Advanced Studies in Pure Mathematics 20, 1992
Aspects of Low Dimensional Manifolds
pp. 301–330

On 4-Manifolds Homotopy Equivalent to the 2-Sphere

Osamu Saeki

Dedicated to Professor Akio Hattori on his 60th birthday

§1. Introduction

Let V be a compact topological 4-manifold homotopy equivalent to the 2-sphere S^2. We say such a 4-manifold is a *homotopy* S^2. The boundary ∂V of V is always a closed connected 3-manifold with the same integral homology and the same linking pairing as those of the lens space $L(p,1)$ for some p (≥ 0). We say such a 3-manifold is a *homology* $L(p,1)$. For a fixed homology $L(p,1)$, M, the homotopy S^2's bounded by M are classified up to homeomorphism by certain equivalence classes of some elements of $H_1(M;\mathbf{Z})$ ([3]).

In this paper, concerning homotopy S^2's, we consider the following problems.

(A) For a fixed homology $L(p,1)$, M, how many homotopy S^2's does M bound? Furthermore, how many of them admit smooth structures?

(B) Give a lower bound for the genera of topologically locally flatly embedded surfaces in V representing the generator γ of $H_2(V;\mathbf{Z})$. If V is smooth, what is the necessary condition for γ to be represented by a smoothly embedded 2-sphere?

(C) Let K be a tame knot in the boundary of V. Under what condition does K bound a topologically embedded flat 2-disk in V? If V and K are smooth and K bounds such a topologically embedded 2-disk, does K also bound a smoothly embedded 2-disk in V?

(D) Does there exist a homotopy S^2 admitting more than one smooth structures?

In [28], we considered problem (A) and showed that if V is a smooth homotopy S^2 satisfying a certain condition on the order of $H_1(\partial V;\mathbf{Z})$

Received June 27, 1989.
Revised March 4, 1991.

such that the generator of $H_2(V; \mathbf{Z})$ is represented by a smoothly embedded 2-sphere, then every *smooth* homotopy S^2 with the same boundary as V is homeomorphic to V. Furthermore we gave the exact number of homotopy S^2's bounded by the 3-manifold ∂V. In this paper we consider smoothly immersed 2-spheres instead of embedded ones and give a similar result. More precisely, we show that if V is a smooth homotopy S^2 satisfying a certain condition on the order of $H_1(\partial V; \mathbf{Z})$ such that the generator of $H_2(V; \mathbf{Z})$ is represented by a smoothly immersed 2-sphere with relatively few double points, then the same results as above hold (§2). We note that, in some cases, this result can be applied to give a lower bound for the number of double points of smoothly immersed 2-spheres representing the generator of $H_2(V; \mathbf{Z})$.

In §3 we define some topological invariants for homotopy S^2's and use them to attack problem (B). First, we define a "Casson invariant" for a homotopy S^2 using the extension of the usual Casson invariant for homology 3-spheres ([6]) to marked homology lens spaces ([4, 15]). We show that if the generator of the second homology group of a smooth homotopy S^2 is represented by a smoothly embedded 2-sphere, then its Casson invariant modulo 2 must vanish. In fact, this is proved only using the well-known theorem of Rohlin. We do not know what essential properties of a homotopy S^2 this Casson invariant reflects. Next we define Casson-Gordon invariants for a homotopy S^2 (cf. [7]) and use them to give a lower bound for the genera of topologically locally flatly embedded surfaces representing the generator of the second homology group of a homotopy S^2. In our case these invariants are the p-signatures of a certain knot in a homology 3-sphere. If a smooth homotopy S^2 consists of one 0-handle and one 2-handle, then it has already been known that a lower bound for the genera of such smoothly embedded surfaces is given by the p-signatures of the knot along which the 2-handle is attached to the 0-handle ([31]). In this paper, we extend this to general homotopy S^2's employing a method similar to that in [19].

In §4, we give a sufficient condition for certain tame knots in the boundary of a homotopy S^2 to bound topologically embedded flat 2-disks. We see that almost all knots satisfying a certain homological condition bound such 2-disks in the homotopy S^2. Furthermore, we give an example of smooth knots in the boundary of a smooth homotopy S^2 which bound topologically embedded flat 2-disks in the homotopy S^2 but never bound smooth ones. In the case of knots in the boundary of the 4-ball, the same examples have been found first by Casson and, after that, by several authors [8, 9, 18]. Our technique is similar to theirs in that we use the celebrated theorem of Donaldson [10].

As to problem (D), Akbulut [1] has recently found a *compact* homotopy S^2 with more than one smooth structures. In this paper we give an example of infinitely many *open* 4-manifolds homotopy equivalent to S^2 each of which admits at least 3 smooth structures (§5).

Throughout the paper, all homology groups are with integral coefficients unless otherwise indicated.

§2. Smooth homotopy S^2 and immersed 2-spheres

Definition. We say that a non-zero integer p satisfies *property (#)* if -1 is not a quadratic residue modulo p; i.e., if $n^2 \not\equiv -1 \pmod p$ for every integer n.

Lemma 2.1. *Let* $|p| = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ *be the prime decomposition of* $|p|$. *Then* p *satisfies property (#) if and only if* $e \geq 2$ *or* $p_i \equiv 3 \pmod 4$ *for some* p_i.

For the proof of Lemma 2.1, see, for example, the proof of Corollary 3.11 in [28]. Note that, by Lemma 2.1, if $|p| \equiv 0, 3 \pmod 4$ then p satisfies property (#).

Let V be a smooth oriented homotopy S^2. Define $k_+(V)$ (resp. $k_-(V)$) to be the minimum number of positive (resp. negative) double points of smoothly immersed self-transverse 2-spheres representing the generator of $H_2(V)$. We call $k_+(V)$ (resp. $k_-(V)$) the *positive* (resp. *negative*) *kinkiness* of V. Then our main theorem of this section is the following.

Theorem 2.2. *Let* V *be a smooth oriented homotopy* S^2 *bounded by a homology* $L(p,1)$, M, *and let* $\gamma \in H_2(V)$ *be a generator. We assume* $p = |\gamma^2|$ *satisfies property (#). Let* ε *be the sign of* γ^2. *If*

$$k_\varepsilon(V) \leq \begin{cases} (p-6)/4 & (p\text{: even}) \\ (p-1)/4 & (p\text{: odd}, \neq 15, 21) \\ 2 & (p = 15) \\ 4 & (p = 21), \end{cases}$$

then the following holds.

(1) *Every smooth homotopy* S^2 *bounded by* M *is homeomorphic to* V.

(2) *Every homeomorphism* $h : M \to M$ *acts on* $H_1(M)$ *by the multiplication of* ± 1.

(3) *If* $p = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ *is the prime decomposition of* p, *then the number of homeomorphism types of homotopy* S^2*'s*

bounded by M is equal to

$$\begin{cases} 2^{r-1} & (e=1) \\ 2^r & (e=0,2) \\ 2^{r+1} & (e \geq 3). \end{cases}$$

Remark. Even when p does not satisfy property (#), part (2) of Theorem 2.2 does hold if the homeomorphism h is orientation preserving.

Before we prove Theorem 2.2, we describe some of its consequences. The proof will be given at the end of this section.

Let K be a smooth knot in the oriented 3-sphere S^3. For an integer p, we denote by $V(K;p)$ the smooth oriented homotopy S^2 obtained by attaching a 2-handle to the 4-ball D^4 along the knot K with the p-framing. Furthermore we denote by $M(K;p/1)$ the boundary 3-manifold of $V(K;p)$. Note that $M(K;p/1)$ is diffeomorphic to the 3-manifold obtained by performing the $p/1$-Dehn surgery on the knot K ([26]).

We denote by $u(K)$ the unknotting number of a knot K in S^3 (for example, see [16]). Note that K bounds in the 4-ball a smoothly immersed self-transverse 2-disk with $u(K)$ double points ([9]). Taking the union of this immersed 2-disk and the core disk of the 2-handle, we can represent the generator of $H_2(V(K;p))$, for any p, by a smoothly immersed 2-sphere with $u(K)$ double points; i.e., we always have

$$k_\pm(V(K;p)) \leq k_+(V(K;p)) + k_-(V(K;p)) \leq u(K).$$

Then we obtain the following immediately.

Corollary 2.3. *Let K be a smooth knot in S^3 and let p be an integer satisfying property (#). If $|p| \geq 4u(K)+6$, then the following holds.*

(1) *Every smooth homotopy S^2 bounded by $M(K;p/1)$ is homeomorphic to $V(K;p)$.*
(2) *Every homeomorphism $h : M(K;p/1) \to M(K;p/1)$ acts on $H_1(M(K;p/1))$ by the multiplication of ± 1.*
(3) *If $|p| = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ is the prime decomposition of $|p|$, then the number of homeomorphism types of homotopy S^2's bounded by $M(K;p/1)$ is equal to*

$$\begin{cases} 2^{r-1} & (e=1) \\ 2^r & (e=0,2) \\ 2^{r+1} & (e \geq 3). \end{cases}$$

This corollary shows that, *for any knot* K, if $|p|$ is sufficiently large and satisfies property (#), then $M(K;p/1)$ satisfies the above properties. This is the major difference between Theorem 2.2 and the result obtained in [28]; the latter is applicable only to slice knots.

Let a and b be relatively prime integers. Then we denote by $T(a,b)$ the left-hand ($ab > 0$) or the right-hand ($ab < 0$) torus knot of type (a,b). As a corollary of Theorem 2.2 we have the following.

Corollary 2.4. *Every smooth homotopy S^2 bounded by the lens space $L(4n+3,4)$ $(n \geq 0)$ is homeomorphic to the handlebody $V(T(2,2n+1);-(4n+3))$.*

Proof. By Moser [25], $\partial V(T(2,2n+1);-(4n+3)) = M(T(2,2n+1);-(4n+3)/1)$ is homeomorphic to the lens space $L(4n+3,4)$. Note that the integer $p = 4n+3$ always satisfies property (#). On the other hand, it is well-known that $u(T(2,2n+1)) \leq n$. Thus by the previous remark, $k_-(V(T(2,2n+1);-p)) \leq n = (p-3)/4$. Since p is odd, the result follows from Theorem 2.2 unless $p = 15$. If $p = 15$, it can be shown that the number of homotopy S^2's bounded by $L(15,4)$ is 1 (see Example 2.8 of [28]). This completes the proof.

In [28], we remarked that there are exactly 1401 homotopy S^2's bounded by the lens spaces $L(p,q)$ with $2 \leq p \leq 100$ and that among them there exist at least 701 homotopy S^2's which cannot admit any smooth structures and at least 274 homotopy S^2's admitting smooth structures. Using the above Corollary 2.4 and the technique used in the proof of Theorem 2.2, we can find, among the 1401 homotopy S^2's, additional 16 non-smoothable homotopy S^2's. We also note that, using the result of Maruyama [23], we can find additional 14 homotopy S^2's admitting smooth structures.

Next we indicate how Theorem 2.2 can be applied to give lower bounds for the kinkinesses of a homotopy S^2.

Definition. Let V be a homotopy S^2 and let $\delta \in H_2(V,\partial V)$ be a generator. Then we define $\beta(V) = \partial\delta \in H_1(\partial V)$, where $\partial : H_2(V,\partial V) \to H_1(\partial V)$ is the boundary homomorphism. Note that $\beta(V)$ generates $H_1(\partial V)$ and is determined up to the multiplication of ± 1.

Remark. If $H_1(\partial V)$ is finite cyclic of order p, we always have $\mathrm{lk}(\beta(V),\beta(V)) = \pm 1/p$, where lk: $H_1(\partial V) \times H_1(\partial V) \to \mathbf{Q}/\mathbf{Z}$ is the linking pairing of ∂V.

Theorem 2.5. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Suppose there exists a homeomorphism $h : \partial V \to \partial V$*

such that $h_*(\beta(V)) = r\beta(V)$ *with* $r \not\equiv \pm 1 \pmod p$ *and* $r^2 \equiv 1 \pmod p$. *Then we have*

$$k_\varepsilon(V) \geq \begin{cases} (p-4)/4 & (p : \textit{even}) \\ (p+1)/4 & (p : \textit{odd}, \neq 15, 21) \\ 3 & (p = 15) \\ 5 & (p = 21), \end{cases}$$

where ε $(= \pm 1)$ *is the signature of* V.

If p satisfies property $(\#)$, the above Theorem is a direct consequence of Theorem 2.2. The general case can be proved by the same method as in the proof of Theorem 2.2 below.

As a typical example, we can obtain a lower bound for the kinkinesses of the homotopy S^2's obtained by attaching a 2-handle to the 4-ball along some torus knots.

Corollary 2.6. *Let* s *and* r *be relatively prime integers greater than* 1. *Suppose that* $s^2 \not\equiv \pm 1 \pmod{rs + \varepsilon}$ *and that* $rs + \varepsilon$ *divides* $s^4 - 1$, *where* $\varepsilon = \pm 1$. *Then we have*

$$k_-(V(T(r,s); -(rs+\varepsilon))) \geq \begin{cases} (rs+\varepsilon-4)/4 & (rs+\varepsilon : \textit{even}) \\ (rs+\varepsilon+1)/4 & (rs+\varepsilon : \textit{odd} \neq 15, 21) \\ 3 & (rs+\varepsilon = 15) \\ 5 & (rs+\varepsilon = 21). \end{cases}$$

Proof. By Moser [25], $\partial V(T(r,s); -(rs+\varepsilon))$ is diffeomorphic to the lens space $L(rs+\varepsilon, s^2)$. Since $s^4 \equiv 1 \pmod{rs+\varepsilon}$, there exists a homeomorphism $h : L(rs+\varepsilon, s^2) \to L(rs+\varepsilon, s^2)$ which acts on the first homology group by the multiplication of s^2. Now the result follows from Theorem 2.5.

Example 2.7. There do exist r, s and ε satisfying the condition in Corollary 2.6. For example, we have

$$\begin{aligned} k_-(T(2,7); -15) &\geq 3, \\ k_-(T(3,5); -16) &\geq 3, \\ k_-(T(3,13); -40) &\geq 9, \\ k_-(T(4,13); -51) &\geq 13. \end{aligned}$$

We also note that $k_-(T(a,b); p) \leq (|a| - 1)(|b| - 1)/2$ for any p.

Using Corollary 2.6, we can obtain a lower bound for the unknotting numbers of certain torus knots. However, this lower bound is worse than the one obtained in [16]. There is a conjecture that the unknotting number of the torus knot $T(a,b)$ is equal to $(|\,a\,|\,-1)(|\,b\,|\,-1)/2$. We do not know whether there is a smoothly immersed 2-sphere in some $V(T(a,b);p)$ representing the generator of $H_2(V(T(a,b);p))$ with the number of double points strictly fewer than $(|\,a\,|\,-1)(|\,b\,|\,-1)/2$. Note that if $p=\pm 1$, then there always exists a topologically locally flatly embedded 2-sphere or torus representing the generator of the second homology group (see Proposition 3.6).

Now we proceed to the proof of Theorem 2.2. Our method is similar to that in [28].

Let X be a smooth closed 1-connected oriented 4-manifold. Given a homology class $\zeta \in H_2(X)$, one can represent ζ by an immersed 2-sphere whose self-intersections are transverse. Define d_ζ^+ (resp. d_ζ^-) to be the minimum number of positive (resp. negative) double points of such immersed 2-spheres representing ζ. The following is a theorem of Kuga and Suciu which plays a key role in the proof of Theorem 2.2. See also the remark after Theorem 4 in [16].

Theorem 2.8 (Kuga [22], Suciu [30]).

(1) *Let X be a smooth closed oriented 4-manifold homotopy equivalent to $S^2\times S^2$ and let ξ and η be generators of $H_2(X)$ such that $\xi^2=\eta^2=0$ and $\xi\cdot\eta=1$. If $\zeta=a\xi+b\eta\in H_2(X)$ satisfies $\zeta^2\neq 0$, then we have*

$$d_\zeta^\varepsilon \geq \min\left\{(|\,a\,|\,-1)(|\,b\,|\,-1),\ \left[\frac{|\,ab\,|\,+1}{2}\right]\right\},$$

where ε is the sign of ζ^2 and, for $x\in\mathbf{Q}$, $[x]$ denotes the largest integer not exceeding x.

(2) *Let X be a smooth closed oriented 4-manifold homotopy equivalent to $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ and let ξ and η be generators of $H_2(X)$ such that $\xi^2=1, \eta^2=-1$ and $\xi\cdot\eta=0$. If $\zeta=a\xi+b\eta\in H_2(X)$ satisfies $\zeta^2\neq 0$, then we have*

$$d_\zeta^\varepsilon \geq \min\left\{\frac{(|\,a\,|+|\,b\,|\,-2)(|\,|\,a\,|-|\,b\,|\,|-1)}{2},\ \left[\frac{|\,a^2-b^2\,|\,+3}{4}\right]\right\},$$

where ε is the sign of ζ^2.

Proof of Theorem 2.2. Changing the orientation of V if necessary, we may assume V is positive definite (i.e., $\varepsilon=1$). Let V' be a smooth homotopy S^2 with $\partial V'$ diffeomorphic to $M = \partial V$ and let $h : \partial V' \to \partial V$ be a homeomorphism. Changing h by an isotopy, we assume h is a diffeomorphism. Furthermore, we orient V' so that h is an orientation preserving map. Thus V' is not necessarily positive definite. We have, for some $r \in \mathbf{Z}$, $h_*(\beta(V')) = r\beta(V)$. We will show $r \equiv \pm 1 \pmod{p}$. Then the part (1) of Theorem 2.2 follows from a result of Boyer [3]. Furthermore the part (2) is also proved if we set $V' = V$. Then the part (3) follows by the same argument as in [28].

Set $X = V \cup_h (-V')$, which is a smooth closed 1-connected 4-manifold with $H_2(X) \cong \mathbf{Z} \oplus \mathbf{Z}$. X has the orientation induced from those of V and $-V'$. By [28], there are generators θ and τ of $H_2(X)$ with the following properties.

(i) θ is represented by a smoothly immersed 2-sphere with $k_+(V)$ positive double points.

(ii) $\theta \cdot \theta = p$ and $\theta \cdot \tau = r$.

Remember that θ comes from the generator of $H_2(V)$ and τ is defined to be the "union" of the generator of $H_2(V', \partial V')$ and r times the generator of $H_2(V, \partial V)$.

Set $t = \tau \cdot \tau$. Then the intersection matrix of X with respect to θ and τ is

$$Q = \begin{pmatrix} p & r \\ r & t \end{pmatrix}.$$

Since Q is unimodular, $\det Q = pt - r^2 = \pm 1$. Hence $r^2 \equiv \mp 1 \pmod{p}$. Since p satisfies property (#), $r^2 \equiv 1 \pmod{p}$. Thus $\det Q = -1$ and the intersection form of X is indefinite. Hence, there are generators ξ and η of $H_2(X)$ with respect to which the intersection matrix of X is one of the following forms;

$$\text{(A)} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{or}$$

$$\text{(B)} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Case (A). Since Q must be of even type, p is even. Suppose $\theta = a\xi + b\eta$ $(a, b \in \mathbf{Z})$. Since $p = \theta \cdot \theta = 2ab$ is positive, we may assume $a > 0$ and $b > 0$, changing the orientations of ξ and η if necessary. By Theorem 2.8, we have

$$d_\theta^+ \geq \min \left\{ (a-1)(b-1), \left[\frac{ab+1}{2} \right] \right\}. \tag{2.1}$$

On the other hand, by the hypothesis, we have

$$d_\theta^+ \leq k_+(V) \leq \frac{p-6}{4}. \tag{2.2}$$

Combining (2.2) with (2.1), we have

$$\frac{ab-3}{2} \geq \min\left\{(a-1)(b-1), \left[\frac{ab+1}{2}\right]\right\}. \tag{2.3}$$

Since $(ab-3)/2 < [(ab+1)/2]$, we have $(ab-3)/2 \geq (a-1)(b-1)$. This shows that $a \leq 1$ or $b \leq 1$. Then the same argument as in [28] shows $r \equiv \pm 1 \pmod p$.

Case (B). Suppose $\theta = a\xi + b\eta$ $(a, b \in \mathbf{Z})$. We may assume $a \geq 0$ and $b \geq 0$, changing the orientations of ξ and/or η if necessary. Note that $p = \theta \cdot \theta = a^2 - b^2 > 0$. By Theorem 2.8, we have

$$d_\theta^+ \geq \min\left\{\frac{(a+b-2)(a-b-1)}{2}, \left[\frac{a^2-b^2+3}{4}\right]\right\}. \tag{2.4}$$

If p is even, we have

$$d_\theta^+ \leq k_+(V) \leq \frac{p-6}{4}. \tag{2.5}$$

Combining this with (2.4), we obtain $a+b \leq 3$ or $a-b \leq 1$. Since $a \geq 0, b \geq 0$ and a is prime to b, we have $a-b=1$. This contradicts the fact that $p = a^2 - b^2$ is even.

If $p = (a+b)(a-b)$ is odd not equal to 15 or 21, we have, by the hypothesis,

$$d_\theta^+ \leq k_+(V) \leq \frac{p-1}{4}. \tag{2.6}$$

Combining this with (2.4), we obtain

$$(a+b-4)(a-b-2) \leq 3. \tag{2.7}$$

Note that $a+b$ and $a-b$ are odd and that a and b are relatively prime. Then we have $a-b=1$ or $(a,b) = (4,1)$ or $(5,2)$. Since $p = a^2 - b^2 \neq 15, 21$, we have $a - b = 1$. When $p = 15$, we obtain

$$2 \geq \min\left\{\frac{(a+b-2)(a-b-1)}{2}, 4\right\} \tag{2.8}$$

and when $p = 21$,

$$4 \geq \min\left\{\frac{(a+b-2)(a-b-1)}{2}, 6\right\}. \tag{2.9}$$

Both inequalities imply $a - b = 1$. Thus when p is odd we always have $a - b = 1$. Then the same argument as in [28] shows $r \equiv \pm 1 \pmod{p}$. This completes the proof.

Remark. When p is equal to 15 or 21, we cannot omit the special condition that $k_\varepsilon(V) \leq 2$ and $k_\varepsilon(V) \leq 4$ respectively. For example, consider $V = V(T(2,7); -15)$. Then $k_-(V) \leq 3$ (cf. Example 2.7). However $\partial V = L(15,4)$ admits a self-homeomorphism acting on $H_1(L(15,4))$ by the multiplication of ± 4. When $p = 21$, consider $V = V(T(2,11); -21)$. Then $k_-(V) \leq 5$. However $\partial V = L(21,4)$ bounds the smooth homotopy S^2, $V(T(4,5); -21)$, which is not homeomorphic to V.

§3. Topological invariants for homotopy S^2 and embedded surfaces

Let M be a homology $L(p,1)$ $(p > 0)$; i.e., M is a closed oriented 3-manifold such that $H_*(M) \cong H_*(L(p,1))$ and $\mathrm{lk}(\alpha, \alpha) \equiv -\varepsilon/p \pmod{\mathbf{Z}}$ for some generator α of $H_1(M)$ $(\varepsilon = \pm 1)$. We call such a pair (M, α) a *marked homology* $L(p,1)$. It is well-known that M is obtained by (εp)-surgery on a knot K in some homology 3-sphere Σ so that the core of the surgery torus in M represents α. Define

$$\lambda_0(M, \alpha) = p\lambda(\Sigma) + (\varepsilon/2)\Delta''_K(1), \tag{3.1}$$

where $\lambda(\Sigma)$ is the Casson invariant of the homology 3-sphere Σ ([6]) and $\Delta_K(t)$ is the normalized Alexander polynomial of K. For a fixed marked homology $L(p,1)$, Σ and K as above are not uniquely determined. However, by results of Boyer-Lines [4] and Fukuhara [15], $\lambda_0(M, \alpha)$ is an invariant of the marked homology $L(p,1)$, (M, α). We warn the reader that λ_0 here is p times the invariant defined in [4] or [15] and that λ_0 here is integer valued. Note also that if $p = 1$, i.e. if M is a homology 3-sphere, (3.1) with $\lambda_0(M, \alpha)$ replaced by $\lambda(M)$ is nothing but the surgery formula for the usual Casson invariant. Therefore, λ_0 agrees with λ for homology 3-spheres.

Definition. Let V be an oriented homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Note that $(\partial V, \beta(V))$ is a marked homology $L(p,1)$.

Then we define $\overline{\lambda}(V) = \lambda_0(\partial V, \beta(V))$, which we call the *Casson invariant* of V. $\overline{\lambda}(V)$ is a topological invariant of V.

Remark. If ∂V is a homology 3-sphere, then $\overline{\lambda}(V) = \lambda(\partial V)$. In general, however, $\overline{\lambda}(V)$ is not an invariant of ∂V. For example, denoting by U the trivial knot in S^3, we have $\overline{\lambda}(V(U;5)) = 0$ and $\overline{\lambda}(V(T(2,3);-5)) = -1$, though $\partial V(U;5) \cong \partial V(T(2,3);-5) \cong L(5,1)$ ([25]).

The first result of this section is the following.

Theorem 3.1. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite. If the generator of $H_2(V)$ can be represented by a smoothly embedded 2-sphere, then $\overline{\lambda}(V) \equiv 0 \pmod 2$.*

Remark. If V is diffeomorphic to a handlebody $V(K;p)$ for some knot K in S^3, then $\overline{\lambda}(V)$ mod 2 agrees with the Arf invariant of K. In this case, the above result has already been known ([2]).

Proof of Theorem 3.1. Suppose S is a smoothly embedded 2-sphere in Int$V$ which represents the generator of $H_2(V)$. We denote by $N(S)$ the tubular neighborhood of S. Then it is easily seen that the 4-manifold $Y = V - \mathrm{Int}N(S)$ is a homology cobordism between ∂V and $L(p,1)$ ($\cong \partial N(S)$). Furthermore, if $\alpha \in H_1(L(p,1))$ is the homology class corresponding to $\beta(V) \in H_1(\partial V)$ through the homology cobordism Y, then α is represented by the core of the surgery torus of $M(U;p/1)(\cong L(p,1))$, where U is the trivial knot in S^3; hence, $\lambda_0(L(p,1),\alpha) = 0$. Then Theorem 3.1 follows from the following Proposition 3.2.

Proposition 3.2. *The Casson invariant λ_0 modulo 2 for marked homology $L(p,1)$ is a homology cobordism invariant; i.e., if Y is a smooth homology cobordism between the homology $L(p,1)$'s M_0 and M_1 and if $\alpha_0 \in H_1(M_0)$ and $\alpha_1 \in H_1(M_1)$ correspond through this homology cobordism Y, then $\lambda_0(M_0,\alpha_0) \equiv \lambda_0(M_1,\alpha_1) \pmod 2$.*

Remember that, for homology 3-spheres, the Casson invariant modulo 2 is the Rohlin invariant, which is a homology cobordism invariant.

Proof of Proposition 3.2. We may assume $p \geq 2$. Let K_i $(i = 0,1)$ be a smooth knot in M_i representing $\alpha_i \in H_1(M_i)$. Let W be the 4-manifold obtained by attaching two 2-handles h_0 and h_1 to Y along K_0 and K_1 in such a way that ∂W consists of two disjoint homology 3-spheres Σ_0 and Σ_1. Denote by $K_i' \subset \Sigma_i$ the knot which is the boundary of the cocore of the 2-handle h_i (Figure 1). Then M_i is obtained by

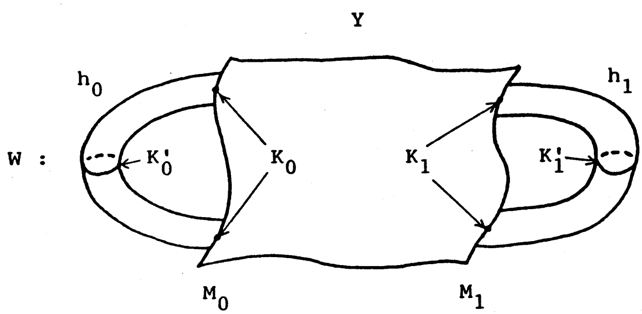

Figure 1

the $(\pm p)$-surgery on the knot K_i'. Furthermore the core of this surgery torus in M_i corresponds to K_i which represents $\alpha_i \in H_1(M_i)$. Hence $\lambda_0(M_i, \alpha_i) = p\lambda(\Sigma_i) \pm (1/2)\Delta''_{K_i'}(1)$. Let $f_i \in H_2(W, \partial W)(\cong H_2(W))$ be the homology class represented by the cocore of the 2-handle h_i. Since Y is a homology cobordism, there is a 2-chain c in Y with boundary K_0 and K_1. Denote by $e \in H_2(W)$ the homology class represented by the 2-cycle which consists of the cores of h_i and the 2-chain c. Then (f_i, e) are generators of $H_2(W, \partial W) \cong H_2(W) \cong \mathbf{Z} \oplus \mathbf{Z}$. Furthermore we have $f_i \cdot f_i = \pm p$, $f_0 \cdot f_1 = 0$, $f_i \cdot e = \pm 1$ and $\pm pe = \pm f_0 \pm f_1$. Hence,

$$\begin{aligned} e \cdot e &= \frac{1}{p^2}(\pm f_0 \pm f_1) \cdot (\pm f_0 \pm f_1) \\ &= \frac{1}{p^2}(\pm p \pm p). \end{aligned}$$

Since this must be an integer, we have

$$\begin{cases} \text{(A)} \quad e \cdot e = 0 \qquad \text{or} \\ \text{(B)} \quad e \cdot e = \pm 1 \quad \text{and} \quad p = 2. \end{cases}$$

Case (A). The homology class $f_0 + f_1 \in H_2(W, \partial W) \cong H_2(W)$ is characteristic and we can represent it by a smoothly embedded annulus A by tubing the cocores of h_0 and h_1.

When p is even, W is spin and its signature is zero; hence, $\lambda(\Sigma_0) \equiv \lambda(\Sigma_1) \pmod 2$. Thus, for the proof of Proposition 3.2, it suffices to prove

$$\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K_0'}(1) \equiv \lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K_1'}(1) \pmod 2. \tag{3.2}$$

Note that $\lambda(\Sigma_i) + (1/2)\Delta''_{K'_i}(1)$ is equal to the Casson invariant of the homology 3-sphere Σ'_i obtained by $(+1)$-surgery on K'_i in Σ_i. Let X be the 4-manifold obtained by attaching two 2-handles h'_0 and h'_1 to W along K'_i with the $(+1)$-framing. Note that ∂X consists of the homology 3-spheres Σ'_0 and Σ'_1. Denote by S' the smoothly embedded 2-sphere in X which consists of the annulus A and the cores of h'_0 and h'_1. Note that $[S'] \in H_2(X)$ is characteristic and that $[S']^2$ is equal to the signature of X. Then using the same method as in [21], one can deduce $\lambda(\Sigma'_0) \equiv \lambda(\Sigma'_1) \pmod 2$. This shows the equality (3.2) holds.

Case (B). The homology classes $f_0, f_1 \in H_2(W, \partial W) \cong H_2(W)$ are both characteristic. Let X_i be the 4-manifold obtained by attaching a 2-handle h'_i to W along K'_i with the $(+1)$-framing. Denote by S_i the smoothly embedded 2-sphere in X_i consisting of the cocore of the 2-handle h_i and the core of the 2-handle h'_i. Note that $[S_i] \in H_2(X_i)$ is characteristic and that $[S_i]^2$ is equal to the signature of X_i. Thus, by the same argument as in Case (A), we have

$$\lambda(\Sigma'_0) \equiv \lambda(\Sigma_1) \pmod 2 \quad \text{and}$$
$$\lambda(\Sigma_0) \equiv \lambda(\Sigma'_1) \pmod 2.$$

Hence,

$$\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K'_0}(1) \equiv \lambda(\Sigma_1) \pmod 2 \quad \text{and} \tag{3.3}$$

$$\lambda(\Sigma_0) \equiv \lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K'_1}(1) \pmod 2. \tag{3.4}$$

Adding (3.3) and (3.4) shows

$$2\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K'_0}(1) \equiv 2\lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K'_1}(1) \pmod 2.$$

This completes the proof of Proposition 3.2 and hence Theorem 3.1.

Remark. There is a smooth homotopy S^2, V, such that $\overline{\lambda}(V) \equiv 0 \pmod 2$ and yet the generator of $H_2(V)$ cannot be represented by a smoothly embedded 2-sphere. For example, consider $V = V(T(3,-11); 34)$. A computation shows $\overline{\lambda}(V) = 40$. However the generator of $H_2(V)$ cannot be represented by a smoothly embedded 2-sphere (see [28] or Example 3.5 below).

Remark. Let (M, α) be an oriented marked homology $L(p,1)$. If p is odd, the Rohlin invariant $\mu(M)(\in \mathbf{Z}/16\mathbf{Z})$ of M is defined and it is a homology cobordism invariant. It can be shown, using the method similar to that in the proof of Proposition 3.2, that

$$8\lambda_0(M,\alpha) \equiv \mu(M) - (p-1)p \cdot \mathrm{lk}(\alpha,\alpha) \pmod{16},$$

where $\mathrm{lk}(\alpha,\alpha) = \pm 1/p$.

Next we define the Casson-Gordon invariants for a homotopy S^2. In our case, they are essentially the p-signatures of a certain knot in a homology 3-sphere.

Let K be a smooth knot in an oriented homology 3-sphere Σ and let L be a Seifert matrix of K. For a positive integer p, set $\omega_p = \exp(2\pi\sqrt{-1}/p)$. Then we define $\sigma_K(\omega_p^r)$ ($r = 1, 2, \cdots, p-1$) to be the signature of the Hermitian matrix $(1-\omega_p^{-r})L + (1-\omega_p^r)L^T$, where L^T is the transpose of L. It is well-known that $\sigma_K(\omega_p^r)$ are invariants of K and they are called *p-signatures* of K. Note that they are independent of the orientation of K, while they depend on the orientation of Σ in general. If we change the orientation of K, its Seifert matrix becomes L^T and the signature of the corresponding Hermitian matrix does not change.

Lemma 3.3. *Let Δ_i $(i = 0,1)$ be a compact oriented contractible topological 4-manifold and let K_i be a tame knot in the homology 3-sphere $\partial\Delta_i$. Let V_i be the oriented 4-manifold obtained by attaching a 2-handle to Δ_i along K_i with the $(\pm p)$-framing $(p \geq 0)$. If V_0 is orientation preservingly homeomorphic to V_1, then when $p > 0$,*

$$\sigma_{K_0}(\omega_p^r) = \sigma_{K_1}(\omega_p^r) \quad \textit{for} \quad 1 \leq r \leq p-1$$

and when $p = 0$,

$$\sigma_{K_0}(\omega_q^s) = \sigma_{K_1}(\omega_q^s) \quad \textit{for every} \quad q > 0 \quad \textit{and} \quad 1 \leq s \leq q-1.$$

Proof. We prove the case of $p > 0$. When $p = 0$, the proof is similar. We may assume V_i are positive definite. Orient K_i arbitrarily and let $\alpha_i \in H_1(\partial V_i)$ be the homology class represented by a meridian loop of K_i. Let $h : V_0 \to V_1$ be a homeomorphism. By the restriction $h|\partial V_0$, we identify ∂V_0 and ∂V_1 and denote it by M. We may assume that, by this identification, $\alpha_0 = \alpha_1$ in $H_1(M)$, changing orientations of K_i if necessary. Define the homomorphism $\varphi : H_1(M) \to \mathbf{Z}/p\mathbf{Z}$ by $\varphi(\alpha_i) = 1$ ($1 \in \mathbf{Z}/p\mathbf{Z}$ is the generator). Let L_i be a Seifert matrix for

K_i. Then by [7, Lemma 3.1],

$$\sigma_r(M,\varphi) = \mathrm{sign}V_i - \mathrm{sign}((1-\omega_p^{-r})L_i + (1-\omega_p^r)L_i^T) - \frac{2r(p-r)}{p} \quad (i=0,1), \tag{3.5}$$

where $\sigma_r(M,\varphi)$ is the Casson-Gordon invariant of M associated with φ and r. Note that in [7] everything is assumed to be smooth. However, their method is easily extended to the topological category, since the G-signature theorem holds also in the topological case (see [32]). Since $\mathrm{sign}V_0 = \mathrm{sign}V_1$, we have $\sigma_{K_0}(\omega_p^r) = \sigma_{K_1}(\omega_p^r)$ by (3.5).

Definition. Let V be an oriented homotopy S^2 with $H_1(\partial V)$ isomorphic to $\mathbf{Z}/p\mathbf{Z}$ $(p \geq 0)$. Then by [3], V is obtained by attaching a 2-handle to a compact contractible 4-manifold $\varDelta$ along some tame knot K in the homology 3-sphere $\partial\varDelta$ with the $(\pm p)$-framing. If $p > 0$, we define

$$\sigma_V(\omega_p^r) = \sigma_K(\omega_p^r) \quad (1 \leq r \leq p-1),$$

which we call the *p-signatures* of V. Similarly if $p = 0$, we define, for every $q > 1$,

$$\sigma_V(\omega_q^s) = \sigma_K(\omega_q^s) \quad (1 \leq s \leq q-1).$$

By Lemma 3.3, this is well-defined.

As the equation (3.5) in the proof of Lemma 3.3 shows, the p-signatures of a homotopy S^2 are essentially the Casson-Gordon invariants of the boundary 3-manifold.

Next we use these invariants to attack problem (B) in §1.

Definition. Let V be a homotopy S^2. We define $g(V)$ to be the minimal genus of topologically locally flatly embedded surfaces representing the generator of $H_2(V)$.

Remark. Even if V itself is not smooth, $\mathrm{Int}V$ admits a smooth structure (see [14, §8.2]). Thus the generator of $H_2(V)(\cong H_2(\mathrm{Int}V))$ is always represented by a locally flatly embedded surface.

Theorem 3.4. *Let V be a homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p $(p > 0)$. Then for every prime power d dividing p, we have*

$$2g(V) \geq | \sigma_V(\omega_d^s) | \quad (s = 1, 2, \cdots, d-1).$$

Remark. This lower bound has already been known for the case that V consists of one 0-handle and one 2-handle and the embedded surfaces considered are smooth ([31]). In this case, when $H_1(\partial V)$ is infinite cyclic, the above inequality holds for all prime power d. We do not know whether Theorem 3.4 also holds for V with $H_1(\partial V)$ infinite cyclic.

Proof of Theorem 3.4. We may assume V is positive definite. Let F be a topologically locally flatly embedded surface in Int$V$ of genus $g = g(V)$ representing the generator of $H_2(V)$. There exists a d-fold cyclic branched covering $\pi : \widetilde{V} \to V$ branched along F such that the d-fold covering $\pi|\partial\widetilde{V} : \partial\widetilde{V} \to \partial V$ corresponds to the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$ defined by $\varphi(\beta(V)) = 1$. Let $\tau : \widetilde{V} \to \widetilde{V}$ be the canonical covering translation. Define $E_s \subset H_2(\widetilde{V}) \otimes \mathbf{C}$ to be the ω_d^s-eigenspace of $\tau_* : H_2(\widetilde{V}) \otimes \mathbf{C} \to H_2(\widetilde{V}) \otimes \mathbf{C}$ (note that $\tau_*^d =$id). Furthermore define $\varepsilon_s(\widetilde{V})$ to be the signature of the restriction to E_s of the intersection pairing on $H_2(\widetilde{V}) \otimes \mathbf{C}$. Then by the definition of the Casson-Gordon invariants [7],

$$\sigma_s(\partial V, \varphi) = \mathrm{sign} V - \varepsilon_s(\widetilde{V}) - \frac{2ps(d-s)}{d^2}.$$

Combining this with [7, Lemma 3.1] and the definition of p-signatures, we have

$$\varepsilon_s(\widetilde{V}) = \sigma_V(\omega_d^s).$$

Since $|\varepsilon_s(\widetilde{V})| \leq \dim_{\mathbf{C}} E_s$, it suffices to show that $\dim_{\mathbf{C}} E_s = 2g$.

It is easily verified, using a method similar to that in [19, §4], that

$$\dim_{\mathbf{C}} H_2(\widetilde{V}) \otimes \mathbf{C} = 2g(d-1) + 1$$

(note that d is a prime power by the assumption). It is well-known that $E_0 = \pi^*(H_2(V) \otimes \mathbf{C})$. Hence,

$$\sum_{s=1}^{d-1} \dim_{\mathbf{C}} E_s = 2g(d-1).$$

Using this equation and the linear algebra together with the assumption that d is a prime power, we easily deduce $\dim_{\mathbf{C}} E_s = 2g$ ($s = 1, 2, \cdots, d-1$). This completes the proof.

Example 3.5. Consider $V = V(T(a,b); p)$, where $|a|, |b| \geq 2$ and $p \neq 0, \pm 1$. Then the generator of $H_2(V)$ cannot be represented by a topologically locally flatly embedded 2-sphere since the p-signatures of the torus knot $T(a,b)$ do not vanish. Remember that if p is even, the

p-signature $\sigma_{T(a,b)}(-1)$ is the usual signature of the torus knot $T(a,b)$. Thus the lower bound for the 4-ball genera of torus knots given in [29] is also valid for $g(V(T(a,b);p))$ if p is even.

Definition. For a homotopy S^2, V, we denote by $KS(V)$ ($\in \mathbf{Z}/2\mathbf{Z}$) the Kirby-Siebenmann obstruction to extending the product smooth structure on $\partial V \times \mathbf{R}$ across $V \times \mathbf{R}$.

If ∂V is a homology 3-sphere, Theorem 3.4 gives no restrictions on $g(V)$. In that case, we have the following.

Proposition 3.6. *Let V be a homotopy S^2 with ∂V a homology 3-sphere. Then $g(V) = 0$ if $\mu(\partial V) = KS(V)$ and $g(V) = 1$ if $\mu(\partial V) \neq KS(V)$, where $\mu(\partial V)$ is the Rohlin invariant of ∂V.*

Proof. Let $P = \mathbf{C}P^2 -\ \mathrm{Int}D^4$ and $Q = Ch -\ \mathrm{Int}D^4$, where Ch is the Chern manifold ([12]). Furthermore let Δ be the contractible 4-manifold bounded by ∂V ([12]). Then by [3] V is homeomorphic to $P \natural \Delta$ if $\mu(\partial V) = KS(V)$ and $Q \natural \Delta$ if $\mu(\partial V) \neq KS(V)$, where $\natural$ denotes the boundary connected sum. Since P is homeomorphic to a D^2-bundle over S^2, the generator of $H_2(P \natural \Delta)$ is represented by a locally flatly embedded 2-sphere. Furthermore, Ch is homeomorphic to $V(T(2,3);1) \cup \Delta'$, where Δ' is the contractible 4-manifold bounded by $M(T(2,3);1/1)$. Thus the generator of $H_2(Q \natural \Delta)$ is represented by a locally flatly embedded torus. Hence, $g(V) = 0$ if $\mu(\partial V) = KS(V)$ and $g(V) \leq 1$ if $\mu(\partial V) \neq KS(V)$.

Next we show that $g(V) \neq 0$ if $\mu(\partial V) \neq KS(V)$. Suppose $g(V) = 0$ and let S be a locally flatly embedded 2-sphere in $\mathrm{Int}V$ representing the generator of $H_2(V)$. By [13], S has a neighborhood N ($\subset \mathrm{Int}V$) which is a 2-disk bundle over S. Note that ∂N is homeomorphic to S^3. Set $\Delta'' = (V -\ \mathrm{Int}N) \cup_{\partial N} D^4$. Then $\mu(\partial V)$ is equal to the Kirby-Siebenmann obstruction of Δ'', which in turn is equal to $KS(V)$. This contradicts the assumption that $\mu(\partial V) \neq KS(V)$. This completes the proof.

In Theorem 3.4, only the p-signatures of the form $\sigma_V(\omega_d^s)$ with d a prime power are handled. For the general p-signatures, we have the following.

Proposition 3.7. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. For an integer d (> 0) dividing p, suppose $H_1(\widetilde{\partial V}; \mathbf{Q}) = 0$, where $\widetilde{\partial V} \to \partial V$ is the d-fold cyclic covering associated with the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$ defined by $\varphi(\beta(V)) = 1$. If* *the generator of $H_2(V)$ is represented by a smoothly embedded 2-sphere, then*

$$\sigma_V(\omega_d^s) = 0 \quad \textit{for} \quad 1 \le s \le d-1.$$

Remark. Suppose V is obtained (topologically) by attaching a 2-handle to a contractible 4-manifold Δ along a knot K in the homology 3-sphere $\partial\Delta$. Then $H_1(\widetilde{\partial V};\mathbf{Q}) = 0$ if and only if $\Delta_K(\omega_d^s) \neq 0$ for $1 \le s \le d-1$, where $\Delta_K(t)$ is the Alexander polynomial of K. In particular, if d is a prime power, we always have $H_1(\widetilde{\partial V};\mathbf{Q}) = 0$.

Proof of Proposition 3.7. Let S be a smoothly embedded 2-sphere in $\mathrm{Int}V$ representing the generator of $H_2(V)$ and let $Y = V-\mathrm{Int}N(S)$, where $N(S)$ is the tubular neighborhood of S. Then Y is a smooth homology cobordism between ∂V and $L(p,1)$. Let $\varphi' : H_1(L(p,1)) \to \mathbf{Z}/d\mathbf{Z}$ be the homomorphism defined by composing the isomorphism $H_1(L(p,1)) \to H_1(\partial V)$ induced by Y and the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$. Then by Matic [24] and Ruberman[27],

$$\sigma_1(\partial V, \varphi^s) = \sigma_1(L(p,1), \varphi'^s) \quad (1 \le s \le d-1).$$

Since $\sigma_s(\partial V,\varphi) = \sigma_1(\partial V,\varphi^s)$ and $\sigma_s(L(p,1),\varphi') = \sigma_1(L(p,1),\varphi'^s)$, we have $\sigma_s(\partial V,\varphi) = \sigma_s(L(p,1),\varphi')$. Combining this with [7, Lemma 3.1], we obtain

$$\sigma_V(\omega_d^s) = 0 \quad (1 \le s \le d-1)$$

(see also the proof of Theorem 3.1 in this section). This completes the proof.

In [4], Boyer-Lines extended the Casson invariant for homology 3-spheres to homology lens spaces. Furthermore, they gave a relation of this invariant to the Casson-Gordon invariants. Using this result, Theorem 3.1, and Proposition 3.7, we have the following.

Proposition 3.8. *Let V be an ε-definite ($\varepsilon = \pm 1$) smooth homotopy S^2 satisfying the condition on $H_1(\widetilde{\partial V};\mathbf{Q})$ as in Proposition* 3.7 *for $d = p$. If the generator of $H_2(V)$ is represented by a smoothly embedded 2-sphere, then*

$$12p\lambda(\partial V) \equiv -\varepsilon\frac{(p-1)(p-2)}{2} \pmod{24},$$

where $\lambda(\partial V)$ is the Boyer-Lines' Casson invariant for the homology lens space ∂V.

Remark. $12p\lambda(\partial V)$ is always an integer ([4, Theorem 2.8]).

Proof of Proposition 3.8. Suppose V is obtained (topologically) by attaching a 2-handle to a contractible 4-manifold $\varDelta$ along a knot K in $\partial\varDelta$ with (εp)-framing. Let $\alpha \in H_1(\partial V)$ be the homology class represented by a meridian of K. Then by [4, Proposition 2.23],

$$p\lambda(\partial V) = \lambda_0(\partial V, \alpha) + \frac{1}{8}\sigma(K,p) + \frac{1}{8}\tau(\partial V),$$

where

$$\sigma(K,p) = \sum_{r=1}^{p-1} \sigma_K(\omega_p^r)$$

and

$$\tau(\partial V) = \sum_{r=1}^{p-1} \sigma_r(\partial V, \varphi).$$

(Here, $\varphi : H_1(\partial V) \to \mathbf{Z}/p\mathbf{Z}$ is the homomorphism defined by $\varphi(\beta(V)) = 1$.) By Theorem 3.1, we have $\lambda_0(\partial V, \alpha) \equiv 0 \pmod 2$ and by Proposition 3.7, $\sigma(K,p) = 0$. Furthermore, by the proof of Proposition 3.7, we have $\sigma_r(\partial V, \varphi) = \sigma_r(L(p,\varepsilon), \varphi')$; hence,

$$\tau(\partial V) = \tau(L(p,\varepsilon)) = -\varepsilon\frac{(p-1)(p-2)}{3}.$$

Thus,

$$8p\lambda(\partial V) \equiv -\varepsilon\frac{(p-1)(p-2)}{3} \pmod{16}.$$

Multiplying 3/2 gives the result.

§4. Knots in the boundary of a homotopy S^2

Definition. Let V be a homotopy S^2 and let K be a tame knot in ∂V representing $\beta(V) \in H_1(\partial V)$. We say K is a *boundary knot* if there is a topologically embedded proper flat 2-disk D in V such that $\partial D = K$ and D represents the generator of $H_2(V, \partial V)$. Here, a properly embedded 2-disk D is *flat* if it is the core of an embedded open 2-handle $D \times \mathbf{R}^2$ in V with $(D \times \mathbf{R}^2) \cap \partial V = \partial D \times \mathbf{R}^2$.

Definition. Let K be a tame knot in a homology $L(p,1)$, M, with $\mathrm{lk}([K],[K]) = \pm 1/p$ $(p > 0, p \neq 2)$. Let Σ be the homology 3-sphere obtained by a surgery on K such that the coefficient of the corresponding surgery by which M is obtained from Σ is an integer. Note that Σ

depends only on M and K if $p \geq 3$. Then we denote by $\mu(K)(\in \mathbf{Z}/2\mathbf{Z})$ the Rohlin invariant $\mu(\Sigma)$ of Σ. Note that, when $p = 1$, $\mu(K)$ is still well-defined, though Σ is not. When $p = 2$, $\mu(K)$ cannot be defined.

Our first result of this section is the following.

Proposition 4.1. *Let V be a homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Suppose K is a tame knot in ∂V representing $\beta(V) \in H_1(\partial V)$. Then we have the following.*
(1) *When p is even, K is always a boundary knot.*
(2) *When p is odd, K is a boundary knot if and only if $\mu(K) = KS(V)$.*

Proof. First, we construct an embedded flat 2-disk bounded by K. Set $M = \partial V$ and let X be the 4-manifold obtained by attaching a 2-handle h_0 to $M \times [0,1]$ along $K \times \{1\}$ in such a way that ∂X consists of M and a homology 3-sphere Σ. By [12], there exists a contractible 4-manifold $\varDelta$ with $\partial \varDelta = \Sigma$. Denote by V' the 4-manifold $X \cup_\Sigma \varDelta$. Note that V' is a homotopy S^2 with $\partial V' = M$. By [3] and the hypothesis on $\mu(K) = \mu(\Sigma)$, we see that there is a homeomorphism $h : V \to V'$ such that $h|\partial V = \mathrm{id}_M$. Since $h(K)$ bounds in V' an embedded flat 2-disk (the core of the 2-handle h_0), K also bounds one in V.

Conversely, suppose p is odd and K bounds in V a topologically embedded proper flat 2-disk D in V which represents the generator of $H_2(V, \partial V)$. Then there exists a closed neighborhood N of D in V homeomorphic to $D^2 \times D^2$. Denote by B the closure of $V - N$. Note that ∂B is a homology 3-sphere and that $\mu(K) = \mu(\partial B)$. Then it is easily shown that B is a homology 4-ball. Thus V is obtained by attaching a 2-handle N to B. By [3], $KS(V)$ is equal to $\mu(\partial B) = \mu(K)$. This completes the proof.

Remark. There exists a knot in the boundary of a homotopy S^2, V, which is not a boundary knot but bounds in V an embedded flat 2-disk. For example, consider the knot K in $\partial V(U;p)$ as in Figure 2, where U is the trivial knot and p is odd. Then it is easily seen $\mu(K) = 1$. (To see this, observe that $\mu(K)$ is equal to the Rohlin invariant of the homology 3-sphere obtained by the surgery along the framed link in S^3 as in Figure 3. Then we can apply a formula of Kaplan [20, Theorem 4.2].) Since the Kirby-Siebenmann obstruction of $V(U;p)(= V)$ vanishes, K is not a boundary knot by Proposition 4.1. However, K bounds in V a smoothly embedded 2-disk. This can be constructed as follows. There is a smoothly immersed 2-disk D' in D^4 with one self-intersection such that $\partial D' = K$. D' intersects in V the smoothly embedded 2-sphere S (the union of the core of the 2-handle and the cone over U in D^4)

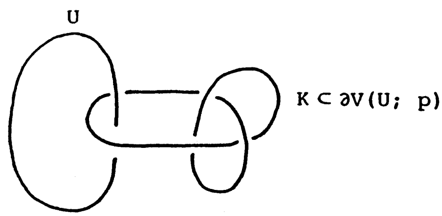

Figure 2

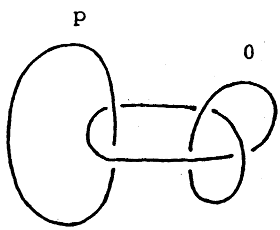

Figure 3

transversely in one point. Piping S and D' along an arc on D' which connects the self-intersection point of D' with the intersection point of D' and S as in Figure 4, we obtain the desired embedded 2-disk D. Of course, D does not represent the generator of $H_2(V, \partial V)$.

Next we give an example of knots in the boundary of a smooth homotopy S^2 which are boundary knots but never bound smoothly embedded 2-disks.

Definition. Let K be a smooth knot in the boundary of a smooth homotopy S^2, V. Then we denote by $k(K)$ the minimal number of double points of smooth properly immersed self-transverse 2-disks in V bounded by K. Following [17], we call $k(K)$ the *kinkiness* of K.

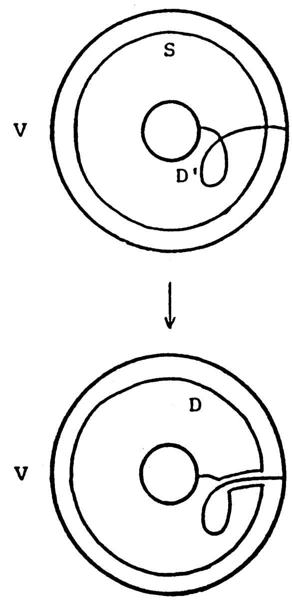

Figure 4

Let n and l be odd integers with $n, l \geq 3$ and set $p = l(nl+2)$ and $q = nl+1$. Note that p and q are relatively prime integers. Set $V = V(T(p,q); -pq)$. Note that by [25], ∂V is diffeomorphic to $L(p,q)\#L(q,p)$. Set $r = q^2 - q - 1$ and let $m_1, m_2, \cdots, m_r$ be the r oriented knots in ∂V represented by the meridians of $T(p,q)$ (see Figure 5). Here we make the orientation convention $[m_i] = -\beta(V)$ in $H_1(\partial V)$. Let K' be any knot in ∂V obtained by performing the oriented band connected sum operations to $m_1 \cup m_2 \cup \cdots \cup m_r$ between distinct components $(r-1)$ times. Note that $[K'] = -r\beta(V)$ in $H_1(\partial V)$. Since $q^2 \equiv 1 \pmod p$, there is a diffeomorphism $h' : L(p,q) \to L(p,q)$ which acts on $H_1(L(p,q))$ by the multiplication of q. Set $h = h'\#\mathrm{id} : L(p,q)\#L(q,p) \to L(p,q)\#L(q,p)$. It is easily seen that h acts on $H_1(\partial V)$ by the multiplication of $-r$ (note that $H_1(\partial V) \cong H_1(L(p,q)\#L(q,p)) \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z} \cong \mathbf{Z}/pq\mathbf{Z}$). Note that $r^2 \equiv 1 \pmod{pq}$. Then we denote by K the smooth knot $h(K')$ in ∂V.

Note that K represents $\beta(V) \in H_1(\partial V)$.

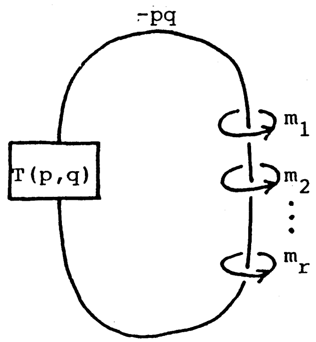

Figure 5

Proposition 4.2. *The knot K bounds in V a topologically embedded proper flat 2-disk. However,*

$$k(K) \geq \left[\frac{n(nl-1)+2}{4}\right].$$

In particular, K cannot bound in V any smoothly embedded 2-disks.

Proof. Since pq is even, K is a boundary knot by Proposition 4.1.

Now set $X = V_0 \cup_h (-V_1)$ ($V_0 = V_1 = V, h : \partial V_1 \to \partial V_0$), which is a smooth closed 1-connected 4-manifold with $H_2(X) \cong \mathbf{Z} \oplus \mathbf{Z}$. Let D be a smoothly immersed self-transverse 2-disk in $V_0(\cong V)$ bounded by K with $k(K)$ double points. Let S_i be the topologically embedded (not locally flat) 2-sphere in V_i which consists of the core of the 2-handle of $V_i = V(T(p,q); -pq)$ and the cone over $T(p,q)$ in D^4. Furthermore let S_2 be the smoothly immersed 2-sphere in X which consists of the r cocores of the 2-handle in V_1 corresponding to $m_1 \cup m_2 \cup \cdots \cup m_r$, the $(r-1)$ bands used to make K', and the immersed 2-disk D. Note that S_2 is a smoothly immersed 2-sphere with $k(K)$ double points. Furthermore let $\tau \in H_2(X)$ be the homology class represented by the union of the cocore of the 2-handle of V_0, the r cocores of the 2-handle of V_1 and a 2-chain in $\partial V_0 = h(\partial V_1)$ connecting their boundaries (note that $h_*(r\beta(V_1)) = -\beta(V_0)$). Then by [28], $\theta = [S_1]$ and τ generate $H_2(X)$. Furthermore $[S_2] = \tau + j[S_0]$ for some integer j.

It is easily seen that the intersection matrix of X with respect to the basis θ and τ is

$$Q = \begin{pmatrix} -l(nl+1)(nl+2) & n^2l^2+nl-1 \\ n^2l^2+nl-1 & -n(nl-1) \end{pmatrix}.$$

($\tau \cdot \tau = -n(nl-1)$ is the consequence of the fact that $\det Q = \pm 1$ and that $|l(nl+1)(nl+2)| > 2$.) Furthermore Q is isomorphic to the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus there are generators ξ and η of $H_2(X)$ with $\xi \cdot \xi = \eta \cdot \eta = 0$ and $\xi \cdot \eta = 1$. Furthermore, we may assume

$$\begin{aligned} \theta &= \frac{l(nl+1)}{2}\xi - (nl+2)\eta \quad \text{and} \\ \tau &= \frac{1-nl}{2}\xi + n\eta. \end{aligned}$$

Then we have

$$[S_0] = \frac{l(nl+1)}{2}\xi + (nl+2)\eta.$$

Thus

$$\begin{aligned} [S_2] &= \tau + j[S_0] \\ &= \left\{\frac{1-nl}{2} + \frac{l(nl+1)}{2}j\right\}\xi + \{n + (nl+2)j\}\eta. \end{aligned}$$

Set

$$\begin{aligned} \alpha &= \frac{1-nl}{2} + \frac{l(nl+1)}{2}j \quad \text{and} \\ \beta &= n + (nl+2)j. \end{aligned}$$

By Theorem 2.8, the number of double points of S_2 $(= k(K))$ is greater than or equal to

$$\min\left\{(|\alpha|-1)(|\beta|-1), \left[\frac{|\alpha\beta|+1}{2}\right]\right\}.$$

Since both $|\alpha|$ and $|\beta|$ attains its minimum when $j=0$, we have

$$\begin{aligned} k(K) &\geq \min\left\{\left(\frac{nl-1}{2} - 1\right)(n-1), \left[\frac{n(nl-1)+2}{4}\right]\right\} \\ &= \left[\frac{n(nl-1)+2}{4}\right]. \end{aligned}$$

This completes the proof.

Remark. The smooth knot $K' = h^{-1}(K)$ does bound in V a smoothly embedded 2-disk. Thus $h : \partial V \to \partial V$ does not extend to a self-diffeomorphism of V. In fact, it is easily seen that h does not extend even to a self-homeomorphism of V.

We end this section by posing a problem.

Problem. Is there a smooth homotopy S^2, V, such that some $l \in \pi_1(\partial V)$ cannot be represented by any knot which is the boundary of a smoothly embedded 2-disk in V?

If V admits a handlebody decomposition without 3-handles, every $l \in \pi_1(\partial V)$ is so represented ([11]). Thus if the problem above is affirmative, the homotopy S^2 needs a 3-handle in any of its handlebody decomposition. Note also that if l represents $\pm\beta(V)$ in $H_1(\partial V)$, then, by Proposition 4.1, l can be represented by a knot which bounds a topologically embedded flat 2-disk in V.

§5. Exotic open homotopy S^2

For an integer p, let $D(p)$ denote the D^2 bundle over S^2 with euler number p. $D(p)$ is a (compact) homotopy S^2. Our result of this section is the following.

Proposition 5.1. *Let l and m be relatively prime odd integers greater than 2. Then the open 4-manifold* $\mathrm{Int}D(4lm)$ *admits at least 2 smooth structures other than the canonical one.*

To prove Proposition 5.1, we need the following.

Lemma 5.2. *Let a and b be relatively prime integers and set $\zeta = a\xi + b\eta \in H_2(S^2 \times S^2)$, where $\xi = \{*\} \times [S^2]$ and $\eta = [S^2] \times \{*\}$ are the standard generators. Then ζ can be represented by a smoothly immersed 2-sphere in $S^2 \times S^2$ with simply connected complement and with $(|a|-1)(|b|-1)$ double points.*

Proof. We may assume $a \geq 0$ and $b \geq 0$. If $(a, b) = (1, 0)$ or $(0, 1)$, the assertion is trivial. Hence, we may assume $a \geq 1$ and $b \geq 1$. We construct a desired immersed 2-sphere by the "standard" method. Take distinct $(a+1)$ points $x_0, x_1, \cdots, x_a \in S^2$ and distinct $(b+1)$ points $y_0, y_1, \cdots, y_b \in S^2$. Set $R = (\cup_{i=1}^{a}\{x_i\} \times S^2) \cup (\cup_{j=1}^{b} S^2 \times \{y_j\})$ and $X = S^2 \times S^2 - R$. Here, we orient $\{x_i\} \times S^2$ and $S^2 \times \{y_j\}$ so that R

represents ζ. Note that $X = (S^2 - \{x_1, \cdots, x_a\}) \times (S^2 - \{y_1, \cdots, y_b\})$. Set $z = (x_0, y_0) \in X$ and define $\alpha_1, \alpha_2, \cdots, \alpha_a, \beta_1, \beta_2, \cdots, \beta_b \in \pi_1(X, z)$ as follows. Connect x_0 and a point near to x_i $(i = 1, 2, \cdots, a)$ by an arc in $(S^2 - \{x_1, \cdots, x_a\}) \times \{y_0\}$ as in Figure 6. Then α_i is represented by a loop in $(S^2 - \{x_1, \cdots x_a\}) \times \{y_0\} (\subset X)$ which starts at x_0, goes along the arc toward x_i, goes around x_i once counterclockwise, and goes back to x_0 along the same arc. β_i can be defined using an arc in $\{x_0\} \times (S^2 - \{y_1, \cdots, y_b\})$ in a similar way. Note that $\alpha_1 \alpha_2 \cdots \alpha_a = \beta_1 \beta_2 \cdots \beta_b = 1$ and that $\pi_1(X, z) \cong <\alpha_1, \cdots, \alpha_a \mid \alpha_1 \cdots \alpha_a = 1> \times <\beta_1, \cdots, \beta_b \mid \beta_1 \cdots \beta_b = 1>$.

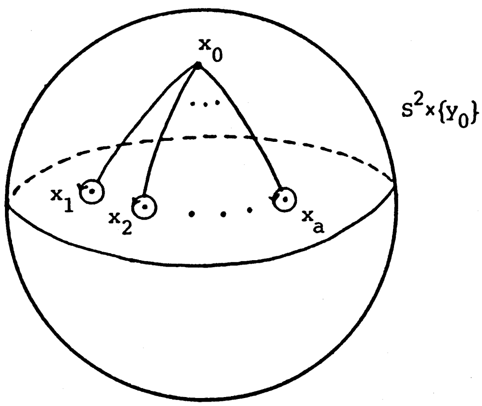

Figure 6

Next we do the "smoothing operations" to R at $(x_1, y_1), (x_1, y_2), \cdots, (x_1, y_b), (x_2, y_1), (x_3, y_1), \cdots, (x_a, y_1)$ as follows. Set $(D^4, B) = (D^2 \times D^2, D^2 \times \{0\} \cup \{0\} \times D^2)$ $(\partial(D^4, B) = (S^3, \text{Hopf link}))$. Furthermore let A' be the annulus embedded in S^3 as in Figure 7 and denote by A the properly embedded annulus in D^4 which is obtained by pushing IntA' into IntD^4. Note that $\partial(D^4, A) \cong \partial(D^4, B)$ and that $\pi_1(D^4 - A) \cong \mathbf{Z}$. The smoothing operation at a double point q of R means that we replace $(D^4(q), D^4(q) \cap R) \cong (D^4, B)$ by (D^4, A) or $-(D^4, A)$, where $D^4(q)$ is a sufficiently small 4-ball in $S^2 \times S^2$ centered at q. Here we choose (D^4, A) or $-(D^4, A)$ so that the orientation is consistent with that of R. Denote by S the immersed oriented surface which results from the $(a + b - 1)$ smoothing operations. It is easily seen that S is an immersed 2-sphere representing ζ with $(a - 1)(b - 1)$ double points.

Furthermore, by van Kampen's Theorem we see that $\pi_1(S^2 \times S^2 - S, z)$ is isomorphic to $\pi_1(X, z)$ with additional relations $\alpha_i = \beta_j^{-1}$ $((i,j) = (1,1), (1,2), \cdots, (1,b), (2,1), (3,1), \cdots, (a,1))$. Thus $\pi_1(S^2 \times S^2 - S, z)$ is generated by α_1 and we have the relations $\alpha_1^a = \alpha_1^b = 1$. Since a and b are relatively prime, we see that the complement of S is simply connected. This completes the proof.

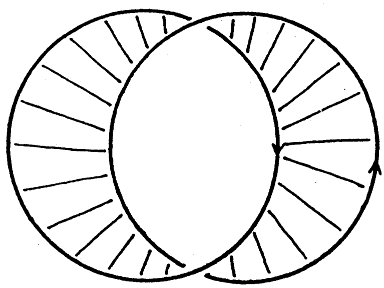

Figure 7

Proof of Proposition 5.1. Set

$$\zeta_1 = \xi + 2lm\eta$$
$$\zeta_2 = 2\xi + lm\eta$$
$$\zeta_3 = 2l\xi + m\eta.$$

By Lemma 5.2, ζ_i is represented by a smoothly immersed 2-sphere S_i in $S^2 \times S^2$ with simply connected complement with the number of double points equal to

$$\begin{cases} 0 & (i=1) \\ lm-1 & (i=2) \\ (2l-1)(m-1) & (i=3). \end{cases}$$

Let B_i^4 be a 4-ball in $S^2 \times S^2$ which avoids double points of S_i such that $(B_i^4, B_i^4 \cap S_i)$ is the standard disk pair. Then by [5], ζ_i is represented by a Casson handle which has S_i- $\mathrm{Int}B_i^4$ as its first stage core; i.e., ζ_i is represented by $V_i = B_i^4 \cup CH_i$, where CH_i is a Casson handle which is attached to B_i^4 along the trivial knot. Since CH_i is homeomorphic to $D^2 \times \mathbf{R}^2$ by [12], $V_i' = \mathrm{Int}V_i$ is homeomorphic to $\mathrm{Int}D(4lm)$ (note that

$\zeta_i^2 = 4lm$). Note that V_i' has the smooth structure as an open set of $S^2 \times S^2$ and that V_1' is diffeomorphic to $\mathrm{Int}D(4lm)$.

Next we show V_i' are mutually non-diffeomorphic. Let $k(V_i')$ be the minimum number of double points of smoothly immersed 2-spheres representing the generator of $H_2(V_i')$. Note that $k(V_i')$ is an invariant of the smooth manifold V_i'. By the construction of V_i', we have

$$k(V_1') = 0 \quad \text{and}$$
$$k(V_2') \leq lm - 1.$$

On the other hand by Theorem 2.8,

$$\begin{aligned} k(V_2') &\geq lm - 1 \quad \text{and} \\ k(V_3') &\geq \min\left\{(2l-1)(m-1), \left[\frac{2lm+1}{2}\right]\right\} \\ &= lm, \end{aligned}$$

since V_i' are submanifolds of $S^2 \times S^2$ representing ζ_i. Hence, $k(V_1'), k(V_2')$ and $k(V_3')$ are all distinct. Thus V_i' are not diffeomorphic to each other. This completes the proof.

In [22], Kuga showed that $D^2 \times \mathbf{R}^2$ has infinitely many smooth structures. Our method above is similar to Kuga's.

We conjecture that for a given positive integer N, there exists an open homotopy S^2 admitting at least N smooth structures. This conjecture is true if, in Theorem 2.8 in §2, $d_\zeta^\varepsilon = (|a|-1)(|b|-1)$.

Remark. Akbulut [1] has recently found a compact homotopy S^2 with at least two smooth structures. The generator of the second homology group of this homotopy S^2 can be represented by an embedded 2-sphere which is smooth with respect to one of the smooth structures, while it can never be smooth with respect to the other smooth structure. Thus the interior of this homotopy S^2 also has at least two smooth structures. We note that this open homotopy S^2 is not homeomorphic to the interior of a D^2 bundle over S^2.

References

[1] S. Akbulut, An exotic 4-manifold, preprint.

[2] S. Boyer, Shake-slice knots and smooth contractible 4-manifolds, Math. Proc. Camb. Phil. Soc., **98** (1985), 93–106.

[3] ———, Simply-connected 4-manifolds with a given boundary, Trans. Amer. Math. Soc., **298** (1986), 331–357.

[4] S. Boyer and D. Lines, Surgery formulae for Casson's invariant and extensions to homology lens spaces, J. reine angew. Math., **405** (1990), 181–220.

[5] A.J. Casson, Three lectures on new infinite constructions in 4-dimensional manifolds, in "A la recherche de la topologie perdue", Birkhäuser, Boston-Basel-Stuttgart, 1986, pp. 201–244.

[6] ———, Lecture at MSRI, March 1985.

[7] A.J. Casson and C.McA. Gordon, On slice knots in dimension three, Proc. Symp. Pure Math., **32** (1978), 39–53.

[8] T.D. Cochran and R. Gompf, An application of Donaldson's theorem to classical knot concordance, homology 3-spheres and property P, Topology, **27** (1988), 495–512.

[9] T.D. Cochran and W.B.R. Lickorish, Unknotting information from 4-manifolds, Trans. Amer. Math. Soc., **297** (1986), 125–142.

[10] S.K. Donaldson, An application of gauge theory to the topology of 4-manifolds, J. Diff. Geom., **18** (1983), 269–316.

[11] R. Fenn, On Dehn's lemma in 4 dimensions, Bull. London Math. Soc., **3** (1971), 79–81.

[12] M. Freedman, The topology of 4-manifolds, J. Diff. Geom., **17** (1982), 357–453.

[13] ———, The disk theorem for four-dimensional manifolds, in "Proc. Internat. Cong. Math., Warszawa", 1983, pp. 647–663.

[14] M. Freedman and F. Quinn, "Topology of 4-manifolds", Princeton Mathematical Series 39, Princeton Univ. Press, Princeton, N.J., 1990.

[15] S. Fukuhara, Casson's invariant for homology lens spaces, Lecture at Tokyo Institute of Technology, January 1989.

[16] S. Fukuhara, Y. Matsumoto and O. Saeki, An estimate for the unknotting numbers of torus knots, Topology Appl., **38** (1991), 293–299.

[17] R. Gompf, Infinite families of Casson handles and topological disks , Topology, **23** (1984), 395–400.

[18] ———, Smooth concordance of topologically slice knots, Topology, **25** (1986), 353–373.

[19] W.C. Hsiang and R.H. Szczarba, On embedding surfaces in four-manifolds, Proc. Symp. Pure Math., **22** (1971), 97–103.

[20] S. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc., **254** (1979), 237–263.

[21] M.A. Kervaire and J.W. Milnor, On 2-spheres in 4-manifolds, Proc. Nat. Acd. Sci. U.S.A., **47** (1961), 1651–1657.

[22] K. Kuga, On immersed 2-spheres in $S^2 \times S^2$, in "A Fete of Topology (Papers dedicated to Itiro Tamura)", Academic Press, 1988, pp. 525–541.

[23] N. Maruyama, On Dehn surgery along a certain family of knots, J. Tsuda College, **19** (1987), 261–280.

[24] G. Matic, $SO(3)$-connections and rational homology cobordisms, J. Diff. Geom., **28** (1988), 277–307.

[25] L. Moser, Elementary surgery along a torus knot, Pacific J. Math., **38** (1971), 737–745.

[26] D. Rolfsen, "Knots and links", Publish or Perish Inc., Berkeley, 1976.

[27] D. Ruberman, Rational homology cobordisms of rational space forms, Topology, **27** (1988), 401–414.

[28] O. Saeki, Simply connected 4-manifolds of second betti number 1 bounded by homology lens spaces, J. Math. Soc. Japan, **41** (1989), 593–606.

[29] T. Shibuya, A lower bound of 4-genus for torus knots, Memoirs of the Osaka Inst. of Technology, Series A, **31** (1986), 11–17.

[30] A. Suciu, Immersed spheres in $\mathbf{C}P^2$ and $S^2 \times S^2$, Math. Z., **196** (1987), 51–57.

[31] A.G. Tristram, Some cobordism invariants for links, Proc. Camb. Phil. Soc., **66** (1969), 251–264.

[32] C.T.C. Wall, "Surgery on compact manifolds", Academic Press, New York, 1970.

Department of Mathematics
Faculty of Science
Yamagata University
Yamagata 990
Japan