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Billiards without Boundary and Their Zeta Functions

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In this article we consider a simple dynamical system in R^2 with elastic reflection. As it is shown in [2], the zeta function

$$\zeta(s) = \prod_{\gamma} (1 - \exp[-sT_\gamma])^{-1} \tag{1}$$

of such a dynamical system satisfies nice properties which enable us to apply the results in [3] and [4], where $\prod_\gamma$ is taken over all prime periodic orbits of the dynamical system and T_γ denotes their period. Our present purpose is to summarize auxiliary results which do not appear in [2].

First we recall definitions and notations in [2]. Let $O_1, O_2, ..., O_L$ $(L \geq 3)$ a finite number of bounded domains in R^2, which will be called scatterers, with smooth boundary. We impose the following hypotheses on scatterers:

(H.1) (dispersing) For each j, O_j is strictly convex, i.e., the boundary ∂O_j is a simple closed curve with nonvanishing curvature.

(H.2) (no eclipse) For any triple of distinct indices (j, k, l), the convex hull of $\overline{O_j}$ and $\overline{O_k}$ does not intersect $\overline{O_l}$.

Under these hypotheses it is clear that the boundary ∂O of $O = \cup_{j=1}^{L} O_j$ equals $\cup_{j=1}^{L} \partial O_j$.

Let $SR^2 = R^2 \times S^1 = \{(q, v)\,; |v| = 1\}$ be the unit tangent bundle of R^2, and let $\pi : SR^2 \to R^2$ be the natural projection. Choose a point $q_j \in \partial O_j$ and fix it for each j. We define the following quantities for $x = (q, v) \in \partial O$.

$$\begin{aligned} &\xi_0(x) = \xi_0(q) = j \ \text{ if } \pi(x) \in \partial O_j, \text{ and} \\ &r(x) = r(q),\ \phi(x) = \phi(q, v),\ k(x) = k(q), \end{aligned} \tag{2}$$

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where $r(x)$ is the arclength between $q_{\xi_0(x)}$ and q measured clockwise along the curve $\partial O_{\xi_0(x)}$, $\phi(x)$ is the angle between the vector v and the unit outer normal $n(q)$ of $\partial O_{\xi_0(x)}$ at q measured anticlockwise, and $k(x)$ is the curvature of $\partial O_{\xi_0(x)}$ at q.

Then $\pi^{-1}(\partial O)$ is parametrized as

$$\pi^{-1}(\partial O) = \{(j, r, \phi)\,; 1 \leq j \leq L, 0 \leq r < \text{ the perimeter of } \partial O_j, \text{ and } 0 \leq \phi < 2\pi\}. \tag{3}$$

Put

$$M = \pi^{-1}(\partial O) \cup M_-, \text{ where } M_- = \{x \in \pi^{-1}(\partial O)\,; \frac{\pi}{2} \leq \phi(x) \leq \frac{3\pi}{2}\} \tag{4}$$

Now we consider the motion of a particle in the exterior domain $Q = R^2 - \bar{O}$, which moves with the unit velocity in Q and reflects at $\partial O = \partial Q$ so that the angle of incidence equals the angle of reflection. As in [1] and [6], such a motion of a particle determines a dynamical system (or a flow) S_t on M. We call it a billiard without boundary in the light of the Sinai's billiard. We define the first collision time τ_+ and the last collision time τ_- by

$$\begin{cases} \tau_+(x) = \inf\{t > 0\,;\, \pi(S_t x) \in \partial Q\} \\ \tau_-(x) = \sup\{t < 0\,;\, \pi(S_t x) \in \partial Q\}. \end{cases} \tag{5}$$

Here we regard $\tau_+(x)$ (resp. $\tau_-(x)$) as $+\infty$ (resp. $-\infty$) if the set in the definition is empty. Put

$$\begin{cases} \Omega = \{x \in M\,;\, \pi(S_t x) \in \partial Q \text{ for infinitely many } t > 0 \\ \qquad\qquad \text{and infinitely many } t < 0\}, \\ \Omega_0 = \pi^{-1}(Q) \cap \Omega, \text{ and} \\ \Omega_- = M_- \cap \Omega. \end{cases} \tag{6}$$

We define the map T and T^{-1} by

$$\begin{cases} T(x) = S_{\tau_+(x)}(x) & \text{if } \tau_+(x) < +\infty \\ T^{-1}(x) = S_{\tau_-(x)}(x) & \text{if } \tau_-(x) > -\infty. \end{cases} \tag{7}$$

The notation T^{-1} is compatible with the notation of the inverse of T. We note that if $x = (q, v) \in M$ and if $0 \leq t < \tau_+(x)$, then the flow S_t

can be expressed as

$$S_t x = \begin{cases} (q + tv, v) & \text{if } x \in \pi^{-1}Q \\ (q + t\tilde{v}, \tilde{v}) & \text{if } x \in M_- \end{cases} \tag{8}$$

in our formulation, where $\tilde{v} \in S^1$ is determined so that the point $(q, \tilde{v}) \in M_-$ has the coordinate $(\xi_0(x), r(x), \pi - \phi(x))$ in the parametrization of $\pi^{-1}(\partial Q) = \pi^{-1}(\partial O)$ in (3).

The zeta function of the flow S_t is defined by the formal product (1). In [2] the author obtains:

Theorem 1 (Theorem 0, Theorem 1, and Theorem 2 in [2]). *Assume the hypotheses* (H.1) *and* (H.2). *There exists a positive number* H *such that* (1) $\zeta(s)$ *converges and is nonzero analytic in* $\Re s > H$, *and* (2) $\zeta(s)$ *can be extended meromorphically beyond the axis* $\Re s = H$ *and* $s = H$ *is the unique pole on the axis* $\Re s = H$ *and is simple.*

Corollary. *Under the hypotheses* (H.1) *and* (H.2) *the prime periodic orbits of the billiard without boundary satisfies an analogue of the prime number theorem:*

$$\#\{\gamma\,;\, \exp[HT_\gamma] \le t\}\frac{\log t}{t} \longrightarrow 1 \text{ as } t \to \infty, \tag{9}$$

where γ *denotes the prime periodic orbit of* S_t.

We give a brief sketch of the proof. As in the case of the Axiom A flows, we reduce our problem to the case of the symbolic flows and this enable us to apply the results in [3] and [4]. Let $A = (A(i,j))_{1 \le i,j \le L}$ be the $(L \times L)$-matrix with entry $A(i,j) = 1 - \delta(i,j)$ and $\Sigma_A = \{\xi = (\xi_j)_{j=-\infty}^{\infty}\,;\, A(\xi_j \xi_{j+1}) = 1$ for any $j \in Z\}$, where $\delta(i,j)$ is the Kronecker's delta. Then for any $\xi \in \Sigma_A$, there is a unique $x = x(\xi) \in \Omega_-$ such that $\xi(x) = \xi$, where $\xi_j(x) = \xi_0(T^j x)$ for $j \in Z$. Moreover if $\xi_j(x) = \xi_j(y)$ for $-n \le j \le n$, then we have

$$d(x, y) < C\rho^n, \tag{10}$$

where C is a positive constant independent of x and y, $d(x,y)$ denotes the euclidean distance $\{(r(x) - r(y))^2 + (\phi(x) - \phi(y))^2\}^{\frac{1}{2}}$, and ρ is given by

$$\rho = \{1 + (\min_{j \ne k}(\text{dist}(O_j, O_k))(\min_{q \in \partial Q} k(q))\}^{-1}. \tag{11}$$

Now the function f below is well-defined and it belongs to the space F_ρ in virtue of the estimate (10) (for the definition of F_ρ, see [3]).

$$f(\xi) = \tau_+(x(\xi)), \text{ for } \xi \in \Sigma_A. \tag{12}$$

Put $\Sigma^f = \{(\xi, s)\,;\, \xi \in \Sigma_A, 0 \leq s < f(\xi)\}$. Define σ_t by

$$\sigma_t(\xi, s) = (\sigma^k \xi, u), \text{ if } \sum_{j=0}^{k-1} f(\sigma^j \xi) \leq t + s < \sum_{j=0}^{k} f(\sigma^j \xi), \tag{13}$$

and $h : \Sigma^f \to \Omega_-$ by

$$h(\xi, s) = S_s x(\xi).$$

Then h gives a conjugacy between the dynamical systems $(\Omega, S_t|_\Omega)$ and (Σ^f, σ_t). Using the estimate (10), we can also prove that f can not be expressed as

$$f(\xi) = aK(\xi) + \Phi(\sigma\xi) - \Phi(\xi) \tag{14}$$

for any positive number $a > 0$, a Z-valued function $K(\cdot)$ and a real valued function $\Phi(\cdot)$ on Σ_A.

From now on, we summarize the auxiliary facts. Since we obtain the fact that the function f is in F_ρ, and ρ is given by the formula (11), we have immediately:

Proposition 1. *The number H in Theorem 1 depends analytically on the C^2 perturbation of the scatterers.*

If the diameter of the scatterers are small enough compared with the distance between the scatterers we can regard the zeta function for the flow as a small perturbation of the zeta function corresponding to a directed graph. For instance, we can show:

Proposition 2. *Let $p_1, p_2, \ldots, p_L$ $(L \geq 3)$ be a family of points such that any distinct three of them are not located on the same line. Let $O_j(\lambda)$ be an open disc with radius λ centered at p_j. We denote by ζ_λ the zeta function corresponds to the family $\{O_j(\lambda)\}_{j=1}^{L}$ and ζ_0 the zeta function of the symbolic flow (Σ^f, σ_t^0) over the subshift of finite type (Σ_A, σ) with the ceiling function (the first collision time) defined by $f_0(\xi) = |p_{\xi_0} - p_{\xi_1}|$. Then for any bounded domain $K \subset C$, there exists $\lambda_K > 0$ such that ζ_λ is meromorphic and nonzero in K for any $\lambda \leq \lambda_K$*

and ζ_λ^{-1} converges uniformly to ζ_0^{-1} on K as $\lambda \to \infty$. In particular, each pole of ζ_λ in K converges to the pole of ζ_0.

Proof. Put $\rho(\lambda) = (1 + \min_{j \neq k} \mathrm{dist}(O_j(\lambda), O_k(\lambda))\lambda^{-1})^{-1}$. From Pollicott [4], ζ_λ can be extended meromorphically at least to the domain $s(\lambda) < \Re s$, where $s(\lambda)$ is the unique real number satisfying

$$P(-s(\lambda) f_\lambda) = \frac{|\log \rho(\lambda)|}{2} \frac{|\log \rho(\lambda)|}{|\log \rho(\lambda)| + 2\log(L-1)} \tag{15}$$

and $P(g)$ denotes the so-called topological pressure of g. The right hand side of (15) goes to $+\infty$ as λ goes to 0. Note that $P(g_1) \leq P(g_2)$ holds if g_1 and g_2 are real valued functions in F_ρ with $g_1 \leq g_2$. Clearly there exists positive constants $A_1 < A_2$ such that $A_1 < f_\lambda < A_2$ holds for any sufficiently small λ. Thus we have

$$\begin{aligned} -A_2 s + \log(L-1) &= P(-sA_2) \leq P(-sf_\lambda) \leq P(-sA_1) \\ &= -A_1 s + \log(L-1). \end{aligned} \tag{16}$$

Therefore $s(\lambda) \to -\infty$ as $\lambda \to 0$ in virtue of (15) and (16). Combining these facts and Proposition 1, we obtain Proposition 2.

Example. Consider the regular triangle with sides of length l. Let p_1, p_2, and p_3 be its vertices. Then it is easy to see

$$\frac{1}{\zeta_0(s)} = (1 - 2e^{-3sl})(1 + e^{-3sl})^2. \tag{17}$$

Therefore $\zeta_0(s)$ is meromorphic in the entire complex plane. It is holomorphic in the domain $\Re s > H_0 = \frac{\log 2}{3l}$, and on the axis $\Re s = H_0$, simple poles are located periodically. For $\zeta_\lambda(s)$, the estimate

$$\frac{\log 2}{3l} \leq H_\lambda \leq \frac{\log 2}{3(l - \frac{\sqrt{3}}{2}\lambda)}$$

holds, where H_λ denotes the value corresponding to H in Theorem 1.

Concerning the dynamical system (Ω_-, T) we obtain:

Proposition 3. *Assume the hypotheses* (H.1) *and* (H.2). *For $x \in \Omega_-$, define*

$$\begin{aligned} W^s(x,T) &= \{y \in M_- \, ; \, d(T^n x, T^n y) \to 0, \; n \to \infty\} \\ (\textit{resp. } W^u(x,T) &= \{y \in M_- \, ; \, d(T^{-n} x, T^{-n} y) \to 0, \; n \to \infty\}). \end{aligned}$$

Then $W^s(x,T)$ *(resp.* $W^u(x,T)$*) is a smooth curve of the form* $\phi=\psi(r)$ *with* $-K_2 \le \frac{d\psi}{dr} \le -K_1$ *(resp.* $K_1 \le \frac{d\psi}{dr} \le K_2$*), where* $K_1 = \min_{q\in\partial Q} k(q)$ *and* $K_2 = \max_{q\in\partial Q} k(q) + \{\min_{j\neq k} \mathrm{dist}(O_j, O_k)\}^{-1}$.

Sketch of Proof. Denote by D_n the definition domain of the first collision map T^n for $n\in Z$. For $t\in R$, we put

(18)

$$\begin{cases} V_1(x,t) = k(Tx) - \dfrac{\cos\phi(Tx)}{\cos\phi(x)} \dfrac{1}{\dfrac{\tau_+(x)}{\cos\phi(x)} - \dfrac{1}{t+k(x)}}, & \text{if } x\in D_1, \\ V_0(x,t) = t, & \\ V_{-1}(x,t) = -k(T^{-1}x) - \dfrac{\cos\phi(T^{-1}x)}{\cos\phi(x)} \dfrac{1}{\dfrac{\tau_-(x)}{\cos\phi(x)} - \dfrac{1}{t-k(x)}}, & \\ & \text{if } x\in D_-. \end{cases}$$

Inductively we can define

$$\text{(19)} \qquad \begin{cases} V_{n+1}(x,t) = V_1(T^n x, V_n(x,t)), & \text{if } x\in D_{n+1}, \\ V_{-n-1}(x,t) = V_{-1}(T^{-1}x, V_{-n}(x,t)) & \text{if } x\in D_{-n-1}. \end{cases}$$

By the same way as in [1] and [6], we can show that the connected component of $D_\infty = \cap_{n=1}^{\infty} D_n$ containing $x\in\Omega_-$ is a curve $\gamma = \{(r,\phi)\,;\, \phi = \psi(r), -K_2 \le \frac{d\psi}{dr} \le -K_1\}$. Moreover $\gamma_n = T^n\gamma$ is also a curve $\{(r_n,\phi_n)\,;\, -K_2 \le \frac{d\psi_n}{dr_n} \le -K_1\}$, and satisfies

$$\text{(20)} \qquad \frac{d\psi}{dr} = V_{-n}\Big(T^n x, \frac{d\psi_n}{dr_n}\Big).$$

By using the definition (19) of V_{-n} and the induction in k, we can also show that $\dfrac{d^k V_{-n}(T^n x, t)}{dr^k}$ converges locally uniformly in r and uniformly in t to a function which is independent of t. Thus γ turns out to be a smooth curve. It is clear that $\gamma = W^s(T,x)$.

Proposition 3 reminds us the results in Ruelle [5] and Tangerman [7]. In these papers it is shown that a certain dynamical system with smooth local stable and unstable manifolds has a zeta function meromorphic in the entire complex plane. Therefore we reach the following:

Conjecture. The zeta function of a billiard without boundary can be extended meromorphically to the entire complex plane.

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