

Spectral Zeta Functions

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Abstract.

This article gives a survey of various generalizations of Riemann's ζ -function, associated with operator spectra and which may be generically called spectral zeta functions. Areas of application include Riemannian geometry (the spectrum of the Laplacian) and quantum mechanics. We review one example of each class in concrete detail: the Laplacian on a compact surface of constant negative curvature, and the Schrödinger operator on the real line with a homogeneous potential q^{2M} (M a positive integer).

§1. Introduction

This review is concerned with spectral theory for linear operators of the following specific type: partial differential operators on a (complex) Hilbert space, which are moreover *positive* (as operators) and have a *compact resolvent*. The spectrum of such an operator has the form

$$(1.1) \quad 0 \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k \leq \dots, \quad \lambda_k \uparrow +\infty$$

(eigenvalues being counted with their multiplicities).

Examples include the Laplacian ($-\Delta$) on a compact Riemannian manifold, and the Schrödinger operator of a bound system in quantum mechanics (in this case, the spectrum gives the set of discrete, or *quantized*, energy values allowed to the system).

Like the roots of an algebraic equation, the eigenvalues are more easily approached through symmetric functions associated with them, or *spectral functions*.

Several such generating functions are built for dynamical systems by borrowing specific features from the Riemann zeta function $\zeta(s)$. All such functions tend to be indistinctly called zeta functions, although they ultimately have nothing in common.

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- an *Euler product* defines the *Ruelle zeta function* of a *classical* (i.e., non-quantal) dynamical system,

$$(1.2) \quad R(\sigma) = \prod_{\{p\}} \left(1 - e^{-\sigma\tau(p)}\right)$$

(product over the primitive periodic orbits p , $\tau(p)$ being the length of p); Euler products do not *directly* appear in spectral theory, but the zeta function (1.2) can emerge in the limit of *classical mechanics* for quantum eigenvalue problems (see Section 4);

- the *Hadamard product* form of $\zeta(s)$ (over the Riemann zeros) is copied in the definition of *functional determinants* (Section 3);

- the *Dirichlet series* inspires the *Minakshisundaram-Pleijel (M-P) zeta function*,

$$(1.3) \quad Z(s) = \sum_{k=0}^{\infty} \lambda_k^{-s},$$

which is the spectral zeta function *par excellence*.

Our main concern will be to examine how the classical properties of Riemann's zeta function $\zeta(s)$ carry over (or not) to the various spectral functions modelled upon $\zeta(s)$; or: what is the *spectral* (rather than purely *arithmetical*) content of Riemann's function?

For the purpose of the discussion, we shall divide the elementary properties of $\zeta(s)$ into three classes of (mildly) increasing difficulty:

- a) the meromorphic structure of $\zeta(s)$ and the values $\zeta(-n)$, $n = 0, 1, 2, \dots$;
- b) the values of $\zeta'(0)$ and $\zeta(2n)$, $n = 1, 2, \dots$ (*special values*, in brief);
- c) Riemann's functional equation.

After recalling the basic techniques illustrated upon $\zeta(s)$ (Section 2) and reviewing functional determinants (Section 3), we shall arrive at the core of the paper, namely the specific treatment of two very different classes of differential operators: the Laplacian on a compact surface of constant negative curvature, involving Selberg's zeta function (Section 4), and the Schrödinger operator $(-d^2/dq^2 + q^{2M})$ on the real line (Section 5). Section 6 gives a concluding discussion.

We will use the standard notations of special function handbooks [1].

The spirit of this survey will be to emphasize the main ideas on a global level at the expense of technicalities, even though these are quite important; we refer the reader to our earlier papers for more detailed discussions and references.

§2. Basic techniques

2.1. General spectral functions [2]

Spectral functions are symmetric functions of the eigenvalues together with an auxiliary continuous variable. Substantial flexibility is achieved by building spectral functions not only over the original spectrum $\{\lambda_k\}$, but also over some *distorted* spectra $\{\rho_k\}$, of the form

$$(2.1) \quad \rho_k = \rho(\lambda_k), \quad \text{with } \rho_k \text{ deleted if } \rho_k = 0,$$

where $\rho(\lambda)$ is some smooth function such that $\rho(\lambda) \uparrow +\infty$ for $\lambda \uparrow +\infty$. For a fixed operator, different distortions of its spectrum will disclose inequivalent, complementary spectral properties; accordingly, we keep separate in our notation the basic operator and its distortions. At the start, one obviously uses $\rho_k \equiv \lambda_k$ (always *removing zero values*), but we shall also resort to some quite contrived (at first sight) distortions!

The basic *types* of spectral functions are then Theta, Zeta and Det (determinant),

$$(2.2) \quad \text{Theta}(t) = \sum_k e^{-t\rho_k},$$

$$(2.3) \quad \text{Zeta}(s) = \sum_k \rho_k^{-s},$$

$$(2.4) \quad \text{Det}(\rho) = \prod_k^* (\rho_k + \rho)$$

(\prod^* denoting a convergent Hadamard product, obtained by zeta-regularization of the divergent expression $\prod (\rho_k + \rho)$, see Section 3).

The fundamental *transformations* linking the three types are, formally,

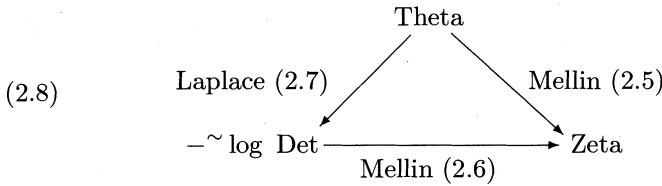
$$(2.5) \quad \text{Zeta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Theta}(t) t^{s-1} dt$$

$$(2.6) \quad = \frac{s}{\pi} \sin \pi s \int_0^\infty \sim \log \text{Det}(\rho) \rho^{-s-1} d\rho,$$

$$(2.7) \quad -\sim \log \text{Det}(\rho) = \int_0^\infty \sim \frac{\text{Theta}(t)}{t} e^{-t\rho} dt.$$

The symbols \sim and \sim denote *regularizations*: we had to subtract the terms causing divergences, at $\rho \rightarrow +\infty$ and $t \rightarrow 0^+$ respectively; moreover, restrictions apply to the domains of validity of the formulae (details

in the references [2]). We summarize by a diagram:



By experience, spectral information is most pregnant in *Theta-type functions*, hence these play the main initial role. Theta-type functions may carry information of two very distinct types. One type is exhibited by the heat operator techniques, and the other will be illustrated through the genesis of the functional equation for $\zeta(s)$.

2.2. Heat operator techniques

The basic Theta-type function is the *trace of the heat operator*,

(2.9)

$$\theta(t) = \sum_k e^{-t\lambda_k}.$$

If the operator under study is either a differential operator on a compact manifold, or a Schrödinger operator with a positive polynomial potential, then $\theta(t)$ admits an asymptotic series

(2.10)

$$\theta(t) \sim \sum_{n=0}^{\infty} c_{i_n} t^{i_n} \quad (t \rightarrow 0^+)$$

with $i_0 < 0$. This expansion is computable in principle, so it expresses the *analytic structure of $\theta(t)$ in the small ($t \rightarrow 0$)*.

Information like Eq. (2.10) about a Theta-type function is then transferable in two directions (cf. diagram (2.8)):

- towards $\log \text{ Det}(\lambda)$, by Laplace transformation: to the series (2.10) there corresponds term by term an *asymptotic power expansion* of $\log \text{ Det}(\lambda)$ for $\lambda \rightarrow +\infty$; the leading, divergent terms of this series are subtracted to get the finite part $\sim \log \text{ Det}(\lambda)$, used in Eqs. (2.6-7);

- towards $\text{Zeta}(s)$, by Mellin transformation: it follows that $\int_0^\infty \text{Theta}(t)t^{s-1}dt (= \Gamma(s)\text{Zeta}(s))$ has a meromorphic extension to the whole s plane with only simple poles, one per power of t in the expansion like (2.10).

Specifically, now, to $\theta(t)$ in Eq. (2.9) corresponds the *M-P zeta function* $Z(s)$ of Eq. (1.3),

$$(2.11) \quad \begin{cases} Z(s) = \Gamma(s)^{-1} \eta(s), \\ \eta(s) = \int_0^\infty \theta(t) t^{s-1} dt \quad (\operatorname{Re} s > -i_0). \end{cases}$$

The expansion (2.10) implies that $\eta(s)$ is meromorphic in all of \mathbb{C} and has only simple poles, located at $s = -i_n$ with residues c_{i_n} . Hence,

(i) $Z(s)$ is *meromorphic in the whole plane*, and has only *simple poles*, located at $s = -i_n$ with residues $c_{i_n}/\Gamma(i_n)$ (a zero residue meaning a regular point);

(ii) $s = 0, -1, -2, \dots$ are always regular points, and

$$(2.12) \quad \begin{cases} Z(-m) = (-1)^m m! c_m & (m \in \mathbb{N}), \\ c_m = \text{coefficient of } t^m & \text{in the expansion (2.10)}. \end{cases}$$

Such closed expressions for $Z(-m)$ are called *trace identities*.

Thus, the results of class a) have found immediate analogs for the M-P zeta function, consisting of its explicit meromorphic structure (i) and its trace identities (ii).

2.3. The functional equation for $\zeta(s)$ [3]

One definition of the Riemann zeta function is *privileged* from the standpoint of spectral theory. It is the following Mellin representation, (which comes in two related forms, cf. [1] and Fig. 1),

$$(2.13) \quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \Theta(\tau) \tau^{s-1} d\tau,$$

$$(2.13') \quad \zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_C \Theta(\tau) (-\tau)^{s-1} d\tau,$$

with

$$(2.14) \quad \Theta(\tau) = \sum_{k=1}^{\infty} e^{-k\tau} = (e^\tau - 1)^{-1}.$$

In other words, $\zeta(s)$ is the M-P zeta function of the *integer spectrum* $\{k\} = \{1, 2, \dots\}$.

Thus, starting from the classical expansion

$$(2.15) \quad \Theta(\tau) = \sum_{n=0}^{\infty} B_n \tau^{n-1} / n!,$$

Eq. (2.13) generates the well-known meromorphic structure of $\zeta(s)$ and the trace identities (2.12), which here read as

$$(2.16) \quad \zeta(-m) = (-1)^m B_{m+1}/(m+1).$$

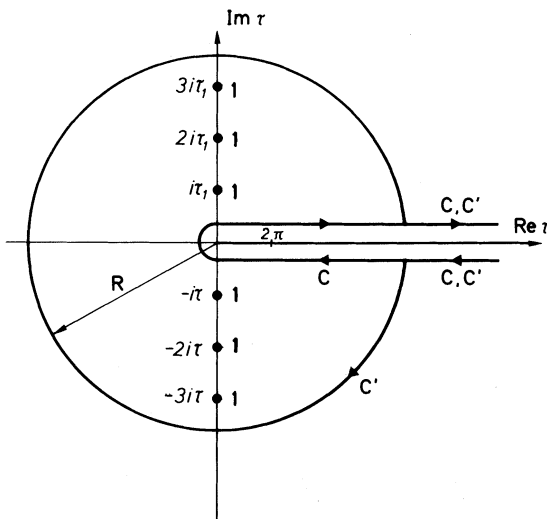


Fig. 1. The Mellin representations of $\zeta(s)$, Eq. (2.13'). The dots are the poles of $\Theta(\tau) = (e^\tau - 1)^{-1}$ and the boldface numbers are residues which contribute to $\zeta(s)$.

Now, the functional equation itself follows most readily from Eq. (2.13'). For $\text{Re } s < 0$ we may inflate the contour C away from $s = 0$ (as C' on Fig. 1); then, as the radius R of C' is increased, the integral over C' tends to zero. Consequently, the value of the integral in (2.13') results entirely from the contributions of the (non-zero) poles of $\Theta(\tau)$. If we denote these poles as $\pm i\tau_r$, with

$$(2.17) \quad \tau_r = r\tau_1, \quad \tau_1 = 2\pi; \quad r = 1, 2, \dots,$$

then the residue calculus gives

$$(2.18) \quad \zeta(s) = \Gamma(1-s) \sum_{r=1}^{\infty} \left[(i\tau_r)^{s-1} + (-i\tau_r)^{s-1} \right]$$

$$(2.19) \quad \begin{aligned} &= \Gamma(1-s) 2 \sin \frac{\pi s}{2} \sum_{r=1}^{\infty} (2\pi r)^{s-1} \\ &= 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2} \zeta(1-s), \end{aligned}$$

Q.E.D.

We now stress that this output was generated *entirely by the polar decomposition* of $\Theta(\tau)$. The key ingredient was thus the *analytic structure in the large* of $\Theta(\tau)$, and precisely the *singular part* of $\Theta(\tau)$. A generalization of this property is expressed in the *Poisson formula for a compact Riemannian manifold* M , which computes the real singularities of a Theta-type function, $\Theta(it) = \text{Tr} \exp(-it\sqrt{-\Delta_M})$, and finds them at the *lengths of periodic geodesics* (i.e., in the *length spectrum* of the manifold) [4]. The particular case (2.14) corresponds to the manifold $S^1 : \{k\}$ is the spectrum of $\sqrt{-\Delta_{S^1}}$ while Eq. (2.17) gives the length spectrum of the circle. From this general point of view, the right-hand side of Eq. (2.18) arises as a *zeta function over the length spectrum*.

However, this case is also *very special* in the sense that the general Poisson formula for manifolds is replaced here by the stronger classical *Poisson summation formula*. Now, the operator spectrum on one side and the length spectrum on the other are *identical* (up to a scale factor), making the Poisson formula *self-reciprocal*. A coincidence of *this nature* must be viewed as totally accidental from the perspective of general operator theory, hence non-generalizable. On the other hand, this coincidence is what causes the *same* ζ -function to emerge on both sides of Eq. (2.19), whence the *functional equation*.

All this suggests the following attitudes towards the properties of $\zeta(s)$ still awaiting generalization:

- class b) (special values): although $\zeta'(0)$ and $\zeta(2n)$ are directly given by the functional equation, we should seek an alternative, more general computation method;

- class c) (functional equation for $\zeta(s)$): consider this as a particular *manifestation of the singular decomposition of $\Theta(\tau)$* ; then, use only this broader concept as a starting point for generalizations, instead of the functional equation itself.

In the next section we shall achieve both goals indeed, within the framework of pure spectral theory.

§3. Functional determinants

3.1. General notions [2]

A useful preliminary step is the introduction of a “two-prong” function,

$$(3.1) \quad Z(s, \lambda) = \sum_{k=0}^{\infty} (\lambda_k + \lambda)^{-s} \quad (\lambda \notin (-\infty, -\lambda_0]),$$

analogous to the generalized (Hurwitz) zeta function of the integer spectrum [1b],

$$(3.2) \quad \zeta(s, \lambda) = \sum_{k=0}^{\infty} (k + \lambda)^{-s}.$$

As a function of s , $Z(s, \lambda)$ is the M-P zeta function for a shifted spectrum; hence

$$(3.3) \quad Z(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^{\infty} \theta(t) e^{-\lambda t} t^{s-1} dt$$

and its meromorphic structure and trace identities follow as before.

The new feature in $Z(s, \lambda)$ is its λ -dependence, embodied in a functional relation,

$$(3.4) \quad Z(s, \lambda) = s \int_{\lambda}^{+\infty} Z(s + 1, \lambda') d\lambda' \quad (\text{Re } s > -i_0),$$

which constitutes an *effective* tool for the analytic continuation in s . In particular, repeated uses of Eq. (3.4) establish the following results.

The function $D(\lambda)$, defined by

$$(3.5) \quad D(\lambda) = \exp[-Z'_s(0, \lambda)],$$

has a convergent Hadamard product expansion with the monomial factors $(1 + \lambda/\lambda_k)$. (It is thus a Det-type function for the spectrum $\{\lambda_k\}$: the *zeta-regularized*, or *functional determinant*). Moreover,

$$(3.6) \quad \begin{cases} \log D(\lambda) = -Z'(0) - \sum_{n=1}^{\infty} (-1)^n \frac{\text{FP}' Z(n)}{n} \lambda^n \\ \text{FP}' Z(n) = \text{Finite Part } [Z(s)]_{s=n} \text{ [+ anomalous term} \\ \hspace{15em} \text{if } n \geq 2 \text{ and } c_{-n} \neq 0 \end{cases}$$

and, following from Eqs. (3.3) and (2.7),

$$(3.7) \quad -\sim \log D(\lambda) = \int_0^\infty \sim \frac{\theta(t)}{t} e^{-\lambda t} dt$$

3.2. The Riemann case

For the spectrum of integers, the determinant formulae become explicitly

$$(3.8) \quad \log D(\lambda) = -\zeta'(0) + \gamma\lambda - \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)}{n} \lambda^n,$$

$$(3.9) \quad -\sim \log D(\lambda) = \int_0^\infty \sim \frac{\Theta(\tau)}{\tau} e^{-\lambda\tau} d\tau,$$

with

$$(3.10) \quad \sim \log D(\lambda) = \log D(\lambda) + \lambda(\log \lambda - 1) + \frac{1}{2} \log \lambda,$$

$$(3.11) \quad \sim \Theta(\tau) = \frac{1}{e^\tau - 1} - \frac{1}{\tau} + \frac{1}{2}.$$

Now, we can exhibit *another* manifestation of the *singular structure* of $\Theta(\tau)$, this time upon the Laplace transform (3.9) (the functional equation for $\zeta(s)$ was a manifestation upon the Mellin transform (2.13')). A contribution of the singular part of $\Theta(\tau)$ has to be a *contour* integral in order to lend itself to a residue calculation. To achieve this starting from (3.9), we perform coupled rotations of both the argument λ and the integration path, in their respective planes, and both ways (Fig. 2):

$$(3.12) \quad -\sim \log D(-i\lambda) = \int_L \sim \frac{\Theta(\tau)}{\tau} e^{+i\lambda\tau} d\tau,$$

$$(3.13) \quad -\sim \log D(+i\lambda) = \int_{L'} \sim \frac{\Theta(\tau)}{\tau} e^{-i\lambda\tau} d\tau \\ = - \int_{L''} \sim \frac{\Theta(\tau)}{\tau} e^{i\lambda\tau} d\tau$$

(we have exploited the *parity* of $\sim \Theta(\tau)$, actually *not* an accidental property). Since the integrands have been regularized at $\tau = 0$, the contribution (3.12)+(3.13) is a *contour integral* around the upper poles of $\Theta(\tau)$,

$$(3.14) \quad \sim \log D(i\lambda) + \sim \log D(-i\lambda) = - \int_C \sim \frac{\Theta(\tau)}{\tau} e^{i\lambda\tau} d\tau, \quad C = L - L'';$$

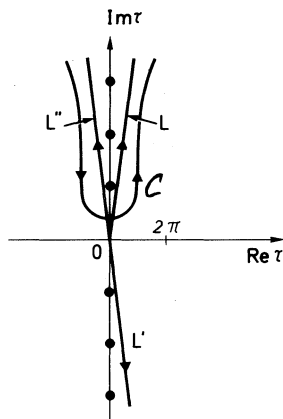


Fig. 2. The Laplace transformation paths in the plane of the function $\Theta(\tau) = (e^\tau - 1)^{-1}$, as used in Eqs. (3.12–3.14).

it is thus a quantity solely dependent on the singular structure of $\Theta(\tau)$. We compute the right-hand side by the method of residues, using $\tau_1 = 2\pi$ and Eq. (3.10), exponentiate the result and obtain

$$(3.15) \quad \begin{aligned} D(i\lambda)D(-i\lambda) &= e^{\pi\lambda}(1 - e^{\tau_1\lambda})/\lambda \\ &= 2 \frac{\sinh \pi\lambda}{\lambda}, \end{aligned}$$

i.e., a functional equation for the determinant.

After that, expanding the logarithms of both sides with the help of Eq. (3.8), we find

$$(3.16) \quad 2 \left[-\zeta'(0) - \sum_{n=1}^{\infty} (-1)^n \zeta \left(\frac{2n}{2n} \right) \lambda^{2n} \right] \equiv \log 2\pi + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} \frac{B_{2n}}{2n} \lambda^{2n};$$

thus, the special values $\zeta'(0)$ and $\zeta(2n)$ have now come out of the *determinantal* functional equation (3.15).

Finally, the functional determinant for the integer spectrum has the explicit expression [1b]

$$(3.17) \quad D(\lambda) = \sqrt{2\pi}/\Gamma(1 + \lambda),$$

hence its functional equation is none other than the *reflection formula* for $\Gamma(z)$. Our point, however, is that we never came to invoke any *particular*

property of the special function $\Gamma(z)$ (such as its other functional relation $\Gamma(1+z) = z\Gamma(z)$, which certainly reflects the regularity of the integers). All our arguments were of a purely “spectral” nature, and as such they will generalize. Up to minor qualifications, the following final picture will result in general:

$$(3.18) \quad \left\{ \begin{array}{l} \text{Certain } \Theta\text{-functions have a remarkable global analytic structure.} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad \text{Functional determinants satisfy functional relations.} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad \text{Special zeta values obey algebraic identities.} \end{array} \right.$$

The central difficulty in the whole scheme will be to implement the first statement in each case, *i.e.*, to identify an *adequate* Theta-type function $\Theta(\tau)$ offering access to the details of its analytic structure; in general it will be much richer than meromorphic: *resurgent*, which means indefinitely ramified, with implicit (*analytic bootstrap*) relations linking its discontinuity functions.

§4. The spectrum of a compact hyperbolic surface

4.1. The Selberg trace formula [5]

Let X be a compact surface of constant negative curvature (-1) and genus $g (\geq 2)$. The spectrum $\{\lambda_k\}$ of its Laplacian $-\Delta_X$ is very special; although it is unknown, it is completely determined through one closed relation, the *Selberg trace formula*. We can then directly resort to this trace formula to construct and make explicit the spectral functions for $-\Delta_X$.

The structure of the trace formula favors the auxiliary spectrum

$$(4.1) \quad \rho_k = \sqrt{\lambda_k - 1/4},$$

and for simplicity we shall assume that no λ_k equals $1/4$. The Selberg trace formula, with a general test function $h(\rho)$ and its Fourier transform $\hat{h}(\tau) = (2\pi)^{-1} \int_{-\infty}^{+\infty} h(\rho) e^{-i\tau\rho} d\rho$, reads as

$$(4.2) \quad \left\{ \begin{aligned} \sum_{k=0}^{\infty} h(\rho_k) &= (g-1) \int_{-\infty}^{+\infty} h(\rho) \rho \tanh \pi \rho \, d\rho + \sum_{\{p\}} \sum_{r=1}^{\infty} W_{p,r} \hat{h}(r\tau(p)), \\ W_{p,r} &= \frac{\tau(p)}{2 \sinh[r\tau(p)/2]}, \end{aligned} \right.$$

where the summation $\sum_{\{p\}}$ runs over the *primitive periodic geodesics* p on X , and $\tau(p)$ denotes the *length* of p . Convergence primarily demands that $h(\rho)$ be an *even function analytic in a strip* $|\operatorname{Im} \rho| < 1/2 + \varepsilon$.

For instance, the choice $h(\rho) = e^{-t\rho^2}$, $\hat{h}(\tau) = \frac{1}{2}(\pi t)^{-1/2} e^{-\tau^2/4t}$, leads to the Theta-type function

$$(4.3) \quad \theta_X(t) = \sum_{k=0}^{\infty} e^{-t\rho_k^2}.$$

Then the right-hand side of the trace formula immediately reveals the full $t \downarrow 0$ asymptotic expansion of $\theta_X(t)$ (only the integral term contributes),

$$(4.4) \quad \theta_X(t) \sim (g-1) \sum_{n=0}^{\infty} \frac{(2^{1-2n} - 1)}{n!} B_{2n} t^{n-1}.$$

The usual heat trace $\sum e^{-t\lambda_k} = e^{-t/4} \theta_X(t)$ will have a more involved expansion than $\theta_X(t)$ itself, suggesting that $\{\lambda_k - 1/4\}$ is a more basic spectrum than $\{\lambda_k\}$.

Accordingly, we would like to discard the usual M-P zeta function [6],

$$(4.5) \quad Z_X(s) = \sum_k \lambda_k^{-s} \quad (\operatorname{Re} s > 1),$$

in favor of the Mellin transform of $\theta_X(t)$, formally

$$(4.6) \quad Z_X(s, -1/4) = \sum_k (\lambda_k - 1/4)^{-s} = \sum_k \rho_k^{-2s};$$

but the presence of *low eigenvalues* (i.e., $\lambda_k < 1/4$) makes this ill-defined. The Selberg trace formula can be of no help here, since the function $h(\rho) = |\rho|^{-2s}$ grossly violates the analyticity condition.

By contrast, in the “mirror” case of the sphere S^2 (constant curvature +1), the analogous shift $\lambda_k \rightarrow \lambda_k + 1/4$ is permitted and simplifies

the spectral analysis. We have

$$(4.7) \quad \theta_{S^2}(t) = \sum_{\ell=0}^{\infty} (2\ell + 1) e^{-t(\ell+1/2)^2} \sim \sum_{n=0}^{\infty} (-1)^n \frac{(2^{1-2n} - 1)}{n!} B_{2n} t^{n-1}$$

(compare with Eq. (4.4)), and $Z_{S^2}(s, +1/4)$ is expressible in closed form:

$$(4.8) \quad Z_{S^2}(s, +1/4) = \sum_{\ell=0}^{\infty} (2\ell + 1)(\ell + 1/2)^{-2s} = (2^{2s} - 2)\zeta(2s - 1).$$

We shall consequently attempt again to define Eq. (4.6) properly, later.

Another choice of test function, $h(\rho) = \cos t\rho$, displays the content of the Poisson formula for the manifold X , similarly to Section 2:

$$(4.9) \quad \sum_k \cos t\rho_k = \frac{(1-g) \cosh t/2}{2 \sinh^2 t/2} + \sum_{\{p\}} \sum_{r=1}^{\infty} \frac{W_{p,r}}{2} [\delta(t - r\tau(p)) + \delta(t + r\tau(p))].$$

The essential feature is the explicit, isolated τ -singularities of the right-hand side, located in the length spectrum. Here, moreover, the singularities are elementary, thanks to the special choice of distorted spectrum $\{\rho_k\}$ instead of $\sqrt{\lambda_k}$ as in Section 2.

All this suggests that a Theta function of interest should be

$$(4.10) \quad \Theta_X(\tau) = \sum_k e^{-\tau\rho_k}.$$

However, Eq. (4.10) is ill-defined for the same reasons as Eq. (4.6). Fortunately, we can find a different trace formula which will allow and determine singular spectral functions like (4.6) and (4.10).

4.2. The sectorial trace formula [7]

Suppose instead that $h(\rho)$ is analytic in a sector $|\text{Arg } \rho| < \pi/2 + \varepsilon$ (with mild decay conditions at 0 and ∞); then,

$$(4.11) \quad \sum_k h(\rho_k) = 2(g-1) \int_0^{+\infty} h(\rho)\rho \tanh \pi\rho \, d\rho + \int_0^{\infty} \left[\frac{h(-i\kappa) - h(+i\kappa)}{2\pi i} \right] d \log \mathcal{Z}_X \left(\frac{1}{2} + \kappa \right),$$

where

$$(4.12) \quad \mathcal{Z}_X(\sigma) = \prod_{\{p\}} \sum_{m=0}^{\infty} \left(1 - e^{-(\sigma+m)\tau(p)} \right)$$

defines the *Selberg zeta function*; $\mathcal{Z}_X(1/2 + \kappa)$ vanishes at $\pm i\rho_k$, and the second integration path must avoid those zeros which lie in the interval $[0, 1/2]$; it can do so by the right (or left), in connection with the choice of each corresponding $\text{Arg } \rho_k$ in the left-hand side of (4.11), $-\pi/2$ (or $+\pi/2$).

With such a choice made, we can define and compute unambiguously a function $\Theta_X(\tau)$ as in Eq. (4.10). Putting $h(\rho) = e^{-\tau\rho}$ in the sectorial trace formula, we obtain

$$(4.13) \quad \Theta_X(\tau) = \sum_k e^{-\tau\rho_k} = (2 - 2g) \left[\frac{1}{\tau^2} + 2 \sum_{r'=1}^{\infty} \frac{(-1)^{r'}}{(\tau + 2\pi i r')^2} \right] + \frac{1}{\pi} \int_0^{\infty} \sin \tau \kappa \, d \log \mathcal{Z}_X \left(\frac{1}{2} + \kappa \right).$$

The right-hand side then reveals the global analytic structure of $\Theta_X(\tau)$ (Fig. 3). This function is *meromorphic*, and has:

- α) double poles at $\tau = -2\pi r'$, with principal part coefficients $4(1 - g)(-1)^{r'}$ (halved for $\tau = 0$);
- β) simple poles at $\tau = \pm i r \tau(p)$, with residues $W_{p,\tau}/2\pi$;
- γ) a functional equation,

$$(4.14) \quad \Theta_X(\tau) + \Theta_X(-\tau) = (g - 1) \frac{\cos \tau/2}{\sin^2 \tau/2}.$$

Like the function $\Theta(\tau) = (e^\tau - 1)^{-1}$ used for $\zeta(s)$, the function $\Theta_X(\tau)$ has a nice *global meromorphic structure*. We can therefore perform the exact analogs of the contour integrations made in Sections 2-3, in order to exhibit consequences of this global structure.

On the other hand, the *local analytic structure* (or *complete expansion* at $\tau = 0$) is no longer available for $\Theta_X(\tau)$, but only for the *other* Theta-type function $\theta_X(t)$, in Eqs. (4.3-4). The information which for $\zeta(s)$ was carried by the *single* function $\Theta(\tau)$ is now distributed among *two distinct functions*.

4.3. The functional determinant [8,2,9]

We first describe the effect of the poles of $\Theta_X(\tau)$ upon its Laplace transform. Manipulations similar to those leading from Eq. (3.9) to

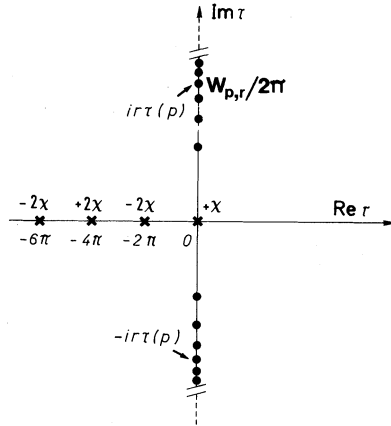


Fig. 3. The analytic structure of $\Theta_X(\tau)$ for a compact hyperbolic surface X . Dots are simple poles and crosses are double poles. The boldface values are the coefficients of the principal pole terms; for the simple poles they are the residues $W_{p,r}/2\pi$; for the double poles, the notation is $\chi = 2 - 2g$.

(3.15) give here a *determinant formula*,

$$\begin{aligned}
 (4.15) \quad \text{Det}(-\Delta_X - 1/4 + \kappa^2) & \left[e^{\kappa^2} \text{Det} \left(\sqrt{-\Delta_{S^2} + 1/4 + \kappa} \right) \right]^{2g-2} \\
 & = \mathcal{Z}_X \left(\frac{1}{2} + \kappa \right).
 \end{aligned}$$

Here, the functional determinant of $-\Delta_X$ occupies the place which $\lambda D(i\lambda)D(-i\lambda)$ had in Eq. (3.15). The Selberg zeta function \mathcal{Z}_X is contributed by the simple poles of $\Theta_X(\tau)$: like the right-hand side of (3.15), it originates from a geodesic length spectrum (notice the same Euler factors in both!). Finally, the other determinant factor comes from the double poles of $\Theta_X(\tau)$, which did not exist in $\Theta(\tau)$.

So there is, as announced, a functional relation for the functional determinant, Eq. (4.15), similar to the reflection formula for the Euler Gamma function, Eqs. (3.15), (3.17). (In ref. [7] we proceeded backwards, using Eq. (4.15) to actually *derive* the sectorial trace formula).

Selberg's functional equation for the zeta function \mathcal{Z}_X [5] is contained in Eq. (4.15) (divide it by the same with $\kappa \rightarrow -\kappa$, thus eliminate $\text{Det}(-\Delta_X - 1/4 + \kappa^2)$ by parity). Thus, it also lies *in parallel with the reflection formula* for $\Gamma(z)$.

Therefore, *from the point of view of spectral theory*, the Selberg functional equation is of a totally different nature from Riemann's, and Z_X has none of the properties of a Zeta-type function; the Selberg zeta function is fundamentally a *Det-type function*. (Naturally, other points of view on these two functions may uncover different connections, see these Proceedings).

There remains the remarkable coincidence that the Selberg zeta function, like Riemann's, has an Euler factorization (Eq. (4.12)), with the skeleton now being provided by the Ruelle zeta function (1.2) of the geodesic flow on X . This Euler factorization may receive a semi-classical interpretation, as a manifestation in the spectrum of the limiting wave dynamics of wave-length tending to zero [10] which is governed by the geodesic flow (geometrical optics or classical limit). Since for this specific problem the asymptotic or semi-classical formulae are exact [11], the factorization of the functional determinant over the Ruelle zeta function can be exact as well.

4.4. The "zeta function of the zeta function" [7,9]

The spectral zeta function in this problem is not the Selberg zeta function, but the Dirichlet series of its (non-trivial) zeros. Thanks to the sectorial trace formula we now have an interpretation and an evaluation method for Eq. (4.6): we set $h(\rho) = \rho^{-2s}$ in Eq. (4.11) and find

$$(4.16) \quad \begin{aligned} Z_X(s, -1/4) &= \sum_k \rho_k^{-2s} \\ &= (1-g) \frac{Z_{S^2}(s, +1/4)}{\cos \pi s} + \frac{\sin \pi s}{\pi} \int_0^\infty \kappa^{-2s} d \log \mathcal{Z}_X \left(\frac{1}{2} + \kappa \right) \end{aligned}$$

(cf. the rule of determination provided with Eq. (4.11), and Eq. (4.8) for Z_{S^2}).

All at once, this formula gives the meromorphic continuation of $Z_X(s, -1/4)$ and displays many of its properties, by mere inspection of the zeros and poles of its components: trace identities,

$$(4.17) \quad \begin{aligned} &Z_X(-n, -1/4) \\ &= (-1)^{n+1} (g-1) (1-2^{-2n-1}) B_{2n+2} / (n+1) \quad (n \in \mathbb{N}), \end{aligned}$$

and special values (Z_X at $s = +n$, and Z'_X at $s = 0$, in terms of higher derivatives of $\log \mathcal{Z}_X$ at $1/2$) [9]. These are the results of classes a) and b).

Now, we turn to class c), finding the substitute for Riemann's functional equation.

If we replace Z_X by its Euler product (4.12) in Eq. (4.16) and expand term by term, we obtain *formally*:

$$(4.18) \quad Z_X(s, -1/4) = (1 - g) \frac{Z_{S^2}(s, +1/4)}{\cos \pi s} + \frac{\pi^{-1/2} \Gamma(\frac{1}{2} - s)}{2^{2s} \Gamma(s)} \sum_{\{p\}} \sum_{r=1}^{\infty} W_{p,\tau} [r\tau(p)]^{2s-1},$$

which arises in the same way as Riemann's functional equation in the form (2.18): the sum in (4.18) is a (weighted) zeta series over the simple poles of $\Theta_X(\tau)$, while the Z_{S^2} term is the resummed contribution of the double poles.

It is obvious that Eq. (4.18) can in no way represent a *functional equation* for $Z_X(s, -1/4)$, because now the eigenvalue spectrum in the left-hand side and the length spectrum in the right-hand side are so different. Moreover, the term by term expansion of the integral in (4.16) was illegitimate, and the series (4.18) *diverges for all s* (this is also directly implied by the exponential proliferation of the length spectrum, or positive topological entropy of the geodesic flow). Nevertheless, Eq. (4.18) holds true as the *asymptotic expansion of $Z_X(s, -1/4)$ for $s \rightarrow -\infty$* .

In conclusion, the substitute to Riemann's functional equation for the spectral zeta function of this problem is, strictly speaking, the representation (4.16) for $Z_X(s, -1/4)$, which among other things yields its analytic continuation and its asymptotic behavior for $s \rightarrow -\infty$.

§5. Homogeneous Schrödinger operators

We now consider the spectral problem for a homogeneous Schrödinger operator on the real line,

$$(5.1) \quad \hat{H}\psi = \left(-\frac{d^2}{dq^2} + q^{2M} \right) \psi = \lambda\psi,$$

with M a positive integer. The spectrum $\{\lambda_k\}$ is purely discrete; the obvious dependences on M will not be explicitly indicated.

Only for $M = 1$ (harmonic oscillator) is the problem solvable; then $\lambda_k = 2k + 1$, hence the spectral functions are essentially those of Sections 2-3. Here we shall survey results for $M \geq 2$; all examples will focus on $M = 2$ (quartic oscillator), which already contains (almost) the full analytical complexity of the general case, *i.e.*, of linear differential equations with polynomial coefficients. At the same time, algebraic or

combinatorial complexity is kept minimal: only one parameter is relevant, λ .

5.1. Basic results [12]

The trace of the *heat operator* $\exp(-t\hat{H})$ can be evaluated in powers of t ($t \rightarrow 0^+$) by global symbol calculus *à la* Weyl, resulting in an explicit asymptotic expansion,

$$(5.2) \quad \begin{cases} \theta(t) (\equiv \sum_k e^{-t\lambda_k}) \sim \sum_{n=0}^{\infty} c_{i_n} t^{i_n} & (i_n = \frac{(M+1)}{2M}(2n-1)), \\ e.g., & c_{i_0} = (2M\sqrt{\pi})^{-1}\Gamma[1/(2M)]. \end{cases}$$

Then, as explained in Section 2.2, the M-P zeta function,

$$(5.3) \quad Z(s) = \sum_k \lambda_k^{-s}, \quad \text{Re } s > (M+1)/(2M),$$

has a meromorphic continuation to $s \in \mathbb{C}$ with explicit poles, residues, and trace identities (for example, if M is even, $Z(-m) = 0 \quad \forall m \in \mathbb{N}$).

We also gave a few special values in [12b], notably

$$(5.4) \quad Z'(0) = \log \sin \frac{\pi}{2(M+1)},$$

but the arguments were *ad hoc* and perhaps misleading as to the deep origin of those results. Here, we shall present a unified viewpoint in accordance with the diagram (3.18) [13].

The main initial obstacle comes from an ‘‘uncertainty relation’’ between δ_∞ , the asymptotic density of a spectrum $\{\rho_k\}$ for $\rho \rightarrow +\infty$, and d^{-1} , the minimal inverse distance between any two singularities of the function $\sum e^{-t\rho_k}$ on the imaginary axis:

$$(5.5) \quad \delta_\infty d^{-1} \geq (2\pi)^{-1}.$$

Now, standard (WKB) estimates for the spectrum of Eq. (5.1) give

$$(5.6) \quad \lambda_k \sim \left[\Gamma \left(\frac{3}{2} + \frac{1}{2M} \right) (2c_{i_0})^{-1} (2k+1) \right]^{2M/(M+1)}$$

The case $M = 1$ is special: Eq. (5.6) is exact, and the function $\theta(t)$ of Eq. (5.2) is described by the Poisson summation formula: it has equidistant poles, and with spacing exactly $2\pi\delta_\infty$. But as soon as $M \geq 2$, $\delta_\infty = 0$, and Eq. (5.5) implies a natural barrier for the function $\theta(t)$: the imaginary axis is densely covered by its singularities, which then cannot be resolved.

5.2. Resurgent Theta-type function [13]

As in the preceding section, our only hope is to find *another* Theta-type function with better global analytic properties. This time, however, we have no exact trace formula to produce such a function immediately, and in fact there is no guarantee that it will exist at all.

A first approach is based on Eq. (5.6) and on the limiting case $M = 1$. We distort the spectrum into an asymptotically uniform one, by setting

$$(5.7) \quad \rho_k = \lambda_k^{(M+1)/(2M)}$$

and

$$(5.8) \quad \Theta_{\frac{M+1}{2M}}(\tau) = \sum_k e^{-\tau \rho_k}.$$

Unfortunately this function cannot be *directly* and fully resolved into singular terms even for $M = 2$; nevertheless, this attempt gave encouraging partial results [14], like the prediction of a *regular lattice of branch points* for the function $\Theta_{3/4}(\tau)$ of the case $M = 2$.

Further progress requires a more thorough investigation of the differential equation (5.1) itself. One can associate to it (for each fixed M) an infinite *algebra of ramified analytic functions* whose singularities obey simple recombination laws under convolution. All singularities are then governed by this algebraic structure, which is of the general type known as *resurgence algebra* [15]. The singular structure of the function $\Theta_{\frac{M+1}{2M}}(\tau)$ then follows from its actual membership within the algebra, *i.e.*, from its *resurgent* character. We now describe the main lines of this analysis (cf. [13] for details).

5.3. Analytic bootstrap

We introduce a spectral function $a(\lambda)$ directly coupled to the differential equation (5.1) itself. By general theorems [16], this equation admits *asymptotic* solutions (the *WKB* expressions)

$$(5.9) \quad \begin{cases} \psi_{\pm}(\lambda, q) = \pi(q)^{-1/2} \exp \pm \int^q \pi(q') dq', \\ \pi(q) = (q^{2M} - \lambda)^{1/2}, \end{cases}$$

which become exact in *either* limit $|q| \rightarrow \infty$ or $\lambda \rightarrow -\infty$. For $\lambda < 0$ and for the (unique) *true* solution $\psi(\lambda, q)$ of (5.1) having a decreasing

behavior at $q \rightarrow -\infty$ prescribed by

$$(5.10) \quad \lim_{q \rightarrow -\infty} \psi_+(\lambda, q)^{-1} \psi(\lambda, q) = 1,$$

we define the *Jost function*

$$(5.11) \quad a(\lambda) = \lim_{q \rightarrow +\infty} \psi_+(\lambda, q)^{-1} \psi(\lambda, q).$$

On the one hand, $a(\lambda) \rightarrow 1$ for $\lambda \rightarrow -\infty$ since WKB solutions then become exact; on the other hand, $a(\lambda) = 0$ whenever an exact solution decreases for both $q \rightarrow \pm\infty$, which happens precisely at the eigenvalues; hence, $a(-\lambda)$ should be proportional to $\text{Det}(-d^2/dq^2 + q^{2M} + \lambda) = D(\lambda)$. Both facts are reflected in the following rigorous identity,

$$(5.12) \quad \begin{aligned} \log a(-\lambda) &\equiv \sim \log D(\lambda) \\ &(\equiv \log D(\lambda) + \Gamma(i_0) c_{i_0} \lambda^{\frac{M+1}{2M}} \text{ for } M > 1) \end{aligned}$$

(valid at first for $\lambda > 0$, then analytically continued everywhere). We have thus linked the functional determinant $D(\lambda)$ to the special solution $\psi(-\lambda, q)$, via Eqs. (5.9–11).

Then, noting that the corresponding equation $(\hat{H} + \lambda)\psi = 0$ can be rescaled (for $\lambda > 0$) as

$$(5.13) \quad \left(-x^{-2} \frac{d^2}{d\tilde{q}^2} + \tilde{q}^{2M} + 1 \right) \psi(\tilde{q}) = 0,$$

where

$$(5.14) \quad x = \lambda^{\frac{M+1}{2M}} \quad (\text{and } \tilde{q} = \lambda^{-\frac{1}{2M}} q),$$

we can expand the solution $\psi(-\lambda, \tilde{q})$ *semi-classically* (in powers of x^{-1} , around the WKB approximation $\psi_+(-\lambda, \tilde{q})$; x^{-1} is also the quantum mechanical Planck's constant \hbar). This results in an expansion for the function $a(-\lambda)$, of the form

$$(5.15) \quad \begin{cases} \log a(-\lambda) \sim \sum_{n=1}^{\infty} a_n x^{1-2n}, & (x \rightarrow +\infty) \\ a_n = -\Gamma(i_n) c_{i_n}. \end{cases}$$

This series should diverge for all x since we are using a singular perturbation technique. We then consider the *Borel-transformed* series,

$$(5.16) \quad (\log a)_B(\tau) = \sum_{n=1}^{\infty} a_n \tau^{2n-1} / (2n-1)! .$$

If this series has a finite radius of convergence, and if its sum $\Theta_0(\tau)$ extends analytically to the positive τ -axis, we may tentatively write a *Borel resummation formula*,

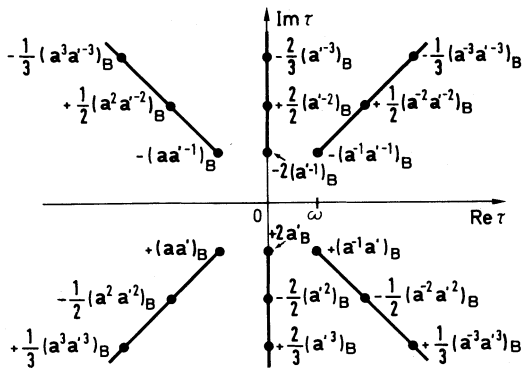
$$(5.17) \quad \begin{cases} \log a(-\lambda) &= x \int_0^\infty (\log a)_B(\tau) e^{-x\tau} d\tau, \\ \text{i.e., } \sim \log D(\lambda) &= x \int_0^\infty \Theta_0(\tau) e^{-x\tau} d\tau. \end{cases}$$

We are using Eqs. (5.12), (5.14); our notation $\Theta_0(\tau)$ for the Borel transform $(\log a)_B$ is meant to recall the Laplace relationship (2.7). (However, $\Theta_0(\tau)$ cannot have a Dirichlet series like (2.2), otherwise it would not be analytic around $\tau = 0$).

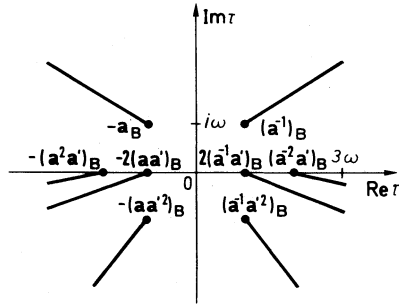
Now, all this can be *justified* (semi-rigorously) using the complex WKB method, which moreover establishes that $\Theta_0(\tau)$ belongs to a particular *algebra of ramified analytic functions with isolated singularities* [13]. The *locations* of these singularities have a geometric interpretation reminiscent of the Poisson formula for manifolds: they lie at the values of the *action integrals* over all cycles, *i.e.*, of $\oint (q^{2M} - \lambda)^{1/2} dq$ taken along all real or complex *periods of classical trajectories*.

In the simplest case $M = 2$, we display the analytic structure of $\Theta_0(\tau)$ in Fig. 4a. The action values (elliptic integrals) form the discrete lattice

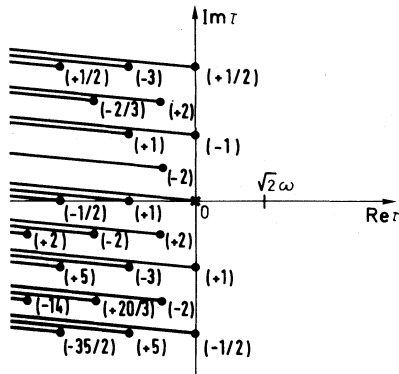
$$(5.18) \quad \omega(\mathbb{Z} + i\mathbb{Z}), \quad \omega = \Gamma(1/4)^2 / (3\sqrt{\pi}).$$



4a



4b



4c

Fig. 4. Analytic bootstrap structure of resurgent spectral functions for the quartic oscillator ($M = 2$): a) the Borel-transform series (5.16), or $\Theta_0(\tau)$; b) the function $a'_B(\tau) = a_B(-i\tau)$, also in the resurgence algebra; c) the function $\tau^{-1}\Theta_{3/4}(\tau)$, with $\Theta_{3/4}(\tau) = \sum_k e^{-t\lambda_k^{3/4}}$. Discontinuity functions across the cuts are marked for a), b) (see [13] for details).

All discontinuity functions across cuts are the Borel transforms, nor-

malized as in Eq. (5.16), of monomials $c_{mm'} a(x)^m a(ix)^{m'}$ ($m, m' \in \mathbb{Z}$); the x -series $a(x)$ and $a'(x) \equiv a(ix)$ are computed by exponentiation of Eq. (5.15).

Figure 4b shows the analytic structure of $a'_B(s)$. In fact, the function $\Theta_0(\tau)$ and all monomials $(a^m a'^{m'})_B$ ($m, m' \in \mathbb{Z}$) form a closed system under monodromy operations (property of analytic bootstrap); they generate a special type of convolution algebra called a resurgence algebra [15].

Finally, figure 4c sketches the similar analytic structure found for the function $\tau^{-1}\Theta_{3/4}(\tau)$, where $\Theta_{3/4}(\tau)$ is an actual Theta-type function (Eq. (5.8), with $M = 2$). The pattern is rotated by $\pi/4$ from the previous figure and the discontinuity formulae suffer slight additional complications (hence we have simply marked a few weights $c_{mm'}$), but the relationship with the previous algebra is visible.

5.4. The M-P zeta function

The function $Z(s) = \sum \lambda_k^{-s}$, already defined in Eq. (5.3), has three Mellin representations: the usual one, Eq. (2.11), another one differing from it by the rescaling of the powers (Eqs. (5.7-8)),

$$(5.19) \quad Z\left(\frac{M+1}{2M}s\right) = \frac{1}{\Gamma(s)} \int_0^\infty \Theta_{\frac{M+1}{2M}}(\tau) \tau^{s-1} d\tau \quad (\text{Re } s > 1),$$

and finally the representation obtained by composing the Laplace transform (5.17) and the Mellin transform (2.6),

$$(5.20) \quad \begin{aligned} & Z\left(\frac{M+1}{2M}s\right) \\ &= \frac{\sin\left(\frac{M+1}{2M}\pi s\right)}{\sin \pi s} \frac{s}{\Gamma(s)} \int_0^\infty \Theta_0(\tau) \tau^{s-1} d\tau \quad (-1 < \text{Re } s < 0). \end{aligned}$$

Either of the two displayed equations can be used to investigate the effects on the zeta function of isolated singularities of Θ -functions. Eq. (5.19) was used in [12b], but (5.20) is actually much simpler to use because it is readily turned into a contour integral (Fig. 5 for $M = 2$):

$$(5.20') \quad \begin{aligned} & Z\left(\frac{M+1}{2M}s\right) \\ &= \frac{\sin\left(\frac{M+1}{2M}\pi s\right)}{\sin \pi s} \frac{s\Gamma(1-s)}{2i\pi} \int_C \Theta_0(\tau)(-\tau)^{s-1} d\tau \quad (\text{Re } s < 0). \end{aligned}$$

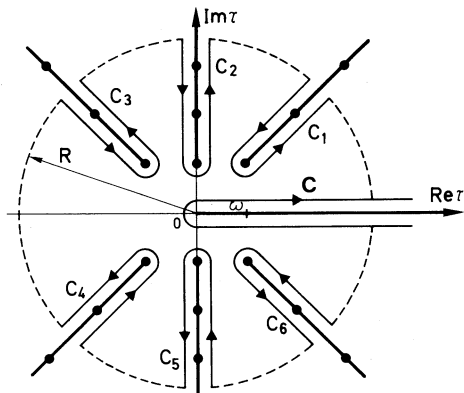


Fig. 5. The Mellin representations (5.20) for the spectral zeta function $Z(s) = \sum_k \lambda_k^{-s}$ of the quartic oscillator ($M = 2$).

We can now mimic exactly the contour deformation which led from Eq. (2.13') to Riemann's functional equation, ... except that the contributions of the singularities are now *integrals* over the (radial) cuts (Fig. 5),

$$\begin{aligned}
 (5.21) \quad & Z\left(\frac{M+1}{2M}s\right) \\
 &= \frac{\sin\left(\frac{M+1}{2M}\pi s\right)}{\sin \pi s} \frac{s\Gamma(1-s)}{2i\pi} \sum_j \int_{C_j} \Theta_0(\tau)(-\tau)^{s-1} d\tau \quad (\text{Re } s < 0),
 \end{aligned}$$

and we cannot resum this in closed form. Clearly, anyhow, the structure of the right-hand side is quite different from that of $Z(s)$ itself, so that any local functional equation for $Z(s)$ of Riemann's type can be excluded.

However, if we order the summation terms in Eq. (5.21) according to the increasing distance of the cuts' origins from $s = 0$, and expand term by term (using the information like in Fig. 4a for $M = 2$), the result is the explicit *asymptotic expansion* of $Z(s)$ for $\text{Re } s \rightarrow -\infty$, in analogy with Eq. (4.18) for the zeta function of a hyperbolic surface. Here, however, it will be a double series in exponentials and powers of s .

For instance, for $M = 2$, the leading expansion terms are

$$(5.22) \quad Z\left(\frac{3s}{4}\right) \sim \frac{s \sin 3\pi s/4}{\pi \cos \pi s/2} \left[\omega^s \sum_{j=0}^{\infty} \alpha_j \Gamma(-s-j) + \cos \frac{\pi s}{4} (\sqrt{2}\omega)^s \sum_{j=0}^{\infty} \gamma_j \Gamma(-s-j) + \dots \right] (s \rightarrow -\infty).$$

The first asymptotic series in the brackets is the contribution of the two cuts at the nearest branch points, namely $\tau = \pm i\omega$, and the second one is the contribution from the four next nearest branch points having $|\tau| = \sqrt{2}\omega$ (Fig. 5). The coefficients α_j and γ_j are specified by generating functions which *invoke the a_n again* (Eq. (5.15); this is a manifestation of analytic bootstrap):

$$(5.23) \quad \sum_{\ell=0}^{\infty} \alpha_{\ell} z^{\ell} \equiv \exp \left[- \sum_{j=1}^{\infty} a_j (\omega z)^{2j-1} \right]; \quad \text{e.g., } \alpha_0 = 1,$$

$$\sum_{\ell=0}^{\infty} \gamma_{\ell} z^{\ell} \equiv \exp \left[2 \sum_{j=1}^{\infty} a_j \cos \left[(2j-1) \frac{3\pi}{4} \right] (\sqrt{2}\omega z)^{2j-1} \right]; \quad \text{e.g., } \gamma_0 = 1.$$

We recall that the asymptotic expansion (5.22) was constructed using the same mechanism which produced Riemann's functional equation for $\zeta(s)$; here, this constitutes the *only* property of this type for $Z(s)$ which we can write out in a fully *explicit* closed form.

5.5. The functional determinants [13]

Through the integral representation (5.17), the rigid analytic structure of $\Theta_0(\tau)$ imposes a severe constraint upon the functional determinant $D(\lambda)$. Indeed, generalizing Eq. (3.12-14), we may form all integrals like (5.17) (cf. Fig. 6 for $M = 2$),

$$(5.24) \quad I_{\varphi}(x) = x \int_0^{e^{i\varphi}\infty} \Theta_0(\tau) e^{-x\tau} d\tau.$$

This is a piecewise constant function of the angle φ , which has finite jumps at the phases of branch points of $\Theta_0(\tau)$ in the sheet of integration. The successive jumps are functions of I_{φ} itself through the analytic bootstrap relations, moreover $I_{\varphi+\pi}(x) \equiv -I_{\varphi}(-x)$ by the parity property of

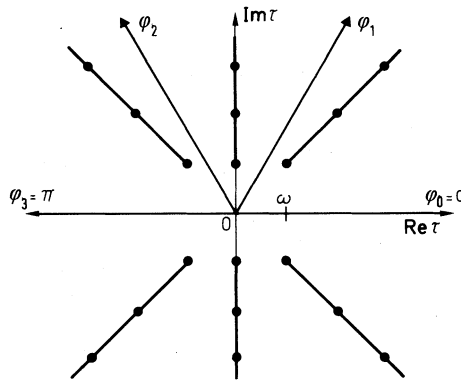


Fig. 6. The Laplace transforms (5.24) of the function $\Theta_0(\tau)$ for the quartic oscillator ($M = 2$).

$\Theta_0(\tau)$ in Eq. (5.16). Exploiting all this, we can deduce a symmetric *polynomial functional equation* for the functional determinant,

$$(5.25) \quad P [D(\lambda), D(\Omega^2\lambda), \dots, D(\Omega^{2M}\lambda)] \equiv 0, \quad \Omega = e^{i\pi/(M+1)}.$$

For $M = 1$, $D(\lambda) = \sqrt{2\pi} 2^{-\lambda/2} / \Gamma((1 + \lambda)/2)$, and we recover the *reflection formula* for the Gamma function. Eq. (5.25) *generalizes* this to higher M . For $M = 2$, the functional equation is

$$(5.26) \quad D(\lambda) D(j\lambda) D(j^2\lambda) \equiv D(\lambda) + D(j\lambda) + D(j^2\lambda) + 2 \quad (j = e^{2i\pi/3}).$$

For $M > 2$ the equation is intractable in polynomial form; an alternative form is the system

$$(5.27) \quad \begin{cases} \sum_{k=0}^M \phi(\Omega^{2k}\lambda) = \pi/2, \\ \phi(\lambda) \equiv \text{Arc sin} [D(\Omega\lambda)D(\Omega^{-1}\lambda)]^{-1/2}. \end{cases}$$

These equations are of the same type as the functional equations satisfied by the Stokes multipliers of polynomial Sturm-Liouville equations [16]. (In fact, the Borel transforms of Stokes multipliers belong to the same resurgent algebra already discussed). All studies indicate that these functional equations are very mysterious and that their solutions *within entire functions* (the only relevant ones in the present case) are quite elusive.

The same argument as Eq. (3.16) for $\zeta(s)$ can now be applied. If we expand the functional equation in powers around $\lambda = 0$, we obtain information about special values of $Z(s) = \sum \lambda_k^{-s}$, namely,

$$(5.28) \quad Z'(0) = \log \sin[\pi/(2M + 2)],$$

and countably many polynomial identities of the form

$$(5.29) \quad Z[(M + 1)n] = P_n\{Z(m) \mid 1 \leq m < (M + 1)n\}, \quad n = 1, 2, \dots$$

where P_n is homogeneous of total degree $(M + 1)n$ if $Z(m)$ is assigned degree m . For instance, the first identity for $M = 2$ is

$$(5.30) \quad Z(3) = Z(1)^3/6 - Z(1)Z(2)/2.$$

These identities are to be viewed as generalizations of the classical zeta-values $\zeta(2n)$; here they are only implicit, like the analytic bootstrap relations which generated them.

Further progress is achieved by splitting all spectral functions into even and odd parts; for instance,

$$(5.31) \quad D(\lambda) = D^+(\lambda)D^-(\lambda),$$

where $D^+(\lambda)$ (resp. $D^-(\lambda)$) is the functional subdeterminant over the eigenvalues $\{\lambda_{2k}\}$ belonging to the *even* eigenfunctions (resp. $\{\lambda_{2k+1}\}$ belonging to the *odd* eigenfunctions).

A *functional relation* is then found between D^+ and D^- , one form of which is, for $M > 1$,

$$(5.32) \quad \Omega^{1/2}D^+(\Omega^{-1}\lambda)D^-(\Omega\lambda) - \Omega^{-1/2}D^+(\Omega\lambda)D^-(\Omega^{-1}\lambda) \equiv 2i \\ (\Omega = e^{i\pi/(M+1)}).$$

This relation not only implies Eq. (5.27) but contains much more information: its power series expansion around $\lambda = 0$ now gives one identity for each power of λ , instead of only one identity like (5.29) for every $(M + 1)$ -th power of λ in Eq. (5.27):

$$(5.33) \quad Z'(0) + \sum_{n=1}^{\infty} \cos \frac{n\pi}{M+1} \frac{Z(n)}{n} (-\lambda)^n \\ \equiv \log \sin \left[\frac{\pi}{2(M+1)} + \sum_{m=1}^{\infty} \sin \frac{m\pi}{M+1} \frac{Z^P(m)}{m} (-\lambda)^m \right]$$

($M > 1$), where

$$(5.34) \quad Z^P(s) = \sum_{k=0}^{\infty} (-1)^k \lambda_k^{-s}.$$

The first two identities besides Eq. (5.28) read as

(5.35)

$$\begin{aligned} \cot \frac{\pi}{2(M+1)} \sin \frac{\pi}{M+1} Z^P(1) - \cos \frac{\pi}{M+1} Z(1) &= 0 \quad (M > 1), \\ \cot \frac{\pi}{2(M+1)} \sin \frac{2\pi}{M+1} Z^P(2) - \cos \frac{2\pi}{M+1} Z(2) &= \\ &= [2 \cos \frac{\pi}{2(M+1)} Z^P(1)]^2. \end{aligned}$$

Eq. (5.33) thus subsumes the class b) results for this problem. The supplementary information gained upon going from Eq. (5.27) to (5.33) corresponds for $M = 1$ to the special values

$$(5.36) \quad \beta(2n+1) = \frac{(\pi/2)^{2n+1}}{2(2n)!} |E_{2n}|$$

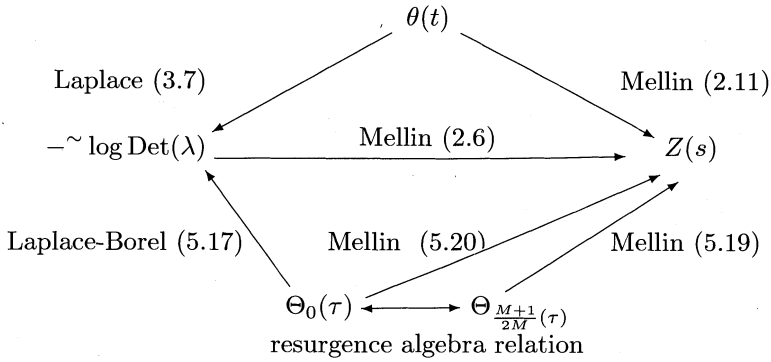
of the alternating zeta-function $\beta(s) = \sum_0^{\infty} (-1)^k (2k+1)^{-s}$ (which is $Z^P(s)$ for $M = 1$), in terms of Euler numbers [1].

Now, a fundamental open problem is to understand the structure and solvability of functional equations like (5.26–27) or (5.32), which can take a bewildering variety of equivalent forms, and the degree to which these relations determine the spectrum of Eq. (5.1).

Conclusion

We have obtained, for the spectrum of homogeneous Schrödinger operators, the following diagram of the type (2.8):

(5.37)



The top-half arrows provide the results of class a), and the bottom-half arrows those of classes b) and c), for the spectral (M-P) zeta function $Z(s)$.

§6. Concluding discussion

In the preceding two sections we have surveyed two spectral generalizations of some classical results about the Riemann zeta functions and its spectral (trigonometric, Gamma) functions.

The two examples were chosen to have very distinct features. The spectrum of a compact hyperbolic surface is described by a *partial* differential operator, which is special but genuinely two-dimensional (non-separable, mixing in the classical limit). The spectrum of a homogeneous Schrödinger operator is a prototype of a non-solvable problem in one degree of freedom, described by an *ordinary* differential equation with variable coefficients.

Each case has special properties of its own which were exploited in its analysis. In the former case we used the Selberg trace formula which expresses some form of exact solvability for the problem; in the latter we used techniques for ordinary differential equations, which are much more developed than for partial differential operators. At present we cannot directly analyze the spectrum of general partial differential operators, which is the remote goal of such studies.

We can conjecture, however, that the identical structures found in those two contrasted examples will appear *in general*: resurgent Theta-type functions; functional relations for functional determinants; identities on special values and some asymptotic ($s \rightarrow -\infty$) information for certain spectral zeta functions. Solvable problems (cf. Section 4, and

the harmonic oscillator in Section 5) should lead to *meromorphic* Theta functions (a degenerate case of resurgence).

A candidate for a resurgent Theta function in a general spectral Schrödinger problem

$$(6.1) \quad \hat{H}(\hbar)\psi = E\psi$$

is, for instance,

$$(6.2) \quad \Theta_E(\tau) = \sum_k e^{-\tau x_k(E)},$$

where $x_k(E)$ are the “characteristic values” of \hbar^{-1} for which E (fixed) is an eigenvalue in Eq. (6.1). We have conjectured that the complex singularities of $\Theta_E(\tau)$ should lie at the *action periods* of the classical (bicharacteristic) Hamiltonian flow, real and complex – assuming that its periodic orbits are isolated [17]. This conjecture essentially re-expresses, in a divergence-free manner, Gutzwiller’s *trace formula for quantum mechanics* [18], adding the contributions of the *complexified orbits* in compliance with Balian and Bloch’s analysis [19]. All in all, the function (6.2) provides one possible starting point for further generalizations of the results described here, but all the computing algorithms remain to be elaborated.

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