Scalar Product of Hecke L-Functions and Its Application

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Introduction

This is a slightly revised version of my notes prepared for the lectures given in Genova (Italy) during the second fortnight of February 1987.

Let us start by recalling a few classical notions to fix our notations. Let k be an algebraic number field of finite degree over $\mathbb{Q}$. The (absolute) Weil group $W(k)$ is defined, [23], [24], as the projective limit

$$W(k) = \varprojlim W(E|k)$$

of relative Weil groups $W(E|k)$, where E varies over all the finite Galois extensions of k. Each of the relative Weil groups $W(E|k)$ is *quasi-compact*:

$$W(E|k) \cong \mathbb{R}_+ \times W_1(E|k),$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers and $W_1(E|k)$ is a compact group, being an extension of the finite Galois group $G(E|k)$ of E over k by a compact group C_E^1 of idèle-classes having unit volume. Any *continuous* finite dimensional representation

(1) $$\rho \colon W(k) \to GL(\ell, \mathbb{C})$$

factors through $W(E|k)$ for some E. We say that ρ is *normalised* if it factors through $W_1(E|k)$ for a finite Galois extension $E|k$. Let $S_0(F)$, $S_1(F)$ and $S_2(F)$ denote the sets of prime ideals, of real places and of complex places of a number field F, respectively; sometimes we write $S_\infty(F) := S_1(F) \cup S_2(F)$ and $S(F) := S_0(F) \cup S_\infty(F)$. For $\mathfrak{p} \in S(F)$, let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$ and let $F_\mathfrak{p}^a$ denote the maxmal abelian extension of $F_\mathfrak{p}$. With a finite Galois extension $E|k$ and a pair of primes $\mathfrak{p} \in S_0(E)$, $p \in S_0(k)$ such that $\mathfrak{p}|p$ one associates the inertia subgroup

$$I_\mathfrak{p} = \{\, \tau \mid \tau \in G(E_\mathfrak{p}^a|k_p),\ \tau\alpha = \alpha \ (\bmod \mathfrak{p})\ \text{ for } \alpha \in \mathbf{v}_\mathfrak{p} \,\}$$

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and the Frobenius class

$$\sigma_{\mathfrak{p}} = \{\, \tau \mid \tau \in G(E_{\mathfrak{p}}^a|k_p),\ \tau\alpha = \alpha^{|p|}\ (\bmod\ \mathfrak{p}\,)\ \ \text{for}\ \alpha \in \mathbf{v}_{\mathfrak{p}}\,\}$$

in $G(E_{\mathfrak{p}}^a|k_p)$, the Galois group of $E_{\mathfrak{p}}^a$ over k_p, here $\mathbf{v}_{\mathfrak{p}}$ stands for the ring of integers in $E_{\mathfrak{p}}^a$ and we write, for brevity,

$$|\mathfrak{a}| := N_{F/\mathbb{Q}}\mathfrak{a},$$

where $\mathfrak{a}$ ranges over the fractional ideals of a number field F. The Galois group $G(E_{\mathfrak{p}}^a|k_p)$ is regarded as a subgroup of $W(E|k)$. Let V be the representation space of ρ and suppose that ρ factors through $W(E|k)$; one defines a vector space

$$V_p = \{\, x \mid x \in V,\ \tau x = x \text{ for } \tau \in I_{\mathfrak{p}}\,\}$$

and proves (cf., e.g., [14, p. 21] that the set

$$\{\, p \mid p \in S_0(k),\ V_p \neq V\,\}$$

is finite. Let

(2) $$L(s,\chi) = \prod_{p \in S_0(k)} \det(I - \rho(\sigma_{\mathfrak{p}})|p|^{-s})^{-1},$$

where $\chi := \operatorname{tr}\rho$, $\rho(\sigma_{\mathfrak{p}}) = \rho(\tau)|_{V_p}$ for $\tau \in \sigma_{\mathfrak{p}}$ (the restriction $\rho(\tau)|_{V_p}$ of the operator $\rho(\tau)$ to V_p is easily seen to depend only on $\sigma_{\mathfrak{p}}$ but not on the choice of τ; moreover, the Euler factor

$$\det(I - \rho(\sigma_{\mathfrak{p}})|p|^{-s})^{-1}$$

depends only on p but not on the choice of $\mathfrak{p}$ above p). If ρ is normalised (or if it is unitary) then the Euler product (2) converges absolutely in the half plane $\operatorname{Re} s > 1$. Continuing the function

$$s \mapsto L(s,\chi)$$

meromorphically to $\mathbb{C}$, one defines the Weil L-function, [24], associated to ρ. If the image of ρ is *finite*, it can be shown to factor through the Galois group $G(F|k)$ of a finite Galois extension $F|k$; on the other hand, since

$$W(k|k) \cong C_k,$$

any one-dimensional representation of $W(k)$ may be identified with a Grössencharakter of k (here C_k denotes the idèle-class group of k). Thus the class of Weil L-function contains both any Artin L-function and any

Hecke L-function "mit Grössencharakteren". Let $X(k)$ be the set of all the continuous finite-dimensional normalised representations of the form (1) and let $\mathrm{gr}(F)$ be the group of normalised grossencharacters of an algebraic number field F. We shall view an element of $\mathrm{gr}(F)$ both as a character of C_F (trivial on $\mathbb{R}_+$ embedded diagonally in the connected component of C_F) and as a multiplicative function on the monoid $I_o(F)$ of integral ideals of F. By a theorem of R. Brauer, [1], the Weil L-function (2) may be decomposed in a product of abelian L-functions (cf. [24]):

$$(2')\qquad L(s,\chi) = \prod_{i=1}^{\mu} L(s,\varphi_i)^{e_i},\ e_i \in \{-1,1\},\ \varphi_i \in \mathrm{gr}(E_i),$$

where $k \subseteq E_i \subseteq E$ (assuming that ρ factors through $W_1(E|k)$, say). It has been conjectured, [24], that the function

$$s \mapsto (s-1)^{g(\chi)} L(s,\chi),$$

where $g(\chi)$ denotes the multiplicity of the identical representation in ρ, is holomorphic in $\mathbb{C}$ (*Artin-Weil conjecture*).

Lecture 1   Estimates for character sums in number fields

Let $\lambda \in \mathrm{gr}(F)$ and let $\mathcal{F}(\lambda)$ denote the conductor of λ, then

$$\lambda((\alpha)) = \prod_{\mathfrak{p} \in S_\infty(F)} |\alpha_{\mathfrak{p}}|^{it_{\mathfrak{p}}(\lambda)} \left(\frac{\alpha_{\mathfrak{p}}}{|\alpha_{\mathfrak{p}}|}\right)^{a_{\mathfrak{p}}(\lambda)} \quad \text{for } \alpha = 1(\mathcal{F}(\lambda)),\ \alpha \in \mathbf{v}_F,$$

where $\mathbf{v}_F$ denotes the ring of integers of a number field F and $\alpha_{\mathfrak{p}} = \sigma_{\mathfrak{p}}(\alpha)$, $\sigma_{\mathfrak{p}} \colon F \to F_{\mathfrak{p}}$ being the natural embedding of F into its completion $F_{\mathfrak{p}}$ at the place $\mathfrak{p}$. Moreover,

$$t_{\mathfrak{p}}(\lambda) \in \mathbb{R},\ a_{\mathfrak{p}}(\lambda) \in \mathbb{Z},\ a_{\mathfrak{p}}(\lambda) \in \{0,1\} \text{ for } \mathfrak{p} \in S_1(F).$$

We write

$$a(\lambda) = \prod_{\mathfrak{p} \in S_1} (2 + |t_{\mathfrak{p}}(\lambda)|)^{1/2} \prod_{\mathfrak{p} \in S_2} \left(2 + \frac{|t_{\mathfrak{p}}(\lambda)| + |a_{\mathfrak{p}}(\lambda)|}{2}\right),$$

and

$$b(\lambda) = \left(|D_F| N_{F/\mathbb{Q}} \mathcal{F}(\lambda)\right)^{1/2},$$

where D_F is the discriminant of F. One develops the Weil L-function (2) in a Dirichlet series:

$$L(s, \chi) = \sum_{\mathfrak{a} \in I_o(k)} c(\mathfrak{a}, \chi) |\mathfrak{a}|^{-s}$$

and remarks that

$$c(\mathfrak{a}, \chi) = \chi(\mathfrak{a}) \quad \text{for } \chi \in \mathrm{gr}(k).$$

Let

$$\rho_j \colon W(k) \to GL(d_j, \mathbb{C}), \quad 1 \leqq j \leqq r,$$

be a continuous normalised representation of the Weil group of k and let

$$(3) \qquad\qquad L(s, \vec{\chi}) = \sum_{\mathfrak{a} \in I_o(k)} |\mathfrak{a}|^{-s} \prod_{j=1}^{r} c(\mathfrak{a}, \chi_j),$$

where $\vec{\chi} := (\chi_1, \ldots, \chi_r)$, $\chi_j := \mathrm{tr}\,\rho_j$. On the other hand, let k_j be a finite extension of k and let $d_j = [k_j : k]$, $1 \leqq j \leqq r$. For $\lambda_j \in \mathrm{gr}(k_j)$, let

$$L(s, \lambda_j) = \sum_{\mathfrak{a} \in I_o(k)} |\mathfrak{a}|^{-s} c_{\mathfrak{a}}(\lambda_j),$$

so that

$$c_{\mathfrak{a}}(\lambda_j) = \sum_{\substack{\mathfrak{b} \in I_o(k_j) \\ N_{k_j/k}\mathfrak{b} = \mathfrak{a}}} \lambda_j(\mathfrak{b}),$$

and let

$$(4) \qquad\qquad L(s, \vec{\lambda}) = \sum_{\mathfrak{a} \in I_o(k)} |\mathfrak{a}|^{-s} \prod_{j=1}^{r} c_{\mathfrak{a}}(\lambda_j),$$

where $\vec{\lambda} := (\lambda_1, \ldots, \lambda_r)$. We call the function (3) the scalar product of Weil L-functions $L(s, \chi_j)$, $1 \leqq j \leqq r$, while the function (4) is called the scalar product *over* k of the Hecke functions $L(s, \lambda_j)$, $1 \leqq j \leqq r$. Regarding λ_j as a one-dimensional representation of $W(k_j)$ we deduce from the basic properties of the Weil L-functions that

$$c_{\mathfrak{a}}(\lambda_j) = c(\mathfrak{a}, \chi_j) \quad \text{when } \rho_j = \mathrm{Ind}_{W(k_j)}^{W(k)} \lambda_j$$

and, in particular,

(5) $L(s, \vec{\lambda}) = L(s, \vec{\chi})$ when $\rho_j = \text{Ind}_{W(k_j)}^{W(k)} \lambda_j$, $1 \leqq j \leqq r$.

Proposition 1.   *Let* $\rho = \rho_1 \otimes \cdots \otimes \rho_r$, $\chi = \text{tr}\,\rho$. *Then*

(6) $L(s, \vec{\chi}) = L(s, \chi) \prod_{p \in S_0(k)} \Phi_p(|p|^{-s}) \prod_{p \in S_0(\vec{\chi})} l_p(|p|^{-s}),$

where $\Phi_p(t) \in \mathbb{C}[t]$, $\Phi_p(t) = 1 \,(\text{mod}\ t^2)$, $S_0(\vec{\chi})$ *is a finite set,* $l_p(t) \in$ $\mathbb{C}[t]$. *Moreover, the degrees of* $\Phi_p(t)$ *and* $l_p(t)$ *are bounded by* $d - 1$, $d := \prod_{j=1}^{r} d_j$.

Equation (6) defines a meromorphic continuation of the function $s \mapsto L(s, \vec{\chi})$ to the half-plane $\text{Re}\,s > 1/2$ since $L(s, \chi)$ is meromorphic in $\mathbb{C}$, $S_0(\chi)$ is the finite and the product

$$L(s, \Phi) = \prod_{p \in S_0(k)} \Phi_p(|p|^{-s})^{-1}$$

converges absolutely for $\text{Re}\,s > 1/2$. Thus, by (5), we obtain a meromorphic continuation of the scalar product $L(s, \vec{\lambda})$ of Hecke L-functions to the half-plane $\text{Re}\,s > 1/2$. Moreover, by a theorem of G. Mackey, [12], a tensor product of monomial representations, is equivalent to a direct sum of monomial representations, so that $L(s, \chi)$ satisfies the Artin-Weil conjecture when each of ρ_j, $1 \leqq j \leqq r$, is induced by a λ_j. To be more precise, we have the following result.

Corollary 1.   *There are number fields* E_i, $1 \leqq i \leqq \nu$, *and grossencharacters* ψ_i *such that*

(7) $L(s, \vec{\lambda}) = \prod_{i=1}^{\nu} L(s, \psi_i) L(s, \Phi)^{-1} L_o(\vec{\lambda}, s)$ *for* $\text{Re}\,s > \dfrac{1}{2}$,

where $L_o(\vec{\lambda}, s) = \prod_{p \in S_o(\vec{\chi})} l_p(|p|^{-s})$ *in notations of* (6), $\psi_i \in \text{gr}(E_i)$, $k \subseteq E_i \subseteq K$, K *being the smallest Galois extension of* k *containing each of the fields* k_j, $1 \leqq j \leqq r$.

In particular, one remarks that

$$L(s, \vec{\lambda}) = \frac{\omega(\vec{\lambda}, s)}{(s-1)^w} + f(s), \quad \omega(\vec{\lambda}, s) \in \mathbb{C}[s],$$

where $f(s)$ is holomorphic for $s = 1$ and

$$w = \mathrm{card}\{\, i \mid \psi_i = 1 \,\}.$$

The methods of classical analytic number theory, [10], lead to an asymptotic estimate for the sum

$$A(x, \vec{\lambda}) := \sum_{|\mathfrak{a}| < x} \prod_{j=1}^{r} c_{\mathfrak{a}}(\lambda_j),$$

and, more generally, for the sum

$$A(x, \vec{\chi}) := \sum_{|\mathfrak{a}| < x} \prod_{j=1}^{r} c(\mathfrak{a}, \chi_j).$$

We write

(8) $$A(x, \vec{\lambda}) = x P_{\vec{\lambda}}(\log x) + R(\vec{\lambda}, x),$$

and

(9) $$A(x, \vec{\chi}) = x P_{\vec{\chi}}(\log x) + R(\vec{\chi}, x),$$

where $P_{\vec{\lambda}}(t)$ (respectively, $P_{\vec{\chi}}(t)$) is a polynomial in $\mathbb{C}[t]$ of degree $w - 1$ (respectively, $g(\chi) - 1$) whose coefficients are effectively computable in terms of the behaviour of $L(s, \vec{\lambda})$ (respectively, $L(s, \vec{\chi})$) in the neighbourhood of the point $s = 1$. If $w = 0$ (respectively, $g(\chi) = 0$) we let $P_{\vec{\lambda}}(t) = 0$ (respectively, $P_{\vec{\chi}}(t) = 0$).

Theorem 1. *Estimate (9) holds with*

$$R(\vec{\chi}, x) = O(x \exp(-c_1(\vec{\chi})\sqrt{\log x})), \qquad c_1(\vec{\chi}) > 0,$$

as $x \to \infty$, with effectively computable (in terms of $\vec{\chi}$) constants. Moreover, if $L(s, \chi)$ in (6) satisfies the Artin-Weil conjecture, then

$$R(\vec{\chi}, x) = O(C_2(\varepsilon; \delta_1) B(\chi) x^{1 - \frac{2}{N+2} + \varepsilon} (\log x)^{nd}), \quad \varepsilon > 0,$$

where, in notations of (2′),

$$N = \sum_{j=1}^{\mu} [E_j : \mathbb{Q}] \frac{e_j + 1}{2}, \ \delta_1 = d \, \mathrm{card}(S_o(\vec{\chi})), \ n = [k : \mathbb{Q}],$$

$$B(\chi) = \prod_{i=1}^{\mu} a(\varphi_i)^{e_i + 1} b(\varphi_i)^2 B_o$$

with B_o depending on $g(\chi)$ and $\sum_{j=1}^{\mu}(e_j - 1)$ only. Finally, if each of the functions $L(s, \varphi_i)$ in $(2')$ satisfies Riemann hypothesis we have

$$R(\vec{\chi}, x) = O_\varepsilon \left(x^{1/2+\varepsilon} \prod_{i=1}^{\mu} a(\varphi_i)^\varepsilon b(\varphi_i)^\varepsilon \right), \qquad \varepsilon > 0.$$

In view of Corollary 1, one obtains as a consequence of Theorem 1 the following result.

Corollary 2. *Estimate* (8) *holds with*

$$R(\vec{\lambda}, x) = O \left(C_2(\varepsilon; \delta_1) \left(\prod_{i=1}^{\nu} a(\psi_i)^2 b(\psi_i)^2 \right) 6^w x^{1 - \frac{2}{N+2} + \varepsilon} (\log x)^{nd} \right),$$

where $N = \sum_{j=1}^{\nu} [E_j : \mathbb{Q}]$, $\varepsilon > 0$.

We remark that an estimate of the form

$$R(\chi, x) = O(x^{1/2-\gamma}), \qquad \text{with } \gamma > 0$$

would imply the Artin-Weil conjecture for $L(s, \chi)$ and that, on the other hand, the well-known Ω-theorem, [2], for grossencharacters gives

$$R(\chi, x) = O(x^\gamma) \implies \gamma \geqq \frac{1}{2} - \frac{1}{2n}$$

when $\chi \in \mathrm{gr}(k)$.

Corollary 2 will be used in the last lecture to obtain an equidistribution theorem for integral ideals having equal norms.

Theorem 2. *Suppose that* ρ *is normalised and* $L(s, \chi)$ *satisfies* $(2')$. *Then*

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O \left(\sum_{j=1}^{\mu} x^{\alpha_j} \right.$$

$$\left. + x \sum_{j=1}^{\mu} n_j \exp \left(-c_3 \frac{\log x}{\log(a(\varphi_j)b(\varphi_j)) + \sqrt{n_j \log x}} \right) \right)$$

with $c_3 > 0$, *where* α_j *denotes the possible (real) Siegel's zero of* $L(s, \varphi_j)$ *when* $\varphi_j^2 = 1$; *here* $n_j := [E_j : \mathbb{Q}]$ *and* $\chi(p) := \mathrm{tr}\, \rho(\sigma_p)$ *for* $p \in S_0(k)$.

If each of the $L(s, \varphi_j)$, $1 \leqq j \leqq \mu$, satisfies Riemann hypothesis, then

$$\sum_{|p| < x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O\left(\sqrt{x} \log\left[\prod_{j=1}^{\mu} x^{n_j} \alpha(\varphi_j) b(\varphi_j)\right]\right).$$

The implied O-constants and c_3 are numerical and can be explicitly evaluated.

This theorem is the starting point of our investigation in the next lecture. It may be proved along the classical lines (cf., for instance, [15, Theorem 4] and note that in the estimate (17) of this paper the first term is redundant as soon as one restores the $n(E_j)$, which have been mistakenly omitted there, in the second term).

Lecture 2.   A new prime number theorem

Let $|S_j(k)| = r_j$, $j = 1, 2$ so that $n = r_1 + 2r_2$, and let E_k be the group of units in k. By the Dirichlet's unit theorem, we can write

$$E_k \cong \mathbb{Z}^{r_1 + r_2 - 1} \times W,$$

where W is a finite cyclic group. Let

$$X = \prod_{p \in S_\infty} k_p \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

and let X^* be the group of invertible elements in the $\mathbb{R}$-algebra X; let i denote the diagonal embedding of k^* in X^*. Obviously,

$$X^* / i(E_k) \cong \mathbb{R}_+ \times \mathcal{T}$$

where $\mathcal{T}$ is an $(n-1)$-dimensional real torus. Finally, consider a homomorphism

$$f \colon I(k) \to \mathcal{T}$$

subject to the condition

$$f((\alpha)) = \pi(i(\alpha)) \quad \text{for } \alpha \in k^*,$$

where π denotes the natural projection of X^* on $\mathcal{T}$ and $I(k)$ stands for the group of fractional ideals of k.

Theorem 3. *Let* $\mathcal{U}$ *be a "smooth" subset* of* $\mathcal{T}$ *and let* A *be a conjugacy class in the Galois group of a finite Galois extension* E/k. *The following formula holds:*

$$\mathrm{card}\{\,p \mid p \in S_o(k),\ |p| < x,\ f(p) \in \mathcal{U},\ \sigma_p = A\,\} =$$

$$\frac{\mu(\mathcal{U})|A|}{[E:k]} \int\limits_2^x \frac{du}{\log u} + O(x\exp(-C(\mathcal{U})\sqrt{\log x})), \quad C(\mathcal{U}) > 0,$$

where μ *is the Haar measure on* $\mathcal{T}$ *normalised by the condition* $\mu(\mathcal{T}) = 1$.

As a special case of this theorem, one obtains Chebotarev's density theorem (just take $\mathcal{U} = \mathcal{T}$); on the other hand, if E is chosen to be Abelian over k, Theorem 3 reduces to an equidistribution theorem in the spirit of Hecke's multidimensional arithmetic, [4], [5], [13]. Theorem 3 is an easy consequence of the estimates of Theorem 2 (see, for instance, [14, p. 68]).

To state the main theorem of this paragraph, consider a *finite* subset $\mathfrak{N}$ of $X(k)$ and let

$$\mathfrak{M} = \{\,\chi \mid \chi = \mathrm{tr}\,\rho,\ \rho \in \mathfrak{N}\,\}$$

be the corresponding set of characters.

Theorem 4. *Let* $g_o \in W(k)$ *and let* $0 < \varepsilon < 1$. *The following asymptotic formula holds:*

$$(10) \quad \mathrm{card}\{\,p \mid p \in S_o(k), |p| < x, |\chi(p) - \chi(g_o)| < \varepsilon \ \text{for each } \chi \text{ in } \mathfrak{M}\,\}$$

$$\alpha(\mathfrak{M}, g_o, \varepsilon) \int\limits_2^x \frac{du}{\log u} + O(x\exp(-c_4\sqrt{\log x})), \quad c_4 > 0.$$

Moreover, $\alpha(\mathfrak{M}, g_o, \varepsilon) > c_5\varepsilon^{c_6}$, $c_5 > 0$, $c_6 > 0$. *Here the constants* c_j, $j = 4, 5, 6$, *depend at most on* $\mathfrak{M}$ *(but not on* ε, g_o, x*).*

Theorem 4 can also be viewed as a generalisation of both Chebotarev's density theorem and Hecke's type equidistribution theorems: one obtains the former one when each ρ in $\mathfrak{N}$ has a finite image, while the latter ones follow if each ρ in $\mathfrak{N}$ is one-dimensional. To sketch the

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* This notion will be analysed in the fourth lecture. For the time being one may picture $\mathcal{U}$ as a rectangular subset of $\mathcal{T}$.

proof of Theorem 4 let us note first that there is a finite Galois extension $E|k$ such that each ρ in $\mathfrak{N}$ factors through $W_1(E|k)$ since $\mathfrak{N}$ is finite. Consider a subset

$$A(\mathfrak{M}, g_o, \varepsilon) = \{\, g \mid g \in W_1(E|k),\ |\chi(g) - \chi(g_o)| < \varepsilon \text{ for } \chi \in \mathfrak{M}\,\}$$

of $W_1(E|k)$ and let μ be the Haar measure on $W_1(E|k)$ normalised by the condition $\mu(W_1(E|k)) = 1$. One can actually take

(11) $$\alpha(\mathfrak{M}, g_o, \varepsilon) = \mu(A(\mathfrak{M}, g_o, \varepsilon))$$

and prove that

$$\mu(A(\mathfrak{M}, g_o, \varepsilon)) > c_5 \epsilon^m, \qquad m := [E : \mathbb{Q}] - 1.$$

To deduce the asymptotic formula (10) with α defined by (11) one notes that, in fact, each ρ in $\mathfrak{N}$ factors through a certain group G which fits in the exact sequence

$$1 \to T \to G \to H \to 1,$$

where T is a finite-dimensional real torus and H is a finite group. Moreover, conditions $|\chi(g) - \chi(g_o)| < \epsilon$, $\chi \in \mathfrak{M}$, define a semialgebraic subset $\mathcal{U}$ of G. The asymptotic formula to be proved would follow now from Theorem 2 and the general equidistribution principles if one could estimate from above the volume of the δ-neighbourhood of the boundary of $\mathcal{U}$ uniformly in the interval, say, $0 < \delta < 1$. Such an estimate can indeed be proved as a consequence of recent results on volumes of tubes around semialgebraic sets, [25] (cf. also [3]). This concludes our sketch of proof of Theorem 4 (cf., however, [16] and [17, §5] where this proof has been carried out in detail). We close this lecture by suggesting an open problem: can one prove a general theorem on equidistribution of Frobenius classes in a Weil group (cf. [14, p. 69–71]) that, in particular, would imply both Theorem 3 and Theorem 4?

Lecture 3.   Analytic continuation and the natural boundary of scalar products

The object of this lecture is the following theorem.

Theorem 5.
   (i) *The function $L(s, \vec{\chi})$ defined by (3) can be meromorphically continued to the half-plane $\mathbb{C}_+ = \{\, s \mid \operatorname{Re} s > 0 \,\}$.*

   (ii) *Suppose that $r \geq 2$ and $d_1 \geq \ldots \geq d_r \geq 2$. The line $\mathbb{C}^o = \{\, s \mid \operatorname{Re} s = 0 \,\}$ is the natural boundary of $L(s, \vec{\chi})$ unless $r = d_1 = d_2 = 2$.*

(iii) *If $r = d_1 = d_2 = 2$, then the function $L(s, \vec{\chi})$ is equal to a ratio of two Weil L-functions (up to a finite number of Euler factors) and therefore it is meromorphic in $\mathbb{C}$.*

Identity (5) shows that assertions (i)–(iii) hold true when one replaces $L(s, \vec{\chi})$ by $L(s, \vec{\lambda})$. Let us describe now the main steps in the proof of Theorem 5 . In view of (6), the function $L(s, \vec{\chi})L(s, \Phi)^{-1}$ is meromorphic in $\mathbb{C}$; moreover, the polynomials $\Phi_p(t)$ can be explicitly evaluated when $r = d_1 = d_2 = 2$ and this evaluation proves (iii). In general, the sequence of polynomials $\{\, \Phi_p(t) \mid p \in S_o(k)\,\}$ can be parametrized as follows. Let $\mathcal{Y}$ be the ring of virtual characters of $W(k)$ and let

$$H(t) = \sum_{j=0}^{l} t^j a_j, \quad a_j \in \mathcal{Y},$$

be a polynomial with coefficients in this ring. We extend the definition $\chi(p) = \operatorname{tr} \rho(\sigma_p)$ by linearity to $\mathcal{Y}$ and write

$$H_p(t) = \sum_{j=0}^{l} t^j a_j(p) \quad \text{for } p \in S_o(k),$$

and

$$H_g(t) = \sum_{j=0}^{l} t^j a_j(g) \quad \text{for } g \in W(k).$$

Lemma 1. *There are a polynomial $H(t)$ in $\mathcal{Y}[t]$ and a finite subset $S_o'(\vec{\chi})$ of $S_o(k)$ such that*

(12) $$H_p(t) = \Phi_p(t) \quad \text{for } p \in S_o(k) \setminus S_o'(\vec{\chi}).$$

This lemma shows that it suffices to investigate the analytic properties of the Euler product

(13) $$L(s, H) = \prod_{p \in S_o(k)} H_p(|p|^{-s})^{-1}$$

for $H(t) \in \mathcal{Y}[t]$.

Definition. *Let $H(t) \in \mathcal{Y}[t]$ and suppose that $H(0) = 1$. We say that H is unitary if*

$$H_g(\alpha) = 0 \implies |\alpha| = 1$$

for each g in $W(k)$.

   Lemma 2.   *If $2 \leqq d_r \leqq \ldots \leqq d_1$ and $d_1 d_2 r > 8$, then the polynomial H defined by (12) and (6) is <u>not</u> unitary.*

   In view of Lemma 2, the statements (i) and (ii) follow from the following proposition.

   Proposition 2.   *Let $H(t) \in \mathcal{Y}[t]$ and $H(0) = 1$. The function*

$$s \mapsto L(s, H)$$

defined by (13) in the half-plane $\operatorname{Re} s > 1$ can be meromorphically continued to $\mathbb{C}_+$; if H is not unitary, then $\mathbb{C}^o$ is the natural boundary of this function.

   We sketch the proof of Proposition 2. Without loss of generality, we may assume that

$$H(t) = 1 + \sum_{j=1}^{l} a_j t^j, \quad a_j \in \mathcal{Y}_0,$$

where $\mathcal{Y}_0$ denotes the ring of virtual characters of $W_1(E|k)$ for a finite Galois extension $E|k$. Let

$$H_g(t) = \prod_{i=1}^{l} (1 - \alpha_i(g)t), \quad g \in W_1(E|k),$$

and let
$$\gamma = \sup\{\, |\alpha_i(g)| \,|\, 1 \leqq i \leqq l, \ g \in W_1(E|k) \,\}.$$

   Lemma 3.   (i) *Formally, in $\mathcal{Y}_0[[t]]$, we have*

$$H(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X_n(H)} \det(1 - t^n \varphi)^{b_n(\varphi)}, \quad b_n(\varphi) \in \mathbb{Z},$$

where $X_n(H)$ is a finite subset of X. Moreover,

$$\left| \sum_{\varphi \in X_n(H)} b_n(\varphi) \operatorname{tr} \varphi(g) \right| \leqq l\gamma^n \frac{1}{n} \sum_{d|n} 1 \quad \text{for } g \in W_1(E|k).$$

   Making use of Lemma 3, one can prove the following statement.

Lemma 4. *There is M_0 such that if $M > M_0$ and $N > (\gamma + 1)^M$ then*

$$(14) \qquad L(s, H) = Z_N(s)U_M(s)g(s)$$

where

$$U_M(s) = \prod_{1 \leqq n < M} \prod_{\varphi \in X_n(H)} L(ns, \operatorname{tr} \varphi)^{-b_n(\varphi)},$$

$$Z_N(s) = \prod_{|p| < N} H_p(|p|^{-s})^{-1},$$

and $g(s)$ is holomorphic and has no zeros in

$$\mathbb{C}_{1/M} = \{ s \mid \operatorname{Re} s > \frac{1}{M} \}.$$

Since both $Z_N(s)$ and $U_M(s)$ are meromorphic in $\mathbb{C}$ and since

$$\mathbb{C}_+ = \bigcup_{M > M_0} \mathbb{C}_{1/M},$$

equations (14) provide a meromorphic continuation of $L(s, H)$ to $\mathbb{C}_+$. Moreover, as an easy consequence of Lemma 3 one proves that $L(s, H)$ is a meromorphic function in $\mathbb{C}$ when H is unitary (that is, if $\gamma = 1$). If $\gamma > 1$ one can prove that the closure of the set

$$\{ s \mid \operatorname{Re} s > 0, \ Z_N(s)^{-1} = 0 \text{ for some } N \}$$

contains the line $\mathbb{C}^o$. To show that this line is the natural boundary of $L(s, H)$ it remains to prove that the poles of this function coming from the first factor in (14) cannot be cancelled by the zeros of the second factor. Making use of Theorem 4 one can estimate the number of poles of $Z_N(s)$ in a neighbourhood of a fixed point in $\mathbb{C}^o$. On the other hand, a careful analysis of the structure of the zero-set of $U_M(s)$ provides an upper bound for the number of such zeros which shows that complete cancellation cannot occur. We have to refer to [17] for the details of this argument; an alternative proof of Theorem 5 may be found in [9].

Theorem 5 is of considerable interest for the general theory of L-functions having Euler product. For the history of its proof and for some related results we refer the reader to two short notes, [21], [22], and our final report on this problem, [17]. This exposition (as well as the article [17]) owes much to the early work of N. Kurokawa, [6]–[8], where Theorem 5 has been proved for representations ρ_j, $1 \leqq j \leqq r$, of Galois type.

Lecture 4.   Integral points on algebraic sets
defined by a system of norm-forms

The following problem has stimulated much research in number theory and arithmetic algebraic geometry. Let $f_j(x_j) \in \mathbb{Z}[x_j]$, $1 \leqq j \leqq r$, and consider the algebraic set U given by the system of equations $f_j(x_j) = 0$, $1 \leqq j \leqq r$. One chooses a compact subset of $U(\mathbb{R})$ and asks for an estimate of the number of integer points in this subset (here $x_j = (x_{j1}, \dots, x_{jn_j})$ is an array of n_j variables). When the number of variables $n = \sum_{j=1}^{r} n_j$ is relatively small compared to the degrees and the number of equations the analytic methods must be supplemented by arithmetical considerations. We study here the simplest problem of this type: dealing with norm-form equations allows one to avoid algebro-geometric considerations and to work in the framework of classical algebraic number theory. To give a precise statement of our results we need a notion of "smoothness" generalizing the notion of a plane domain with a boundary satisfying Lipschitz condition. Consider a triple (W, E, μ) consisting of a set W, a Borel measure μ and a system E of measurable subsets of W; a measurable subset V of W is said to be *n-smooth* if for each Δ in the interval $0 < \Delta < 1$ one can find a finite subset $E_o(\Delta)$ of E satisfying the following conditions:

(i)   card $E_o(\Delta) \leqq \Delta^{-n}$;
(ii) $\rho \cap \rho' = \emptyset$ when $\rho \neq \rho'$, $\rho \in E_o(\Delta)$, $\rho' \in E_o(\Delta)$;
(iii)   let $V_+ = \bigcup\limits_{\rho \in E_o(\Delta)} \rho$, $V_- = \bigcup\limits_{\substack{\rho \in E_o(\Delta) \\ \rho \subseteq V}} \rho$, then

$V_+ \supseteq V$ and $\mu(V_+ \setminus V_-) < C(V)\Delta$ (for some constant $C(V)$ independent of Δ and called the *smoothness* constant of V).

The elements of E are referred to as *elementary sets*. Consider, secondly, a triple (S, π, N) consisting of a set S, a map $\pi \colon S \to W$ and a map $N \colon S \to \mathbb{R}_+$; this triple is said to be *(E, μ)-equidistributed* if

$$\operatorname{card}\{\, s \mid s \in S,\ \pi(s) \in \rho,\ Ns < x \,\} = \mu(\rho)a(x) + O(b(x))$$

for $\rho \in E$, where $\dfrac{b(x)}{a(x)} \to 0$ as $x \to \infty$.

Lemma 5.   *If (S, π, N) is (E, μ)-equidistributed, then*

$$\operatorname{card}\{\, s \mid s \in S,\ \pi(s) \in V,\ Ns < x \,\} = \mu(V)a(x) + O(b_1(x))$$

with $b_1(x)/a(x) \to 0$ (and b_1 exactly expressible in terms of b, n, $C(V)$) for any n-smooth set V.

Let $k_j|\mathbb{Q}$, $1 \leq j \leq r$, be a finite Galois extension of degree d_j and let $K = k_1 \cdots k_r$ be the composite field of k_j, $d := [K : \mathbb{Q}]$. To simplify our exposition we impose the following condition on these fields (cf. [11]):

$$(15) \qquad (e_i(p), e_j(p)) = 1 \qquad for \quad i \neq j, \ p \in S_o(\mathbb{Q}),$$

where $e_i(p)$ denotes the ramification index of p in k_i. Condition (15) implies, in particular, that $d = \prod_{j=1}^{r} d_j$, so that the fields $k_1, \ldots, k_r$ are linearly disjoint over $\mathbb{Q}$. Let $\mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_r$ and let $H = H_1 \times \cdots \times H_r$, where $\mathcal{T}_j$ denotes the $(d_j - 1)$-dimensional torus assigned to k_j as in Lecture 2 and where H_j is the ideal class group of k_j; let $f \colon I(k_1) \times \cdots \times I(k_r) \to \mathcal{T}$ be the product of homomorphisms $f_j \colon I(k_j) \to \mathcal{T}_j$ defined in Lecture 2 and consider two sets of r-tuples of ideals

$$I_o = \{ \vec{\mathfrak{a}} \mid \vec{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r), \ \mathfrak{a}_j \in I_o(k_j), \ N\mathfrak{a}_1 = \cdots = N\mathfrak{a}_r < x \}$$

and

$$S_o = \{ \vec{\mathfrak{p}} \mid \vec{\mathfrak{p}} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_r), \ \mathfrak{p}_j \in S_o(k_j), \ N\mathfrak{p}_1 = \cdots = N\mathfrak{p}_r < x \}.$$

Theorem 6. *Take the rectangular subsets of $\mathcal{T}$ as elementary and let μ be the Haar measure on $\mathcal{T}$ normalised by the condition $\mu(\mathcal{T}) = 1$. Let $\vec{A} \in H$ and suppose that τ is a $\sum_{j=1}^{r}(d_j - 1)$-smooth subset of $\mathcal{T}$. Then*

$$\mathrm{card}\{ \vec{\mathfrak{p}} \mid \vec{\mathfrak{p}} \in S_o \cap \vec{A}, \ f(\vec{\mathfrak{p}}) \in \tau \} = \frac{\mu(\tau)}{|H|} \int_{2}^{x} \frac{du}{\log u} + O(x \exp(-c\sqrt{\log x}))$$

with $c > 0$, and

$$\mathrm{card}\{ \vec{\mathfrak{a}} \mid \vec{\mathfrak{a}} \in I_o \cap \vec{A}, \ f(\vec{\mathfrak{a}}) \in \tau \} = \frac{\mu(\tau) w_o}{|H|} x + O(x^{1-\gamma})$$

with $\gamma > 0$, where $w_o = w(K) L(1, \Phi)^{-1}$, $w(K)$ denotes the residue of the ζ-function $Z_K(s)$ of K as $s = 1$ and $L(s, \Phi)$ is defined by the equation $\sum_{n=1}^{\infty} n^{-s} \prod_{j=1}^{r} a_n^{(j)} = Z_K(s) L(s, \Phi)^{-1}$, $a_n^{(j)} := \mathrm{card}\{ \mathfrak{a} \mid \mathfrak{a} \in I_o(k_j), \ N\mathfrak{a} = n \}$.

Theorem 6 shows that both integral and prime divisors with equal norms are equidistributed in the sense of E. Hecke; it can be deduced from the estimates (8) and the prime number theorem for grossencharacters with the help of Lemma 5 and Equation (6). Choose an integral

ideal $\mathfrak{a}_{oj}$ in A_j and let $\{\, w_{ji} \mid 1 \leqq i \leqq d_j \,\}$ be a $\mathbb{Z}$-basis of $\mathfrak{a}_{oj}$; one defines a norm-form f_j associated to A_j as follows:

$$f_j(x_j) = N_{k_j(x_j)/\mathbb{Q}(x_j)} \left(\sum_{1 \leqq i \leqq d_j} x_{ji} w_{ji} \right) N\mathfrak{a}_{oj}^{-1};$$

obviously, $f_j(x_j) \in \mathbb{Z}[x_j]$. Consider the algebraic set W defined by a system of equations:

$$f_1(x_1) = \cdots = f_r(x_r)$$

and let $W_o = W_{o1} \times \cdots \times W_{or}$ with W_{oj}, $1 \leqq j \leqq r$, be defined by the equation $f_j(x_j) = 1$. In what follows we assume, for simplicity, that k_j is *totally complex*, then f_j is positive definite and we can define a projection

$$\pi \colon W(\mathbb{R}) \to W_o(\mathbb{R}), \ \pi \colon a_j \mapsto a_j f_j(a_j)^{-1/d_j}, \ a_j \in \mathbb{R}^{d_j}$$

(this map is not defined on the subset $f_j(x_j) = 0$, $1 \leqq j \leqq r$, of smaller dimension and containing no integer points except the origin). Moreover, one can define a natural projection map $h \colon W_o(\mathbb{R}) \to \mathcal{T}$ of $W_o(\mathbb{R})$ on $\mathcal{T}$ and take as elementary the subsets of $W(\mathbb{R})$ of the shape:

$$\{\, \vec{a} \mid \vec{a} = (a_1, \ldots, a_r), \ t_1 < f_j(a_j) \leqq t_2, \ \pi(\vec{a}) \in \mathcal{U} \,\},$$

where $0 \leqq t_1 < t_2$ and where $h(\mathcal{U})$ is a smooth subset of $\mathcal{T}$.

Theorem 7.   *There is a Borel measure μ on $W(\mathbb{R})$ such that*

$$\mathrm{card}(V \cap W(\mathbb{Z})) = \mu(V)\frac{w(K)}{|H|L(1,\Phi)} + O(C(V)t(V)^{1-\gamma})$$

with $\gamma > 0$, where $t(V) = \max\{\, f_j(a_j) \mid \vec{a} \in V \,\}$, for any smooth subset V of $W(\mathbb{R})$ (i.e. smooth with respect to the system of elementary sets we have just described).

Corollary 3.   *The following asymptotic formula holds true:*

$$\mathrm{card}\{\, \vec{a} \mid \vec{a} \in W(\mathbb{Z}), \ \|a_j\| < x^{1/d_j}, \ 1 \leqq j \leqq r \,\} = \frac{m(x)w(K)}{|H|L(1,\Phi)} + O(x^{1-\gamma})$$

with $\gamma > 0$, where $\|a_j\| := \max_{1 \leqq i \leqq d_j} |a_{ij}|$ and where $m(x) := \mu(\{\, \vec{y} \mid \vec{y} \in W(\mathbb{R}), \ \|y_j\| < x^{1/d_j}, \ 1 \leqq j \leqq r \,\})$. Moreover, there are two constants C_1, C_2 such that $C_1 x \leqq m(x) \leqq C_2 x$.

Theorem 7 can be deduced (with the help of Lemma 5) from Theorem 6. The results described in this lecture appear in [18]–[20] (cf. also [11]

Concluding remarks and acknowledgements

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Postscript

On submitting this paper for the proceedings of the conference on
Zeta functions in geometry I should like to heartily thank the organizers
for inviting me to take part in this conference and for their exceptional
hospitality in Tokyo. I use this opportunity to enlarge my bibliography
by listing the following articles closely related to the problems discussed
in my lectures:

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