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Selberg Zeta Functions and Jacobi Forms

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§0. Introduction

0.1. The purpose of this paper is to generalize our previous work [Ar2, 3] to the case of the spaces of Jacobi forms of a more general type. We define Selberg zeta functions associated with certain theta multiplier systems of $\mathrm{SL}_2(\mathbb{Z})$. Those Selberg zeta functions can be continued to meromorphic functions in the whole complex plane satisfying certain functional equations in virtue of the general theory of Selberg trace formula for $\mathrm{SL}_2(\mathbb{R})$ due originally to Selberg [Se1, 2], and recently to Hejhal [He1, 2], Fischer [Fi]. A remarkable thing to be stressed is that our Selberg zeta functions have close relations with the dimensions of the spaces of Jacobi forms. We show that the dimensions of the spaces of certain Jacobi forms of lower weights can be explicitly expressed in a closed form with the use of the orders of the zeros of our Selberg zeta functions via the Selberg trace formula. To describe the Selberg trace formula we need the theory of real analytic Eisenstein series associated with the theta multiplier systems of $\mathrm{SL}_2(\mathbb{Z})$. As a byproduct we show that the functional equation satisfied by the real analytic Eisenstein series for the Jacobi group can be obtained from the one satisfied by the real analytic Eisenstein series associated with the theta multiplier systems.

0.2. We explain our results more precisely. Let l be a positive integer and S a half-integral positive definite symmetric matrix of size l. For ν a positive integer and Γ, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index having the element -1_2, we denote by $J_{\nu,S}(\Gamma)$ (resp. $J^*_{\nu,S}(\Gamma)$) the space of holomorphic Jacobi forms (resp. skew-holomorphic Jacobi forms) of weight ν and index S with respect to the Jacobi group Γ^J (for the precise definition see (5.3) in § 5). Let $J^{\mathrm{cusp}}_{\nu,S}(\Gamma)$ and $J^{*\,\mathrm{cusp}}_{\nu,S}(\Gamma)$ denote the subspaces consisting of cusp forms in $J_{\nu,S}(\Gamma)$ and $J^*_{\nu,S}(\Gamma)$, respectively. If $l = 1$ (i.e., S is a positive integer), Jacobi forms of $J_{\nu,S}(\Gamma)$ have been

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studied by Eichler-Zagier [E-Z] from various points of view. In that case ($l = 1$) skew-holomorphic Jacobi forms have been introduced and studied by Skoruppa (cf. [Sk1], [S-Z1], [Sk2, 3]). Set $R_S = (2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l$. Denote by d the cardinality of the set R_S ($d = \det(2S)$). Let $V = \mathbb{C}^d$, the $\mathbb{C}$-vector space of column vectors $(x_r)_{r\in R_S}$ ($x_r \in \mathbb{C}$) equipped with the scalar product $\langle x, y\rangle_S = \sum_{r\in R_S} x_r\overline{y_r}$ for $x = (x_r)$, $y = (y_r) \in V$. For each $r \in R_S$ the theta series $\theta_r(\tau, z)$ is defined by (1.1) in § 1. It is known that these theta series satisfy the theta transformation formula:

$$\theta_r(M(\tau, z)) = e\left(\frac{c\,{}^t zSz}{c\tau + d}\right)(c\tau + d)^{l/2}\sum_{q\in R_S} u_{rq}(M)\theta_q(\tau, z)$$

$$\text{with } u_{rq}(M) \in \mathbb{C} \text{ for any } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where $M(\tau, z) = (M\tau, \dfrac{z}{c\tau + d})$ and the branch of $(c\tau + d)^{l/2}$ is chosen so that $-\pi < \arg(c\tau + d) \leq \pi$. The $d \times d$ matrix $U(M) = (u_{rq}(M))_{r,q\in R_S}$ is a unitary matrix (or a unitary transformation of V) with respect to the scalar product $\langle\ ,\ \rangle_S$. Denote by $\chi(M)$ the complex conjugate of $U(M)$. Since $\chi(-1_2)$ has the eigen values $\pm e^{\pi i l/2}$, the vector space V has the decomposition $V = V_+ \oplus V_-$, V_+ (resp. V_-) being the $e^{\pi i l/2}$-eigen subspace (resp. $-e^{\pi i l/2}$-eigen subspace) of V. The spaces V_+ and V_- are $\chi(M)$-invariant subspaces of V for any $M \in \mathrm{SL}_2(\mathbb{Z})$. Denote by $\chi_+(M)$ (resp. $\chi_-(M)$) the restriction of $\chi(M)$ on V_+ (resp. V_-). We define the Selberg zeta functions $Z_{\Gamma,S,+}(s)$, $Z_{\Gamma,S,-}(s)$ associated with χ_+, χ_- by the identity (1.8) in § 1. The zeta functions $Z_{\Gamma,S,\pm}(s)$ are absolutely convergent for $Re(s) > 1$. We formulate the resolvent trace formula (Theorem 4.1 in § 4) associated with Γ and χ following Fischer [Fi]. It can be shown that via the resolvent trace formula the zeta functions $Z_{\Gamma,S,\pm}(s)$ are analytically continued to meromorphic functions of s in the whole s-plane satisfying the functional equation (4.7). The dimensions of the spaces of Jacobi forms can be computed in a closed form thanks to the resolvent trace formula. In particular we obtain certain relations between the dimensions of the spaces of Jacobi forms of lower weights and the orders of the zeros of our Selberg zeta functions. Our main results are the following:

$$\text{(i)}\qquad \begin{aligned} \dim_{\mathbb{C}} J_{k,S}^{\mathrm{cusp}}(\Gamma) &= \lambda_\Gamma(k; S) && \text{if } k > l/2 + 2, \\ \dim_{\mathbb{C}} J_{l-k,S}^{*\,\mathrm{cusp}}(\Gamma) &= \mu_\Gamma(k; S) && \text{if } k < l/2 - 2, \end{aligned}$$

where $\lambda_\Gamma(k; S)$, $\mu_\Gamma(k; S)$ ($k \in \mathbb{Z}$) are the numbers given by (5.9).

(ii) Assume that l is odd. Let ε denote the sign $+$ or $-$ according as $l \equiv 1 \mod 4$ or $l \equiv 3 \mod 4$. Then,

$$\begin{aligned}
\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{(l+3)/2,S}(\Gamma) &= \mathrm{Ord}_{s=3/4}\, Z_{\Gamma,S,\varepsilon}(s) + \lambda_\Gamma((l+3)/2;S),\\
\dim_{\mathbb{C}} J^{*}_{(l+1)/2,S}(\Gamma) &= \mathrm{Ord}_{s=3/4}\, Z_{\Gamma,S,\varepsilon}(s),\\
\dim_{\mathbb{C}} J^{*\,\mathrm{cusp}}_{(l+3)/2,S}(\Gamma) &= \mathrm{Ord}_{s=3/4}\, Z_{\Gamma,S,-\varepsilon}(s) + \mu_\Gamma((l-3)/2;S),\\
\dim_{\mathbb{C}} J_{(l+1)/2,S}(\Gamma) &= \mathrm{Ord}_{s=3/4}\, Z_{\Gamma,S,-\varepsilon}(s).
\end{aligned}$$

(iii) Assume that l is even. Let ε denote the sign $+$ or $-$ according as $l \equiv 0 \mod 4$ or $l \equiv 2 \mod 4$. Then,

$$\begin{aligned}
\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{l/2+2,S}(\Gamma) &= \mathrm{Ord}_{s=1}\, Z_{\Gamma,S,\varepsilon}(s) + \lambda_\Gamma(l/2+2;S),\\
\dim_{\mathbb{C}} J^{*\,\mathrm{cusp}}_{l/2+2,S}(\Gamma) &= \mathrm{Ord}_{s=1}\, Z_{\Gamma,S,\varepsilon}(s) + \mu_\Gamma(l/2-2;S),\\
\dim_{\mathbb{C}}\{\, v \in V \mid \chi(M)v = v \text{ for any } M \in \Gamma\} &= \mathrm{Ord}_{s=1}\, Z_{\Gamma,S,\varepsilon}(s),
\end{aligned}$$

and

$$\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{l/2+1,S}(\Gamma) + \dim_{\mathbb{C}} J^{*\,\mathrm{cusp}}_{l/2+1,S}(\Gamma) = \mathrm{Ord}_{s=1/2}\, Z_{\Gamma,S,-\varepsilon}(s).$$

In the last paragraph we obtain the functional equation satisfied by the real analytic Eisenstein series for the Jacobi group.

We make two significant remarks.

Remark 1. Murase-Sugano in their note [M-S] have already obtained a dimension formula for the space $J^{\mathrm{cusp}}_{k,S}(\Gamma)$ under the condition $k > l/2+2$ (the first identity in the above (i)). They have used the trace formula involving the Bergman kernel function for the space $J^{\mathrm{cusp}}_{k,S}(\Gamma)$.

Remark 2. In [Sk1] and [S-Z1] Skoruppa-Zagier calculated the dimensions of the spaces $J_{1,m}(\mathrm{SL}_2(\mathbb{Z}))$ and $J^*_{1,m}(\mathrm{SL}_2(\mathbb{Z}))$ in the case of $l = 1$ and $S = m$, a positive integer:

$$\begin{aligned}
\dim_{\mathbb{C}} J_{1,m}(\mathrm{SL}_2(\mathbb{Z})) &= 0,\\
\dim_{\mathbb{C}} J^*_{1,m}(\mathrm{SL}_2(\mathbb{Z})) &= \frac{1}{2}\{\sum_{d|m,\, d>0} 1 + \delta(m = \square)\},
\end{aligned}$$

where the symbol $\delta(m = \square)$ indicates 1 or 0 according as m is a square of some integer or not. If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then by these identities and the above (ii), the Selberg zeta function $Z_{\Gamma,m,+}(s)$ has a zero at $s = 3/4$ of a strictly positive order, and moreover, $Z_{\Gamma,m,-}(3/4) \neq 0$.

0.3. Hejhal [He1, 2] and Fischer [Fi] discussed the Selberg trace formula for $\mathrm{SL}_2(\mathbb{R})$ associated with Γ, a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ of finite covolume, and χ, a unitary multiplier system of Γ, in full details with proofs. In the present paper we have employed Fischer's resolvent trace formula which fits very well to the calculation of the dimensions of the spaces of Jacobi forms.

This work was done while the author was staying at Max-Planck Institut für Mathematik in Bonn in 1989–1990, to which he expresses his gratitude.

Notation. As usual we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, the complex number field, the real number field, the ring of rational integers, and the set of positive integers, respectively. Denote by $\mathbb{C}^l$ (resp. $\mathbb{R}^l$) the $\mathbb{C}$-vector (resp. $\mathbb{R}$-vector) space of complex (resp. real) column vectors of size l and by $\mathbb{Z}^l$ the $\mathbb{Z}$-lattice of integral column vectors of size l in $\mathbb{R}^l$. Denote by $\mathrm{Sym}_l(\mathbb{R})$ (resp. $\mathrm{Sym}_l(\mathbb{Z})$) the set of symmetric matrices of size l with entries in $\mathbb{R}$ (resp. $\mathbb{Z}$). Let $\Gamma(s)$ and $\zeta(s)$ be the gamma function and the Riemann zeta function, respectively. For a meromorphic function $f(z)$, denote by $\mathrm{Res}_{z=\alpha} f(z)$ (resp. $\mathrm{Ord}_{z=\beta} f(z)$) the residue of $f(z)$ at the pole $z = \alpha$ (resp. the order of the zero at $z = \beta$). For $z \in \mathbb{C}$, denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ (or $\mathrm{Im}\, z$) the real part of z and the imaginary part of z, respectively. For a finite set A, $\#(A)$ denotes the cardinality of the set A. We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i\alpha)$.

§1. Theta multiplier systems of $\mathrm{SL}_2(\mathbb{Z})$ and Selberg zeta functions

We choose the branch of $z^\alpha = \exp(\alpha \log z)$ ($z \neq 0$, $\alpha \in \mathbb{R}$) with $-\pi < \arg z \leq \pi$ throughout the paper.

First of all we recall theta transformation formulas for classical theta series. Let l be a positive integer. Let S be a positive definite half-integral symmetric matrix of size l and fix it once and for all. Let R_S denote the $\mathbb{Z}$-module $(2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l$. We set

$$d = \det(2S),$$

which is a positive integer. Then, $\#(R_S) = d$. We write

$$S(u, v) = {}^t uSv \quad \text{and} \quad S[u] = {}^t uSu \quad \text{for } u, v \in \mathbb{C}^l.$$

Denote by $V = \mathbb{C}^d$ the $\mathbb{C}$-vector space consisting of column vectors $(x_r)_{r \in R_S}$ ($x_r \in \mathbb{C}$). Let $\langle x, y \rangle_S$ be the positive definite hermitian scalar product given by

$$\langle x, y \rangle_S = \sum_{r \in R_S} x_r \overline{y_r} \quad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

Denote by $\mathfrak{H}$ the upper half plane on which the group $\mathrm{SL}_2(\mathbb{R})$ acts in a usual manner. For each $r \in (2S)^{-1}\mathbb{Z}^l$, the classical theta series $\theta_r(\tau, z)$ is defined by

$$(1.1) \quad \theta_r(\tau, z) = \sum_{q \in \mathbb{Z}^l} e(\tau S[q+r] + 2S(q+r, z)) \qquad (\tau \in \mathfrak{H},\ z \in \mathbb{C}^l).$$

By abuse of notation one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S,\tau}$ denote the space of holomorphic functions $\theta\colon \mathbb{C}^l \to \mathbb{C}$ satisfying

$$\theta(z + \lambda\tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z) \quad \text{for any } \lambda, \mu \in \mathbb{Z}^l.$$

Obviously, $\theta_r(\tau, z) \in \Theta_{S,r}$. We set, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathfrak{H}$, $z \in \mathbb{C}^l$,

$$A\tau = \frac{a\tau + b}{c\tau + d}, \quad J(A, \tau) = c\tau + d, \quad \text{and} \quad A(\tau, z) = \left(A\tau, \frac{z}{J(A, \tau)}\right).$$

For a real number μ, denote by $\sigma_\mu(A, B)$ (A, $B \in \mathrm{SL}_2(\mathbb{R})$) the number defined by

$$\sigma_\mu(A, B) = \exp(i\mu(\arg J(A, B\tau) + \arg J(B, \tau) - \arg J(AB, \tau))),$$

where $\arg w$ ($w \neq 0$) is chosen so that $-\pi < \arg w \leq \pi$. The number $\sigma_\mu(A, B)$ is independent of the choice of $\tau \in \mathfrak{H}$ in the above definition. Necessary properties of $\sigma_\mu(A, B)$ for later use are referred to (1.3.3)–(1.3.9), p.18 of [Fi]. We arrange the theta series $\theta_r(\tau, z)$ as a column vector:

$$\Theta(\tau, z) = (\theta_r(\tau, z))_{r \in R_S} \in V.$$

It is known that the theta series $\theta_r(\tau, z)$ satisfy the theta transformation formula:

$$(1.2) \quad \begin{gathered} \Theta(M(\tau, z)) = J(M, \tau)^{l/2} e\left(\frac{c}{J(M, \tau)} S[z]\right) U(M)\Theta(\tau, z) \\ \text{for any} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \end{gathered}$$

where $U(M)$ is a certain unitary matrix of size d with respect to the scalar product $\langle\ ,\ \rangle_S$. For the proof of (1.2) we refer for instance to Shintani [Sh]. For convenience we consider the complex conjugate of $U(M)$. Set

$$(1.3) \qquad \chi(M) = \overline{U(M)} \qquad (M \in \mathrm{SL}_2(\mathbb{Z})).$$

Then the formula (1.2) implies the following property of $\chi(M)$ as a unitary multiplier system:

$$(1.4)\quad \chi(M_1M_2)=\sigma_{l/2}(M_1,M_2)\chi(M_1)\chi(M_2)\qquad (M_1,\ M_2\in \mathrm{SL}_2(\mathbb{Z})).$$

By this identity χ becomes a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ if l is a positive even integer. For each $r\in R_S$, denote by e_r the column vector of V whose q-th component equals one or zero according as $q=r$ or not. Namely,

$$e_r=(\delta_{rq})_{q\in R_S},\ \delta_{rq} \text{ standing for the Kronecker symbol.}$$

Denote by L the matrix of size d (or a linear transformation of V) characterized by

$$(1.5)\qquad Le_r=e_{-r}\quad \text{for any}\quad r\in R_S.$$

Substituting $M=-1_2$ in (1.2), we easily have

$$(1.6)\qquad \chi(-1_2)=e^{\pi il/2}L.$$

Since all the eigen values of L are ±1, one can define V_+ (resp. V_-) to be the $\mathbb{C}$-subspace of V consisting of all vectors $v\in V$ satisfying

$$Lv=v\qquad (\text{resp. } Lv=-v).$$

We set

$$(1.7)\qquad R_S^0=\{\,r\in R_S\mid r\equiv -r\mod \mathbb{Z}^l\,\}\quad\text{and}\quad d_0=\#(R_S^0).$$

Then,

$$\dim_{\mathbb{C}} V_+=(d+d_0)/2\quad\text{and}\quad \dim_{\mathbb{C}} V_-=(d-d_0)/2.$$

Since every $\chi(M)$ ($M\in\mathrm{SL}_2(\mathbb{Z})$) commutes with L via (1.4), one can define $\chi_+(M)$ (resp. $\chi_-(M)$) to be the restriction of $\chi(M)$ onto the subspace V_+ (resp. V_-). Then, $\chi_+(M)$ and $\chi_-(M)$ are unitary endomorphisms of V_+ and V_-, respectively. Under these preparations we can define two Selberg zeta functions associated with $\chi_\pm$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index having the element -1_2. Every hyperbolic element P of Γ has an expression

$$P=\pm A\begin{pmatrix} N(P)^{1/2} & 0\\ 0 & -N(P)^{1/2}\end{pmatrix}A^{-1}$$
with $N(P)>1$ and some $A\in\mathrm{SL}_2(\mathbb{R})$.

Then, $N(P)$ is uniquely determined and called the norm of P. Denote by $\{P\}_\Gamma$ (resp. $\{P_0\}_\Gamma$) the Γ-conjugacy classes of hyperbolic (resp. primitive hyperbolic) elements of Γ. Let $\mathrm{id}(V_\varepsilon)$ denote the identity map of V_ε ($\varepsilon = +$ or $-$). We set

$$(1.8) \qquad Z_{\Gamma,S,\varepsilon}(s) = \prod_{\{P_0\}_\Gamma, \mathrm{tr}\, P_0 > 2} \prod_{n=0}^{\infty} \det(\mathrm{id}(V_\varepsilon) - \chi_\varepsilon(P_0) N(P_0)^{-s-n}),$$

where ε is the sign $+$ or $-$ and where the first product indicates that P_0 runs over the Γ-conjugacy classes of primitive hyperbolic elements of Γ with $\mathrm{tr}\, P_0 > 2$. The infinite products on the right hand side of (1.8) are absolutely and uniformly convergent on any compact sets in the half-plane $\mathrm{Re}(s) > 1$ (see, for instance, Corollary 2.2.6 of [Fi]). Thus the zeta function $Z_{\Gamma,S,+}(s)$, $Z_{\Gamma,S,-}(s)$ indicate holomorphic functions of s in $\mathrm{Re}(s) > 1$. Then the logarithmic derivatives of the zeta functions have the form

$$(1.9) \qquad \begin{aligned} \frac{Z'_{\Gamma,S,\varepsilon}}{Z_{\Gamma,S,\varepsilon}}(s) = \sum_{\{P\}_\Gamma, \mathrm{tr}\, P > 2} & \mathrm{tr}(\chi_\varepsilon(P)) \log N(P_0) \cdot \\ & \frac{N(P)^{-s}}{1 - N(P)^{-1}} \quad (\mathrm{Re}(s) > 1), \end{aligned}$$

where $\varepsilon = +$ or $-$ and P_0 is the primitive hyperbolic element associated to P with $\mathrm{tr}\, P_0 > 2$.

§2. Certain L^2-spaces of automorphic forms and Eisenstein series

We recall some basic facts of Roelcke [Ro1, 2] and Fischer [Fi] concerning certain automorphic forms and Eisenstein series associated with χ in a manner convenient to our situation.

Let k be a rational integer and set

$$(2.1) \qquad \kappa = (k - l/2)/2.$$

In the sequel let $\varepsilon(k)$ denote the sign $+$ or $-$ according as k is even or odd. Then it is easy to see from (1.4), (1.6) that

$$(2.2) \qquad \begin{cases} \chi_\pm(M_1 M_2) = \sigma_{2\kappa}(M_1, M_2) \chi_\pm(M_1) \chi_\pm(M_2) \\ \qquad\qquad\qquad\qquad (M_1, M_2 \in \mathrm{SL}_2(\mathbb{Z})) \\ \chi_{\varepsilon(k)}(-1_2) = e^{-2\pi i \kappa}\, \mathrm{id}(V_{\varepsilon(k)}), \end{cases}$$

where in the first equality the signs $+$, $-$ are chosen in the corresponding manner. Notice that in (2.2), $\sigma_{2\kappa}(M_1, M_2)$, which takes the values ± 1, depends on l and not on k. We set

$$j_M(\tau, \kappa) = \exp(2i\kappa \arg J(M, \tau)) \quad (M \in \mathrm{SL}_2(\mathbb{R}),\ \tau \in \mathfrak{H}).$$

We write $j_M(\tau)$ in place of $j_M(\tau, \kappa)$ if there is no fear of confusion. This factor of automorphy satisfies the property

$$\text{(2.3)} \qquad j_A(B\tau) j_B(\tau) = \sigma_{2\kappa}(A, B) j_{AB}(\tau) \quad (A, B \in \mathrm{SL}_2(\mathbb{R}),\ \tau \in \mathfrak{H}).$$

We set

$$d\omega(\tau) = \eta^{-2} d\xi d\eta \qquad (\xi = \mathrm{Re}(\tau),\ \eta = \mathrm{Im}(\tau)).$$

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index with the element -1_2. Let $\mathcal{H}_{\kappa,S,\Gamma}$ (resp. $\mathcal{H}^{\varepsilon(k)}_{\kappa,S,\Gamma}$) be the space of measurable functions $f: \mathfrak{H} \to V$ (resp. $f: \mathfrak{H} \to V_{\varepsilon(k)}$) satisfying

(2.4)

i) $f(M\tau) = \chi(M) j_M(\tau) f(\tau)$ (resp. $f(M\tau) = \chi_{\varepsilon(k)}(M) j_M(\tau) f(\tau)$) for all $M \in \Gamma$

ii) $\displaystyle\int_{\Gamma\backslash\mathfrak{H}} \langle f(\tau), f(\tau)\rangle_S \, d\omega(\tau) < +\infty,$

where $\Gamma\backslash\mathfrak{H}$ is a fundamental domain of Γ in $\mathfrak{H}$. For instance, the space $\mathcal{H}^{+}_{\kappa,S,\Gamma}$ is defined, only if k is even. If f is an element of $\mathcal{H}_{\kappa,S,\Gamma}$, then by (i) of (2.4) and (1.6), (2.3), the equality

$$Lf(\tau) = (-1)^k f(\tau)$$

holds and accordingly $f(\tau)$ is contained in $V_{\varepsilon(k)}$. Thus each element f of $\mathcal{H}_{\kappa,S,\Gamma}$ is canonically identified with an element of $\mathcal{H}^{\varepsilon(k)}_{\kappa,S,\Gamma}$. Via this identification we may set

$$\mathcal{H}_{\kappa,S,\Gamma} = \mathcal{H}^{\varepsilon(k)}_{\kappa,S,\Gamma}.$$

Then the space $\mathcal{H}_{\kappa,S,\Gamma}$ forms a Hilbert space via the Petersson scalar product (f, g):

$$(f, g) = \int_{\Gamma\backslash\mathfrak{H}} \langle f(\tau), g(\tau)\rangle_S \, d\omega(\tau) \qquad (f, g \in \mathcal{H}_{\kappa,S,\Gamma}).$$

We set

$$\Delta_\kappa = \eta^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) - 2i\kappa\eta \frac{\partial}{\partial \xi} \qquad (\tau = \xi + i\eta)$$

as in Definition 1.4.3 of [Fi]. Denote by $\mathcal{D}_\kappa$ the set of all twice continuously differentiable functions $f \in \mathcal{H}_{\kappa,S,\Gamma}$ with $\Delta_\kappa f \in \mathcal{H}_{\kappa,S,\Gamma}$ (notice that the condition (i) for $\Delta_\kappa f$ in (2.4) is automatically fulfilled). According to Satz 3.1 of [Ro1] the linear operator $-\Delta_\kappa : \mathcal{D}_\kappa \to \mathcal{H}_{\kappa,S,\Gamma}$ satisfies the property

$$(-\Delta_\kappa f, g) = (f, -\Delta_\kappa g) \qquad \text{for } f, g \in \mathcal{D}_\kappa$$

and moreover it can be extended to the unique self-adjoint operator $-\Delta_\kappa^\frown : \mathcal{D}_\kappa^\frown \to \mathcal{H}_{\kappa,S,\Gamma}$ with $\mathcal{D}_\kappa^\frown$ as its domain (Satz 3.2 of [Ro1]).

For each cusp ζ of Γ, denote by Γ_ζ the stabilizer of ζ in Γ: $\Gamma_\zeta = \{ M \in \Gamma \mid M\zeta = \zeta \}$. Denote by h the cardinality of the Γ-equivalence classes of cusps of Γ and let $\zeta_1, \dots, \zeta_h$ be a complete set of representatives of the Γ-equivalence classes of cusps of Γ. Choose $A_1^*, \dots, A_h^* \in \mathrm{SL}_2(\mathbb{Z})$ such that $A_j^* \zeta_j = \infty$ $(1 \le j \le h)$. Then for each j there exists a positive integer l_j such that

$$A_j^* \Gamma_{\zeta_j} A_j^{*-1} = \left\{ \pm \begin{pmatrix} 1 & l_j n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\} \qquad (1 \le j \le h).$$

The positive integer l_j is uniquely determined by the Γ-equivalence class of the cusp ζ_j. Put, for each j $(1 \le j \le h)$,

$$A_j = l_j^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & l_j \end{pmatrix} A_j^* \quad \text{and} \quad T_j = A_j^{-1} U A_j, \tag{2.5}$$
$$\text{where} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

Then the stabilizer Γ_{ζ_j} is generated by the elements -1_2 and T_j.

We have to consider a further decomposition of the set $R_S = (2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l$. Let R_S^0 be the subset of R_S given by (1.7). Then there exists a subset $R_S^\sim$ of R_S such that

$$R_S = R_S^0 \cup R_S^\sim \cup \{-R_S^\sim\} \qquad \text{(disjoint union)},$$

where $-R_S^\sim$ denotes the subset $\{ -r \mid r \in R_S^\sim \}$ of R_S. We fix such a subset $R_S^\sim$ once and for all. Moreover for the fixed integer k we define the subset $R_{S,k}$ as follows:

$$R_{S,k} = \begin{cases} R_S^\sim \cup R_S^0 & \text{if } k \text{ is even,} \\ R_S^\sim & \text{if } k \text{ is odd.} \end{cases}$$

Set, for each j $(1 \leq j \leq h)$ and $r \in R_S$,

$$v_{jr} = \chi(A_j^*)^{-1} e_r. \tag{2.6}$$

Then for each j, $\{v_{jr}\}_{r \in R_S}$ forms an orthonormal basis of V. Set, for each j $(1 \leq j \leq h)$,

$$w_{jr} = \begin{cases} v_{jr} & \text{if } r \in R_S^0 \\ (v_{jr} + (-1)^k v_{j,-r})/\sqrt{2} & \text{if } r \in R_S - R_S^0. \end{cases} \tag{2.7}$$

By the property (1.5), we have $Lv_{jr} = v_{j,-r}$. Therefore, $\{w_{jr}\}_{r \in R_{S,k}}$ forms an orthonormal basis of $V_{\varepsilon(k)}$.

It is easy to see from the definition (1.1) of $\theta_r(\tau, z)$ that

$$\theta_r(\tau + n, z) = e(nS[r])\theta_r(\tau, z) \quad \text{for any } n \in \mathbb{Z} \text{ and } r \in R_S.$$

In view of the relation

$$\Theta(\tau, z) = \sum_{r \in R_S} \theta_r(\tau, z) e_r$$

and by the linearly independence of the theta series $\{\theta_r(\tau, z)\}_{r \in R_S}$, we thus have

$$\chi\left(\begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix}\right) e_r = e(-l_j S[r]) e_r \quad \text{for any } r \in R_S.$$

This identity is changed into the form

$$\chi(T_j) v_{jr} = e(-l_j S[r]) v_{jr} \qquad (1 \leq j \leq h,\ r \in R_S), \tag{2.8}$$

since one can prove

$$\chi(A_j^* T_j A_j^{*-1}) = \chi(A_j^*)\chi(T_j)\chi(A_j^*)^{-1}$$

similarly as in Corollary 1.3.8 of [Fi]. We further decompose the set $R_{S,k}$ into two parts. Set, for each j $(1 \leq j \leq h)$,

$$\begin{cases} R_{S,j,k}^n = \{r \in R_{S,k} \mid l_j S[r] \in \mathbb{Z}\} \\ R_{S,j,k}^* = \{r \in R_{S,k} \mid l_j S[r] \notin \mathbb{Z}\}. \end{cases} \tag{2.9}$$

Let $f \colon \mathfrak{H} \to V$ be a twice continuously differentiable function satisfying the condition (i) of (2.4) and $-\Delta_\kappa f = s(1-s)f$ with some $s \in \mathbb{C}$.

We discuss the Fourier expansion of f at each cusp ζ_j. As we have seen before, $f(\tau)$ is a $V_{\varepsilon(k)}$-valued function. Therefore the function

$$f_{A_j}(\tau) = j_{A_j}(A_j^{-1}\tau)f(A_j^{-1}\tau)$$

can be written in a linear combination of w_{jr} ($r \in R_{S,k}$):

$$f_{A_j}(\tau) = \sum_{r \in R_{S,k}} f_{A_j}^r(\tau) w_{jr}.$$

For any real number x, let $\langle x \rangle$ denote the real number with the conditions $x - \langle x \rangle \in \mathbb{Z}$ and $0 \leq \langle x \rangle < 1$. We set

$$\beta_{jr} = \langle -l_j S[r] \rangle \qquad (1 \leq j \leq h,\ r \in R_S). \tag{2.10}$$

Then we see easily from (i) of (2.4) and (2.8) that

$$f_{A_j}^r(\tau + 1) = e(\beta_{jr}) f_{A_j}^r(\tau) \qquad (r \in R_{S,k}).$$

As is discussed by Roelcke [Ro1, pp.300–301] and Fischer in Proposition 1.5.4 of [Fi] in a general situation, $f_{A_j}^r(\tau)$ has a Fourier expansion of the form:

$$f_{A_j}^r(\tau) = u_j^r(\eta) + q_j^r(\tau) \qquad (\tau \in \mathfrak{H},\ \xi = \mathrm{Re}(\tau),\ \eta = \mathrm{Im}(\tau)), \tag{2.11}$$

where

$$\left.\begin{aligned} q_j^r(\tau) &= \sum_{n \in \mathbb{Z}} q_{j,n}^r(\eta) e((n + \beta_{jr})\xi) \\ u_j^r(\eta) &= 0 \end{aligned}\right. \qquad \text{if } r \in R_{S,j,k}^*$$

and

$$\left.\begin{aligned} q_j^r(\tau) &= \sum_{n \in \mathbb{Z} - \{0\}} q_{j,n}^r(\eta) e(n\xi) \\ u_j^r(\eta) &= \begin{cases} b_{rj}\eta^s + c_{rj}\eta^{1-s} & s \neq 1/2 \\ b_{rj}\eta^{1/2} + c_{rj}\eta^{1/2}\log\eta & s = 1/2 \end{cases} \end{aligned}\right. \qquad \text{if } r \in R_{S,j,k}^n$$

with some functions $q_{j,n}^r(\eta)$ and some constants b_{rj}, c_{rj}. Thus $f(\tau)$ has a Fourier expansion at each cusp ζ_j of the form:

$$\begin{aligned} f(\tau) &= j_{A_j}(\tau)^{-1}\{u_j(\mathrm{Im}(A_j\tau)) + q_j(A_j\tau)\} \\ u_j(\eta) &= \sum_{r \in R_{S,k}} u_j^r(\eta) w_{jr}, \qquad q_j(\tau) = \sum_{r \in R_{S,k}} q_j^r(\tau) w_{jr}, \end{aligned} \tag{2.12}$$

where $u_j^r(\eta)$, $q_j^r(\tau)$ are the same as in (2.11).

Now we define the real analytic Eisenstein series at each cusp ζ_j associated with Γ, χ following Definition 1.5.3 of [Fi]. We set, for each j $(1 \leq j \leq h)$ and $r \in R^n_{S,j,k}$,

(2.13)
$$E_{jr}(\tau, s) = \sum_{M \in \Gamma_{\zeta_j} \backslash \Gamma} \sigma_{2\kappa}(A_j, M)^{-1} \chi(M)^{-1} w_{jr} j_{A_j M}(\tau)^{-1} (\operatorname{Im} A_j M\tau)^s.$$

The infinite series on the right hand side is well-defined and absolutely convergent for $\mathrm{Re}(s) > 1$. Since the vectors w_{jr} are in $V_{\varepsilon(k)}$, on the right hand side of the definition (2.13), $\chi(M)^{-1}$ may be replaced with $\chi_{\varepsilon(k)}(M)^{-1}$ according to the definition of $\chi_\pm$. It is easy to see that

$$E_{jr}(M\tau, s) = \chi(M) j_M(\tau) E_{jr}(\tau, s) \quad \text{for any } M \in \Gamma$$

and that

$$-\Delta_\kappa E_{jr}(*, s) = s(1-s) E_{jr}(*, s).$$

The Eisenstein series $E_{jr}(\tau, s)$ has the Fourier expansion at each cusp ζ_l of the form:

$$E_{jr}(\tau, s) = j_{A_l}(\tau)^{-1} \{u_{jr,l}(\operatorname{Im} A_l\tau, s) + q_{jr,l}(A_l\tau, s)\},$$

where $u_{jr,l}(\eta, s)$ $(\eta > 0)$ is the constant term (i.e., the zeroth Fourier coefficient) and where $u_{jr,l}(\eta, s)$, $q_{jr,l}(\tau, s)$ have the forms similar as in (2.12). The constant term $u_{jr,l}(\eta, s)$ has the following expression:

$$u_{jr,l}(\eta, s) = \delta_{jl} \eta^s w_{jr} + p_{jr,l}(s) \eta^{1-s} \qquad (\eta > 0)$$

with a certain $V_{\varepsilon(k)}$-valued function $p_{jr,l}(s)$. Set

$$\varphi_{jr,lp}(s) = \langle p_{jr,l}(s), w_{lp} \rangle_S \qquad (1 \leq j \leq h,\ r \in R^n_{S,j,k},\ p \in R_{S,k}).$$

Notice that $\varphi_{jr,lp}(s) = 0$ if $p \in R^*_{S,l,k}$. These functions $\varphi_{jr,lp}(s)$ are holomorphic in $\mathrm{Re}(s) > 1$. We set

$$t_\infty = \sum_{j=1}^{h} \#(R^n_{S,j,k}).$$

We note that t_∞ depends on $k \mod 2$. This number t_∞ represents the degree of singularity of $\chi_{\varepsilon(k)}$ (see Notation 1.5.1 of [Fi]). The $t_\infty \times t_\infty$ matrix $\Phi(s)$ is defined by

$$(2.14) \qquad \Phi(s) = (\varphi_{jr,lp}(s)) \qquad (1 \leq j, l \leq h,\ r \in R^n_{S,j,k},\ p \in R^n_{S,l,k}),$$

jr being the line index, lp being the column index, both in lexicographic order. By (10.30) in [Ro2], we have

$$\overline{\varphi_{jr,lp}(s)} = \varphi_{lp,jr}(\bar{s}) \qquad (\text{i.e.,} \quad {}^{\mathrm{t}}\overline{\Phi(s)} = \Phi(\bar{s})).$$

Let $E(\tau, s)$ denote the $d \times t_\infty$-matrix whose column vectors are the Eisenstein series $E_{jr}(\tau, s)$:

$$E(\tau, s) = (E_{jr}(\tau, s))_{j=1,\dots,h; r \in R^n_{S,j,k}}.$$

The main theorem for the Eisenstein series due to Selberg [Se1,2] and Roelcke [Ro1,2] is formulated in our terminology as follows (see also 1.5, pp.28–35 in [Fi]). The special case of $l = 1$ (l being the size of the matrix S) and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is discussed in [Ar2].

Theorem 2.1. *The Eisenstein series $E_{jr}(\tau, s)$ and $\varphi_{jr,lp}(s)$ are analytically continued to meromorphic functions of s in the whole complex plane that are holomorphic in the half plane $\mathrm{Re}(s) \geq 1/2$ except for the real segment $(1/2, 1]$. They satisfy the functional equations*

$$E(\tau, 1-s) = E(\tau, s)\,{}^{\mathrm{t}}\Phi(1-s) \quad \textit{and} \quad \Phi(s)\Phi(1-s) = 1_{t_\infty}.$$

Namely,

$$E_{jr}(\tau, 1-s) = \sum_{l=1}^{h} \sum_{p \in R^n_{S,l,k}} E_{lp}(\tau, s)\varphi_{jr,lp}(1-s),$$

$$\sum_{l=1}^{h} \sum_{p \in R^n_{S,l,k}} \varphi_{jr,lp}(s)\varphi_{lp,iq}(1-s) = \delta_{jr,iq}$$

$$(1 \leq i, j \leq h,\ r \in R^n_{S,j,k},\ q \in R^n_{S,i,k}),$$

$\delta_{jr,iq}$ being the Kronecker symbol. Moreover the Eisenstein series $E_{jr}(\tau, s)$ $(1 \leq j \leq h,\ r \in R^n_{S,j,k})$ are linearly independent for $s \neq 1/2$, if all $E_{jr}(\tau, s)$ are holomorphic at s.

This theorem has an application to the determination of the functional equation satisfied by real analytic Eisenstein series for the Jacobi group, on which we shall discuss in the last paragraph.

§3. Computation of $\operatorname{tr}\chi(M)$

To describe the resolvent trace formula for $\mathcal{H}^{\pm}_{\kappa,S,\Gamma}$ as explicit as possible with the help of the special properties of the theta multiplier systems $\chi_\pm$, one has to calculate the traces of $\chi(M)$ ($M \in \mathrm{SL}_2(\mathbb{Z})$) in a convenient form. The calculation of $\operatorname{tr}\chi(M)$ has been done by Skoruppa-Zagier [S-Z2] in the case of l, the size of S, being one. They extended $U(M)$ to certain linear operators of the space $\Theta_{S,\tau}$ of the theta series and computed the traces of those operators by analyzing their explicit actions on theta series. In the present paper we select another method which is based on the theory of Bergman kernel function of the space $\Theta_{S,\tau}$. It seems that such a method is essentially known, but for the convenience of the reader we exhibit it here briefly.

Let $G = \mathrm{SL}_2(\mathbb{R})$. First we define the Jacobi group G^J. Set

$$G^J = \{\, (M, (\lambda, \mu), \rho) \mid M \in G,\ \lambda, \mu \in \mathbb{R}^l,\ \rho \in \mathrm{Sym}_l(\mathbb{R}) \,\}. \tag{3.1}$$

Then, G^J forms a group by the composition law

$$\begin{aligned} g_1 g_2 = (M_1 M_2, (\lambda_1, \mu_1)M_2 + (\lambda_2, \mu_2),& \\ \rho_1 + \rho_2 - \mu_1{}^{\mathrm{t}}\lambda_1 + \mu^*{}^{\mathrm{t}}\lambda^* + \lambda^*{}^{\mathrm{t}}\mu_2 + \mu_2{}^{\mathrm{t}}\lambda^*)& \\ (g_j = (M_j, (\lambda_j, \mu_j), \rho_j) \in G^J, \quad j = 1, 2)& \\ \text{with} \quad (\lambda^*, \mu^*) = (\lambda_1, \mu_1)M_2.& \end{aligned}$$

The center of the group G^J is given by the subgroup

$$\{\, (1_2, (0,0), \rho) \mid \rho \in \mathrm{Sym}_l(\mathbb{R}) \,\}.$$

Then any element M of G and any Y of $\mathbb{R}^l \times \mathbb{R}^l$ may be canonically identified with the elements $(M, (0,0), 0)$ and $(1_2, Y, 0)$ of G^J, respectively. We often write M (resp. Y) in place of the corresponding element $(M, (0,0), 0)$ (resp. $(1_2, Y, 0)$) of G^J. Denote by D the product of the upper half plane $\mathfrak{H}$ and $\mathbb{C}^l$: $D = \mathfrak{H} \times \mathbb{C}^l$. The Jacobi group G^J acts on D in the manner

$$\begin{aligned} g(\tau, z) = \left(M\tau, \frac{z + \lambda\tau + \mu}{c\tau + d} \right)& \\ (g = (M, (\lambda, \mu), \rho) \in G^J,\ (\tau, z) \in D).& \end{aligned} \tag{3.2}$$

A factor of automorphy $J_{k,S}(g, (\tau, z))$ associated with the given S and an integer k is defined by

$$\begin{aligned} &J_{k,S}(g, (\tau, z)) \\ &= J(M, \tau)^k e\left(-\operatorname{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M, \tau)} S[z + \lambda\tau + \mu] \right) \\ &\qquad (g = (M, (\lambda, \mu), \rho) \in G^J, (\tau, z) \in D). \end{aligned}$$

Then it satisfies

(3.3)
$$\begin{aligned}&J_{k,S}(g_1g_2,(\tau,z))\\ &=J_{k,S}(g_1,g_2(\tau,z))J_{k,S}(g_2,(\tau,z)) \quad (g_1,g_2\in G^J).\end{aligned}$$

For each $\tau\in\mathfrak{H}$, denote by L_τ the $\mathbb{Z}$-lattice $\mathbb{Z}^l\tau+\mathbb{Z}^l$ in $\mathbb{C}^l$ and by $\mathbb{C}^l/L_\tau$ a fundamental domain of L_τ in $\mathbb{C}^l$. For any $z\in\mathbb{C}^l$ we write $z=u\tau+v$ with $u,v\in\mathbb{R}^l$ and define a volume form $d\omega(z)$ on $\mathbb{C}^l$ by

$$d\omega(z)=\prod_{j=1}^{l}du_j\prod_{j=1}^{l}dv_j \qquad \text{with } u=(u_j),\ v=(v_j).$$

For $\theta_1,\theta_2\in\Theta_{S,\tau}$, a scalar product $\langle\theta_1,\theta_2\rangle$ is defined by

$$\langle\theta_1,\theta_2\rangle=\int_{\mathbb{C}^l/L_\tau}\theta_1(z)\overline{\theta_2(z)}\exp(-4\pi\eta s[u])\,d\omega(z) \qquad (\eta=\operatorname{Im}\tau).$$

It is easy to show the following orthogonality relation for the theta series $\theta_r(\tau,z)$ $(r\in R_S)$:

(3.4)
$$\langle\theta_a(\tau,*),\theta_b(\tau,*)\rangle=\begin{cases}0 & a\not\equiv b \mod \mathbb{Z}^l\\ 2^{-l}\det(S)^{-1/2}\eta^{-l/2} & a\equiv b \mod \mathbb{Z}^l.\end{cases}$$

It is known that $\{\theta_r(\tau,z)\}_{r\in R_S}$ forms an orthogonal basis of the space $\Theta_{S,\tau}$. We set, following (1), § 2 in [S-Z2],

$$h_S((\tau,z),(\tau',z'))=e(-S[z-\overline{z'}]/(\tau-\overline{\tau'})).$$

We easily have

(3.5)
$$\overline{h_S((\tau,z),(\tau',z'))}=h_S((\tau',z'),(\tau,z)),$$

(3.6)
$$\begin{aligned}&h_S(g(\tau,z),g(\tau',z'))\\ &=J_{0,S}(g,(\tau,z))h_S((\tau,z),(\tau',z'))\overline{J_{0,S}(g,(\tau',z'))} \qquad (g\in G^J).\end{aligned}$$

Lemma 3.1. *Let $\tau,\tau'\in\mathfrak{H}$ and $z\in\mathbb{C}^l$. Then,*

$$\begin{aligned}&\int_{\mathbb{C}^l}h_S((\tau,z),(\tau',z'))\theta_r(\tau',z')\exp(-4\pi\eta' S[u'])\,d\omega(z')\\ &\quad=\det(2S)^{-1}\left(\frac{\tau-\overline{\tau'}}{2i\eta'}\right)^{l/2}\theta_r(\tau,z) \qquad \textit{for any } r\in R_S,\end{aligned}$$

where $\eta' = \operatorname{Im}\tau'$ *and* $z' = u'\tau' + v'$ *with* $u', v' \in \mathbb{R}^l$.

Since the proof is done by a straightforward computation with the use of (1.1), we omit it.

Via Lemma 3.1, a linear map $\iota: \Theta_{S,\tau'} \to \Theta_{S,\tau}$ can be defined by

$$\begin{aligned}&(\iota_{\tau,\tau'}\theta)(z)\\&= \int_{\mathbb{C}^l} h_S((\tau, z), (\tau', z'))\theta(z') \exp(-4\pi\eta' S[u'])\, d\omega(z').\end{aligned}$$

We set

$$\begin{aligned}&H_S((\tau, z), (\tau', z'))\\&= \sum_{\lambda,\mu\in\mathbb{Z}^l} e(-\overline{\tau'}S[\lambda] - 2S(\lambda, \overline{z'}))h_S((\tau, z), (\tau', z' + \lambda\tau' + \mu)),\end{aligned} \tag{3.7}$$

which is absolutely convergent for any $(\tau, z), (\tau', z') \in D$ and indicates a holomorphic (resp. anti-holomorphic) function in (τ, z) (resp. (τ', z')). Then we get immediately, for $\theta \in \Theta_{S,\tau'}$,

$$\begin{aligned}&(\iota_{\tau,\tau'}\theta)(z)\\&= \int_{\mathbb{C}^l/L_{\tau'}} H_S((\tau, z), (\tau', z'))\theta(z') \exp(-4\pi\eta' S[u'])\, d\omega(z').\end{aligned} \tag{3.8}$$

The following lemma shows the essential feature of the function $H_S((\tau, z), (\tau', z'))$.

Lemma 3.2. *Let* $\tau, \tau' \in \mathfrak{H}$ *and* $z, z' \in \mathbb{C}^l$. *Then,*

$$H_S((\tau, z), (\tau', z')) = (\det S)^{-1/2} \left(\frac{\tau - \overline{\tau'}}{2i}\right)^{l/2} \sum_{r\in R_S} \theta_r(\tau, z)\overline{\theta_r(\tau', z')}.$$

Proof. As a function of z', $\overline{H_S((\tau, z), (\tau', z'))}$ belongs to the space $\Theta_{S,\tau'}$ and therefore,

$$H_S((\tau, z), (\tau', z')) = \sum_{r\in R_S} A_r \cdot \overline{\theta_r(\tau', z')}$$

with some functions A_r in τ, z, τ'. On the other hand Lemma 3.1 immediately implies

$$\iota_{\tau,\tau'}(\theta_r(\tau', *))(z) = \det(2S)^{-1} \left(\frac{\tau - \overline{\tau'}}{2i\eta'}\right)^{l/2} \theta_r(\tau, z).$$

Thus getting this identity together with (3.8), we can calculate A_r and hence have the assertion of Lemma 3.2.

q.e.d.

We have, immediately by Lemma 3.2,

$$H_S((\tau,z),(\tau',z')) = \overline{H_S((\tau',z'),(\tau,z))}. \tag{3.9}$$

It follows from (3.5), (3.7), and (3.9) that this function has another expression:

$$\begin{aligned} &H_S((\tau,z),(\tau',z')) \\ &= \sum_{X\in\mathbb{Z}^l\times\mathbb{Z}^l} J_{0,S}(X,(\tau,z))^{-1} h_S(X(\tau,z),(\tau',z')). \end{aligned} \tag{3.10}$$

The following integral expression for $\operatorname{tr} U(M)$ easily follows from the theta transformation formula (1.2), Lemma 3.2, and the orthogonality relation (3.4).

Proposition 3.3. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{H}$. Then,*

(3.11)

$$\begin{aligned} &\det(2S)^{-1}\left(\frac{M\tau-\overline{\tau}}{2i\eta}\right)^{l/2} J(M,\tau)^{l/2}\operatorname{tr} U(M) \\ &= \int_{\mathbb{C}^l/L_\tau} J_{0,S}(M,(\tau,z))^{-1} H_S(M(\tau,z),(\tau,z))\exp(-4\pi\eta S[u])\,d\omega(z). \end{aligned}$$

For simplicity set $L = \mathbb{Z}^l \times \mathbb{Z}^l$, on which any integral matrix of size two acts by right multiplication. We have, by the expression (3.10),

$$\begin{aligned} &J_{0,S}(M,(\tau,z))^{-1} H_S(M(\tau,z),(\tau,z)) \\ &= \sum_{X\in L} J_{0,S}(XM,(\tau,z))^{-1} h_S(XM(\tau,z),(\tau,z)). \end{aligned} \tag{3.12}$$

Lemma 3.4. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ with $\det(M-1_2) \neq 0$. For each $X \in L$, the element XM in G^J can be expressed as*

$$XM = Y^{-1}MX'Y\cdot\mathfrak{z}$$

with $X', Y \in L$ and $\mathfrak{z} = (1_2,(0,0),\rho)$, $\rho \in \mathrm{Sym}_l(\mathbb{Z})$, where we notice that $XM = X' + Y(1_2 - M)$ in L. Moreover by this expression, X runs over all elements of L, if and only if Y runs over all elements of L and X' over all residue classes of L modulo $L(M-1_2)$.

We omit the proof, which is easy.

Lemma 3.5. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ with $\det(M - 1_2) \neq 0$. Then,*

$$(3.13)\quad \begin{aligned} &\det(2S)^{-1}\left(\frac{M\tau - \overline{\tau}}{2i\eta}\right)^{l/2} J(M,\tau)^{l/2}\operatorname{tr} U(M) \\ &= \sum_{X \in L/L(M-1_2)} I(\tau, M; X), \end{aligned}$$

where we put

$$\begin{aligned} &I(\tau, M; X) \\ &= \int_{\mathbb{C}^l} J_{0,S}(MX, (\tau, z))^{-1} h_S(MX(\tau, z), (\tau, z)) \exp(-4\pi\eta S[u])\, d\omega(z). \end{aligned}$$

Proof. It is easy to see from (3.12), Lemma 3.4 and (3.6) that

$$\begin{aligned} &J_{0,S}(M, (\tau, z))^{-1} H_S(M(\tau, z), (\tau, z)) \\ &= \sum_{Y \in L} \sum_{X \in L/L(M-1_2)} J_{0,S}(Y^{-1}MXY, (\tau, z))^{-1} \cdot \\ &\qquad\qquad h_S(Y^{-1}MXY(\tau, z), (\tau, z)) \\ &= \sum_{Y \in L} \sum_{X \in L/L(M-1_2)} J_{0,S}(MX, Y(\tau, z))^{-1} \\ &\qquad\qquad \cdot h_S(MXY(\tau, z), Y(\tau, z)) |J_{0,S}(Y, (\tau, z))|^{-2}. \end{aligned}$$

Since we have

$$|J_{0,S}(Y, (\tau, z))|^{-2} \exp(-4\pi\eta S[u]) = \exp(-4\pi\eta S[u + \lambda]) \qquad (Y = (\lambda, \mu) \in L),$$

the integral on the right hand side of the equality (3.11) turns out the expression on the right hand side of the equality (3.13).

q.e.d.

Now we compute $\operatorname{tr} U(R)$ for elliptic elements R of $\mathrm{SL}_2(\mathbb{Z})$.

Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element of $\mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{H}$ a fixed point of R (we may choose $\tau = i$ or $e^{2\pi i/3}$). Set, for simplicity,

$$\omega = c\tau + d, \qquad \zeta = \tau - \overline{\tau}\,(= 2i\eta).$$

We have $RX(\tau,z)=(\tau,(z+\lambda\tau+\mu)/\omega)$ for $X=(\lambda,\mu)\in L$. Noticing that $z=u\tau+v$ ($u,v\in\mathbb{R}^l$) and $c/\omega+1/(\omega^2\zeta)=1/\zeta$, we see easily that

(3.14)

$$\begin{aligned}
&J_{0,S}(RX,(\tau,z))^{-1}h_S(RX(\tau,z),(\tau,z))\exp(-4\pi\eta S[u])\\
&=e(\tau S[\lambda]+2S(\lambda,z)-cS[z+\lambda\tau+\mu]/\omega)\\
&\quad\cdot e(-S[(z+\lambda\tau+\mu)\omega^{-1}-\overline{z}]/\zeta)e(S[z-\overline{z}]/\zeta)\\
&=e(-{}^{\mathrm{t}}(\lambda\overline{\tau}+\mu)S(\lambda\tau+\mu)/\zeta)\\
&\quad\cdot e\left(\frac{2(1-\omega)}{\zeta\omega}S(z,\overline{z})+2S\left(\frac{-(\lambda\overline{\tau}+\mu)}{\zeta},z\right)+2S\left(\frac{\lambda\tau+\mu}{\omega\zeta},\overline{z}\right)\right)\\
&=e(-{}^{\mathrm{t}}(\lambda\overline{\tau}+\mu)S(\lambda\tau+\mu)/\zeta)e\left(({}^{\mathrm{t}}u\,{}^{\mathrm{t}}v)\mathfrak{z}\begin{pmatrix}u\\v\end{pmatrix}+2({}^{\mathrm{t}}u\,{}^{\mathrm{t}}v)Px\right)
\end{aligned}$$

where we put

$$P=\begin{pmatrix}\tau 1_l & \overline{\tau}1_l\\ 1_l & 1_l\end{pmatrix}\ (\in \mathrm{GL}_{2l}(\mathbb{C})),$$

$$\mathfrak{z}=\frac{1-\omega}{\zeta\omega}\begin{pmatrix}2|\tau|^2S & (\tau+\overline{\tau})S\\ (\tau+\overline{\tau})S & 2S\end{pmatrix}\ (\in \mathrm{M}_{2l}(\mathbb{C}))$$

and

$$x=\begin{pmatrix}-S(\lambda\overline{\tau}+\mu)/\zeta\\ S(\lambda\tau+\mu)/\omega\zeta\end{pmatrix}\ (\in\mathbb{C}^{2l}).$$

The matrix $\mathfrak{z}$ is a point of the Siegel upper half plane of degree $2l$. If we put $A=-2\pi i\mathfrak{z}$ and note that $((\omega-1)/\omega)^2=(t-2)/\omega$ with $t=a+d$, then by a standard calculation of the integral,

$$\begin{aligned}
I(\tau,R;X)=&e(-{}^{\mathrm{t}}(\lambda\overline{\tau}+\mu)S(\lambda\tau+\mu)/\zeta)\left(\frac{-\omega}{4(2-t)}\right)^{l/2}\\
&\cdot\det(S)^{-1}e(2\pi i\,{}^{\mathrm{t}}x\,{}^{\mathrm{t}}PA^{-1}Px).
\end{aligned}$$

Since τ is a root of the quadratic equation: $c\tau^2+(d-a)\tau-b=0$, an elementary calculation shows that the integral $I(\tau,R;X)$ equals

$$\begin{aligned}
&\det(2S)^{-1}(2-t)^{-l/2}(-\omega)^{l/2}\\
&\quad\cdot e\left(\left(-\frac{1}{\zeta}+\frac{2}{\zeta(1-\omega)}\right){}^{\mathrm{t}}(\lambda\overline{\tau}+\mu)S(\lambda\tau+\mu)\right)\\
&=\det(2S)^{-1}(2-t)^{-l/2}(-\omega)^{l/2}\\
&\quad\cdot e\left(\frac{1}{t-2}(bS[\lambda]-(a-d)S(\lambda,\mu)-cS[\mu])\right).
\end{aligned}$$

Each elliptic element R of $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SL}_2(\mathbb{R})$-conjugate to some element $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $0 < \theta < 2\pi$. We write $\theta(R)$ for this θ if R is to be specified. Thus the following expressions for $\operatorname{tr} U(R)$ for elliptic elements R of $\mathrm{SL}_2(\mathbb{Z})$ immediately follows from Lemma 3.5 and the above last identity, if we notice that $R\tau = \tau$ and $\omega = J(R, \tau)$.

Proposition 3.6. *Let* $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an elliptic element of* $\mathrm{SL}_2(\mathbb{Z})$ *and set* $t = a + d$. *Then,*

$$\operatorname{tr} U(R) = \frac{\varepsilon(R)}{(2-t)^{l/2}} \sum_{(\lambda,\mu) \in L/L(R-1_2)} e\left(\frac{1}{t-2}(bS[\lambda] - (a-d)S(\lambda,\mu) - cS[\mu])\right),$$

where

$$\varepsilon(R) = \begin{cases} e^{-\pi i l/2} & 0 < \theta(R) < \pi \\ e^{\pi i l/2} & \pi < \theta(R) < 2\pi. \end{cases}$$

Remark. In the case of $l = 1$, this proposition is nothing but only a special case of Theorem 2 in [S-Z2].

We set, for $\nu = 2$ or 3,

$$G_\nu(S) = \nu^{-l/2} \sum_{\mathbf{x} \in (\mathbb{Z}/\nu\mathbb{Z})^l} e({}^t\mathbf{x} S \mathbf{x}/\nu).$$

Denote by J (resp. W) the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$) of $\mathrm{SL}_2(\mathbb{Z})$. It is well-known that $\{J, J^3, W, W^2, W^4, W^5\}$ forms a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-conjugacy classes of elliptic elements of $\mathrm{SL}_2(\mathbb{Z})$. Obviously, $\theta(J) = \pi/2$, $\theta(W) = \pi/3$. We have, immediately by Proposition 3.6,

$$\begin{aligned} &\operatorname{tr} U(J) = e^{-\pi i l/2} G_2(S), && \operatorname{tr} U(J^3) = e^{\pi i l/2} G_2(S), \\ &\operatorname{tr} U(W) = e^{-\pi i l/2}, && \operatorname{tr} U(W^2) = e^{-\pi i l/2} G_3(S), \\ &\operatorname{tr} U(W^4) = e^{\pi i l/2} \cdot \overline{G_3(S)}, && \operatorname{tr} U(W^5) = e^{\pi i l/2}. \end{aligned} \tag{3.15}$$

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index having the element -1_2. Denote by $\{R\}_\Gamma$ the Γ-conjugacy classes of elliptic elements of Γ and for each elliptic element R of Γ denote by $2\nu(R)$ the order of the centralizer

$Z_\Gamma(R)$ of R in Γ. Let $\psi(z)$ denote the logarithmic derivative of the gamma function:

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

Let k be an integer and κ the number given by (2.1). Now to describe the resolvent trace formula in an explicit form we compute the following function of s in a reasonable form (see Theorem 2.5.1 of [Fi]):

$$\frac{1}{2s-1} \sum_{\{R\}_\Gamma\, 0<\theta<\pi} C(R,k;s) \tag{3.16}$$

where R runs over a complete set of representatives of the Γ-conjugacy classes of elliptic elements of Γ with $0 < \theta(R) < \pi$ and where we put

(3.17)
$$C(R,k;s) = \operatorname{tr}\chi_{\varepsilon(k)}(R) \cdot \frac{ie^{2i\kappa\theta(R)}}{2\nu(R)^2 \sin\theta(R)}$$
$$\cdot \sum_{j=0}^{\nu(R)-1} \left(e^{i\theta(R)(2j+1)}\psi\left(\frac{s+\kappa+j}{\nu(R)}\right) - e^{-i\theta(R)(2j+1)}\psi\left(\frac{s-\kappa+j}{\nu(R)}\right)\right).$$

Denote by $e_\nu(\Gamma)$ ($\nu = 2$ or 3) the number of Γ-conjugacy classes of elliptic elements of Γ of order 2ν. Since any elliptic element R of Γ is $\mathrm{SL}_2(\mathbb{Z})$-conjugate to one of J, J^3, W, W^2, W^4, W^5, and by Corollary 1.3.8 of [Fi], $\chi_\pm(URU^{-1}) = \chi_\pm(U)\chi_\pm(R)\chi_\pm(U)^{-1}$ if $U \in \mathrm{SL}_2(\mathbb{Z})$ and $0 < \theta(R) < \pi$, the quantity (3.16) equals

$$\frac{1}{2s-1}\{e_2(\Gamma)C(J,k;s) + e_3(\Gamma)\sum_{j=1}^{2} C(W^j,k;s)\}. \tag{3.18}$$

For any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, denote by $c(A)$ the $(2,1)$-entry c of A. We have, by the identities (1.4), (1.6), and the definition of $\chi_\pm$,

$$\left.\begin{aligned} \operatorname{tr}\chi(M) &= \operatorname{tr}\chi_+(M) + \operatorname{tr}\chi_-(M) \\ \operatorname{tr}\chi(-M) &= \sigma_{2\kappa}(-1_2, M)e^{\pi i l/2}(\operatorname{tr}\chi_+(M) - \operatorname{tr}\chi_-(M)) \end{aligned}\right. \quad (M \in \mathrm{SL}_2(\mathbb{Z})).$$

We notice that, if $c(M) > 0$, then, $\sigma_{2\kappa}(-1_2, M) = (-1)^l$. In this case we get

$$\operatorname{tr}\chi_{\varepsilon(k)}(M) = \frac{1}{2}(\operatorname{tr}\chi(M) + e^{\pi i(k+l/2)}\operatorname{tr}\chi(-M)) \qquad \text{if } c(M) > 0.$$

Therefore with the help of (3.15) and the equality (1.3), the traces which we need are expressed as follows:

$$
\begin{aligned}
&\operatorname{tr}\chi_{\varepsilon(k)}(J) = G_2(S)(e^{\pi il/2} + e^{\pi ik})/2, \\
(3.19)\qquad &\operatorname{tr}\chi_{\varepsilon(k)}(W) = (e^{\pi il/2} + e^{\pi ik}G_3(S))/2, \\
&\operatorname{tr}\chi_{\varepsilon(k)}(W^2) = (e^{\pi il/2}\cdot\overline{G_3(S)} + e^{\pi ik})/2.
\end{aligned}
$$

We define the numbers $\epsilon_\nu(n;S)$ ($\nu = 2$ or 3, $n \in \mathbb{Z}$) by

$$
\begin{aligned}
&\epsilon_2(n;S) = -G_2(S)\cos((n+l/2)\pi/2) \\
(3.20)\quad &\epsilon_3(n;S) = \frac{\sin((n+l-1)\pi/3)}{2\sin(\pi/3)} + \\
&\qquad \frac{i}{4\sin(\pi/3)}\left(G_3(S)e^{-2\pi i(n+l/4-1)/3} - \overline{G_3(S)}e^{2\pi i(n+l/4-1)/3}\right).
\end{aligned}
$$

Then it easily follows from (3.17), (3.19), and (3.20) that

$$
\begin{aligned}
&C(J,k;s) = \\
&\frac{1}{8}\sum_{j=0}^{1}\epsilon_2(k-2j;S)(\psi((s-\kappa+j)/2) - \psi((s+\kappa+1-j)/2)) \\
(3.21)\quad &\sum_{a=1}^{2}C(W^a,k;s) = \\
&\frac{1}{9}\sum_{j=0}^{2}\epsilon_3(k-2j;S)(\psi((s-\kappa+j)/3) - \psi((s+\kappa+2-j)/3)).
\end{aligned}
$$

It is worth while to mention that $\epsilon_2(n;S)$, $\epsilon_3(n;S)$ take only the values $0, \pm 1$, if $l = 1$ (i.e., S is a positive integer) or $S = 1_l$.

§4. Selberg trace formula

Let k be an integer and κ the number given by (2.1). Let $\varepsilon(k)$ denote the sign $+$ or $-$ according as k is even or odd as before. The theta multiplier system $\chi_{\varepsilon(k)}$ defined in § 1 satisfies the property (2.2) and hence forms a unitary multiplier system of $\mathrm{SL}_2(\mathbb{Z})$ in the sense of Definition 1.3.4 in [Fi]. Therefore the general theory of the resolvent trace formula (Theorem 2.5.1, p.106 in [Fi]) can be applicable to our space $\mathcal{H}^{\varepsilon(k)}_{\kappa,S,\Gamma}$ which is associated with $\chi_{\varepsilon(k)}$. Recall that $\mathcal{H}^{\varepsilon(k)}_{\kappa,S,\Gamma}$ is canonically identified with the space $\mathcal{H}_{\kappa,S,\Gamma}$ as we have observed in § 2.

Let λ_0, λ_1, …, λ_n, … be all the eigen values of the self-adjoint operator $-\Delta_\kappa^\sim : \mathcal{D}_\kappa^\sim \to \mathcal{H}_{\kappa,S,\Gamma}$ counted with multiplicities. Since they are all real numbers, one can arrange them in the manner

$$\lambda_0 \le \lambda_1 \le \ldots \le \lambda_n \le \ldots .$$

Moreover one can write

$$\lambda_n = 1/4 + r_n^2 \quad \text{with} \qquad r_n \in [0,\infty) \cup \frac{1}{i}(0,\infty) \quad (n \in \mathbb{N} \cup \{0\}).$$

Set, for $s, a \in \mathbb{C}$,

$$D_{k,S,\Gamma}(s,a) = \sum_{n=0}^{\infty} \left(\frac{1}{(s-1/2)^2 + r_n^2} - \frac{1}{(a-1/2)^2 + r_n^2} \right). \tag{4.1}$$

It is known for instance by Theorem 1.6.5 in [Fi] that the infinite series on the right hand side of (4.1) is absolutely convergent and that $D_{k,S,\Gamma}(s,a)$ indicates a meromorphic function of s, a.

We note that

$$\dim_{\mathbb{C}} V_{\varepsilon(k)} = (d + (-1)^k d_0)/2.$$

Denote by $v(\Gamma\backslash\mathfrak{H})$ the volume of a fundamental domain $\Gamma\backslash\mathfrak{H}$ with respect to $d\omega(\tau)$: $v(\Gamma\backslash\mathfrak{H}) = \int_{\Gamma\backslash\mathfrak{H}} d\omega(\tau)$.

In view of the results in § 2 and the expressions (1.9), (3.16), (3.18), (3.21), the resolvent trace formula of Fischer ([Fi], Theorem 2.5.1) can be reformulated in our situation as follows.

Theorem 4.1 (Resolvent trace formula for $\mathcal{H}_{\kappa,S,\Gamma}$). *Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index having the element -1_2. Let k be an integer and κ the real number given by* (2.1). *Denote by $\varphi(s)$ the determinant of the scattering matrix $\Phi(s)$ given by* (2.14): $\varphi(s) = \det\Phi(s)$. *Assume that $s, a \in \mathbb{C}$ with* $\mathrm{Re}(s), \mathrm{Re}(a) > 1$ *and* $|\kappa| - s, |\kappa| - a \notin \mathbb{N} \cup \{0\}$. *Then,*

(4.2)

$$
\begin{aligned}
&D_{k,S,\Gamma}(s,a)= \\
&-\frac{1}{8\pi}(d+(-1)^k d_0)v(\Gamma\backslash\mathfrak{H})(\psi(s+\kappa)+\psi(s-\kappa)) \\
&+\frac{1}{2s-1}\cdot\frac{Z'_{\Gamma,S,\varepsilon(k)}}{Z_{\Gamma,S,\varepsilon(k)}}(s) \\
&+\frac{1}{2s-1}\{\frac{e_2(\Gamma)}{8}\sum_{j=0}^{1}\epsilon_2(k-2j;S) \\
&\qquad\qquad\cdot\left(\psi\left(\frac{s-\kappa+j}{2}\right)-\psi\left(\frac{s+\kappa+1-j}{2}\right)\right) \\
&\qquad+\frac{e_3(\Gamma)}{9}\sum_{j=0}^{2}\epsilon_3(k-2j;S) \\
&\qquad\qquad\cdot\left(\psi\left(\frac{s-\kappa+j}{3}\right)-\psi\left(\frac{s+\kappa+2-j}{3}\right)\right)\} \\
&+\frac{1}{2s-1}[-\log 2\cdot\frac{h(d+(-1)^k d_0)}{2}-\log\prod_{j=1}^{h}\prod_{r\in R^*_{S,j,k}}\sin(\pi\beta_{jr}) \\
&\qquad+(\psi(s+\kappa)-\psi(s-\kappa))\left(\frac{h(d+(-1)^k d_0)}{4}-\sum_{j=1}^{h}\beta_j\right) \\
&\qquad+t_\infty(\psi(s-\kappa)-\psi(s)-\psi(s+1/2))] \\
&+\frac{1}{2s-1}\cdot\xi_{\mathrm{par},\Phi}(s) \\
&-\{\text{the same expression with } s \text{ being replaced by } a\},
\end{aligned}
$$

where we set

(4.3)
$$
\begin{aligned}
\xi_{\mathrm{par},\Phi}(s)=&\frac{1}{2s-1}\,\mathrm{tr}(1_{t_\infty}-\Phi(1/2))+ \\
&\frac{2s-1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(s-1/2)^2+t^2}-\frac{1}{1/4+t^2}\right)\cdot\frac{\varphi'}{\varphi}(1/2+it)\,dt,
\end{aligned}
$$

and

$$\beta_j=\sum_{r\in R^*_{S,j,k}}\beta_{jr}\qquad(1\le j\le h).$$

Moreover the integral on the right hand side of (4.3) *is absolutely convergent for* $\mathrm{Re}(s) > 1/2$ *and* $\xi_{\mathrm{par},\Phi}(s)$ *is analytically continued to a meromorphic function in the whole* s*-plane* (*see also* Lemma 2.4.19 *and* (2.4.6) *of* [Fi]).

It has been verified by the expression (2.4.6), p.103 of [Fi] that $\xi_{\mathrm{par},\Phi}(s)$ has at most simple poles whose residues are rational integers and that in particular it has a simple pole at $s = 1/2$ with the residue $(t_\infty - \mathrm{tr}\,\Phi(1/2))/2 \in \mathbb{Z}$ (note that $\Phi(1/2)$ is a hermitian matrix with $\Phi(1/2)^2 = 1_{t_\infty}$). Proposition 2.17 of [He2], p.440 implies the functional equation of $\xi_{\mathrm{par},\Phi}(s)$:

$$\xi_{\mathrm{par},\Phi}(s) + \xi_{\mathrm{par},\Phi}(1-s) = \frac{\varphi'}{\varphi}(s), \tag{4.4}$$

which can be verified also by Lemma 2.4.16 and (2.4.6) of [Fi].

Now we determine briefly the explicit form of the functional equation satisfied by our Selberg zeta functions $Z_{\Gamma,S,\pm}(s)$. Denote by $G(z)$ the Barnes G-function which is an entire function of z satisfying $G(z+1) = \Gamma(z)G(z)$. For the precise definition of $G(z)$ we refer to Definition 3.1.1 of [Fi]. Following Fischer [Fi], we define the functions $\Xi_{\mathrm{I}}(s)$, $\Xi_{\mathrm{hyp}}(s)$, $\Xi_{\mathrm{ell}}(s)$, and $\Xi^*_{\mathrm{par}}(s)$ as follows:

$$\begin{aligned}
\Xi_{\mathrm{I}}(s) = \exp(&\frac{d+(-1)^k d_0}{4\pi} \cdot v(\Gamma\backslash\mathfrak{H})\{s\log(2\pi) + s(1-s)\\
&+ (1/2+\kappa)\log\Gamma(s+\kappa) + (1/2-\kappa)\log\Gamma(s-\kappa)\\
&- \log G(s+\kappa+1) - \log G(s-\kappa+1)\}),\\
\Xi_{\mathrm{hyp}}(s) = {}& Z_{\Gamma,S,\varepsilon(k)}(s),\\
\Xi_{\mathrm{ell}}(s) = {}& \left\{\prod_{j=0}^{1}\left(\frac{\Gamma((s-\kappa+j)/2)}{\Gamma((s+\kappa+1-j)/2)}\right)^{\epsilon_2(k-2j;S)}\right\}^{e_2(\Gamma)/4}\\
&\cdot\left\{\prod_{j=0}^{2}\left(\frac{\Gamma((s-\kappa+j)/3)}{\Gamma((s+\kappa+2-j)/3)}\right)^{\epsilon_3(k-2j;S)}\right\}^{e_3(\Gamma)/3}
\end{aligned}$$

and

$$\Xi^*_{\mathrm{par}}(s) = 2^{-h(d+(-1)^k d_0)s/2} \cdot \prod_{j=1}^{h} \left(\frac{\Gamma(s+\kappa)}{\Gamma(s-\kappa)} \right)^{(d+(-1)^k d_0)/4-\beta_j}$$
$$\cdot \left(\frac{\Gamma(s-\kappa)}{\Gamma(s)\Gamma(s+1/2)} \right)^{t_\infty} \cdot \prod_{j=1}^{h} \prod_{r \in R^*_{S,j,k}} (\sin \pi \beta_{jr})^{-s}.$$

Then, $\Xi_{\mathrm{I}}(s)$, $\Xi_{\mathrm{ell}}(s)$, $\Xi^*_{\mathrm{par}}(s)$ define holomorphic functions in $\mathbb{C} - (-\infty, |\kappa|)$. Set

$$\Xi^*(s) = \Xi_{\mathrm{I}}(s)\Xi_{\mathrm{hyp}}(s)\Xi_{\mathrm{ell}}(s)\Xi^*_{\mathrm{par}}(s).$$

Then the resolvent trace formula (Theorem 4.1) turns out

$$D_{k,S,\Gamma}(s,a) = \tag{4.5}$$
$$\frac{1}{2s-1}\left(\frac{\Xi^{*\prime}}{\Xi^*}(s) + \xi_{\mathrm{par},\Phi}(s) \right) - \frac{1}{2a-1}\left(\frac{\Xi^{*\prime}}{\Xi^*}(a) + \xi_{\mathrm{par},\Phi}(a) \right).$$

Via this formula and by (4.4) the function $(\Xi^{*\prime}/\Xi^*)(s)$ is continued analytically to a meromorphic function in the whole s-plane which satisfies the functional equation

$$(\Xi^{*\prime}/\Xi^*)(s) + (\Xi^{*\prime}/\Xi^*)(1-s) + (\varphi'/\varphi)(s) = 0. \tag{4.6}$$

By the same formula (4.5), $(\Xi^{*\prime}/\Xi^*)(s)$ has only simple poles with the residues all rational integers. Therefore, $\Xi^*(s)$ itself can be continued analytically to a meromorphic function in the whole s-plane and the equation (4.6) implies the functional equation of $\Xi^*(s)$:

(4.7) $\Xi^*(1-s) = \varphi(s)\Xi^*(s)$ (see [He2], Ch.10, (5.7) and [Ar3]).

This functional equation follows also from (3.1.3), (3.2.1) of [Fi]. It has been proved in (3.1.4), p.116 of [Fi] that $\Xi_{\mathrm{hyp}}(s)$ itself can be continued to a meromorphic function in the whole s-plane.

Let $\mathcal{H}_{\kappa,S,\Gamma}$, Δ_κ, $\mathcal{D}_\kappa$, $\Delta_\kappa^\sim$, and $\mathcal{D}_\kappa^\sim$ be the same as in § 2. For $s \in \mathbb{C}$, denote by $\mathcal{H}_{\kappa,S,\Gamma}(s)$ the subspace consisting of $f \in \mathcal{D}_\kappa^\sim$ with $-\Delta_\kappa^\sim f = s(1-s)f$. Let $d_{k,S,\Gamma}(s)$ denote the multiplicity of the eigenvalue $s(1-s)$ of the self-adjoint operator $-\Delta_\kappa^\sim : \Delta_\kappa^\sim \to \mathcal{H}_{\kappa,S,\Gamma}$. It has been proved by Roelcke in Satz 5.6, 5.7 of [Ro1] that

$$\mathcal{H}_{\kappa,S,\Gamma}(s) = \{ f \in \mathcal{D}_\kappa \mid -\Delta_\kappa f = s(1-s)f \}. \tag{4.8}$$

Obviously,

$$d_{k,S,\Gamma}(s) = \dim_{\mathbb{C}} \mathcal{H}_{\kappa,S,\Gamma}(s). \tag{4.9}$$

Furthermore by the definition of the numbers r_n $(n \in \mathbb{N} \cup \{0\})$,

$$\mathcal{H}_{\kappa,S,\Gamma}(s) \neq \{0\} \quad \text{if and only if} \quad s = 1/2 \pm ir_n \text{ for some } n.$$

We are much concerned with the subspace $\mathcal{H}_{\kappa,S,\Gamma}(\kappa)$ (resp. $\mathcal{H}_{\kappa,S,\Gamma}(-\kappa)$) if $\kappa > 0$ (resp. $\kappa < 0$), by evaluating at $s = \kappa$ (resp. $s = -\kappa$). As is easily seen from the results of Roelcke in § 2-5, [Ro1], we have

(4.10)

$$\mathcal{H}_{\kappa,S,\Gamma}(\kappa) = \{\, f \in \mathcal{H}_{\kappa,S,\Gamma} \mid \eta^{-\kappa} f(\tau) \text{ is holomorphic in } \tau \,\} \quad \text{if } \kappa > 0,$$
$$\mathcal{H}_{\kappa,S,\Gamma}(-\kappa) = \{\, f \in \mathcal{H}_{\kappa,S,\Gamma} \mid \eta^{\kappa} f(\tau) \text{ is anti-holomorphic in } \tau \,\} \quad \text{if } \kappa < 0,$$

η denoting $\operatorname{Im}\tau$.

§5. Dimension formulas for the spaces of Jacobi forms

We recall the definition of Jacobi forms and skew-holomorphic Jacobi forms following Eichler-Zagier [E-Z] and Skoruppa [Sk2].

Let G^J be the real Jacobi group given by (3.1) acting on $D = \mathfrak{H} \times \mathbb{C}^l$ via (3.2). Assume that Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index with $-1_2 \in \Gamma$. Set

$$\Gamma^J = \{\, (M, (\lambda, \mu), \rho) \mid M \in \Gamma,\ \lambda, \mu \in \mathbb{Z}^l,\ \rho \in \mathrm{Sym}_l(\mathbb{Z}) \,\}.$$

Then Γ^J forms a discrete subgroup of G^J. Let ν be a positive integer. For any function $\phi\colon D \to \mathbb{C}$ and $g = (M, (\lambda, \mu), \rho) \in G^J$, we set

$$(\phi|_{\nu,S}\, g)(\tau, z) = J_{\nu,S}(g, (\tau, z))^{-1} \phi(g(\tau, z)),$$
$$(\phi|^{*}_{\nu,S}\, g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1} \cdot (\overline{J(M, \tau)})^{-\nu+l} |J(M, \tau)|^{-l} \phi(g(\tau, z)).$$

It is immediate to see from (3.3) that, for $g_1, g_2 \in G^J$,

$$\phi|_{\nu,S}\, g_1 g_2 = \phi|_{\nu,S}\, g_1|_{\nu,S}\, g_2 \quad \text{and} \quad \phi|^{*}_{\nu,S}\, g_1 g_2 = \phi|^{*}_{\nu,S}\, g_1|^{*}_{\nu,S}\, g_2. \tag{5.1}$$

For each element M of $\mathrm{SL}_2(\mathbb{Z})$, put $M\infty = \zeta$, which is a cusp of Γ. Denote by Γ_ζ the stabilizer of ζ in Γ. Then the subgroup $M^{-1}\Gamma_\zeta M$ of $\mathrm{SL}_2(\mathbb{Z})$ is generated by

(5.2)

$$-1_2 \text{ and } \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \text{ with a uniquely determined positive integer } N.$$

The space $J_{\nu,S}(\Gamma)$ (resp. $J^*_{\nu,S}(\Gamma)$) of holomorphic Jacobi forms (resp. skew-holomorphic Jacobi forms) of index S and weight ν is defined to be the space consisting of all functions $\phi: D \to \mathbb{C}$ which satisfy the following three conditions:

(5.3)

(i) $\phi(\tau, z)$ is holomorphic in τ and z
(resp. $\phi(\tau, z)$ is a smooth function in τ and holomorphic in z).

(ii) $\phi(\tau, z)$ satisfies the identity

$$\phi|_{\nu,S}\gamma = \phi \qquad (\text{resp. } \phi|^*_{\nu,S}\gamma = \phi) \qquad \text{for any } \gamma \in \Gamma^J.$$

(iii) The function $\phi|_{\nu,S}M$ (resp. $\phi|^*_{\nu,S}M$) for any $M \in \mathrm{SL}_2(\mathbb{Z})$ has a Fourier Jacobi expansion of the form

$$(\phi|_{\nu,S}M)(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\, r\in\mathbb{Z}^l \\ 4n-N{}^{\mathrm{t}}rS^{-1}r\geq 0}} c(n,r)e(n\tau/N + {}^{\mathrm{t}}rz)$$

$$(\text{resp. } (\phi|^*_{\nu,S}M)(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\, r\in\mathbb{Z}^l \\ 4n-N{}^{\mathrm{t}}rS^{-1}r\leq 0}} c(n,r)e\left(\frac{n\overline{\tau}}{N} + \frac{i\eta}{2}({}^{\mathrm{t}}rS^{-1}r) + {}^{\mathrm{t}}rz\right)),$$

where $\eta = \mathrm{Im}\,\tau$ and we choose a positive integer N as in (5.2) for each M. Furthermore in the above (iii), M ($\in \mathrm{SL}_2(\mathbb{Z})$) is identified with the corresponding element of G^J.

Denote by $J^{\mathrm{cusp}}_{\nu,S}(\Gamma)$ (resp. $J^{*\,\mathrm{cusp}}_{\nu,S}$) the subspace of cusp forms of $J_{\nu,S}(\Gamma)$ (resp. $J^*_{\nu,S}(\Gamma)$). Namely, $J^{\mathrm{cusp}}_{\nu,S}(\Gamma)$ (resp. $J^{*\,\mathrm{cusp}}_{\nu,S}(\Gamma)$) consists of all Jacobi forms $\phi \in J_{\nu,S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J^*_{\nu,S}(\Gamma)$) whose Fourier coefficients $c(n,r)$ in the above (iii) equals zero if $4n - N{}^{\mathrm{t}}rS^{-1}r = 0$.

Let k be a rational integer and κ the number given by (2.1). Now we consider the relation between the spaces of Jacobi forms and the spaces

$\mathcal{H}_{\kappa,S,\Gamma}(\kappa)$ $(\kappa > 0)$ or $\mathcal{H}_{\kappa,S,\Gamma}(-\kappa)$ $(\kappa < 0)$. First assume that $\kappa > 0$ (i.e., $k > l/2$). We set, for each element $f = (f_r)_{r\in R_S} \in \mathcal{H}_{\kappa,S,\Gamma}(\kappa)$,

$$\phi(\tau, z) = \sum_{r\in R_S} \eta^{-\kappa} f_r(\tau)\theta_r(\tau, z) \quad ((\tau, z) \in D\,, \eta = \operatorname{Im}\tau). \tag{5.4}$$

This identity is written of the form

$$\phi(\tau, z) = \langle \eta^{-\kappa} f(\tau), \overline{\Theta(\tau, z)}\rangle_S. \tag{5.5}$$

In this case it is not difficult to see from (1.2) and the identity (i) of (2.4) that $\phi|_{k,S}M = \phi$ for any $M \in \Gamma$. Moreover the identity $\phi|_{k,S}X = \phi$ for any $X \in L$ follows directly from (5.4). In the Fourier expansion (2.11) of $f \in \mathcal{H}_{\kappa,S,\Gamma}(\kappa)$ at each cusp ζ_j $(1 \le j \le h)$, the functions $\eta^{-\kappa}q_j(\tau)$ and $\eta^{-\kappa}u_j(\eta)$ are found to be holomorphic functions in τ and constant functions, respectively, since $\eta^{-\kappa}f(\tau)$ is holomorphic in τ according to (4.10). Taking the square integrability (2.4), (ii) of $f(\tau)$ into account, we see from (2.11), (2.12) that $\eta^{-\kappa}f(\tau)$ has the following Fourier expansion at the cusp ζ_j:

$$\begin{aligned} \eta^{-\kappa} f(\tau) &= J(A_j, \tau)^{-2\kappa} \sum_{r\in R_S} f_{jr}^{\sim}(A_j\tau)v_{jr}, \\ f_{jr}^{\sim}(\tau) &= \sum_{n=0}^{\infty} a_{jrn} e((n + \beta_{jr})\tau) \quad \text{with } a_{jrn} \in \mathbb{C}, \end{aligned} \tag{5.6}$$

where $\{v_{jr}\}_{r\in R_S}$ is an orthonormal basis of V given by (2.6) and β_{jr}'s are the constants defined by (2.10). Moreover by the square integrability of $f(\tau)$ again, we observe that in the case of $\kappa \ge 1/2$,

$$a_{jrn} = 0 \ \text{ if } \ n = 0 \ \text{ and } \ \beta_{jr} = 0. \tag{5.7}$$

Substituting the expression (5.6) for $\eta^{-\kappa}f(\tau)$ in (5.5) and then using (2.5) and the theta transformation formula (1.2), we have

$$\phi(\tau, z) = l_j^{-\kappa} J_{k,S}(A_j^*, (\tau, z))^{-1} \sum_{r\in R_S} f_{jr}^{\sim}((A_j^*\tau)/l_j)\theta_r(A_j^*(\tau, z)),$$

which turns out the identity

$$(\phi|_{k,S}A_j^{*-1})(\tau, z) = l_j^{-\kappa} \sum_{r\in R_S} f_{jr}^{\sim}(\tau/l_j)\theta_r(\tau, z).$$

Thus we see easily from (5.6) that $\phi|_{k,S}A_j^{*-1}$ has a Fourier-Jacobi expansion of the form

$$(\phi|_{k,S}A_j^{*-1})(\tau,z) = \sum_{\substack{n\in\mathbb{Z},\, r\in\mathbb{Z}^l \\ 4n-l_j{}^t rS^{-1}r\geq 0}} c_j(n,r)e(n\tau/l_j + {}^t rz)$$

Moreover if $\kappa \geq 1/2$, then by the property (5.7), the Fourier coefficients $c_j(n,r)$ are necessarily zero if $4n - l_j{}^t rS^{-1}r = 0$. Consequently we have proved that, by the correspondence (5.4), for $f \in \mathcal{H}_{\kappa,S,\Gamma}(\kappa)$,

$$\phi \in J_{k,S}(\Gamma) \quad \text{if} \quad \kappa > 0 \quad \text{and} \quad \phi \in J_{k,S}^{\text{cusp}}(\Gamma) \quad \text{if} \quad \kappa \geq 1/2.$$

Conversely, since each $\phi(\tau,z) \in J_{k,S}(\Gamma)$ belongs to the space $\Theta_{S,\tau}$ as a function of z, ϕ has the expression of the form (5.4) with holomorphic functions $f_r(\tau)$ $(r \in R_S)$ on $\mathfrak{H}$. It is easy to see from the condition (ii) of (5.3) and (1.2) that $f = (f_r)_{r\in R_S}$ satisfies the condition (i) of (2.4). Moreover it follows from the Fourier-Jacobi expansion of ϕ ((iii) of (5.3)) that the function $\eta^{-\kappa}f(\tau)$ has a Fourier expansion of the form (5.6) at each cusp ζ_j. In this case as is easily seen, if $0 < \kappa < 1/2$ (resp. $\kappa \geq 1/2$), then, $f = (f_r)_{r\in R_S}$ corresponding to $\phi \in J_{k,S}(\Gamma)$ (resp. $\phi \in J_{k,S}^{\text{cusp}}(\Gamma)$) via the correspondence (5.4) is an element of $\mathcal{H}_{\kappa,S,\Gamma}(\kappa)$.

Next assume $\kappa < 0$. Let $g(\tau) = (g_r(\tau))_{r\in R_S}$ be a V-valued function on $\mathfrak{H}$. We consider the correspondence $g = (g_r)_{r\in R_S} \to \phi$ by

$$\phi(\tau,z) = \sum_{r\in R_S} \eta^\kappa g_r(\tau)\theta_r(\tau,z) = \langle \eta^\kappa g(\tau), \overline{\Theta(\tau,z)}\rangle_S. \tag{5.8}$$

In a manner similar to the case of $\kappa > 0$, it can be shown with the use of (1.2) that the space $\mathcal{H}_{\kappa,S,\Gamma}(-\kappa)$ corresponds one to one onto the space $J_{l-k,S}^*(\Gamma)$ (resp. $J_{l-k,S}^{*\,\text{cusp}}(\Gamma)$) if $-1/2 < \kappa < 0$ (resp. $\kappa \leq -1/2$) via the correspondence (5.8): $g \to \phi$. Thus we obtain

Proposition 5.1. *Let k be an integer and $\kappa = (k - l/2)/2$.*

(i) *If $k \geq 1 + l/2$ (resp. $l/2 < k < 1 + l/2$), then the space $J_{k,S}^{\text{cusp}}(\Gamma)$ (resp. $J_{k,S}(\Gamma)$) is isomorphic to the space $\mathcal{H}_{\kappa,S,\Gamma}(\kappa)$ via the correspondence $\phi \to (f_r)_{r\in R_S}$ in (5.4) as $\mathbb{C}$-vector spaces.*

(ii) *If $k \leq l/2 - 1$ (resp. $l/2 - 1 < k < l/2$), then the space $J_{l-k,S}^{*\,\text{cusp}}(\Gamma)$ (resp. $J_{l-k,S}^*(\Gamma)$) is isomorphic to the space $\mathcal{H}_{\kappa,S,\Gamma}(-\kappa)$ via the correspondence $\phi \to (g_r)_{r\in R_S}$ in (5.8) as $\mathbb{C}$-vector spaces.*

In the final step of this paragraph we employ Theorem 4.1 (the resolvent trace formula for $\mathcal{H}_{\kappa,S,\Gamma}$) to calculate the dimensions of the spaces of Jacobi forms. Set, for any integer k,

(5.9)

$$\begin{aligned}\lambda_\Gamma(k;S) =&\frac{d+(-1)^k d_0}{8\pi}(k-l/2-1)v(\Gamma\backslash\mathfrak{H}) - \frac{e_2(\Gamma)}{4}\cdot\epsilon_2(k;S)\\&-\frac{e_3(\Gamma)}{3}\cdot\epsilon_3(k;S)+\frac{h(d+(-1)^k d_0)}{4}-\sum_{j=1}^{h}\beta_j - t_\infty,\\ \mu_\Gamma(k;S) =&\frac{d+(-1)^k d_0}{8\pi}(-k+l/2-1)v(\Gamma\backslash\mathfrak{H}) + \frac{e_2(\Gamma)}{4}\cdot\epsilon_2(k-2;S)\\&+\frac{e_3(\Gamma)}{3}\cdot\epsilon_3(k-4;S)-\frac{h(d+(-1)^k d_0)}{4}+\sum_{j=1}^{h}\beta_j.\end{aligned}$$

Since the scattering matrix $\Phi(s)$ given by (2.14) depends on Γ, S, and the weight k, we write $\Phi_k(s)$ in place of $\Phi(s)$ if the weight k is to be specified. Our main theorem is the following.

Theorem 5.2. *Let S be a positive definite half-integral symmetric matrix of size l and k an integer. Assume that Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index having the element -1_2.*

(i) *If $k > l/2+2$ (i.e., $\kappa > 1$), then,* $\dim_{\mathbb{C}} J_{k,S}^{\mathrm{cusp}}(\Gamma) = \lambda_\Gamma(k;S)$.

(ii) *If $k < l/2-2$ (i.e., $\kappa < -1$), then,* $\dim_{\mathbb{C}} J_{l-k}^{*\,\mathrm{cusp}}(\Gamma) = \mu_\Gamma(k;S)$.

(iii) *Suppose that l is odd. Let ε denote the sign $\varepsilon((l-1)/2)$. Namely, ε takes the sign $+$ or $-$ according as $l\equiv 1 \mod 4$ or $l \equiv 3 \mod 4$. Then,*

$$\begin{aligned}\dim_{\mathbb{C}} J_{(l+3)/2,S}^{\mathrm{cusp}}(\Gamma) &= \mathrm{Res}_{s=3/4}(Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s)+\lambda_\Gamma((l+3)/2;S),\\ \dim_{\mathbb{C}} J_{(l+1)/2,S}^{*}(\Gamma) &= \mathrm{Res}_{s=3/4}(Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s),\\ \dim_{\mathbb{C}} J_{(l+3)/2,S}^{*\,\mathrm{cusp}}(\Gamma) &= \mathrm{Res}_{s=3/4}(Z'_{\Gamma,S,-\varepsilon}/Z_{\Gamma,S,-\varepsilon})(s)+\mu_\Gamma((l-3)/2;S),\\ \dim_{\mathbb{C}} J_{(l+1)/2,S}(\Gamma) &= \mathrm{Res}_{s=3/4}(Z'_{\Gamma,S,-\varepsilon}/Z_{\Gamma,S,-\varepsilon})(s).\end{aligned}$$

(iv) *Suppose that l is even. Let ε denote the sign $\varepsilon(l/2)$. Namely, ε takes the sign $+$ or $-$ according as $l\equiv 0 \mod 4$ or $l\equiv 2 \mod 4$. Then,*

$$\begin{aligned}\dim_{\mathbb{C}} J_{l/2+2,S}^{\mathrm{cusp}}(\Gamma) &= \mathrm{mult}(1_{\Gamma,\chi})+\lambda_\Gamma(l/2+2;S),\\ \dim_{\mathbb{C}} J_{l/2+2,S}^{*\,\mathrm{cusp}}(\Gamma) &= \mathrm{mult}(1_{\Gamma,\chi})+\mu_\Gamma(l/2-2;S),\end{aligned}$$

and

$$\begin{aligned}
\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{l/2+1,S}(\Gamma) &= \frac{1}{2}\operatorname{Res}_{s=1/2}(Z'_{\Gamma,S,-\varepsilon}/Z_{\Gamma,S,-\varepsilon})(s), \\
&\quad + \frac{1}{2}\lambda_\Gamma(l/2+1;S) + \frac{1}{4}(t_\infty - \operatorname{tr}\Phi_{l/2+1}(s)), \\
\dim_{\mathbb{C}} J^{*\,\mathrm{cusp}}_{l/2+1,S}(\Gamma) &= \frac{1}{2}\operatorname{Res}_{s=1/2}(Z'_{\Gamma,S,-\varepsilon}/Z_{\Gamma,S,-\varepsilon})(s), \\
&\quad + \frac{1}{2}\mu_\Gamma(l/2-1;S) + \frac{1}{4}(t_\infty - \operatorname{tr}\Phi_{l/2-1}(s)), \\
\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{l/2+1,S}(\Gamma) + \dim_{\mathbb{C}} J^{*\,\mathrm{cusp}}_{l/2+1,S}(\Gamma) &= \operatorname{Res}_{s=1/2}(Z'_{\Gamma,S,-\varepsilon}/Z_{\Gamma,S,-\varepsilon})(s),
\end{aligned}$$

where $\mathrm{mult}(1_\Gamma, \chi)$ *denotes the multiplicity of the identity representation* 1_Γ *of* Γ *occuring in the unitary representation* χ.

Proof of Theorem 5.2. In the resolvent trace formula (4.2) we consider $(2s-1)D_{k,S,\Gamma}(s,a)$ as a meromorphic function of s with a being fixed ($\mathrm{Re}(a) > \mathrm{Max}(1, |\kappa|)$). Notice that the residue at $s = \rho$ of this function equals $d_{k,S,\Gamma}(\rho)$ (resp. $2d_{k,S,\Gamma}(\rho)$) if $\rho \neq 1/2$ (resp. $\rho = 1/2$), by the definition (4.1) and that $d_{k,S,\Gamma}(\rho) = d_{k,S,\Gamma}(1-\rho)$. Moreover we note that the function $\psi(z)$ has simple poles at $z = -n$ ($n \in \mathbb{N} \cup \{0\}$) with the residue -1 and has no other poles.

First assume that $|\kappa| > 1$. We see easily from the expression on the right hand side of (4.2) that the residue at $s = \kappa$ (resp. $s = -\kappa$) of the function $(2s-1)D_{k,S,\Gamma}(s,a)$ coincides with $\lambda_\Gamma(k;S)$ (resp. $\mu_\Gamma(k;S)$) if $\kappa > 1$ (resp. $\kappa < -1$). The assertions (i), (ii) immediately follows from (4.9) and Proposition 5.1.

Next assume that $|\kappa| \leq 1$. Suppose that l is odd. Then, $|\kappa| = 3/4$ or $1/4$. If $|\kappa| = 3/4$ (i.e., $k = (l \pm 3)/2$), then the first and third identities in the assertion (iii) easily follows in a manner similar to the case of $|\kappa| > 1$. Let $\kappa = \pm 1/4$ ($k = (l \pm 1)/2$). We calculate the residue at the pole $s = 3/4$ of the function $(2s-1)D_{k,S,\Gamma}(s,a)$ on the both sides of (4.2). We get, by Proposition 5.1 and the relation $d_{k,S,\Gamma}(1/4) = d_{k,S,\Gamma}(3/4)$, the second and fourth identities in (iii).

Suppose that l is even. Then, $|\kappa| = 1$, $1/2$, or 0. If $\kappa = 1$ (i.e., $k = l/2+2$), we have, in a manner similar to the case of $|\kappa| > 1$,

$$\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{l/2+2}(\Gamma) = \operatorname{Res}_{s=1}(Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s) + \lambda_\Gamma(l/2+2;S).$$

If $\kappa = 0$ (i.e., $k = l/2$), we calculate the residue at the pole $s = 1$ ($= 1 - \kappa$) of the function $(2s-1)D_{k,S,\Gamma}(s,a)$ similarly and get

$$d_{k,S,\Gamma}(0) = \operatorname{Res}_{s=1}(Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s).$$

By (4.8), (4.9), we have

$$d_{k,S,\Gamma}(0)=\dim_{\mathbb{C}}\{\, v\in V \mid \chi(M)v=v \ \text{ for any } \ M\in\Gamma\,\}=\mathrm{mult}(1_\Gamma,\chi).$$

The second identity in the assertion (iv) is quite similarly verified. If $|\kappa|=1/2$ (i.e., $k=l/2\pm 1$), again by calculating the residue at the pole $s=1/2$ of the function $(2s-1)D_{k,S,\Gamma}(s,a)$ in two manners via the trace formula (4.2), the third and fourth identities in the assertion (iv) are similarly derived (cf. Proposition 2.2, (i) of [Ar1]). It is easy to see from (5.9) that

$$\lambda_\Gamma(l/2+1;S)+\mu_\Gamma(l/2-1;S)=-t_\infty,$$

since $\epsilon_2(n;S)$ (resp. $\epsilon_3(n;S)$) depends on $n \mod 4$ (resp. $n \mod 6$). Denote by $\varphi^k_{jr,lp}(s)$ the (jr,lp)-entry of the matrix $\Phi_k(s)$ to specify the weight k. Applying Lemma 1.1 of [Ar1] to the present situation, we see easily that

$$\varphi^{l/2+1}_{jr,lp}(s)=-\varphi^{l/2-1}_{jr,lp}(s),$$

since χ, $\chi_\pm$ depends only on S and not on k. Therefore the last identity in (iv) follows.

q.e.d.

§6. Real analytic Eisenstein series for the Jacobi group

In this paragraph we assume that $\Gamma=SL_2(\mathbb{Z})$ for simplicity (we can get rid of this assumption, but it is much more tedious to treat with any subgroup Γ of $SL_2(\mathbb{Z})$ of finite index).

For $r\in(2S)^{-1}\mathbb{Z}^l$, $s\in\mathbb{C}$, and $k\in\mathbb{Z}$, define a function $\phi_{k,r,s}\colon D\to\mathbb{C}$ by

$$\phi_{k,r,s}(\tau,z)=e(\tau S[r]+2S(r,z))\eta^{s-\kappa}\quad(\eta=\operatorname{Im}\tau,\ \kappa=(k-l/2)/2).$$

Let $\Gamma^J_{\infty,+}$ denote the subgroup of the Jacobi group Γ^J given by

$$\Gamma^J_{\infty,+}=\{\left(\begin{pmatrix}1&n\\0&1\end{pmatrix},(0,\mu),\rho\right)\mid n\in\mathbb{Z},\ \mu\in\mathbb{Z}^l,\ \rho\in\mathrm{Sym}_l(\mathbb{Z})\,\}.$$

We set, for each $r\in(2S)^{-1}\mathbb{Z}^l$ with the condition $S[r]\in\mathbb{Z}$,

$$(6.1)\quad E_{k,S,r}((\tau,z),s)=\sum_{\gamma\in\Gamma^J_{\infty,+}\backslash\Gamma^J}(\phi_{k,r,s}|_{k,S}\gamma)(\tau,z)\qquad((\tau,z)\in D).$$

By (5.1) and the property $\phi_{k,r,s}|_{k,S}\gamma_1=\phi_{k,r,s}$ for any $\gamma_1\in\Gamma^J_{\infty,+}$, the infinite series on the right hand side of (6.1) is well-defined and, as we shall see later, absolutely convergent for $\mathrm{Re}(s) > 1 + l/4$. As a complete set of representatives of the right cosets $\Gamma^J_{\infty,+}\backslash\Gamma^J$, we may take the set

$$\{ (M, (\lambda, 0), 0) \mid M \in \Gamma^+_\infty\backslash\Gamma,\ \lambda \in \mathbb{Z}^l \},$$
$$\text{where } \Gamma^+_\infty = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \}.$$

Then we have

$$\begin{aligned} &E_{k,S,r}((\tau, z), s) \\ &= \sum_{M\in\Gamma^+_\infty\backslash\Gamma} \sum_{q\in\mathbb{Z}^l} J(M,\tau)^{-k} e\left(-\frac{c}{J(M,\tau)} S[z]\right) \\ &\quad \cdot e\left(M\tau \cdot S[q+r] + 2S(q+r, \frac{z}{J(M,\tau)})\right) (\mathrm{Im}\, M\tau)^{s-\kappa}, \end{aligned} \tag{6.2}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We divide the first summation into two parts according as $c = 0$ or $c \neq 0$. We denote by $E^I_{k,S,r}((\tau, z), s)$ (resp. $E^{II}_{k,S,r}((\tau, z), s)$) the infinite series obtained by replacing the first summation with $\sum_{M\in\Gamma^+_\infty\backslash\Gamma,\, c=0}$ (resp. $\sum_{M\in\Gamma^+_\infty\backslash\Gamma,\, c\neq0}$) on the right hand side of (6.2). Then we get, in a manner similar to the argument in p.18 of [E-Z] (see also [Ar2], § 3),

$$\begin{aligned} E^I_{k,S,r}((\tau, z), s) &= \eta^{s-\kappa}(\theta_r(\tau, z) + (-1)^k \theta_{-r}(\tau, z)), \\ E^{II}_{k,S,r}((\tau, z), s) &= \sum_{(c,d)=1,\, c\neq0} \sum_{q\in\mathbb{Z}^l} \frac{\eta^{s-\kappa}}{(c\tau+d)^k |c\tau+d|^{2(s-\kappa)}} \\ &\quad \cdot e\left(\frac{-cS[z-(q+r)/c]}{c\tau+d} + \frac{a}{c} S[q+r]\right). \end{aligned}$$

Replacing q with $\lambda - cq$ and d with $d + cp$, we have another expression for $E^{II}_{k,S,r}((\tau, z), s)$:

$$\begin{aligned} E^{II}_{k,S,r}((\tau, z), s) &= \sum_{c=1}^{\infty} \sum_{\substack{d \bmod c \\ (d,c)=1}} \sum_{\lambda\in\mathbb{Z}^l/c\mathbb{Z}^l} \frac{\eta^{s-\kappa}}{c^{2s+l/2}} \cdot e(\frac{a}{c} S[\lambda + r]) \\ &\quad \cdot \big(F((\tau + d/c, z - (\lambda + r)/c), s) + \\ &\quad (-1)^k F((\tau + d/c, z + (\lambda + r)/c), s)\big), \end{aligned} \tag{6.3}$$

where we put

$$F((\tau,z),s)=\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^l}\frac{1}{|\tau+p|^{2(s-\kappa)}(\tau+p)^k}\cdot e\left(-\frac{S[z+q]}{\tau+p}\right).$$

It is easy to see that this infinite series $F((\tau,z),s)$ is absolutely convergent for $\mathrm{Re}(s)>(1+l/2)/2$ and moreover from the expression (6.3) that $E^{II}_{k,S,r}((\tau,z),s)$ is absolutely convergent if $\mathrm{Re}(s)>1+l/4$. Thus we have proved the assertion concerning the absolutely convergence of the infinite series (6.1). By the expression (6.2), the Eisenstein series $E_{k,S,r}((\tau,z),s)$ depends on $r \mod \mathbb{Z}^l$. Hence one can define $E_{k,S,r}((\tau,z),s)$ for $r\in R_S$ with $S[r]\in\mathbb{Z}$. In the case of $\Gamma=\mathrm{SL}_2(\mathbb{Z})$, h, the number of the Γ-equivalence classes of cusps of Γ, equals one. Hence we denote by $R^n_{S,k}$ (resp. $R^*_{S,k}$) for the set $R^n_{S,1,k}$ (resp. $R^*_{S,1,k}$) (see (2.9) in § 2). In this case we may put $A^*_1=1_2$ and $v_{1r}=e_r$ $(r\in R_S)$ in (2.6). Denote by w_r in place of w_{1r} in (2.7). Set

$$\epsilon_r=\begin{cases}2 & r\in R^0_S\\ \sqrt{2} & r\in R_S-R^0_S.\end{cases}$$

Moreover we denote by $E_r(\tau,s)$ in place of the Eisenstein series $E_{1r}(\tau,s)$ given by (2.13).

Proposition 6.1. *Let $r\in R^n_{S,k}$. Then,*

$$E_{k,S,r}((\tau,z),s)=\epsilon_r\cdot\eta^{-\kappa}\cdot{}^{\mathrm{t}}E_r(\tau,s)\cdot\Theta(\tau,z).$$

Proof. In virtue of (6.2) we have

$$\begin{aligned}
&E_{k,S,r}((\tau,z),s)\\
&=\sum_{M\in\Gamma^+_\infty\backslash\Gamma}J(M,\tau)^{-k}(\mathrm{Im}\,M\tau)^{s-\kappa}e\left(-\frac{c}{J(M,\tau)}S[z]\right)\theta_r(M(\tau,z))\\
&=\sum_{M\in\Gamma_\infty\backslash\Gamma}J(M,\tau)^{-k}(\mathrm{Im}\,M\tau)^{s-\kappa}\\
&\qquad\cdot e\left(-\frac{c}{J(M,\tau)}S[z]\right){}^{\mathrm{t}}(e_r+(-1)^ke_{-r})\Theta(M(\tau,z)),
\end{aligned}$$

where Γ_∞ is the stabilizer of the cusp ∞ in $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. Namely,

$$\Gamma_\infty=\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}\mid n\in\mathbb{Z}\}.$$

Thus the theta transformation formula (1.2) implies the assertion of Proposition 6.1.

q.e.d.

We have, in this case of $h = 1$, $t_\infty = \#(R^n_{S,k})$. We arrange the Eisenstein series $E_{k,S,r}((\tau,z),s)$ as a column vector of t_∞-components:

$$E_{k,S}((\tau,z),s) = (E_{k,S,r}((\tau,z),s))_{r\in R^n_{S,k}} \in \mathbb{C}^{t_\infty}.$$

For $r,p \in R^n_{S,k}$, we write simply $\varphi_{rp}(s)$ for the function $\varphi_{1r,1p}(s)$ in (2.14). In this case, $\Phi(s) = (\varphi_{rp}(s))_{r,p\in R^n_{S,k}}$. Denote by $\Phi^*(s)$ the $t_\infty \times t_\infty$-matrix whose (r,p)-entry is given by $\epsilon_r\epsilon_p^{-1}\varphi_{rp}(s)$:

$$\Phi^*(s) = (\epsilon_r\epsilon_p^{-1}\varphi_{rp}(s))_{r,p\in R^n_{S,k}}.$$

The following is a direct consequence of Theorem 2.1 and Proposition 6.1.

Theorem 6.2. *The Eisenstein series $E_{k,S,r}((\tau,z),s)$ can be analytically continued to meromorphic functions of s in the whole s-plane that are holomorphic in the half plane* $\mathrm{Re}(s) \geq 1/2$ *except on the interval* $(1/2,1]$. *They satisfy the functional equation*

$$E_{k,S}((\tau,z),1-s) = \Phi^*(1-s)E_{k,S}((\tau,z),s).$$

Moreover, $E_{k,S,r}((\tau,z),s)$ ($r \in R^n_{S,k}$) are $\mathbb{C}$-linearly independent for $s \neq 1/2$, if they are holomorphic at s.

Remark. Sugano [Su] obtained Theorem 6.2 under a certain condition for S (then, $t_\infty = 1$) by a different method which is based on an explicit calculation of the Fourier coefficients of the Eisenstein series. More information on the Eisenstein series can be obtained by his method.

Example. If $l = 1$ and $S = m$ is a square free positive integer, then $t_\infty = 1$ and $\Phi^*(s) = \Phi(s) = \varphi(s)$. In this case we have computed an explicit form of $\Phi(s)$ in [Ar2]. We exhibit it here again:

$$\Phi(s) = \frac{e^{-\pi ik/2}}{\sqrt{2m}} \cdot \frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)} \cdot \frac{\zeta(4s-2)}{\zeta(4s-1)} \cdot \prod_{p|m} \frac{1+p^{3/2-2s}}{1+p^{1/2-2s}},$$

where $\kappa = (k-1/2)/2$.

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