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Homologically Trivial Smooth Involutions on K3 Surfaces

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Dedicated to Professor Shôrô Araki on his 60th birthday

Abstract.

We will show that any smooth involution on a K3 surface induces a non-trivial action on its homology. In fact, a closed spin 4-manifold M with $H_1(M; \mathbf{Z}_2) = 0$ and $\operatorname{sign} M \neq 0$ will be shown to admit no homologically trivial locally linear involutions. The proof uses only the G-signature theorem and the sublattices and branched coverings arguments.

§1. Introduction

Some complex surfaces including K3 surfaces admit no homologically trivial holomorphic involutions. There posed a question in [12;11.8] whether the same is true for the smooth involutions or not. This paper answers the question affirmatively at least for the smooth involutions on K3 surfaces. Note that a smooth involution is locally linear.

Theorem 1. *Let M be a closed connected oriented spin 4-manifold with $H_1(M; \mathbf{Z}_2) = 0$. Suppose that there is an orientation preserving locally linear involution σ on M which operates as identity on $H_2(M; \mathbf{Q})$. Then,* $\operatorname{sign} M = 0$.

Since a K3 surface is a simply-connected spin 4-manifold with signature -16, it admits no homologically trivial locally linear involutions. According to Edmonds [5] Theorem 1 in the case that M is simply-connected is already proved by D. Ruberman.

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with the collaboration of Y. Matsumoto and A. Kawauchi, which will be published elsewhere.

§2. Preliminary lemmas

We prepare some lemmas which will be used later and may be useful for the other purposes. We begin with a lemma to construct a double covering from two 2-sheet branched coverings.

Lemma 2.1. *Let σ be a locally linear involution on a connected manifold M with fixed point set F. Suppose there is a subunion of connected components $F' \subsetneq F$ with a non-trivial element e_τ of $H^1(M/\sigma - F'; \mathbf{Z}_2)$ which takes non-zero value on the image of $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of F', where $-/\sigma$ stands for the orbit space and $N(x)$ is a fiber at x of an equivariant normal disk bundle $N(U_x)$ for a neighborhood U_x of x in F. Then, there is a locally linear $\mathbf{Z}_2 \times \mathbf{Z}_2$-action with generators $\tilde{\sigma}$ and $\tilde{\tau}$ on a double (= connected 2-sheet unbranched) covering manifold $\widetilde{M}$ of M such that the orbit space $\widetilde{M}/\tilde{\tau}$ is canonically homeomorphic to M and $\tilde{\sigma}$ induces σ with this identification.*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\text{covering}]{\text{unbranched}} & \widetilde{M}/\tilde{\tau} = M \\ \downarrow & & \downarrow \\ \widetilde{M}/\tilde{\sigma} = M' & \longrightarrow & M/\sigma \end{array}$$

Proof. The projection $\pi : M - F \to M/\sigma - F$ is a covering map induced from a non-trivial element e_σ of $H^1(M/\sigma - F; \mathbf{Z}_2) = \mathrm{Hom}(H_1(M/\sigma - F; \mathbf{Z}), \mathbf{Z}_2) = \mathrm{Hom}(\pi_1(M/\sigma - F), \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of F. Let $j : M/\sigma - F \to M/\sigma - F'$ be the inclusion. Then, we have $j^*e_\tau \neq e_\sigma$, since e_τ takes zero value on $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of $F - F'$. So, we get a $\mathbf{Z}_2 \times \mathbf{Z}_2$-covering of $M/\sigma - F$ associated to $(j^*e_\tau, e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$.

Consider the base change $(j^*e_\tau, j^*e_\tau + e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$. The completed 2-sheet branched coverings $\pi' : M' \to M/\sigma$ and $\pi'' : M'' \to M/\sigma$ (resp.) induced by j^*e_τ and $j^*e_\tau + e_\sigma$ (resp.) have the disjoint branch loci F' and $F - F'$ (resp.). So, the completed 2×2-sheet branched covering $\widetilde{\pi} : \widetilde{M} \to M/\sigma$, induced by $(j^*e_\tau, j^*e_\tau + e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$, has the locally linear involutions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ so that $\widetilde{\pi}' : \widetilde{M} \to \widetilde{M}/\widetilde{\sigma} = M'$ and $\widetilde{\pi}'' : \widetilde{M} \to \widetilde{M}/\widetilde{\sigma}' = M''$ are the 2-sheet branched coverings with branch loci $(\pi')^{-1}(F - F')$ and $(\pi'')^{-1}(F')$

respectively. By the definition $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ commute outside $\widetilde{\pi}^{-1}(F)$. Since $\widetilde{M} - \widetilde{\pi}^{-1}(F)$ is dense in $\widetilde{M}$, $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ commute also on whole $\widetilde{M}$.

Put $\widetilde{\tau} = \widetilde{\sigma} \circ \widetilde{\sigma}'$. Then, $\widetilde{\tau}$ has no fixed point either in $\widetilde{M} - \widetilde{\pi}^{-1}(F)$ or in $\widetilde{\pi}^{-1}(F) = (\widetilde{\pi}')^{-1}(\pi')^{-1}(F - F') \cup (\widetilde{\pi}'')^{-1}(\pi'')^{-1}(F')$ and hence in whole $\widetilde{M}$. Moreover, $\widetilde{M}/\widetilde{\tau} \to M/\sigma$ is the branched covering induced by $j^*e_\tau + j^*e_\tau + e_\sigma = e_\sigma$, that is, equivalent to $M \to M/\sigma$.

Since M is connected, M/σ is connected. If $F' = \emptyset$, the covering associated to the non-trivial element of $H^1(M/\sigma; \mathbf{Z}_2)$ is connected. Otherwise the branch locus of $M' \to M/\sigma$ is non-empty and M' is connected. Then, since the branch locus of $\widetilde{M} \to M'$ is non-empty, $\widetilde{M}$ is connected. Q.E.D.

We recall and define some notions about lattices now. A $\mathbf{Z}$-free module L of finite rank with non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbf{Z}$ is called a lattice. Let L^* denote the dual module $\mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ and we have a canonical embedding $L \subset L^*$ defined by $x \mapsto \langle\ , x\rangle$. The factor group L^*/L is finite abelian and its order divides $|\operatorname{discr} L|$ where $\operatorname{discr} L = \det\langle e_i, e_j\rangle$ for some basis $\{e_i\}$. Let p be a prime. For a finite abelian group A we denote the minimal number of generators of A and $A \otimes \mathbf{Z}_p$ by $\ell(A)$ and $\ell_p(A)$ respectively. A lattice is called unimodular or p-unimodular if $L^*/L = 0$ or $\ell_p(L^*/L) = 0$ respectively. A submodule S of L is called primitive or p-primitive if L/S is $\mathbf{Z}$-free or contains no p-torsion respectively. Define the orthogonal complement $S^\perp = \{y \in L;\ \langle y, x\rangle = 0$ for any $x \in S\}$. If L is unimodular and S is a primitive sublattice, i.e., primitive and the pairing $\langle\ ,\ \rangle$ is non-degenerate not only on L but also on S, we have a natural isomorphism $S^*/S \cong S^{\perp *}/S^\perp$. (See [3;I.2.5] and [10] for example.) Moreover, we can prove

Lemma 2.2. *Let p be a prime. Let L be a p-unimodular lattice and S a p-primitive sublattice. Then, the orthogonal complement $K = S^\perp$ is also a sublattice and the p-torsion part $(S^*/S)_{(p)}$ of S^*/S is isomorphic to the p-torsion part of $(K^*/K)_{(p)}$ of K^*/K.*

Proof. Take an element ℓ of L. Then, $\ell^* = \langle\ , \ell\rangle$ can be considered as an element of S^*; $\ell_1^* = \ell_2^*$ in S^* if and only if $\ell_1 - \ell_2 \in K$. If we consider ℓ^* also as an element in K^*, we get a homomorphism $\mathrm{Im}(L \to S^*)/S \to K^*/K$. That S is p-primitive implies $(S^*/\mathrm{Im}(L^* \to S^*))_{(p)} = 0$. Since $(L^*/L)_{(p)} = 0$ by the assumption, we have $(S^*/S)_{(p)} = (\mathrm{Im}(L^* \to S^*)/S)_{(p)} = (\mathrm{Im}(L \to S^*)/S)_{(p)}$ and we get a correlation homomorphism $(S^*/S)_{(p)} \to (K^*/K)_{(p)}$. By the definition it is easy to see that K is a primitive sublattice of L and $K^\perp$ is a minimal primitive sublattice of L containing S. So, $(K^{\perp}/S)_{(p)} = 0$ by the assumption. Then, we get also a homomorphism $(K^*/K)_{(p)} \to (K^{\perp *}/K^{\perp})_{(p)} = (S^*/S)_{(p)}$ which is an inverse of the homomorphism above. Q.E.D.

Next we give a sufficient and nearly necessary condition to get a branched covering in some cases.

Lemma 2.3. *Let p be a prime. Let $S^2_1, \dots, S^2_\ell$ be disjointly embedded 2-spheres in a closed orientable 4-manifold M with normal disk bundles $N(S^2_1), \dots, N(S^2_\ell)$.*

(1) *Suppose that the homology classes $[S^2_1], \dots, [S^2_\ell]$ are linearly dependent in $H_2(M; \mathbf{Z}_p)$. Then, there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S^2_i; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S^2_i); \mathbf{Z})$ for some i.*

(2) *Suppose that $[S^2_1], \dots, [S^2_\ell]$ are linearly independent in $H_2(M; \mathbf{Z})$ and generate a submodule S of $L = H_2(M; \mathbf{Z})/\mathrm{tor}$. Let $\overline{S}$ be the minimal primitive submodule of L containing S, that is, $L/\overline{S}$ is $\mathbf{Z}$-free. Then, $\overline{S}/S$ is a finite (possibly zero) abelian group and we have an isomorphism*

$$\overline{S}/S \cong \mathrm{Ker}(H_1(M - \cup_{i=1}^{\ell} S^2_i; \mathbf{Z}) \to H_1(M; \mathbf{Z})).$$

Note that the torsion part of L/S is $\overline{S}/S$. So, if L/S contains a non-trivial p-torsion, there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S^2_i; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S^2_i); \mathbf{Z})$ for some i. Moreover, when $H_1(M; \mathbf{Z}) \otimes \mathbf{Z}_p = 0$, the converse is also true, that is, if there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S^2_i; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S^2_i); \mathbf{Z})$ for some i, L/S contains a non-trivial p-torsion.

(3) *Suppose $[S^2_i]^2 \equiv 0 \mod p$ for every i and $2\ell > b_2(M)$. Then, either $[S^2_1], \dots, [S^2_\ell]$ are linearly dependent in $H_2(M; \mathbf{Z}_p)$ or linearly independent in $H_2(M; \mathbf{Z}_p)$ and L/S contains a non-trivial p-torsion, where $L = H_2(M; \mathbf{Z})/\mathrm{tor}$ and S is a submodule generated by $[S^2_1], \dots, [S^2_\ell]$ in L. Note that $b_2(M) = \dim H_2(M; \mathbf{Q}) = \mathrm{rank}\, L$.*

Proof. (1) Put $F = S^2_1 \cup \cdots \cup S^2_\ell$ and $N = M - \mathrm{Int}\, N(F)$. Under the hypothesis we have a non-zero element $a_1[S^2_1] + \cdots + a_\ell[S^2_\ell]$ of $H_2(F; \mathbf{Z}_p) = H_2(N(F); \mathbf{Z}_p)$ which sends to zero in $H_2(M; \mathbf{Z}_p)$ in the following commutative diagram:

$$\begin{array}{ccccc}
H_3(M, N(F); \mathbf{Z}_p) & \stackrel{\partial}{\longrightarrow} & H_2(N(F); \mathbf{Z}_p) & \longrightarrow & H_2(M; \mathbf{Z}_p) \\
PD\uparrow\cong & & PD\uparrow\cong & & \\
H^1(N; \mathbf{Z}_p) & \stackrel{\delta}{\longrightarrow} & H^2(M, N; \mathbf{Z}_p) & & \\
\downarrow & & \downarrow\cong & & \\
H^1(\partial N(F); \mathbf{Z}_p) & \stackrel{\delta}{\longrightarrow} & H^2(N(F), \partial N(F); \mathbf{Z}_p) & &
\end{array}$$

Here the horizontal sequences are natural and exact. So, there is an element α' of $H_3(M, N(F); \mathbf{Z}_p)$ such that $\partial\alpha' \neq 0$. By the Poincaré duality we get an element $\alpha \in H^1(N; \mathbf{Z}_p) = H^1(M - F; \mathbf{Z}_p)$ such that $\delta\alpha \neq 0$. Since $\partial N(F) = \cup_{i=1}^{\ell} \partial N(S_i^2)$, α takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i.

(2) Note first that there is an isomorphism $\overline{S}/S \cong S^*/\overline{S}^*$, where A^* stands for the dual $\mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})$. Consider the following commutative diagram whose horizontal sequences are exact and the coefficient is $\mathbf{Z}$:

$$\begin{array}{ccccccc}
 & & H_2(M, N) & \stackrel{\partial}{\longrightarrow} & H_1(N) & \stackrel{j_*}{\longrightarrow} & H_1(M) \\
 & & PD\uparrow\cong & & PD\uparrow\cong & & PD\uparrow\cong \\
H^2(M) & \stackrel{i^*}{\longrightarrow} & H^2(N(F)) & \stackrel{\delta}{\longrightarrow} & H^3(M, N(F)) & \stackrel{j^*}{\longrightarrow} & H^3(M) \\
\| & & \| & & & & \\
L^* \oplus \mathrm{tor} & \longrightarrow & S^* & & & &
\end{array}$$

Since S^* is torsion free, $\mathrm{Im}\, i^* = \mathrm{Im}\, L^*$. Moreover since L is unimodular, $\mathrm{Im}\, L^*$ is $\overline{S}^*$ by the definition of $\overline{S}$. So,

$$\overline{S}/S \cong S^*/\overline{S}^* = \mathrm{Coker}\, i^* \cong \mathrm{Im}\, \delta = \mathrm{Ker}\, j^*$$

By the Poincaré duality we get $\mathrm{Ker}\, j^* \cong \mathrm{Ker}(j_* : H_1(N; \mathbf{Z}) = H_1(M - F; \mathbf{Z}) \to H_1(M; \mathbf{Z}))$.

(3) We may assume that the homology classes $[S_1^2], \ldots, [S_\ell^2]$ are linearly independent in $H_2(M; \mathbf{Z}_p)$ and in particular linearly independent in $H_2(M; \mathbf{Z})$. We divide into two cases : (i) the case that $[S_i^2]^2 \neq 0$ for every i, and (ii) otherwise.

In case (i) the pairing $\langle\ ,\ \rangle$ on S is non-degenerate and $\ell_p(S^*/S) = \ell$. On the other hand rank $S^\perp = b_2(M) - \ell$ implies $\ell_p(S^{\perp *}/S^\perp) \leq b_2(M) - \ell$.

So, if S is p-primitive i.e., $\ell_p(\overline{S}/S) = 0$, then by Lemma 2.2 we have $\ell \leq b_2(M) - \ell$, which contradicts our hypothesis.

In case (ii) we may assume $[S_i^2]^2 = 0$ $(1 \leq i \leq k)$ and $\neq 0$ $(k+1 \leq i \leq \ell)$. Put $\xi_i = [S_i^2] \in H_2(M;\mathbf{Z})$ $(1 \leq i \leq \ell)$. Assume that S is p-primitive. Then, we have a homology class $\eta_1 \in H_2(M;\mathbf{Z})$ p-dual to ξ_1, that is, $\langle \xi_1, \eta_1 \rangle = mp+1$. Now, we put $\xi_i' = (mp+1)\xi_i - \langle \xi_i, \eta_1 \rangle \xi_1$ for $2 \leq i \leq \ell$ so that $\langle \xi_i', \eta_1 \rangle = \langle \xi_i', \xi_1 \rangle = 0, \xi_i'^2 = 0$ $(2 \leq i \leq k)$ and $\neq 0$ $(k+1 \leq i \leq \ell)$ and $\xi_1, \xi_2', \ldots, \xi_\ell'$ are also linearly independent. Let U_1 be a sublattice generated by ξ_1 and η_1. Since $\ell_p(U_1^*/U_1) = 0$, $L_1 = \{x \in L : \langle x, \xi_1 \rangle = \langle x, \eta_1 \rangle = 0\}$ is a p-unimodular lattice by Lemma 2.2. Let S_1 be the submodule of L_1 generated by $\xi_2', \ldots, \xi_\ell'$. Recall we assume that L/S contains no p-torsion. Then, it is equivalent to say that L_1/S_1 contains no p-torsion, because $(U_1 \oplus L_1)/S \cong \mathbf{Z} \oplus L_1/S_1$ and $L/(U_1 \oplus L_1) \subset U_1^*/U_1 \oplus L_1^*/L_1$ in the exact sequence $0 \longrightarrow (U_1 \oplus L_1)/S \longrightarrow L/S \longrightarrow L/(U_1 \oplus L_1) \longrightarrow 0$.

By an induction argument we get a p-unimodular lattice L_k of rank $= \operatorname{rank} L - 2k$ containing modified linearly independent homology classes $\xi_{k+1}, \ldots, \xi_\ell$. If we define S_k by the submodule of L_k generated by these modified $\xi_{k+1}, \ldots, \xi_\ell$, then $\langle\ ,\ \rangle$ on S_k is non-degenerate and L_k/S_k contains no p-torsion, that is, S_k is a p-primitive sublattice of the p-unimodular lattice L_k. Then, by Lemma 2.2 $\ell_p(S_k^*/S_k) = \ell_p(K_k^*/K_k)$, where K_k denotes the orthogonal complement of S_k in L_k. So, by an argument as in the case (i) $\ell - k \leq (b_2(M) - 2k) - (\ell - k)$ or equivalently $2\ell \leq b_2(M)$, which contradicts our hypothesis. This means that, if $[S_1^2], \ldots, [S_\ell^2]$ are linearly independent in $H_2(M;\mathbf{Z}_p)$, then L/S contains a non-trivial p-torsion. Q.E.D.

We want to estimate the first Betti number $b_1(\widetilde{M}) = \dim H_1(\widetilde{M};\mathbf{Q})$ of the 2-sheet branched covering $\widetilde{M}$ of M.

Lemma 2.4. *Let σ be a locally linear involution acting on a compact connected manifold $\widetilde{M}$ with fixed point set F and orbit space M. Suppose that $H_1(M;\mathbf{Q}) = 0$, F admits an equivariant normal disk bundle $\widetilde{N}(F)$ in $\widetilde{M}$ and one of the following three conditions is satisfied: (1) $F = \emptyset$, (2) F contains neither codimension one nor codimension two component, or (3) F contains no codimension one component and any connected component of codimension two part is simply-connected. Then,*

$$b_1(\widetilde{M}) \leq \ell_2(H_1(M - F;\mathbf{Z})) - 1.$$

Here $\ell_2(A)$ stands for the number of minimal generators of $A \otimes \mathbf{Z}_2$.

Proof. Sekine [13;§1] gives a proof in case $M = S^4$ and F has codimension two. Put $\widetilde{N} = \widetilde{M} - \operatorname{Int}\widetilde{N}(F)$. The natural projection $\pi : \widetilde{M} \to M$ induces a double covering $\pi : \widetilde{N} \to N$ of compact manifolds. We define a chain complex $\widehat{C}_*$ by the exact sequence:

$$0 \longrightarrow \widehat{C}_* \longrightarrow C_*(\widetilde{N};\mathbf{Z}) \xrightarrow{\pi_*} C_*(N;\mathbf{Z}) \longrightarrow 0.$$

Let t be a generator of $\mathbf{Z}_2$. Then, $\widehat{C}_* = (1-t)C_*(\widetilde{N};\mathbf{Z})$. So, $\widehat{C}_* \otimes \mathbf{Z}_2$ is isomorphic to $(1+t)C_*(\widetilde{N};\mathbf{Z}_2) \cong C_*(N;\mathbf{Z}_2)$ as chain complex.

Since $0 \longrightarrow \widehat{C}_* \otimes \mathbf{Q} \longrightarrow C_*(\widetilde{N};\mathbf{Q}) \longrightarrow C_*(N;\mathbf{Q}) \longrightarrow 0$ is also exact, we consider the exact sequence:

$$H_1(\widehat{C}_* \otimes \mathbf{Q}) \longrightarrow H_1(\widetilde{N};\mathbf{Q}) \longrightarrow H_1(N;\mathbf{Q}) \longrightarrow H_0(\widehat{C}_* \otimes \mathbf{Q}) \longrightarrow 0.$$

Put $d = \dim H_1(\widetilde{N};\mathbf{Q}) - \dim H_1(N;\mathbf{Q})$. Then, $d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) - \dim H_0(\widehat{C}_* \otimes \mathbf{Q})$.

Because $H_0(\widehat{C}_* \otimes \mathbf{Z}_2) = \mathbf{Z}_2$ and $H_0(\widehat{C}_*)$ is finitely generated, we have two cases: (i) $H_0(\widehat{C}_*)$ is finite and $\ell_2(H_0(\widehat{C}_*)) = 1$ and (ii) $H_0(\widehat{C}_*) \cong \mathbf{Z} \oplus (\text{odd torsion})$. In case (i) we have $H_0(\widehat{C}_*) * \mathbf{Z}_2 = \mathbf{Z}_2$ and $H_1(\widehat{C}_* \otimes \mathbf{Z}_2) = (H_1(\widehat{C}_*) \otimes \mathbf{Z}_2) \oplus \mathbf{Z}_2$ by the universal coefficient theorem. So,

$$d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) \leq \dim_{\mathbf{Z}_2} H_1(\widehat{C}_*) \otimes \mathbf{Z}_2 = \dim_{\mathbf{Z}_2} H_1(\widehat{C}_* \otimes \mathbf{Z}_2) - 1.$$

In case (ii) we have $H_0(\widehat{C}_*) * \mathbf{Z}_2 = 0$. So,

$$d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) - 1 \leq \dim_{\mathbf{Z}_2} H_1(\widehat{C}_*) \otimes \mathbf{Z}_2 - 1 = \dim_{\mathbf{Z}_2} H_1(\widehat{C}_* \otimes \mathbf{Z}_2) - 1.$$

Note that $H_1(\widehat{C}_* \otimes \mathbf{Z}_2) \cong H_1(N;\mathbf{Z}_2) = H_1(M - F;\mathbf{Z}_2) = H_1(M - F;\mathbf{Z}) \otimes \mathbf{Z}_2$. If $F = \emptyset$, then $H_1(N;\mathbf{Q}) = H_1(M;\mathbf{Q}) = 0$. Hence, the result follows from the condition (1).

Under the condition (2) or (3) the natural maps $H_0(\partial\widetilde{N}(F)) \to H_0(\widetilde{N}) \oplus H_0(\widetilde{N}(F))$ and $H_0(\partial N(F)) \to H_0(N) \oplus H_0(N(F))$ are injective with coefficient in $\mathbf{Q}$ due to the condition that F has no codimension one component. Hence, we have the following commutative diagram of Mayer-Vietoris exact sequences with coefficient in $\mathbf{Q}$:

$$\begin{array}{ccccccc}
H_1(\partial\widetilde{N}(F)) & \xrightarrow{(\tilde{\jmath}_*, \tilde{\imath}_*)} & H_1(\widetilde{N}) \oplus H_1(\widetilde{N}(F)) & \longrightarrow & H_1(\widetilde{M}) & \longrightarrow & 0 \\
\pi_* \downarrow & & \pi_* \oplus \downarrow \pi_* & & \pi_* \downarrow & & \\
H_1(\partial N(F)) & \xrightarrow{(j_*, i_*)} & H_1(N) \oplus H_1(N(F)) & \longrightarrow & H_1(M) & \longrightarrow & 0.
\end{array}$$

Note that $\pi_* : H_1(\widetilde{N}(F)) \longrightarrow H_1(N(F))$ is an isomorphism in any coefficient because they are canonically equal to $H_1(F)$. If F has no codimension two component, we have an exact sequence of groups $\mathbf{Z}_2 \longrightarrow \pi_1(\partial N(F)) \xrightarrow{i_*} \pi_1(N(F)) \longrightarrow 0$. So, $i_* : H_1(\partial N(F); \mathbf{Q}) \longrightarrow H_1(N(F); \mathbf{Q})$ is onto. Since $\tilde{i}_* : \pi_1(\partial\widetilde{N}(F)) \cong \pi_1(\widetilde{N}(F))$, $i_* : H_1(\partial N(F); \mathbf{Q}) = H_1(\partial\widetilde{N}(F); \mathbf{Q})^{\sigma_*} \hookrightarrow H_1(\partial\widetilde{N}(F); \mathbf{Q}) \cong H_1(\widetilde{N}(F); \mathbf{Q}) = H_1(N(F); \mathbf{Q})$ is injective. Hence, $i_* : H_1(\partial N(F); \mathbf{Q}) \longrightarrow H_1(N(F); \mathbf{Q})$ is also an isomorphism. So, the condition (2) implies $\dim H_1(\widetilde{M}; \mathbf{Q}) - \dim H_1(M; \mathbf{Q}) = \dim H_1(\widetilde{N}; \mathbf{Q}) - \dim H_1(N; \mathbf{Q}) = d$, which implies the result as before.

Let F_2 be a connected component of codimension two. Assume the condition (3). Then, there is an exact sequence $\mathbf{Z} \to \pi_1(\partial N(F_2)) \to 0$. If $\pi_1(\partial N(F_2))$ is finite, then $H_1(\partial\widetilde{N}(F_2); \mathbf{Q}) = H_1(\partial N(F_2); \mathbf{Q}) = 0$. Otherwise $\tilde{j}_* : H_1(\partial\widetilde{N}(F_2); \mathbf{Q}) \to H_1(\widetilde{N}; \mathbf{Q})$ is injective or zero if and only if $j_* : H_1(\partial N(F_2); \mathbf{Q}) \to H_1(N; \mathbf{Q})$ is injective or zero respectively. So, the condition (3) also implies $\dim H_1(\widetilde{M}; \mathbf{Q}) - \dim H_1(M; \mathbf{Q}) = \dim H_1(\widetilde{N}; \mathbf{Q}) - \dim H_1(N; \mathbf{Q}) = d$, which completes a proof. Q.E.D.

Remark. Probably we need not to assume the existence of equivariant normal disk bundle; it suffices that $F \times CP^2$ has a compact invariant manifold neighborhood $\widetilde{N}'(F \times CP^2)$ in $\widetilde{M} \times CP^2$ so that $F \times CP^2 \hookrightarrow \widetilde{N}'(F \times CP^2)$ is a homotopy equivalence and $\partial\widetilde{N}'(F \times CP^2) \to \widetilde{N}'(F \times CP^2)$ is a spherical homotopy fibration.

The following lemmas are not new but we list them up to quote in the proof of Theorem.

Lemma 2.5. *Let σ be an orientation preserving locally linear involution on an oriented closed 4-manifold M with fixed point set F. Let F^2 denote the 2-dimensional part of F.*

(1) *Any isolated point x of F can be blow up, that is, there is a locally linear involution σ' on $M^* = M\#\overline{CP}^2 = (M - x) \cup CP^1$ such that $\sigma'|M^* - CP^1 = \sigma|M - x$ and $\sigma'|CP^1 = \mathrm{id}$. In particular, σ' operates as identity on the newly introduced homology class represented by CP^1 and $\pi_1(M^*/\sigma') = \pi_1(M/\sigma)$. We may take also $M\#CP^2$ instead of $M\#\overline{CP}^2$; this comes from that we have an orientation reversing diffeomorphism of RP^3.*

(2) (Freedman-Quinn) *F^2 admits an equivariant normal disk bundle $N(F^2)$ in M.*

(3) (G-signature theorem)

$$\mathrm{sign}(-1, M) = e(F^2),$$

where $e(F^2)$ denotes the total Euler number of the normal bundle of F^2 and -1 stands for the involution concerned.

Proof. (1) Since σ is locally linear, we have a local complex coordinate (z_1, z_2) in a disk neighborhood U of x so that $x = (0,0)$ and $\sigma(z_1, z_2) = (-z_1, -z_2)$. Take a homogeneous coordinate $[\zeta_1, \zeta_2]$ of CP^1 and consider on the product space $U \times CP^1$ the subset U^* defined by $z_1\zeta_2 - z_2\zeta_1 = 0$. It is easy to see that U^* is a complex surface in $U \times CP^1$, the projection $\pi : U^* \to U$ gives an identification of $U^* - \pi^{-1}(0,0)$ with $U - (0,0)$, the preimage $(0,0) \times CP^1$ of $(0,0)$ is isomorphic to CP^1. Consider a holomorphic involution $(\sigma|U) \times \mathrm{id}$ on $U \times CP^1$. Then, we get a holomorphic involution $\sigma'|U^*$ on U^* such that $\sigma'|U^* - \pi^{-1}(0,0) = \sigma|U - (0,0)$ and $\sigma'|(0,0) \times CP^1 = \mathrm{id}$. Define $M^* = (M - U) \cup U^*$ and $\sigma'|M^* - U = \sigma|M - U$. Then, $M^* - CP^1 = M - x$ and M^* is diffeomorphic to $M \# \overline{CP}^2$ because $[CP^1]^2 = -1$. Since $\partial U^*/\sigma' = \partial U/\sigma = RP^3$ and $\pi_1(U^*/\sigma') = \pi_1(U/\sigma) = 0$, we have $\pi_1(M^*/\sigma') = \pi_1(M/\sigma)$ by the van Kampen theorem.

(2) Since M/σ is a manifold near F^2 and F^2 is a locally flat submanifold, F^2 admits a normal disk bundle due to Freedman-Quinn [6;9.3]. So, a lifting gives an equivariant normal disk bundle.

(3) In the smooth case G-signature theorem is due to Atiyah-Singer [2] but has many elementary proofs at least in our case of dimension 4 and semi-free, for example, in Gordon [8]. These elementary proofs can apply also to a locally linear involution, because it admits an equivariant tubular neighborhood of F^2 by (2). See also the comments in Edmonds [5;§4]. Q.E.D.

Lemma 2.6 (Edmonds [5;Prop. 3.1&3.2]). *Let M be a connected oriented spin 4-manifold and σ a locally linear involution that preserves orientation and some spin structure. Then, the fixed point set F, if non-empty, consists either of isolated points or of orientable surfaces.*

In the smooth case the codimension homogeneity modulo 4 is proved by Atiyah-Bott [1] and the orientability of surfaces has many proofs including Edmonds [4]. The proof in the locally linear case is given in Edmonds [5].

§3. Proof of Theorem 1

Since $H_1(M; \mathbf{Z}_2) = 0$, the spin structure on M is unique and we may assume that σ preserves the spin structure. Lemma 2.6 implies that the fixed point set F consists either of isolated points or of orientable surfaces. If F consists of isolated points, then by the G-signature theorem described as Lemma 2.5 (3) $\operatorname{sign}(-1, M) = 0$. Hence, $\operatorname{sign} M = 0$ because σ operates as identity on $H_2(M; \mathbf{Q})$. So, we may assume that F consists of orientable surfaces. In particular, M/σ is also a manifold. Note that F has an equivariant normal disk bundle $N(F)$ in M by Lemma 2.5 (2).

Since $H_*(M/\sigma; \mathbf{Q}) = H_*(M; \mathbf{Q})^{\sigma_*}$, $H_1(M; \mathbf{Q}) = 0$ and $\sigma_* | H_2(M; \mathbf{Q}) = \mathrm{id}$, we have the equality $\chi(M/\sigma) = \chi(M)$ of Euler numbers. Put $\chi = \chi(M)$. Then, from the formula $\chi(M) = 2\chi(M/\sigma) - \chi(F)$ we get also $\chi(F) = \chi$. So, F contains at least $\chi/2$ numbers of components of S^2. Note that M has an even intersection form $q_M : H_2(M; \mathbf{Z})/\operatorname{tor} \times H_2(M; \mathbf{Z})/\operatorname{tor} \to \mathbf{Z}$ and hence $\chi = \chi(M)$ is even. Let $F' = S_1^2, \ldots, S_{\chi/2}^2$ be the subset of F consisting of $\chi/2$ numbers of S^2. Since $H_1(M/\sigma; \mathbf{Q}) = H_1(M; \mathbf{Q})^{\sigma_*} = 0$, we have $\chi = 2 + b_2(M/\sigma) > b_2(M/\sigma)$. Taking account of $[S_i^2]_{M/\sigma}^2 = 2[S_i^2]_M^2$ and Lemma 2.5 (2), we can apply Lemma 2.3 (3) for $p = 2$ and $F' \subset M/\sigma$. So, by Lemma 2.3 (1) and (2) there is a subunion F'' of connected components of F' such that we have a branched covering of M/σ with branch locus F'', that is, $(M, \sigma, F'' \subset F)$ satisfies the condition of Lemma 2.1 except $F'' \neq F$. Note here that $H_1(\partial N(x); \mathbf{Z}) \to H_1(\partial N(S_i^2); \mathbf{Z})$ is a surjection for any x of S_i^2. If $F'' \neq F$, then Lemma 2.1 implies that there is a connected 2-sheet unbranched covering of M. But this contradicts the condition that $H^1(M; \mathbf{Z}_2) = \operatorname{Hom}(H_1(M; \mathbf{Z}), \mathbf{Z}_2) = \operatorname{Hom}(\pi_1(M), \mathbf{Z}_2) = 0$. This means $F'' = F$. Hence, $F' = F$, that is, F consists of $\chi/2$ numbers of S^2.

Since the intersection form q_M of M is even, we can also apply Lemma 2.3 (3) for $p = 2$ and $F \subset M$. By Lemma 2.3 (1) and (2) there is a non-trivial element of $H^1(M - F; \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i. This means that there is a branched covering $\tilde{\pi} : \widetilde{M} \to M$ with branch locus $F_1 \subset F$; a locally linear involution τ on $\widetilde{M}$ with fixed point set F_1. So, there is a non-trivial element of $H^1(M - F_1; \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for every $S_i^2 \subset F_1$. Because $H^1(M; \mathbf{Z}_2) = 0$, this implies that (i) the homology classes of the connected components of F_1 are linearly dependent in $H_2(M; \mathbf{Z}_2)$ or (ii) they are independent and generate a submodule S of $L = H_2(M; \mathbf{Z})/\operatorname{tor}$ so that $\overline{S}/S$ contains a non-trivial 2-torsion according to the last part of Lemma 2.3 (2). Assume that $F_1 \neq F$. In case (i) the homology classes of the connected components of F_1 are also linearly dependent in $H_2(M/\sigma; \mathbf{Z}_2)$ and this leads to a contradiction with $H^1(M; \mathbf{Z}_2) = 0$

through Lemma 2.3 (1) and Lemma 2.1 as before. In case (ii) notice that $\pi_* S$ is the submodule generated by the homology classes of the connected components of F_1 in $H_2(M/\sigma; \mathbf{Z})/\operatorname{tor}$ for the projection $\pi : M \to M/\sigma$. Since $\pi_*|S$ is an isomorphism, $\pi_*\overline{S}/\pi_* S$ is isomorphic to $\overline{S}/S$. Note also that $\pi_*\overline{S}/\pi_* S \subset \overline{\pi_* S}/\pi_* S$. Then, $\overline{\pi_* S}/\pi_* S$ contains a non-trivial 2-torsion. We can apply Lemma 2.3 (2) for $p = 2$ and $F_1 \subset M/\sigma$ and we get the same contradiction with $H^1(M; \mathbf{Z}_2) = 0$ by applying Lemma 2.1 for $(M, \sigma, F_1 \subset F)$ since we have assumed $F_1 \neq F$. Hence, $F_1 = F$, that is, the branch locus for $\tilde{\pi} : \widetilde{M} \to M$ is also F and $\chi(\widetilde{M}) = \chi(M)$.

We will show that $\ell_2(H_1(M - F; \mathbf{Z})) = 1$. Since $H^1(M; \mathbf{Z}_2) = 0$, it is equivalent to say $\ell_2(\operatorname{Ker}(H_1(M - F; \mathbf{Z}) \to H_1(M; \mathbf{Z}))) = 1$. Put $N = M - \operatorname{Int} N(F)$ and consider the following commutative diagram:

$$\begin{array}{ccccc} H_1(\partial N(F); \mathbf{Z}) & \longrightarrow & H_1(N; \mathbf{Z}) & \longrightarrow & H_1(N, \partial N(F); \mathbf{Z}) \\ & & \downarrow & & \downarrow\cong \\ & & H_1(M; \mathbf{Z}) & \longrightarrow & H_1(M, N(F); \mathbf{Z}) \end{array}$$

Since the horizontal sequence is exact, any element of $\operatorname{Ker}(H_1(N; \mathbf{Z}) = H_1(M - F; \mathbf{Z}) \to H_1(M; \mathbf{Z}))$ comes from $H_1(\partial N(F); \mathbf{Z})$. We know that there is an element α of $\operatorname{Hom}(H_1(M - F; \mathbf{Z}), \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for every S_i^2 in F. Now we assume that $\ell_2(\operatorname{Ker}(H_1(M - F; \mathbf{Z}) \to H_1(M; \mathbf{Z}))) \geq 2$. Then, we have some element β of $\operatorname{Hom}(H_1(M - F; \mathbf{Z}), \mathbf{Z}_2)$ which is different from α, that is, takes zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for at least one i. Note that we used here the special property of $\mathbf{Z}_2$. Let F' be the subset of F removed such S_i^2 off. Since $F' \neq F$, the same argument as the above paragraph can be applied again and get a contradiction with the condition $H^1(M; \mathbf{Z}_2) = 0$.

Now since $H_1(M; \mathbf{Q}) = 0$ and F consists of $\chi/2$ numbers of S^2, $\ell_2(H_1(M - F; \mathbf{Z})) = 1$ implies $b_1(\widetilde{M}) = 0$ by Lemma 2.4. So, $\chi(\widetilde{M}) = \chi(M)$ implies $b_2(\widetilde{M}) = b_2(M)$. Hence, $H_2(M; \mathbf{Q}) = H_2(\widetilde{M}; \mathbf{Q})^{\tau_*}$ implies $H_2(\widetilde{M}; \mathbf{Q})^{\tau_*} = H_2(\widetilde{M}; \mathbf{Q})$, that is, $\tau_* = \mathrm{id}$ on $H_2(\widetilde{M}; \mathbf{Q})$. Therefore, $\operatorname{sign}(-1, \widetilde{M}) = \operatorname{sign} \widetilde{M}$. Recall that $\operatorname{sign}(-1, M) = \operatorname{sign} M$ and the G-signature theorem says that

$$\operatorname{sign}(-1, M) = \sum_{i=1}^{\chi/2} [S_i^2]_M^2 = \sum_{i=1}^{\chi/2} 2[S_i^2]_{\widetilde{M}}^2 = 2 \operatorname{sign}(-1, \widetilde{M}).$$

On the other hand $\operatorname{sign} M = \operatorname{sign} \widetilde{M}$ because $H_2(M; \mathbf{Q}) = H_2(\widetilde{M}; \mathbf{Q})^{\tilde{\tau}_*} = H_2(\widetilde{M}; \mathbf{Q})$. Hence, $\operatorname{sign} M = 0$. This completes a proof of Theorem 1.

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