$ are relative invariants of $(\mathbf{P}, \rho, \mathbf{V})$ and the rational characters $\chi_0, \ldots, \chi_{n_0}$ corresponding to $P_0(x), \ldots, P_{n_0}(x)$, respectively, are given by*

$$\chi_0(b, g) = \det(g)^{-2},$$

$$\chi_i(b, g) = \det(g)^{-2} \cdot (a_1 \cdots a_i)^{-2} \qquad (1 \le i \le n_0 - 2),$$

$$\chi_{n_0-1}(b, g) = \begin{cases} \det(g)^{-2} \cdot (a_1 \cdots a_{n_0-1})^{-2} & \textit{if } m \ne 2n, \\ \det(g)^{-1} \cdot (a_1 \cdots a_{n_0-1})^{-1} \cdot a_{n_0} & \textit{if } m = 2n, \end{cases}$$

$$\chi_{n_0}(b, g) = \det(g)^{-1} \cdot (a_1 \cdots a_{n_0})^{-1}$$

for $g \in \mathbf{GL}(n)$ and

$$b = b_1(A) b_2(v) b_3(B) \in \mathbf{B}(m) \quad \textit{with} \quad A = \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_\kappa \end{pmatrix}.$$

(ii) *They are irreducible polynomials and the singular set $\mathbf{S_P}$ of $(\mathbf{P}, \rho, \mathbf{V})$ is given by*

$$\mathbf{S_P} = \bigcup_{i=0}^{n_0} \{x \in \mathbf{V} \mid P_i(x) = 0\}.$$

To prove the second part of the lemma, we need some preliminaries.

Lemma 1.4. *The relative invariants $P_0(x), \dots, P_{n_0}(x)$ are polynomials.*

Proof. By [SO; Lemma 1.2], it is enough to prove the lemma in the case $m \geq 2n$. Then $n_0 = n$. It is obvious that $P_0(x)$ and $P_n(x)$ are irreducible polynomials. Denote by x_3 the square matrix of size n formed by the first n rows of x_2. For $i = 1, \dots, n-1$, put

$$f_i(x) = d_i(x_3({}^t xJx)^{-1}\,{}^t x_3).$$

Then we have

$$\begin{aligned} f_i(x) &= \det(x_3({}^t xJx)^{-1}\,{}^t x_3) \cdot d^*_{n-i}({}^t x_3^{-1}({}^t xJx)x_3^{-1}) \\ &= \frac{P_n(x)^2}{P_0(x)} \cdot d^*_{n-i}({}^t x_3^{-1}({}^t xJx)x_3^{-1}), \end{aligned}$$

where $d^*_{n-i}(T)$ stands for the determinant of the lower right $n-i$ by $n-i$ block of a symmetric matrix T. Since every entry of the matrix $P_n(x) \cdot x_3^{-1}$ is a polynomial of the entries of x_3, the function $P_0(x)P_n(x)^{2(n-i-1)} \cdot f_i(x)$ is a polynomial. On the other hand

$$P_0(x)^i f_i(x) = d_i(P_0(x) \cdot x_3({}^t xJx)^{-1}\,{}^t x_3)$$

is a polynomial. Hence $P_0(x)f_i(x)$ is a polynomial for $i = 1, \dots, n-1$. This shows that $P_i(x)$ $(1 \leq i \leq n-1)$ is a polynomial, unless $m = 2n$ and $i = n-1$. Now assume that $m = 2n$. Then we have

$$f_{n-1}(x) = \frac{P_n(x)}{P_0(x)} \cdot d_1^*\left({}^t(P_n(x)x_2^{-1})\,{}^t x_1 + x_1(P_n(x)x_2^{-1})\right).$$

Hence

$$P_{n-1}(x) = d_1^*\left({}^t(P_n(x)x_2^{-1})\,{}^t x_1 + x_1(P_n(x)x_2^{-1})\right)$$

and $P_{n-1}(x)$ is a polynomial. Q.E.D.

For $\lambda = (\lambda_1, \dots, \lambda_\kappa) \in \mathbb{Z}^\kappa$, we define a rational character χ_λ of $\mathbf{B}(m)$ by

$$\chi_\lambda\big(b_1(A)b_2(v)b_3(B)\big) = a_1^{-\lambda_1} \cdots a_\kappa^{-\lambda_\kappa}.$$

Then any rational character χ of $\mathbf{P} = \mathbf{B}(m) \times \mathbf{GL}(n)$ is of the form

$$\chi(h,g) = \chi_{\lambda_0,\lambda}(h,g) = \chi_\lambda(h) \cdot \det(g)^{-\lambda_0}$$

for some $\lambda \in \mathbb{Z}^\kappa$ and $\lambda_0 \in \mathbb{Z}$. Let $X_\rho(\mathbf{P})$ be the multiplicative group of rational characters of $\mathbf{P}$ corresponding to some relative invariants of

$(\mathbf{P}, \rho, \mathbf{V})$. By (1.1), [SK; §4, Proposition 19] and [SO; Lemma 1.2], we have

$$X_\rho(\mathbf{P}) = \left\{ \chi_{\lambda_0,\lambda} \;\middle|\; \begin{array}{c} \lambda_0 \equiv \lambda_1 \equiv \cdots \equiv \lambda_{n_0} \pmod 2 \\ \lambda_{n_0+1} = \cdots = \lambda_\kappa = 0 \end{array} \right\}.$$

We denote by $P_{\lambda_0,\lambda}$ the relative invariant corresponding to $\chi_{\lambda_0,\lambda} \in X_\rho(\mathbf{P})$, namely the rational function satisfying

$$P_{\lambda_0,\lambda}(\rho(b,g)x) = \chi_{\lambda_0,\lambda}(b,g)P_{\lambda_0,\lambda}(x) \qquad ((b,g) \in \mathbf{P}).$$

Recall that $P_{\lambda_0,\lambda}$ is determined by (λ_0, λ) uniquely up to nonzero constant multiple ([SK; §4, Proposition 3]). Put

$$X_\rho(\mathbf{P})^+ = \left\{ \chi_{\lambda_0,\lambda} \in X_\rho(\mathbf{P}) \;\middle|\; P_{\lambda_0,\lambda} \text{ is a polynomial} \right\}.$$

Let R be the ring of polynomial functions on $\mathbf{V}$ invariant under the action of $\mathbf{SL}(n)$ from the right:

$$R = \left\{ Q(x) \in \mathbb{C}[\mathbf{M}(m,n)] \mid Q(xg) = Q(x) \ \left(g \in \mathbf{SL}(n)\right) \right\}.$$

We consider the ring R as a left $\mathbf{G}$-module via

$$\left((h,g) \cdot P\right)(x) = P(\rho(h,g)^{-1}x) \qquad \left(h \in \mathbf{SO}(J), g \in \mathbf{GL}(n)\right).$$

Then a relatively $\mathbf{P}$-invariant polynomial function is nothing but the highest weight vector of a rational representation of $\mathbf{G}$ contained in R. The highest weight corresponding to $P_{\lambda_0,\lambda}$ is $\chi_{\lambda_0,\lambda}^{-1}$. It is known that the character $\chi_{\lambda_0,\lambda}^{-1}$ of $\mathbf{P}$ is a highest weight of some rational representation of $\mathbf{G}$ if and only if

$$\text{(1.2)} \qquad \begin{cases} \lambda_1 \geq \cdots \geq \lambda_\kappa \geq 0 & \text{when } m \text{ is odd,} \\ \lambda_1 \geq \cdots \geq \lambda_{\kappa-1} \geq |\lambda_\kappa| & \text{when } m \text{ is even.} \end{cases}$$

Therefore we obtain the inclusion relation

$$X_\rho(\mathbf{P})^+ \subset \left\{ \chi_{\lambda_0,\lambda} \in X_\rho(\mathbf{P}) \;\middle|\; \lambda \text{ satisfies (1.2)} \right\}.$$

Lemma 1.5. *Put*

$$\Lambda = \left\{ (\lambda_0, \lambda) \in \mathbb{Z} \times \mathbb{Z}^\kappa \;\middle|\; \begin{array}{c} \lambda_0 \equiv \lambda_1 \equiv \cdots \equiv \lambda_{n_0} \pmod 2 \\ \lambda_0 \geq \cdots \geq \lambda_{n_0} \geq \lambda_{n_0+1} = \cdots = \lambda_\kappa = 0 \end{array} \right\}$$

or

$$= \left\{ (\lambda_0, \lambda) \in \mathbb{Z} \times \mathbb{Z}^n \;\middle|\; \begin{array}{c} \lambda_0 \equiv \lambda_1 \equiv \cdots \equiv \lambda_n \pmod 2 \\ \lambda_0 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \end{array} \right\}.$$

according as $m \neq 2n$ *or* $m = 2n$ $(= 2\kappa)$. *Then*

$$X_\rho(\mathbf{P})^+ = \left\{\chi_{\lambda_0,\lambda} \,|\, (\lambda_0, \lambda) \in \Lambda\right\}.$$

Proof. Let $P_{\lambda_0,\lambda}(x)$ be a polynomial relative invariant. Then λ satisfies the condition (1.2) and we obtain

$$\chi_{\lambda_0,\lambda} = \prod_{i=0}^{n_0-1} \chi_i^{(\lambda_i-\lambda_{i+1})/2} \times \begin{cases} \chi_{n_0}^{\lambda_{n_0}} & (m \neq 2n), \\ \chi_n^{(\lambda_{n-1}+\lambda_n)/2} & (m = 2n). \end{cases}$$

This implies that there exists a nonzero constant c such

$$P_{\lambda_0,\lambda} = c \prod_{i=0}^{n_0-1} P_i^{(\lambda_i-\lambda_{i+1})/2} \times \begin{cases} P_{n_0}^{\lambda_{n_0}} & (m \neq 2n), \\ P_n^{(\lambda_{n-1}+\lambda_n)/2} & (m = 2n). \end{cases}$$

Note that the exponents of $P_1, \dots, P_{n_0}$ are non-negative integers. Assume that $\lambda_0 < \lambda_1$. Then, since P_0 is irreducible, P_0 divides some P_i $(1 \leq i \leq n_0)$. This is impossible. Hence $\lambda_0 \geq \lambda_1$. This shows the inclusion relation

$$X_\rho(\mathbf{P})^+ \subset \left\{\chi_{\lambda_0,\lambda} \,\middle|\, (\lambda_0, \lambda) \in \Lambda\right\}.$$

The opposite inclusion relation follows immediately from the expression above of $P_{\lambda_0,\lambda}$ as a product of $P_0, \dots, P_{n_0}$. Q.E.D.

Now we can complete the proof of Proposition 1.3.

Proof of Proposition 1.3 (ii). Let $Q(x)$ be a prime divisor of $P_i(x)$. Then it is also a relative invariant (cf. [SK; Section 4, Proposition 5]). As is shown in the proof of Lemma 1.5, $Q(x)$ is a product of $P_0(x), \dots, P_{n_0}(x)$. This can occur only when $P_i(x)$ is irreducible. An elementary calculation shows that

$$\mathbf{V}' = \mathbf{V} - \bigcup_{i=0}^{n_0} \{x \in \mathbf{V} \,|\, P_i(x) = 0\}$$

is a single $\mathbf{P}$-orbit. This proves Proposition 1.3 (ii). Q.E.D.

Let $R_{\lambda_0,\lambda}$ be the subspace of R spanned by $\{(h,g)\mathbf{.}P_{\lambda_0,\lambda} \mid (h,g)\in\mathbf{G}\}$. Every polynomial in $R_{\lambda_0,\lambda}$ is homogeneous of degree $\lambda_0 n$. Put

$$\Lambda^* = \left\{\lambda\in\mathbb{Z}^\kappa \;\middle|\; \begin{matrix}\lambda_1\equiv\lambda_2\equiv\cdots\equiv\lambda_{n_0} \pmod 2\\ \lambda_1\geq\cdots\geq\lambda_{n_0}\geq\lambda_{n_0+1}=\cdots=\lambda_\kappa=0\end{matrix}\right\} \qquad (m\neq 2n),$$

$$= \left\{\lambda\in\mathbb{Z}^n \;\middle|\; \begin{matrix}\lambda_1\equiv\lambda_2\equiv\cdots\equiv\lambda_n \pmod 2\\ \lambda_1\geq\cdots\geq\lambda_{n-1}\geq|\lambda_n|\end{matrix}\right\} \qquad (m=2n).$$

For $\lambda=(\lambda_1,\ldots,\lambda_\kappa)\in\Lambda^*$, put $R_\lambda=R_{\lambda_1,\lambda}$. Then

$$R_{\lambda_0,\lambda}=P_0(x)^{(\lambda_0-\lambda_1)/2}R_\lambda \qquad \big((\lambda_0,\lambda)\in\Lambda\big).$$

By the relation between relatively $\mathbf{P}$-invariant polynomials and highest weight vectors of simple $\mathbf{G}$-modules contained in R, Lemma 1.5 can be translated into the following Proposition:

Proposition 1.6. *The decomposition of R into direct sum of simple $\mathbf{G}$-modules is given by*

$$R=\bigoplus_{(\lambda_0,\lambda)\in\Lambda} R_{\lambda_0,\lambda}=\bigoplus_{l=0}^{\infty}\bigoplus_{\lambda\in\Lambda^*}P_0(x)^l\,\mathbf{.}\,R_\lambda.$$

Let $P_0\,(D_x)$ be the differential operator with constant coefficients satisfying

$$P_0\,(D_x)\exp\big(\mathrm{tr}({}^t yJx)\big)=P_0(y)\exp\big(\mathrm{tr}({}^t yJx)\big)\,.$$

Proposition 1.7. *The space $\bigoplus_{\lambda\in\Lambda^*}R_\lambda$ is characterized by the differential equation $P_0\,(D_x)\,Q(x)=0$, namely,*

$$\bigoplus_{\lambda\in\Lambda^*}R_\lambda=\{Q(x)\in R\mid P_0\,(D_x)\,Q(x)=0\}\,.$$

Proof. The proposition is an immediate consequence of the formula for the b-function of $(\mathbf{P},\rho,\mathbf{V})$, which will be proved in Section 4 (Theorem 4.2, see also Theorem 3.4 (ii)).

Remark. Propositions 1.6 and 1.7 are due to Hoppe ([H; Satz 7, Korollar 7.1], see also [M6]). However Hoppe's result contains a slight inaccuracy, which results from that he missed the fact that, when $m = 2n$, $P_0(x)f_{n-1}(x)$ can be divided by $P_n(x)$.

Take a $W \in \mathbf{GL}(m, \mathbb{C})$ such that $J = {}^tWW$. For $\lambda \in \Lambda^*$, put

$$\mathcal{R}_\lambda = \left\{ Q(W^{-1}x) \,|\, Q(x) \in R_\lambda \right\}.$$

Then

$$R = \bigoplus_{l=0}^{\infty} \bigoplus_{\lambda \in \Lambda^*} (\det {}^txx)^l \centerdot \mathcal{R}_\lambda \tag{1.3}$$

gives a decomposition of R into simple $\mathbf{SO}(m)$-modules. Put

$$K = SO(m) = \left\{ k \in \mathbf{GL}(m)_\mathbb{R} \,|\, {}^tkk = 1_m \right\}$$

and

$$K_0 = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in SO(m) \,\middle|\, k_1 \in SO(n), k_2 \in SO(m-n) \right\}.$$

Let $C(K/K_0)$ (resp. $L^2(K/K_0)$) be the space of continuous functions (resp. measurable functions square integrable with respect to the normalized K-invariant measure) on K/K_0. Define a mapping $\alpha : \mathcal{R}_\lambda \longrightarrow C(K/K_0)$ by

$$\alpha(Q)(k) = Q(kx_0), \; x_0 = \begin{pmatrix} 1_n \\ 0 \end{pmatrix}. \tag{1.4}$$

We denote the image $\alpha(\mathcal{R}_\lambda)$ by H_λ. Since $\mathcal{R}_\lambda$ is a simple $\mathbf{SO}(m)$-module, the mapping $\alpha : \mathcal{R}_\lambda \longrightarrow H_\lambda$ is an isomorphism and H_λ gives an irreducible unitary subrepresentation of K of $L^2(K/K_0)$. Moreover we have the following proposition:

Proposition 1.8. *The irreducible decomposition of the regular representation of K on K/K_0 is given by*

$$L^2(K/K_0) = \bigoplus_{\lambda \in \Lambda^*} H_\lambda.$$

§2. Integral representation of the Maass zeta functions

2.1. We consider the $\mathbb{R}$-structure of $(\mathbf{G}, \rho, \mathbf{V})$ such that

$$G = \mathbf{G}_{\mathbb{R}} = K \times \mathbf{GL}(n, \mathbb{R}) \quad \text{and} \quad \mathbf{V}_{\mathbb{R}} = \mathbf{M}(m, n; \mathbb{R}).$$

Put

$$\mathbf{GL}(n, \mathbb{R})^+ = \{g \in \mathbf{GL}(n, \mathbb{R}) \mid \det(g) > 0\}$$

and

$$G^+ = \mathbf{G}_{\mathbb{R}}^+ = K \times \mathbf{GL}(n, \mathbb{R})^+.$$

Put $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ and let L be a lattice in $\mathbf{V}_{\mathbb{R}}$ stable under the Γ-action from the right. Set

$$V' = \mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}} = \{x \in \mathbf{V}_{\mathbb{R}} \mid \operatorname{rank} x = n\}$$

and

$$L' = L \cap V'.$$

The set V' is a single G^+-orbit.

For a homogeneous polynomial $Q(x)$ in R of degree d, the Maass zeta function is defined by the Dirichlet series

$$\zeta(Q, L; s) = \sum_{x \in L'/\Gamma} Q(x)(\det {}^t\!xx)^{-s-d/2n},$$

which is absolutely convergent for $\Re(s) > m/2$ (see Corollary to Proposition 2.3 below).

We also consider the local zeta function

$$\Phi(Q, f; s) = \int_{V'} (\det {}^t\!xx)^{s-d/2n-m/2} Q(x) f(x) dx \quad (f \in \mathcal{S}(\mathbf{V}_{\mathbb{R}}), s \in \mathbb{C}),$$

where dx is the standard Euclidean measure on $\mathbf{V}_{\mathbb{R}} = \mathbf{M}(m, n; \mathbb{R})$. The integral $\Phi(Q, f; s)$ is absolutely convergent for $\Re(s) > 0$ and has an analytic continuation to a meromorphic function of s in $\mathbb{C}$.

Let π be an irreducible unitary representation of the compact Lie group K. Denote by H_π the representation space of π equipped with hermitian inner product $\langle\ ,\ \rangle$.

In order to obtain an integral representation of the Maass zeta function, we introduce the following $\operatorname{End}(H_\pi)$-valued integral:

$$Z_\pi(f, L; s) = \int_{G^+/\Gamma} \det(g)^{-2s} \pi(k) \sum_{x \in L'} f(\rho(k, g)x)\, dk\, dg \quad (f \in \mathcal{S}(\mathbf{V}_{\mathbb{R}})),$$

where dg is a Haar measure on $\mathbf{GL}(n,\mathbb{R})^+$ and dk is the Haar measure on K so normalized that the total volume of K is equal to 1.

As in the previous section, put

$$K_0 = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K \,\middle|\, k_1 \in SO(n), k_2 \in SO(m-n) \right\}$$

and $x_0 = \begin{pmatrix} 1_n \\ 0^{(m-n,n)} \end{pmatrix}$. Then one can find a $(k_x, g_x) \in G^+$ such that $\rho(k_x, g_x)x_0 = x$ for any $x \in V'$, since V' is a single G^+-orbit. The isotropy subgroup G_x^+ of G^+ at x is given by

$$\begin{aligned} G_x^+ &= \left\{ \left(k_x \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} k_x^{-1}, g_x k_1 g_x^{-1} \right) \,\middle|\, k_1 \in SO(n), k_2 \in SO(m-n) \right\} \\ &\cong K_0. \end{aligned}$$

Let $d\mu_x$ be the Haar measure on G_x^+ normalized by $\int_{G_x^+} d\mu_x = 1$. The measure $\omega(x) = (\det {}^t x x)^{-m/2}\, dx$ on V' is G^+-invariant. We normalize the Haar measure dg on $\mathbf{GL}(n,\mathbb{R})^+$ so that the following integral formula holds:

$$\begin{aligned} &\int_{G^+} F(k,g)\, dk\, dg \\ (2.1) \qquad &= \int_{V'} \omega(y) \int_{G_x^+} F\big((k_{yx}, g_{yx})h\big)\, d\mu_x(h) \qquad \big(F \in L^1(G)\big), \end{aligned}$$

where (k_{yx}, g_{yx}) is an element of G^+ satisfying

$$\rho(k_{yx}, g_{yx})x = y.$$

Lemma 2.1. *The integral $Z_\pi(f, L; s)$ is absolutely convergent for $\Re(s) > m/2$.*

Proof. We may assume that s is a real number and f is everywhere non-negative. Since the absolute value of any matrix coefficient of $\pi(k)$ is not greater than 1, any matrix coefficient of $Z_\pi(f, L; s)$ is majorized by $Z_{\pi_0}(f, L; s)$, where π_0 is the trivial representation of $SO(m)$. By a routine calculation based on (2.1), we have

$$\begin{aligned} &Z_{\pi_0}(f, L; s) \\ &\quad = \left\{ \sum_{x \in L'/\Gamma} (\det {}^t x x)^{-s} \right\} \cdot \left\{ \int_{V'} (\det {}^t x x)^{s-m/2} f(x)\, dx \right\}. \end{aligned}$$

The first factor of the right hand side is the zeta function considered by Koecher [K] and is absolutely convergent for $\Re(s) > m/2$. It is obvious that the second factor of the right hand side is absolutely convergent for $\Re(s) > m/2$. This proves the lemma. Q.E.D.

Put

$$H_{\pi,0} = \{v \in H_\pi \mid \pi(k)v = v \ (k \in K_0)\}.$$

By the irreducibility of π, we have $\dim H_{\pi,0} \leq 1$. When $\dim H_{\pi,0} = 1$, the representation π is called *of class* 1 (with respect to K_0). The projection pr of H_π onto $H_{\pi,0}$ is given by the integral

$$\mathrm{pr} = \int_{K_0} \pi(k_0) dk_0,$$

where dk_0 is the normalized Haar measure on K_0.

We define an $\mathrm{End}(H_\pi)$-valued function ϕ_π on V' by setting

$$\phi_\pi(x) = \pi(k_x) \circ \mathrm{pr} \qquad (x \in V').$$

Since the coset $k_x K_0$ is uniquely determined by x, the function ϕ_π does not depend on the choice of k_x.

Lemma 2.2. *Assume that* $\Re(s) > m/2$.

(i) The integral $Z_\pi(f, L; s)$ vanishes unless π is of class 1.

(ii) If π is of class 1, then

$$Z_\pi(f, L; s) = \left\{ \int_{V'} (\det {}^t\!xx)^{s-m/2} \phi_\pi(x) f(x)\, dx \right\} \circ \left\{ \sum_{x \in L'/\Gamma} \phi_\pi(x)^* (\det {}^t\!xx)^{-s} \right\},$$

where $\phi_\pi(x)^*$ is the adjoint operator of $\phi_\pi(x)$.

Proof. Note that, for $\Re(s) > m/2$, the integral $Z_\pi(f, L; s)$ is absolutely convergent and the following calculation is justified by the Fubini theorem. By the identity (2.1), we obtain

$$
\begin{aligned}
Z_\pi(f, L; s) &= \sum_{x \in L'/\Gamma} \int_{V'} \left(\frac{\det {}^t yy}{\det {}^t xx}\right)^s f(y)\,\omega(y) \int_{K_0} \pi(k_y k_0 k_x^{-1})\, dk_0 \\
&= \sum_{x \in L'/\Gamma} \int_{V'} \left(\frac{\det {}^t yy}{\det {}^t xx}\right)^s f(y)\,\pi(k_y) \circ \mathrm{pr} \circ \pi(k_x^{-1})\,\omega(y) \\
&= \left\{ \int_{V'} \left(\det {}^t yy\right)^s f(y)\,\pi(k_y) \circ \mathrm{pr}\ \ \omega(y) \right\} \\
&\qquad\qquad \circ \left\{ \sum_{x \in L'/\Gamma} \left(\det {}^t xx\right)^{-s} \mathrm{pr} \circ \pi(k_x^{-1}) \right\}.
\end{aligned}
$$

Since π is unitary, we have $\mathrm{pr} \circ \pi(k_x^{-1}) = (\pi(k_x) \circ \mathrm{pr})^*$. Hence we get

$$
\begin{aligned}
Z_\pi(f, L; s) = &\left\{ \int_{V'} (\det {}^t yy)^s \phi_\pi(y) f(y)\,\omega(y) \right\} \\
&\qquad \circ \left\{ \sum_{x \in L'/\Gamma} \phi_\pi(x)^* (\det {}^t xx)^{-s} \right\}.
\end{aligned}
$$

If π is not of class 1, then pr is the zero-map and hence $Z_\pi(f, L; s) = 0$.

Q.E.D.

2.2. An irreducible unitary representation π of K is contained in the regular representation of K on $L^2(K/K_0)$ if and only if π is of class 1 with respect to K_0 and then the multiplicity of π is equal to 1. For π of class 1, take a unit vector v_0 in $H_{\pi,0}$. Then the mapping

$$
\begin{array}{cccc}
q: H_\pi & \longrightarrow & L^2(K/K_0) \\
v & \longmapsto & q(v;k) = \langle v, \pi(k) v_0 \rangle
\end{array}
$$

gives an embedding of H_π and the image $q(H_\pi)$ coincides with H_λ for some $\lambda \in \Lambda^*$ (cf. Proposition 1.8). In this case we write $\pi = \pi_\lambda$.

Composing the mapping q with the inverse mapping of α defined by (1.4), we define a K-isomorphism $Q: H_\pi \longrightarrow \mathcal{R}_\lambda$ by

$$
Q(v; x) = \alpha^{-1}(q(v;k))(x) \qquad (v \in H_\pi),
$$

namely, $Q(v; x)$ is the polynomial in $\mathcal{R}_\lambda$ satisfying

$$
Q(v; kx_0) = \langle v, \pi(k) v_0 \rangle.
$$

Proposition 2.3. *Let $\pi = \pi_\lambda$ $(\lambda \in \Lambda^*)$ be an irreducible unitary representation of $SO(m)$ of class 1. When $\Re(s) > m/2$, the following identity holds for any $v, w \in H_\pi$:*

$$\langle Z_\pi(f, L; s)v, w\rangle = \zeta(Q(v;*), L; s) \cdot \Phi(\overline{Q(w;*)}, f; s).$$

Proof. For $x, y \in V'$, we get

$$\begin{aligned}\langle \phi_\pi(y)\phi_\pi(x)^* v, w\rangle &= \langle \phi_\pi(x)^* v, \phi_\pi(y)^* w\rangle \\ &= \langle \mathrm{pr} \circ \pi(k_x)^{-1} v, \mathrm{pr} \circ \pi(k_y)^{-1} w\rangle \\ &= \langle \pi(k_x)^{-1} v, v_0\rangle\langle v_0, \pi(k_y)^{-1} w\rangle \\ &= Q(v; k_x x_0)\overline{Q(w; k_y x_0)}.\end{aligned}$$

Since $Q(v;x)$ and $Q(w;x)$ are $\mathbf{SL}(m)$-invariant polynomial of homogeneous degree $\lambda_1 n$, we have

$$\langle \phi_\pi(y)\phi_\pi(x)^* v, w\rangle = \frac{Q(v;x)}{(\det {}^t xx)^{\lambda_1/2}} \cdot \frac{\overline{Q(w;y)}}{(\det {}^t yy)^{\lambda_1/2}}.$$

Now the lemma follows immediately from this identity and Lemma 2.2.
Q.E.D.

Remark. By the decomposition (1.3), it is sufficient for the description of analytic properties of the Maass zeta functions to consider the case where $Q(x)$ is in $\mathcal{R}_\lambda$ for some $\lambda \in \Lambda^*$. Conversely, since the form of the functional equations of the Maass zeta functions depend on λ (see Theorem 3.1 below), it is inevitable to consider the decomposition (1.3).

Corollary to Proposition 2.3. *For any homogeneous polynomial $Q(x)$ in R, the Maass zeta function $\zeta(Q, L; s)$ is absolutely convergent for $\Re(s) > m/2$.*

§3. Functional equations

3.1. For a lattice L in $\mathbf{V}_\mathbb{R}$, let L^* be the lattice dual to L:

$$L^* = \left\{ y \in \mathbf{V}_\mathbb{R} \mid \mathrm{tr}({}^t yx) \in \mathbb{Z} \text{ for all } x \in L \right\}.$$

The following is the main theorem of the present paper:

Theorem 3.1. *Let $Q(x)$ be a polynomial in $\mathcal{R}_\lambda$ $(\lambda \in \Lambda^*)$. Then*

(i) *$\zeta(Q, L; s)$ has an analytic continuation to a meromorphic function of s in $\mathbb{C}$ and the function*

$$\prod_{i=n_0+1}^{n}\left(s-\frac{\lambda_1+i+1}{2}\right)\left(s-\frac{\lambda_1+m-i+1}{2}\right) \times \prod_{i=1}^{n_0}\left(s+\frac{\lambda_i-i-1}{2}\right)\left(s-\frac{\lambda_i+m-i+1}{2}\right)\cdot\zeta(Q,L;s)$$

is an entire function.

(ii) *Put*

$$\xi(Q,L;s) = v(L)^{1/2}\pi^{-ns} \times \prod_{i=n_0+1}^{n}\Gamma\left(s-\frac{i-\sigma-1}{2}\right)\prod_{i=1}^{n_0}\Gamma\left(s+\frac{\lambda_i-i+1}{2}\right)\cdot\zeta(Q,L;s),$$

where $v(L)=\int_{\mathbf{V}_\mathbb{R}/L} dx$ and $\sigma = 0$ or 1 according as λ_1 is even or odd. Then the following functional equation holds:

$$\xi(Q,L^*;m/2-s) = \exp\left(\frac{\pi\sqrt{-1}}{2}\left(\sum_{i=1}^{n_0}\lambda_i+\sigma(n-n_0)\right)\right)\xi(Q,L;s).$$

3.2. The proof of the theorem above is based on the functional equations satisfied by $Z_\pi(f, L; s)$ and $\Phi(Q, f; s)$. First let us consider the functional equation satisfied by $Z_\pi(f, L; s)$.

For $f \in \mathcal{S}(\mathbf{V}_\mathbb{R})$, define the Fourier transform $\hat{f}$ of f by setting

$$\hat{f}(y) = \int_{\mathbf{V}_\mathbb{R}} f(x)\exp\left(2\pi\sqrt{-1}\,\mathrm{tr}({}^t y x)\right)dx.$$

Proposition 3.2. *Suppose that $f \in \mathcal{S}(\mathbf{V}_\mathbb{R})$ satisfies the condition*

$$f(x) = \hat{f}(x) = 0 \quad \textit{for any } x \in \mathbf{V}_\mathbb{R} \textit{ such that } \operatorname{rank} x < n. \tag{3.1}$$

Then $Z_\pi(f, L; s)$ has an analytic continuation to an entire function of s and satisfies the functional equation

$$Z_\pi(\hat{f}, L; s) = v(L)^{-1} Z_\pi(f, L^*; m/2 - s).$$

Proof. . Put

$$Z_\pi^+(f,L;s)=\int_{\substack{G^+/\Gamma\\ \det g\le 1}}\det(g)^{-2s}\pi(k)\sum_{x\in L'}f(\rho(k,g)x)\,dk\,dg$$

and

$$Z_\pi^-(f,L;s)=\int_{\substack{G^+/\Gamma\\ \det g\ge 1}}\det(g)^{-2s}\pi(k)\sum_{x\in L'}f(\rho(k,g)x)\,dk\,dg.$$

Since $Z_\pi^+(f,L;s)$ is absolutely convergent for any $s\in\mathbb{C}$, the integral $Z_\pi^+(f,L;s)$ represents an entire function of s. By the Poisson summation formula

$$\sum_{x\in L}\hat{f}(\rho(k,g)x)=(\det g)^m v(L)^{-1}\sum_{y\in L^*}f(\rho(k,{}^tg^{-1})y),$$

we have the following identity for any f satisfying the condition (3.1):

$$\begin{aligned}
&Z_\pi^-(\hat{f},L;s)\\
&=v(L)^{-1}\int_{\substack{G^+/\Gamma\\ \det g\ge 1}}\det(g)^{-2s+m}\pi(k)\sum_{y\in L^*\cap V'}f(\rho(k,{}^tg^{-1})x)\,dk\,dg\\
&=v(L)^{-1}\int_{\substack{G^+/\Gamma\\ \det g\le 1}}\det(g)^{2s-m}\pi(k)\sum_{y\in L^*\cap V'}f(\rho(k,g)x)\,dk\,dg\\
&=v(L)^{-1}Z_\pi^+(f,L^*;m/2-s).
\end{aligned}$$

Therefore

$$\begin{aligned}
Z_\pi(\hat{f},L;s)&=Z_\pi^+(\hat{f},L;s)+Z_\pi^-(\hat{f},L;s)\\
&=Z_\pi^+(\hat{f},L;s)+v(L)^{-1}Z_\pi^+(f,L^*;m/2-s)\\
&=v(L)^{-1}Z_\pi(f,L^*;m/2-s).
\end{aligned}$$

This proves the proposition. Q.E.D.

Remark. Let f_0 be a function in $C_0^\infty(V')$. Put

$$D_x=\left(\frac{\partial}{\partial x_{ij}}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}}.$$

Then the functions

$$\det\left({}^tD_xD_x\right)f_0(x)\quad\text{and}\quad\det\left({}^txx\right)\widehat{f_0}(x)$$

satisfy the condition (3.1) in the proposition above (cf. [SS; p.169, Additional remark 2], [S1; Lemma 6.2]).

3.3. The local functional equation satisfied by $\Phi(Q, f; s)$ is given in the following theorem:

Theorem 3.3. *Let σ be as in Theorem 3.1. Let $Q(x)$ be a polynomial in $\mathcal{R}_\lambda$. Then the following functional equation holds for any $f \in \mathcal{S}(\mathbf{V}_\mathbb{R})$:*

$$\begin{aligned}
&\Phi(Q, \hat{f}; s) \\
&= \exp\left(\frac{\lambda_1 n\pi\sqrt{-1}}{2}\right)\pi^{-2ns+n(m-2)/2}\prod_{i=1}^{n}\sin\pi\left(s - \frac{\lambda_1 + m - i - 1}{2}\right) \\
&\times \prod_{i=n_0+1}^{n}\Gamma\left(s - \frac{i-\sigma-1}{2}\right)\Gamma\left(s - \frac{m-i+\sigma-1}{2}\right) \\
&\times \prod_{i=1}^{n_0}\Gamma\left(s + \frac{\lambda_i - i + 1}{2}\right)\Gamma\left(s - \frac{\lambda_i + m - i - 1}{2}\right)\Phi(Q, f; m/2 - s).
\end{aligned}$$

The proof of Theorem 3.3 is based on the following theorem, which will be proved in Section 4.

Theorem 3.4. *For $Q(x)$ in $\mathcal{R}_\lambda$ $(\lambda \in \Lambda^*)$, let $Q(D_x)$ be the differential operator with constant coefficients satisfying*

$$Q(D_x)\exp(\operatorname{tr}{}^t yx) = Q(y)\exp(\operatorname{tr}{}^t yx).$$

Then we have

(i) $$Q(D_x)\left(\det{}^t xx\right)^s = b_\lambda(s)Q(x)\left(\det{}^t xx\right)^{s-\lambda_1},$$

where

$$\begin{aligned}
b_\lambda(s) = 2^{\lambda_1 n}&\prod_{i=n_0+1}^{n}\prod_{j=1}^{\lambda_1}\left(s + \frac{i-j}{2}\right) \\
&\times\prod_{i=1}^{n_0}\left\{\prod_{j=1}^{(\lambda_1+\lambda_i)/2}\left(s + \frac{i+1}{2} - j\right)\prod_{j=1}^{(\lambda_1-\lambda_i)/2}\left(s + \frac{m-i+1}{2} - j\right)\right\}
\end{aligned}$$

and

(ii) $$\det\left({}^t D_x D_x\right)\left(Q(x)\left(\det{}^t xx\right)^s\right) = \beta_\lambda(s)Q(x)\left(\det{}^t xx\right)^{s-1},$$

where

$$\beta_\lambda(s) = 2^{2n} \prod_{i=n_0+1}^{n} \left(s + \frac{i-1}{2}\right)\left(s + \frac{i-2}{2}\right)$$
$$\times \prod_{i=1}^{n_0} \left(s + \frac{\lambda_1 - \lambda_i}{2} + \frac{i-1}{2}\right)\left(s + \frac{\lambda_1 + \lambda_i}{2} + \frac{m-i-1}{2}\right).$$

Remark. Local functional equations and b-functions attached to representations on polynomial rings, which are similar to Theorem 3.3 and Theorem 3.4 (i), have been previously considered in [RS] for regular prehomogeneous vector spaces of commutative parabolic type. For a general theory of such an extension of local functional equations and b-functions, see [S4]. Recall that for usual zeta functions associated with prehomogeneous vector spaces, b-functions determine the gamma factor of the functional equations and the location of (possible) poles of zeta functions (see [SS], [S1]). In the present case, as is seen in the proof of Theorem 3.3 below, the function $b_\lambda(s)$ determines the gamma factor, while the proof of Theorem 3.1 shows that $\beta_\lambda(s)$ controls poles of zeta functions.

Proof of Theorem 3.3. First we consider the case where $\lambda = (0, \ldots, 0)$ and $Q(x)$ is a constant function. In this case, we may assume that $Q(x) \equiv 1$ and then write simply $\Phi(f; s)$ instead of $\Phi(Q, f; s)$. By [SS, Theorem 1], we see that $\Phi(f; s)$ satisfies a functional equation of the form

$$\Phi(\hat{f}; s) = \gamma(s)\Phi(f; m/2 - s),$$

where $\gamma(s)$ is a meromorphic function with an elementary expression in terms of exponential functions and the gamma function. The function $\gamma(s)$ can easily be determined by using the formula

$$\int_{V'} \left(\det {}^t x x\right)^{s-m/2} e^{-\pi \mathrm{tr}({}^t x x)} dx$$
$$= \pi^{-ns+mn/2} \prod_{i=1}^{n} \Gamma\left(s - \frac{i-1}{2}\right) \Big/ \Gamma\left(\frac{m-i+1}{2} - s\right).$$

Now we consider the general case. It is easy to see that

$$Q(x)\hat{f}(x) = \left(-2\pi\sqrt{-1}\right)^{-\lambda_1 n} \left(Q\left(D_x\right) f\right)^{\wedge}(x).$$

Therefore, by the functional equation for $\lambda = (0, \dots, 0)$, we obtain

$$\begin{aligned}
&\Phi(Q, \hat{f}; s) \\
&\quad = \left(-2\pi\sqrt{-1}\right)^{-\lambda_1 n} \Phi\left(\left(Q\left(D_x\right) f\right)^{\widehat{}}; s - \lambda_1/2\right) \\
&\quad = \left(-2\pi\sqrt{-1}\right)^{-\lambda_1 n} \pi^{-2n(s-\lambda_1/2)+n(m-2)/2} \Phi\left(Q\left(D_x\right) f; \frac{m+\lambda_1}{2} - s\right) \\
&\qquad \times \prod_{i=1}^{n} \sin \pi \left(s - \frac{\lambda_1 + m - i - 1}{2}\right) \Gamma\left(s - \frac{\lambda_1 + i - 1}{2}\right) \\
&\qquad\qquad \times \Gamma\left(s - \frac{\lambda_1 + m - i - 1}{2}\right).
\end{aligned}$$

By integrating by parts, we have from Theorem 3.4 (i)

$$\Phi\left(Q\left(D_x\right) f; \frac{m+\lambda_1}{2} - s\right) = (-1)^{\lambda_1 n} b_\lambda(\lambda_1/2 - s)\, \Phi(Q, f; m/2 - s).$$

This proves the theorem. Q.E.D.

Proof of Theorem 3.1. As in the remark following Proposition 3.2, let f_0 be a function in $C_0^\infty(V')$ and put

$$f(x) = \det\left({}^tD_x D_x\right) f_0(x).$$

For a $Q(x)$ in $\mathcal{R}_\lambda$, take $v, w \in H_{\pi_\lambda}$ such that $Q(v; x) = Q(x)$ and $\overline{Q(w; x)} = Q(x)$. Then, by Proposition 2.3 and Proposition 3.2, the function

$$\zeta(Q, L; s)\Phi(Q, f; s) = \langle Z_\pi(f, L; s)v, w \rangle$$

is an entire function. By integrating by parts, we have from Theorem 3.4 (ii)

$$\Phi(Q, f; s) = \beta_\lambda(s - (\lambda_1 + m)/2)\, \Phi(Q, f_0; s - 1).$$

We can choose an f_0 so that $\Phi(Q, f_0; s - 1) \neq 0$. Hence the function $\beta_\lambda(s - (\lambda_1 + m)/2)\zeta(Q, L; s)$ has an analytic continuation to an entire function. This proves the first part. By the functional equation of $Z_\pi(f, L; s)$ in Proposition 3.2, we obtain

$$\zeta(Q, L; s)\Phi(Q, \hat{f}; s) = v(L)^{-1}\zeta(Q, L^*; m/2 - s)\Phi(Q, f; m/2 - s).$$

Hence it follows from Theorem 3.3 that

$$\begin{aligned}
&v(L)^{-1}\zeta(Q, L^*; m/2-s)\\
&\qquad = \exp\left(\frac{\lambda_1 n\pi\sqrt{-1}}{2}\right)\pi^{-2ns+n(m-2)/2}\zeta(Q, L; s)\\
&\qquad\quad \times \prod_{i=1}^{n}\sin\pi\left(s-\frac{\lambda_1+m-i-1}{2}\right)\\
&\qquad\quad \times \prod_{i=n_0+1}^{n}\Gamma\left(s-\frac{i-\sigma-1}{2}\right)\Gamma\left(s-\frac{m-i+\sigma-1}{2}\right)\\
&\qquad\quad \times \prod_{i=1}^{n_0}\Gamma\left(s+\frac{\lambda_i-i+1}{2}\right)\Gamma\left(s-\frac{\lambda_i+m-i-1}{2}\right).
\end{aligned}$$

Using the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we can easily rewrite the identity above into the form given in Theorem 3.1 (ii). Q.E.D.

§4. Local functional equation and the b-function of $(\mathbf{P}, \rho, \mathbf{V})$

In this section we retain the notation used in Section 1. Consider the standard $\mathbb{R}$-structure of $(\mathbf{P}, \rho, \mathbf{V})$:

$$\mathbf{P}_{\mathbb{R}} = \mathbf{B}(m)_{\mathbb{R}} \times \mathbf{GL}(n)_{\mathbb{R}}, \ \ \mathbf{V}_{\mathbb{R}} = \mathbf{M}(m, n; \mathbb{R}).$$

We identify the vector space dual to $\mathbf{V}$ with $\mathbf{V}$ itself via the symmetric bilinear form

$$(x, y) = \mathrm{tr}({}^t y J x).$$

Then the representation ρ^* of $\mathbf{P}$ contragredient to ρ is given by

$$\rho^*(b, g)y = \rho(b, {}^t g^{-1})y = by\,{}^t g.$$

Then $P_0, \ldots, P_{n_0}$ are relative invariants of $(\mathbf{P}, \rho^*, \mathbf{V})$ and the corresponding rational characters are given by

$$\begin{aligned}
\chi_0^*(b, g) &= \det(g)^2,\\
\chi_i^*(b, g) &= \det(g)^2 \cdot (a_1 \cdots a_i)^{-2} & (1 \le i \le n_0 - 2),\\
\chi_{n_0-1}^*(b, g) &= \begin{cases} \det(g)^2 \cdot (a_1 \cdots a_{n_0-1})^{-2} & \text{if } m \ne 2n,\\ \det(g) \cdot (a_1 \cdots a_{n_0-1})^{-1} \cdot a_{n_0} & \text{if } m = 2n,\end{cases}\\
\chi_{n_0}^*(b, g) &= \det(g) \cdot (a_1 \cdots a_{n_0})^{-1}
\end{aligned}$$

for $g \in \mathbf{GL}(n)$ and

$$b = b_1(A)b_2(v)b_2(B) \in \mathbf{B}(m) \quad \text{with} \quad A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_\kappa \end{pmatrix} \in \mathbf{Trig}(\kappa).$$

Hence we obtain

$$\tag{4.1} \begin{aligned} \chi_0 &= {\chi_0^*}^{-1} \\ \chi_i &= {\chi_0^*}^{-2} \cdot \chi_i^* \qquad (1 \leq i \leq n_0 - 2), \\ \chi_{n_0-1} &= \begin{cases} {\chi_0^*}^{-2} \cdot \chi_{n_0-1}^* & \text{if } m \neq 2n, \\ {\chi_0^*}^{-1} \cdot \chi_{n_0-1}^* & \text{if } m = 2n, \end{cases} \\ \chi_{n_0} &= {\chi_0^*}^{-1} \chi_{n_0}^* \end{aligned}$$

and

$$\tag{4.2} \det \rho(b, g) = \det(g)^{-m} = \chi_0^{m/2}.$$

For $\eta = (\eta_1, \dots, \eta_{n_0}) \in \{\pm 1\}^{n_0}$, put

$$V_\eta = \left\{ x \in \mathbf{V}_\mathbb{R} \;\middle|\; \begin{array}{ll} \operatorname{sgn} P_i(x) = \eta_{i+1} \cdots \eta_{n_0} & (0 \leq i \leq n_0 - 2) \\ \operatorname{sgn} P_{n_0-1}(x) = \begin{cases} \eta_{n_0} & \\ \eta_{n_0} \operatorname{sgn} P_{n_0}(x) & \end{cases} & \begin{array}{l} (m \neq 2n) \\ (m = 2n) \end{array} \end{array} \right\}.$$

We have

$$\mathbf{V}_\mathbb{R} - \mathbf{S}_{\mathbf{P},\mathbb{R}} = \bigcup_{\eta \in \{\pm 1\}^{n_0}} V_\eta.$$

Define the local zeta functions of $(\mathbf{P}, \rho, \mathbf{V})$ by the integrals

$$\begin{aligned} \Phi_\eta^\epsilon(f; s_0, s) &= \Phi_\eta^\epsilon(f; s_0, s_1, \dots, s_{n_0}) \\ &= \int_{V_\eta} |P_0(x)|^{s_0} \prod_{i=1}^{n_0-1} |P_i(x)|^{(s_i - s_{i+1})/2} \\ &\quad \times |P_{n_0}(x)|^{s_{n_0}^*} (\operatorname{sgn} P_{n_0}(x))^\epsilon f(x)\, dx \end{aligned}$$

$(f \in \mathcal{S}(\mathbf{V}_\mathbb{R}), (s_0, s) \in \mathbb{C}^{n_0+1}, \eta \in \{\pm 1\}^{n_0}, \epsilon = 0, 1)$, where $s_{n_0}^* = s_{n_0}$ or $(s_{n_0-1} + s_{n_0})/2$ according as $m \neq 2n$ or $m = 2n$. Then $\Phi_\eta^\epsilon(f; s_0, s)$ are absolutely convergent for

$$\tag{4.3} \Re(s_0) > 0, \Re(s_1) \geq \cdots \geq \Re(s_{n_0-1}) \geq \begin{cases} \Re(s_{n_0}) \geq 0 & (m \neq 2n) \\ |\Re(s_{n_0})| & (m = 2n) \end{cases}$$

and have analytic continuations to meromorphic functions of (s_0, s) in $\mathbb{C}^{n_0+1}$.

Define the Fourier transform $\hat{f}$ of $f \in \mathcal{S}(\mathbf{V}_\mathbb{R})$ by

$$\hat{f}(y) = \int_{\mathbf{V}_\mathbb{R}} f(x) \exp\left(2\pi\sqrt{-1}\,\mathrm{tr}({}^t yJx)\right) dx.$$

Theorem 4.1. *The following functional equations hold for any $f \in \mathcal{S}(\mathbf{V}_\mathbb{R})$:*

$$\Phi^\epsilon_\eta(\hat{f}; s_0, s) = \sum_{\eta^* \in \{\pm 1\}^{n_0}} \Gamma^\epsilon_{\eta,\eta^*}(s_0, s) \Phi^\epsilon_{\eta^*}(f; -m/2 - s_0 - s_1, s)$$

$$(\epsilon = 0, 1,\ \eta \in \{\pm 1\}^{n_0}),$$

where

$$\Gamma^\epsilon_{\eta,\eta^*}(s_0, s) = \left(2\exp\left(\frac{\epsilon\pi\sqrt{-1}}{2}\right)\right)^{n-n_0} (2\pi)^{-n(2s_0+(m+1)/2)-n_0(s_1+1/2)}$$

$$\times \prod_{i=n_0+1}^{n} \sin\pi\left(s_0 + \frac{i+1-\epsilon}{2}\right)\Gamma(2s_0+i)$$

$$\times \prod_{i=1}^{n_0} \Gamma\left(s_0 + \frac{s_1 - s_i}{2} + \frac{i+1}{2}\right)\Gamma\left(s_0 + \frac{s_1+s_i}{2} + \frac{m-i+1}{2}\right)$$

$$\times \sum_{\nu \in \{\pm 1\}^{n_0}} \left(\prod_{i=1}^{n_0} \nu_i\right)^\epsilon \exp\left(\frac{\pi\sqrt{-1}}{2} \cdot L^\nu_{\eta,\eta^*}(s_0, s)\right),$$

$$L^\nu_{\eta,\eta^*}(s_0, s) = \frac{(-1)^{n_0}\delta}{2}\prod_{i=1}^{n_0}\nu_i + \frac{1}{2}\sum_{i=1}^{n_0}\nu_i\left(\sum_{j=1}^{i-1}\eta_j + \sum_{j=i+1}^{n_0}\eta^*_j\right)$$

$$+ \sum_{i=1}^{n_0}\left\{\eta_{n_0-i+1}\,\nu_{n_0-i+1}\left(s_0 + \frac{s_1-s_i}{2} + \frac{i+1}{2}\right)\right.$$

$$\left. + \eta^*_i\nu_i\left(s_0 + \frac{s_1+s_i}{2} + \frac{m-i+1}{2}\right)\right\}$$

and $\delta = 1$ or 0 according as m is odd or even.

For $(\lambda_0, \lambda) \in \Lambda$, let $P_{\lambda_0,\lambda}(D_x)$ be the differential operator with constant coefficients satisfying

$$P_{\lambda_0,\lambda}(D_x)\exp(\mathrm{tr}({}^t xJy)) = P_{\lambda_0,\lambda}(y)\exp(\mathrm{tr}({}^t xJy)).$$

Then there exists a polynomial $b_{\lambda_0,\lambda}(s_0, s)$ satisfying

(4.4)

$$\begin{aligned} P_{\lambda_0,\lambda}(D_x) & \left(P_0(x)^{s_0} \prod_{i=1}^{n_0-1} P_i(x)^{(s_i - s_{i+1})/2} \cdot P_{n_0}(x)^{s^*_{n_0}} \right) \\ & = b_{\lambda_0,\lambda}(s_0, s)\, P_{\lambda_0,\lambda}(x) \\ & \times P_0(x)^{s_0-\lambda_0} \prod_{i=1}^{n_0-1} P_i(x)^{(s_i - s_{i+1})/2} \cdot P_{n_0}(x)^{s^*_{n_0}}, \end{aligned}$$

which is called the *b-function* of $(\mathbf{P}, \rho, \mathbf{V})$ (see e.g. [S1; Lemma 3.1]). By Theorem 4.1 and the expression of the b-function in terms of the coefficients of the local functional equation (cf. [S1;(5-8)], [SO; Lemma 3.1]), we can easily calculate $b_{\lambda_0,\lambda}(s_0, s)$.

Theorem 4.2. *For* $(\lambda_0, \lambda) \in \Lambda$, *we have*

$$\begin{aligned} b_{\lambda_0,\lambda}(s_0, s) = 2^{\lambda_0 n} & \prod_{i=n_0+1}^{n} \prod_{j=1}^{\lambda_0} \left(s_0 + \frac{i-j}{2} \right) \\ & \times \prod_{i=1}^{n_0} \left\{ \prod_{j=1}^{(\lambda_0+\lambda_i)/2} \left(s_0 + \frac{s_1 - s_i}{2} + \frac{i+1}{2} - j \right) \right. \\ & \qquad \left. \times \prod_{j=1}^{(\lambda_0-\lambda_i)/2} \left(s_0 + \frac{s_1 + s_i}{2} + \frac{m-i+1}{2} - j \right) \right\}. \end{aligned}$$

Proof of Theorem 3.4. Since $Q(x) \in \mathcal{R}_\lambda$ is a linear combination of K-translates of $P_{\lambda_1,\lambda}(W^{-1}x)$, we may assume that $Q(x) = P_{\lambda_1,\lambda}(W^{-1}x)$. By (4.4), we have

$$\begin{aligned} b_\lambda(s) &= b_{\lambda_1,\lambda}(s, \overbrace{0, \ldots, 0}^{n_0}), \\ \beta_\lambda(s) &= b_{2,\underline{0}}(s, \lambda_1, \ldots, \lambda_{n_0}), \end{aligned}$$

where $\underline{0} = (0, ..., 0) \in \Lambda^*$. Hence Theorem 3.4 is an immediate consequence of Theorem 4.3. Q.E.D.

Proof of Theorem 4.1. By (4.1), (4.2), [S1; Theorem 1] and [SO; Lemma 2.2], a functional equation of the form

$$\Phi^\epsilon_\eta(\hat{f}; s_0, s) = \sum_{\eta^* \in \{\pm 1\}^{n_0}} \Gamma^\epsilon_{\eta,\eta^*}(s_0, s) \Phi^\epsilon_{\eta^*}(f; -m/2 - s_0 - s_1, s)$$

holds for any $f \in \mathcal{S}(\mathbf{V}_{\mathbb{R}})$. By [SO; Theorem 2], if $m < 2n$, then we have

$$\Gamma^{\epsilon}_{\eta,\eta^*}(s_0, s) = \widetilde{\Gamma}^{\epsilon}_{\eta,\eta^*}(s_0, s) \cdot \prod_{i=m-n}^{n-1} \Gamma_{\mathbb{R},\epsilon}(2s_0 + m - i),$$

where

$$\Gamma_{\mathbb{R},\epsilon}(z) = e^{\epsilon\pi\sqrt{-1}/2}\, \pi^{(1-2z)/2}\, \Gamma\left(\frac{z+\epsilon}{2}\right) \Big/ \Gamma\left(\frac{1-z+\epsilon}{2}\right)$$

and $\widetilde{\Gamma}^{\epsilon}_{\eta,\eta^*}(s_0, s)$ are the coefficients of the functional equation for $(\mathbf{P}^{(m-n)}, \rho^{(m-n)}, \mathbf{V}^{(m-n)})$. Hence it suffices to consider the case $m \geq 2n$. Then $n = n_0$. To calculate the coefficients $\Gamma^{\epsilon}_{\eta,\eta^*}(s_0, s)$, we may assume that $f \in C_0^{\infty}(\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}})$ and (s_0, s) satisfies the condition (4.3). To give a parametric representation of V_η, we introduce the following notation:

$$\begin{aligned} \mathrm{Alt}(n) &= \left\{ A \in \mathbf{M}(n, \mathbb{R}) \mid {}^tA = -A \right\}, \\ \mathrm{Sym}(n)^*_\eta &= \{ T \in \mathbf{M}(n; \mathbb{R}) \mid T = {}^tT, \operatorname{sgn} d_i^*(T) \\ &\qquad\qquad = \prod_{j=n-i+1}^{n} \eta_j \ (1 \leq i \leq n) \}, \end{aligned}$$

where $d_i^*(T)$ is the determinant of the lower right i by i block of T. Any element x in V_η can be written uniquely as follows:

$$x = \begin{pmatrix} T + A - 2^{-1}\,{}^tbb - {}^tD_2 D_1 \\ D_1 \\ b \\ 1_n \\ D_2 \end{pmatrix} C,$$

where $T \in \mathrm{Sym}(n)^*_\eta, A \in \mathrm{Alt}(n), b \in \mathbf{M}(\delta, n; \mathbb{R}), D_1, D_2 \in \mathbf{M}(\kappa - n, n; \mathbb{R}), C \in \mathbf{GL}(n; \mathbb{R})$. Then we have

$$\begin{aligned} P_i(x) &= 2^{n-i} (\det C)^2 d^*_{n-i}(T) \qquad (0 \leq i \leq n-2), \\ P_{n-1}(x) &= 2\, d_1^*(T) \times \begin{cases} (\det C)^2 & (m > 2n), \\ \det C & (m = 2n), \end{cases} \\ P_n(x) &= \det C \end{aligned}$$

and

$$dx = 2^{n(n-1)/2}\, |\det C|^{m-n}\, dT\, dA\, db\, dD_1\, dD_2\, dC.$$

Here dT, dA, db, dD_1, dD_2 and dC are the standard Euclidean measures on the matrix spaces. Hence

$$\begin{aligned}
&\Phi_{\epsilon,\eta}(\hat{f}; s_0, s) \\
&\quad = 2^{\alpha(s_0,s,n)} \int |\det C|^{2s_0+s_1+m-n} \operatorname{sgn}(\det C)^{\epsilon} \\
&\qquad \times |\det T|_\eta^{s_0} \prod_{i=1}^{n-1} |d_{n-i}^*(T)|_\eta^{(s_i-s_{i+1})/2} \hat{f}(x)\, dT\, dA\, db\, dD_1\, dD_2\, dC,
\end{aligned}$$

where

$$|\det T|_\eta = \begin{cases} |\det T| & \left(T \in \mathrm{Sym}(n)_\eta^*\right), \\ 0 & \left(T \notin \mathrm{Sym}(n)_\eta^*\right), \end{cases}$$

$$|d_i^*(T)|_\eta = \begin{cases} |d_i^*(T)| & \left(T \in \mathrm{Sym}(n)_\eta^*\right), \\ 0 & \left(T \notin \mathrm{Sym}(n)_\eta^*\right), \end{cases}$$

and

$$\alpha(s_0, s, n) = ns_0 + \frac{1}{2}\left(ns_1 - \sum_{i=1}^{n} s_i + n(n-1)\right).$$

For

$${}^t x^* = \left(x_1^{*(n)}, x_2^{*(n,\kappa-n)}, y^{*(n,\delta)}, x_3^{*(n)}, x_4^{*(n,\kappa-n)}\right) \qquad (x^* \in \mathbf{V}_{\mathbb{R}}),$$

we have

$$\begin{aligned}
(x^*, x) &= \operatorname{tr}({}^t x^* J x) \\
&= \operatorname{tr}(x_3^*(T+A)C) + \operatorname{tr}(y^* bC) - \operatorname{tr}(\tfrac{1}{2} x_3^*\, {}^t b b\, C) \\
&\quad + \operatorname{tr}(x_4^* D_1 C) + \operatorname{tr}(x_1^* C) + \operatorname{tr}(x_2^* D_2 C) - \operatorname{tr}(x_3^*\, {}^t D_2 D_1 C).
\end{aligned}$$

Put $\mathbf{e}[z] = \exp(2\pi\sqrt{-1}z)$. The Fourier inversion formula applied to the integrals with respect to dx_4^* and dD_1 yields the identity

$$\begin{aligned}
&\Phi_\eta^\epsilon(\hat{f}; s_0, s)\\
&= 2^{\alpha(s_0,s,n)} \int |\det C|^{2s_0+s_1+m-n} \operatorname{sgn}(\det C)^\epsilon\\
&\quad \times |\det T|_\eta^{s_0} \prod_{i=1}^{n-1} |d_{n-i}^*(T)|_\eta^{(s_i-s_{i+1})/2}\, dT\, dA\, db\, dD_2\, dC\\
&\quad \times \int dx_1^*\, dx_2^*\, dy^*\, dx_3^* \int \left\{ \int f(x^*)\mathbf{e}[(x^*, x)] dx_4^* \right\} dD_1\\
&= 2^{\alpha(s_0,s,n)} \int |\det C|^{2s_0+s_1+m-\kappa} \operatorname{sgn}(\det C)^\epsilon\\
&\quad \times |\det T|_\eta^{s_0} \prod_{i=1}^{n-1} |d_{n-i}^*(T)|_\eta^{(s_i-s_{i+1})/2}\, dT\, dA\, db\, dD_2\, dC\\
&\quad \times \int f\left({}^t(x_1^*, x_2^*, y^*, x_3^*, x_3^*\,{}^tD_2)\right)\\
&\quad \times \mathbf{e}\big[\operatorname{tr}(x_3^*(T+A)C) + \operatorname{tr}(y^*bC) - \tfrac{1}{2}\operatorname{tr}(x_3^*({}^tbb)C)\\
&\qquad + \operatorname{tr}(x_1^*C) + \operatorname{tr}(x_2^*D_2C)\big]\, dx_1^*\, dx_2^*\, dy^*\, dx_3^*.
\end{aligned}$$

Changing the variables x_1^*, x_2^*, y^*, C, b into $x_3^*(x_1^* - x_2^*D_2), x_3^*x_2^*, x_3^*y^*, Cx_3^{*-1}, bC^{-1}$, respectively, we obtain

$$\begin{aligned}
&\Phi_\eta^\epsilon(\hat{f}; s_0, s)\\
&= 2^{\alpha(s_0,s,n)} \int |\det C|^{2s_0+s_1+m-\kappa-\delta} \operatorname{sgn}(\det C)^\epsilon\\
&\quad \times |\det T|_\eta^{s_0} \prod_{i=1}^{n-1} |d_{n-i}^*(T)|_\eta^{(s_i-s_{i+1})/2}\, dT\, dA\, dC\\
&\quad \times \int |\det x_3^*|^{-2s_0-s_1-n} \operatorname{sgn}(\det x_3^*)^\epsilon\\
&\qquad\qquad \times \mathbf{e}\big[\operatorname{tr}\big((T+A+x_1^*)C\big)\big]\, dx_1^*\, dx_2^*\, dx_3^*\, dD_2\\
&\quad \times \int \left\{ \int f\left({}^t(x_1^* - x_2^*D_2, x_2^*, y^*, 1_n, {}^tD_2)\,{}^tx_3^*\right) \mathbf{e}[by^*]\, dy^* \right\}\\
&\qquad\qquad \times \mathbf{e}[-\tfrac{1}{2}b\,{}^tC^{-1}\,{}^tb]\, db.
\end{aligned}$$

For a nondegenerate real symmetric matrix Y of size n, it is known that

$$\int_{\mathbb{R}^n} \mathbf{e}[{}^t uYu]\,\mathbf{e}[{}^t uv]\,du$$
$$= 2^{-n/2}\,|\det Y|^{-1/2}\,\mathbf{e}\left[\tfrac{1}{8}\mathrm{sgn}(\det Y)\right]\mathbf{e}\left[-\tfrac{1}{4}{}^t vY^{-1}v\right]$$

(cf. [W; n°14, Theorem 2]). If $\delta = 1$, namely m is odd, then this formula implies that

$$\int\left\{\int f({}^t(x_1^* - x_2^*D_2, x_2^*, y^*, 1_n, {}^tD_2)\,{}^t x_3^*)\mathbf{e}[by^*]\,dy^*\right\}\mathbf{e}\left[-\tfrac{1}{2}b\,{}^tC^{-1}\,{}^t b\right]db$$
$$= |\det C|\,|\det C_s|^{-1/2}\mathbf{e}\left[\tfrac{1}{8}(-1)^n\mathrm{sgn}(\det C_s)\right]$$
$$\times \int f({}^t(x_1^* - x_2^*D_2, x_2^*, y^*, 1_n, {}^tD_2)\,{}^t x_3^*)\mathbf{e}[\tfrac{1}{2}{}^t y^*(CC_s^{-1}\,{}^tC)y^*]dy^*,$$

where $C_s = \frac{1}{2}(C + {}^tC)$. Note that, if m is even, the corresponding integral does not appear. Hence we obtain

$$\Phi_\eta^\epsilon(\hat{f}; s_0, s)$$
$$= 2^{\alpha(s_0,s,n)}\int |\det T|_\eta^{s_0}\prod_{i=1}^{n-1}|d_{n-i}^*(T)|_\eta^{(s_i - s_{i+1})/2}$$
$$\times |\det C|^{2s_0+s_1+m-\kappa}\,\mathrm{sgn}(\det C)^\epsilon |\det C_s|^{-\delta/2}\,\mathbf{e}\left[\tfrac{1}{8}(-1)^n\delta\,\mathrm{sgn}(\det C_s)\right]$$
$$\times\left\{\int |\det x_3^*|^{-2s_0-s_1-n}\,\mathrm{sgn}(\det x_3^*)^\epsilon\,\mathbf{e}\left[\tfrac{1}{2}{}^t y^*(CC_s^{-1}\,{}^tC)y^*\right]\right.$$
$$\left.\times f\left({}^t(x_1^* - x_2^*D_2, x_2^*, y^*, 1_n, {}^tD_2)\,{}^t x_3^*\right)dx_1^*\,dx_2^*\,dx_3^*\,dD_2\,dy^*\right\}$$
$$\times\,\mathbf{e}[\mathrm{tr}((T + A + x_1^*)C)]\,dT\,dA\,dC.$$

Put

$$T^* = \tfrac{1}{2}(x_1^* + {}^t x_1^*),\ A^* = \tfrac{1}{2}(x_1^* - {}^t x_1^*)\ \text{and}\ C_a = \tfrac{1}{2}(C - {}^tC).$$

Then we have

$$\begin{aligned}
&\Phi^{\epsilon}_{\eta}(\hat{f}; s_0, s)\\
&= 2^{\alpha(s_0,s,n)+n(n-1)} \int |\det T|^{s_0}_{\eta} \prod_{i=1}^{n-1} |d^*_{n-i}(T)|^{(s_i-s_{i+1})/2}_{\eta}\\
&\quad \times |\det C|^{2s_0+s_1+m-\kappa} \operatorname{sgn}(\det C)^{\epsilon} |\det C_s|^{-\delta/2}\, \mathbf{e}\left[\tfrac{1}{8}(-1)^n \delta \operatorname{sgn}(\det C_s)\right]\\
&\quad \times \left\{ \int |\det x^*_3|^{-2s_0-s_1-n} \operatorname{sgn}(\det x^*_3)^{\epsilon}\, \mathbf{e}\left[\tfrac{1}{2}{}^t y^*(C_s - C_a C_s^{-1} C_a) y^*\right] \right.\\
&\quad \left. \times f\left({}^t(T^* + A^* - x^*_2 D_2, x^*_2, y^*, 1_n, {}^t D_2){}^t x^*_3\right) dT^*\, dA^*\, dx^*_2\, dx^*_3\, dD_2\, dy^* \right\}\\
&\quad \times \mathbf{e}[\operatorname{tr}\big((T+T^*)C_s\big) + \operatorname{tr}\big((A+A^*)C_a\big)]\, dT\, dA\, dC_s\, dC_a.
\end{aligned}$$

Since

$$\begin{aligned}
&\int |\det C|^{2s_0+s_1+m-\kappa} \operatorname{sgn}(\det C)^{\epsilon}\, \mathbf{e}\left[\tfrac{1}{2}{}^t y^*(C_a C_s^{-1} C_a) y^*\right] dC_a\\
&\quad \iint f({}^t(T^* + A^* - x^*_2 D_2, x^*_2, y^*, 1_n, {}^t D_2){}^t x^*_3)\, \mathbf{e}[\operatorname{tr}\big((A+A^*)C_a\big)]\, dA^*\, dA\\
&\quad = 2^{-n(n-1)/2} |\det C_s|^{2s_0+s_1+m-\kappa} \operatorname{sgn}(\det C_s)^{\epsilon}\\
&\qquad \times \int_{\mathrm{Alt}(n)} f({}^t(T^* + A^* - x^*_2 D_2, x^*_2, y^*, 1_n, {}^t D_2){}^t x^*_3)\, dA^*,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\Phi^{\epsilon}_{\eta}(\hat{f}; s_0, s)\\
&= 2^{\alpha(s_0,s,n)+n(n-1)/2} \int |\det T|^{s_0}_{\eta} \prod_{i=1}^{n-1} |d^*_{n-i}(T)|^{(s_i-s_{i+1})/2}_{\eta}\\
&\quad \times |\det C_s|^{2s_0+s_1+m/2} \operatorname{sgn}(\det C_s)^{\epsilon}\, \mathbf{e}\left[\tfrac{1}{8}(-1)^n \delta \operatorname{sgn}(\det C_s)\right]\\
&\quad \times \left\{ \int |\det x^*_3|^{-2s_0-s_1-n} \operatorname{sgn}(\det x^*_3)^{\epsilon}\, \mathbf{e}\left[\tfrac{1}{2}{}^t y^* C_s y^*\right] \right.\\
&\quad \left. \times f\left({}^t(T^* + A^* - x^*_2 D_2, x^*_2, y^*, 1_n, {}^t D_2){}^t x^*_3\right) dT^*\, dA^*\, dx^*_2\, dx^*_3\, dD_2\, dy^* \right\}\\
&\qquad\qquad \times \mathbf{e}[\operatorname{tr}((T+T^*)C_s)]\, dT\, dC_s.
\end{aligned}$$

For $\nu = (\nu_1, \dots, \nu_n) \in \{\pm 1\}^n$, put

$$\mathrm{Sym}(n)_\nu$$
$$= \left\{T \in \mathbf{M}(n;\mathbb{R}) \,|\, T = {}^tT,\ \operatorname{sgn} d_i(T) = \nu_1 \cdots \nu_i \quad (1 \le i \le n)\right\},$$

where $d_i(T)$ is the determinant of the upper left i by i block of T. Also put

$$\Gamma_{\mathbb{R}}(z;\epsilon) = (2\pi)^{-z}\mathbf{e}[\epsilon z/4]\Gamma(z).$$

Lemma 4.3 ([S3;Theorem 3.2]). *We have*

$$\int_{\mathrm{Sym}(n)^*_\eta} \prod_{i=1}^n |d_i^*(T)|^{z_i} \hat{f}(T)\, dT$$
$$= \sum_{\nu\in\{\pm 1\}^n} \gamma_{\eta,\nu}(z) \int_{\mathrm{Sym}(n)_\eta} \prod_{i=1}^n |d_i(T^*)|^{z_i^*} f(T^*)\, dT^*,$$

where

$$\hat{f}(T) = \int f(T^*)\, \mathbf{e}[\mathrm{tr}(TT^*)]\, dT^*,$$

$$z^* = (z_1^*, \dots, z_n^*) = (z_{n-1}, \dots, z_1, -(n+1)/2 - z_1 - \cdots - z_n)$$

and

$$\gamma_{\eta,\nu}(z) = 2^{-n(n-1)/4}\, \mathbf{e}\left[\frac{1}{8}\sum_{1\le i<j\le n} \eta_i\nu_j\right]$$
$$\times \prod_{i=1}^n \Gamma_{\mathbb{R}}(z_i + \cdots + z_n + \frac{n-i+2}{2}; \eta_i\nu_i).$$

Using the lemma above twice, we obtain

$$\begin{aligned}
&\Phi_\eta^\epsilon(\hat{f}; s_0, s) \\
&= 2^{\alpha(s_0,s,n)} \sum_{\nu\in\{\pm1\}^n} \left(\prod_{i=1}^n \nu_i\right)^\epsilon \mathbf{e}\left[\frac{(-1)^n\delta}{8}\prod_{i=1}^n \nu_i\right] \\
&\quad \times \gamma_{\eta,\nu}\left(\frac{s_{n-1}-s_n}{2}, \ldots, \frac{s_1-s_2}{2}, s_0\right) \\
&\quad \times \int_{\mathrm{Sym}(n)_\nu} |\det C_s|^{s_0+(s_1+s_n+m-n-1)/2} \prod_{i=1}^{n-1} |d_i(C_s)|^{(s_i-s_{i+1})/2} \\
&\qquad \left\{\int |\det x_3^*|^{-2s_0-s_1-m}\mathrm{sgn}(\det x_3^*)^\epsilon f(x^*)\,dx^*\right\} \mathbf{e}[\mathrm{tr}(T^*C_s)]\,dC_s \\
&= 2^{\alpha(s_0,s,n)} \sum_{\eta^*\in\{\pm1\}^n} \sum_{\nu\in\{\pm1\}^n} \left(\prod_{i=1}^n \nu_i\right)^\epsilon \mathbf{e}\left[\frac{(-1)^n\delta}{8}\prod_{i=1}^n \nu_i\right] \\
&\quad \times \gamma_{\eta,\nu}\left(\frac{s_{n-1}-s_n}{2}, \ldots, \frac{s_1-s_2}{2}, s_0\right) \\
&\quad \times \gamma_{\nu,\eta^*}\left(\frac{s_1-s_2}{2}, \ldots, \frac{s_{n-1}-s_n}{2}, s_0 + \frac{s_1+s_n+m-n-1}{2}\right) \\
&\quad \times \int |\det T^*|_{\eta^*}^{-s_0-s_1-m/2} \prod_{i=1}^{n-1} |d_{n-i}^*(T^*)|_\eta^{(s_i-s_{i+1})/2} \\
&\qquad\qquad \times |\det x_3^*|^{-2s_0-s_1-m}\,\mathrm{sgn}(\det x_3^*)^\epsilon\, f(x^*)\,dx^*.
\end{aligned}$$

It is easy to check that this identity coincides with the functional equation in Theorem 4.1. Q.E.D.

Remark. One can obtain an analogous result to Theorem 4.1 over p-adic number fields. In the case $n = 1$, Theorem 4.1 as well as its analogue over p-adic number fields has been proved in [S3; Theorem 3.6].

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