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Behavior of Knots under Twisting

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§1. Introduction

This paper is a continuation of [6] in the study of the twist move of knots. First we recall some notations. Let K be an unoriented smooth knot in the oriented 3-sphere S^3, and V a solid torus endowed with a preferred framing which contains K in its interior and satisfies $w_V(K) \geq 2$. ($w_V(K)$ denotes the geometric intersection number of K and a meridian disk of V.) Let f_n be an orientation preserving homeomorphism of V satisfying $f_n(\text{meridian}) = (\text{meridian})$ and $f_n(\text{longitude}) = (\text{longitude}) + n(\text{meridian})$ in $H_1(\partial V)$. (We shall not distinguish notationally between a homeomorphism and an isomorphism on a homology group induced by it.) We denote the knot $f_n(K)$ in S^3 by $K_{V,n}$. If there exsists an orientation preserving homeomorphism of S^3 carrying K_1 to K_2, then we write $K_1 \cong K_2$. Note that $K_1 \cong K_2$ is the same as saying that K_1 and K_2 are ambient isotopic in S^3. We note that for a given knot K, a solid torus V and an integer n determine a unique knot type. For a given knot K, we have an abundant solid tori which contain K to carry out a twist move. Sect.2 is directed towards the following question : for a given knot K, is it possible to obtain the same knot by twistings along distinct solid tori from K? Concerning the case when an original knot is trivial, we give Example 2.1 and Theorem 2.2. In the case when both solid tori are knotted, we shall give Theorem 2.6 and Examples (see Figures 4, 5). In Sect.3, the behavior of Gromov invariants under twistings will be studied. In Sect.4, we study the effects of twistings on primeness of knots. Throughout this paper $N(X)$, ∂X and $\operatorname{int} X$ denote the tubular neighborhood of X, the boundary of X and the interior of X respectively.

§2. On twistings along distinct solid tori

Let V_1 and V_2 be solid tori containing a knot K. We write $V_1 \cong V_2$ provided that there exists an orientation preserving homeomorphism f

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of S^3 such that $f(V_1) = V_2$, $f(K) = K$. Note that $K_{V_1,n} \cong K_{V_2,n}$ holds for any integer n when $V_1 \cong V_2$. To begin with, we give an example as follows.

Example 2.1. In Figure 1, $V_1 \not\cong V_2$ because the winding number of O in V_1 equals 2 and that of O in V_2 equals 3. But $O_{V_1,-1} \cong O_{V_2,-1}$.

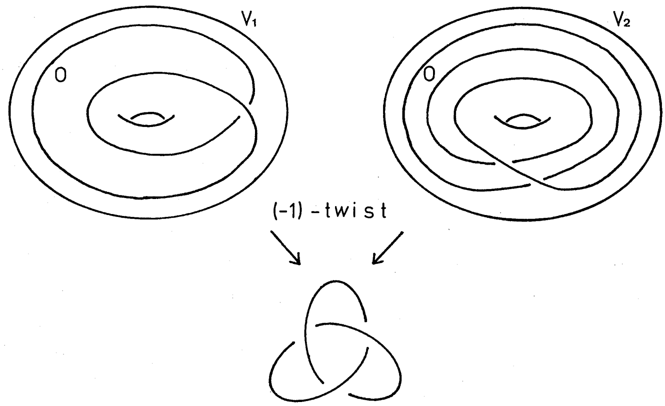

Fig. 1.

For twistings of the unknot, we prove the following theorem.

Theorem 2.2. *Let O be the unknot and V_i $(i = 1, 2)$ a solid torus containing O with $w_{V_i}(O) \geq 1$. If $O_{V_1,n_j} \cong O_{V_2,n_j}$ holds for infinitely many integers n_j, then $V_1 \cong V_2$.*

To prove this, we prepare some lemmas. Let V be a solid torus containing a knot K in its interior with $w_V(K) \geq 1$. Then $V - \operatorname{int} N(K)$ is a boundary irreducible Haken manifold. Consider the torus decomposition of $V - \operatorname{int} N(K)$ in the sense of Jaco-Shalen [3] and Johannson [4]. Combining Thurston's uniformization theorem [7], they assert that $V - \operatorname{int} N(K)$ is uniquely decomposed by a family of tori into pieces each of which is Seifert fibred or admits a complete hyperbolic structure of finite volume in its interior. Moreover each Seifert piece is one of torus knot spaces, cable spaces and composing spaces (see [3]). We denote the piece which contains ∂V by P_0, and the piece containing $\partial N(K)$ by P.

If V is an unknotted solid torus in S^3 which contains K, then $S^3 - \operatorname{int} V$ is also a solid torus, and we denote it by V_J. When we perform $(-1/n)$-Dehn surgery on the unknot J (the core of V_J), then the result is also S^3 and the image of K becomes a new knot K_n^*. The next lemma is an interpretation of a twisting.

Lemma 2.3. $K_{V,n} \cong K_n^*$.

It follows that $S^3 - \operatorname{int} N(K_{V,n})$ is homeomorphic to $V_J \bigcup_{m_J = \ell m^{-n}} (V - \operatorname{int} N(K))$.

Lemma 2.4 ([6]). *If P_0 is a cable space in which a regular fibre is presented by $\ell^p m^q$ $(p \geq 2)$, then $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is a Seifert fibred manifold with two exceptional fibres of indices p, $|pn + q|$. The dual knot of J, J_n^* in $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is a fibre of index $|pn + q|$.*

Lemma 2.5 ([6]). *If P_0 is hyperbolic, then there exists $N_{V,K}$ such that $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is also hyperbolic for $|n| \geq N_{V,K}$. Moreover for any $\varepsilon > 0$, there exsits $N_{V,K}(\varepsilon)$ such that J_n^* is a closed geodesic of length $< \varepsilon$ in $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ for $|n| \geq N_{V,K}(\varepsilon)$.*

Proof of Theorem 2.2. If $w_{V_1}(O) = 1$ (resp. $w_{V_2}(O) = 1$), then by the assumption and Theorem 4.2 in [6], $w_{V_2}(O) = 1$ (resp. $w_{V_1}(O) = 1$) must hold. In this case O is a core of both V_1 and V_2, so we have $V_1 \cong V_2$. Assume $w_{V_i}(O) \geq 2$ and consider the torus decomposition of $V_i - \operatorname{int} N(O)$. Let P_i be the piece containing ∂V_i. Since O is trivial, P_i can not be a composing space. We remark that V_i is necessarily unknotted by the assumption (see [9]), and $S^3 - \operatorname{int} V_i$ is also a solid torus V_{J_i}. Then we can characterize the core of V_{J_i} in $E(O_{V_i,n}) = V_{J_i} \bigcup_{m_{J_i} = \ell_i m_i^{-n}} (V_i - \operatorname{int} N(O))$, which is denoted by $J_{i,n}^*$, as follows. There exists a constant $N_{V_i,O}$ such that $J_{i,n}^*$ is an exceptional fibre of unique maximal index or a unique shortest closed geodesic in $E(O_{V_i,n})$ by Lemmas 2.4 and 2.5 for $|n| \geq N_{V_i,O}$. Now we take n as above. Let f be an orientation preserving homeomorphism of S^3 sending $O_{V_1,n}$ to $O_{V_2,n}$. Then by an ambient isotopy, we may assume f maps $N(O_{V_1,n})$ to $N(O_{V_2,n})$ and maps $J_{1,n}^*$ to $J_{2,n}^*$ (see also [8]). From this, we see that $f|_{V_1}$ is an orientation preserving homeomorphism from V_1 to V_2 with $f|_{V_1}(O) = O$. Moreover $f|_{V_1}$ maps $\ell_1 m_1^{-n}$ to $\ell_2^{\varepsilon} m_2^{-\varepsilon n}$ $(\varepsilon = \pm 1)$. This implies that $f|_{V_1}$ maps ℓ_1 to ℓ_2^{ε}. By extending $f|_{V_1}$ to S^3, we get a required homeomorphism. This completes the proof of Theorem 2.2.

Q.E.D.

If we require both V_1 and V_2 are knotted, the following result holds.

Theorem 2.6. *Let K be a knot in S^3 and V_i a knotted solid torus containing K. Suppose that $V_1 \subset V_2$ and the core C_1 of V_1 satisfies $w_{V_2}(C_1) \geq 2$ and $w_{V_1}(K) \geq 2$. Then $K_{V_1,m} \not\cong K_{V_2,n}$ for any pair $(m,n) \neq (0,0)$ (Figure 2).*

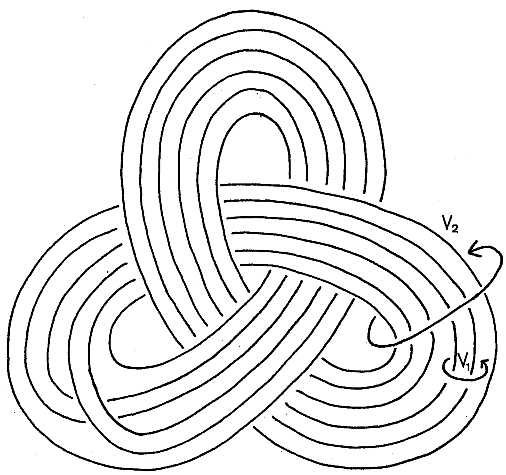

Fig. 2.

Proof. Let $f_m : V_1 \longrightarrow V_1$ and $g_n : V_2 \longrightarrow V_2$ be twist homeomorphisms with m-twist and n-twist respectively. By Theorem 2.1 in [6], $g_n(C_1) \not\cong C_1$ for any integer $n \neq 0$. Meanwhile $f_m(C_1) \cong C_1$ for any integer m. So the composition $g_n \circ f_m^{-1} : V_1 \longrightarrow g_n(V_1)$ sends C_1 to $g_n(C_1) \not\cong C_1$. We remark that C_1 and $g_n(C_1)$ are knotted in S^3, because they are geometrically essential in the knotted solid torus V_2. Also $g_n \circ f_m^{-1}$ satisfies $g_n \circ f_m^{-1}(K_{V_1,m}) = K_{V_2,n}$. Using Theorem [5], we can conclude $K_{V_1,m} \not\cong K_{V_2,n}$, if $n \neq 0$. In the case of $n = 0$, $K_{V_2,n} \cong K$ but $K_{V_1,m} \cong K$ holds only when $m = 0$ by Theorem 2.1 [6]. It follows that $K_{V_1,m} \not\cong K_{V_2,n}$ for any pair $(m,n) \neq (0,0)$. Q.E.D.

Remark. In the above theorem, the condition $w_{V_2}(C_1) \geq 2$ excludes the following trivial example.

Also in general, if both solid tori V_1 and V_2 are knotted then by Schubert's Satz 1 ([12]), we may assume one of the following occurs by

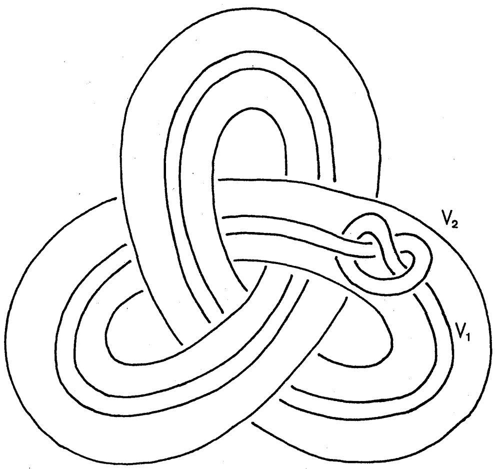

Fig. 3.

an ambient isotopy of S^3 which leaves K fixed. (1) $V_1 \subset V_2$ or $V_2 \subset V_1$, (2) $V_1 \cup V_2 = S^3$, and (3) there exists a solid torus W in $\operatorname{int} V_1 \cap \operatorname{int} V_2$ such that $w_{V_1}(C_W) = w_{V_2}(C_W) = 1$ for the core of C_W of W.

Theorem 2.6 corresponds to the case (1). As for cases (2) and (3), there exist inessential examples as in Figure 4 and Figure 5 respectively.

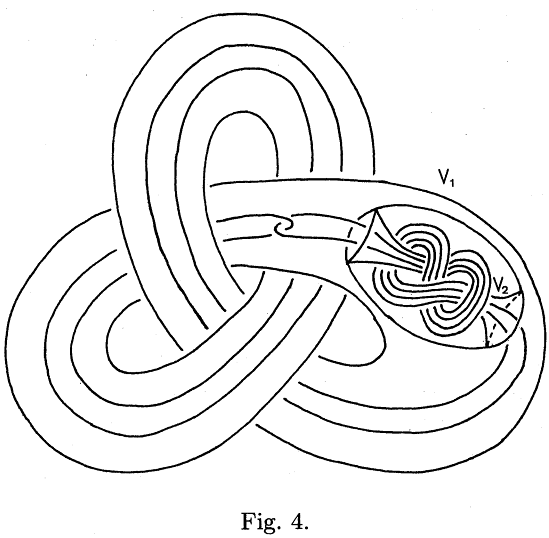

Fig. 4.

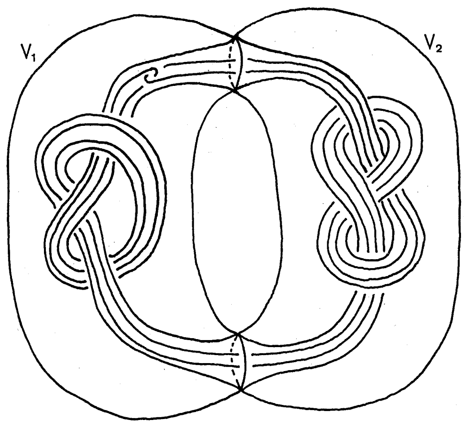

Fig. 5.

§3. Gromov invariants

The notion of the Gromov invariant of closed manifolds was introduced by Gromov [1]. In the 3-dimensional case, Thurston defined the Gromov invariant of compact 3-manifolds whose boundaries consists of tori [14]. In this section we shall study the Gromov invariant of the exterior of a knot K in S^3 which we simply call the Gromov invariant of K and we denote it by $\|K\|$. For the definition of the Gromov invariant, the reader is referred to [1], [14] and [13].

First we prove the following.

Theorem 3.1. *Let K be a knot in S^3 and V a knotted solid torus containing K. Then $\|K_{V,n}\| = \|K\|$ holds for any integer n.*

Proof. If $w_V(K) \leq 1$, then $K_{V,n} = K$ for any integer n. So we assume $w_V(K) \geq 2$. The exterior of $K_{V,n}$ ($K_{V,0} \cong K$) is described as $(S^3 - \operatorname{int} V) \bigcup_{h_n} (V - \operatorname{int} N(K))$ for some gluing homeomorphism h_n. Since V is knotted, $\partial(S^3 - \operatorname{int} V)$ is an incompressible torus. Also ∂V is an incompressible torus in $V - \operatorname{int} N(K)$ because $w_V(K) \geq 2$. Hence we have the following equality independent of n by Soma's theorem [13].

$$\|K_{V,n}\| = \|E((K_{V,n})\| = \|(S^3 - \operatorname{int} V) \amalg (V - \operatorname{int} N(K))\|.$$

It follows that $\|K_{V,n}\| = \|K\|$. Q.E.D.

Hence, in Theorem 2.6, $K_{V_1,m}$ and $K_{V_2,n}$ have the same Gromov invariants for any pair (m, n).

The following is straightforward from Theorem 3.1.

Corollary 3.2. *Suppose that K_1 and K_2 are knots with $\|K_1\| \neq \|K_2\|$. Then K_2 can not be obtained by a sequence of twistings along knotted solid tori from K_1.*

On the other hand, if V is unknotted we have:

Proposition 3.3. *Let O be the unknot in S^3. For any real number r, there exists an unknotted solid torus V containing O such that $\|O_{V,1}\| > r$.*

Proof. Consider a solid torus V as in Figure 6. Then in the exterior of $O_{V,1}$, there exist incompressible tori which decompose it into k figure eight knot spaces, 1 Whitehead link space and 1 composing space. Hence $\|O_{V,1}\| = 1/v_3(k\,\mathrm{Vol}(\text{figure eight knot complement}) + \mathrm{Vol}(\text{Whitehead link complement}))$, where v_3 is the volume of the regular ideal simplex (see [14] [13]). Thus the result holds for some integer $k > 0$. Q.E.D.

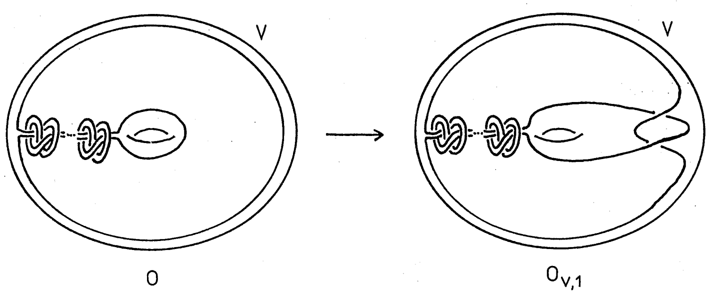

Fig. 6.

This also shows that for any knot K and any real number r, there exists an unknotted solid torus V such that $K_{V,1} > r$.

But the Gromov invariants behave as follows once V is fixed.

Proposition 3.4. *Let K be a knot in S^3 and V an unknotted solid torus containing K. Then $\|K_{V,n}\|$ is less than a constant $C_{V,K}$ for any integer n.*

Proof. We may assume $w_V(K) \geq 2$. If P_0 is a cable space, $\|K_{V,n}\|$ is constant for all but at most two integers n such that a regular fibre is presented by $\ell^p m$ for some p. If P_0 is a composing space, then twisting along V is reduced to that along a knotted solid torus W bounded by the torus ($\subset \partial P_0$) which separates K and ∂V (see Sublemma 3.7 [6]). Hence Theorem 2.1 in [6] implies the result. Suppose that P_0 is hyperbolic, by Lemma 2.3 $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is also hyperbolic for $|n| \geq N_{V,K}$. Then we have $\mathrm{Vol}(\mathrm{int}(V_J \bigcup_{m_J=\ell m^{-n}} P_0)) < \mathrm{Vol}(\mathrm{int}\, P_0)$ by Thurston's theorem (6.5.6 Theorem [14]), and from this we have the following inequality for $|n| \geq N_{V,K}$,

$$\begin{aligned}
\|K_{V,n}\| &= 1/v^3\Big(\sum_{\substack{P_i:\text{hyperbolic}\\ i\neq 0}} \mathrm{Vol}(\mathrm{int}\, P_i) + \mathrm{Vol}(\mathrm{int}(V_J \bigcup_{m_J=\ell m^{-n}} P_0))\Big) \\
&< 1/v^3\Big(\sum_{\substack{P_i:\text{hyperbolic}\\ i\neq 0}} \mathrm{Vol}(\mathrm{int}\, P_i) + \mathrm{Vol}(\mathrm{int}\, P_0)\Big) \\
&= \|K \amalg J\|.
\end{aligned}$$

Now we set $C_1 = \max\{\|K_{V,n}\| : |n| < N_{V,K}\}$ and we take $C_{V,K} = \max\{C_1, \|K \amalg J\|\}$, then $C_{V,K}$ is the required constant. Q.E.D.

Example 3.5 (Thurston [14]).

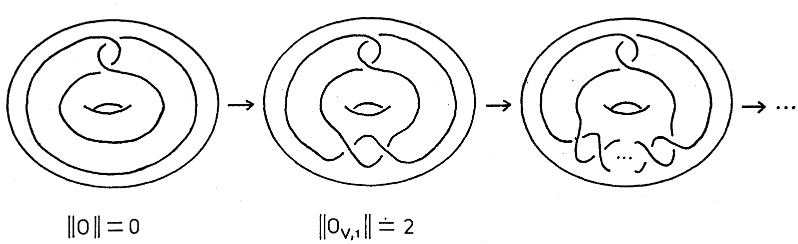

Fig. 7.

The Gromov invariants of these knots tend from below to a finite limit ($\doteq 3.6$).

§4. Primeness of knots under twistings

In this section, we investigate the effects of twistings on primeness of knots. To begin with, we consider the case when a twisting solid torus is knotted.

Theorem 4.1. *Let K be a knot in S^3 and V a knotted solid torus containing K. Then K is prime if and only if $K_{V,n}$ is prime for any integer n.*

Proof. We may assume $w_V(K) \geq 2$. Consider the torus decomposition of $V - \operatorname{int} N(K)$ and denote the piece containing $\partial N(K)$ by P. Suppose that K is a prime knot, then it turns out P is not a composing space. Now we consider the torus decomposition of $E(K_{V,n}) = (S^3 - \operatorname{int} V) \bigcup_{h_n} (V - \operatorname{int} N(K))$. In $E(K_{V,n})$, P is also a decomposing piece. It follows that $K_{V,n}$ is also prime for any integer n. Q.E.D.

If V is unknotted, then the following example exists.

Example 4.2. In Figure 8, K is a prime knot, but $K_{V,n}$ is a composite knot for any nonzero integer n.

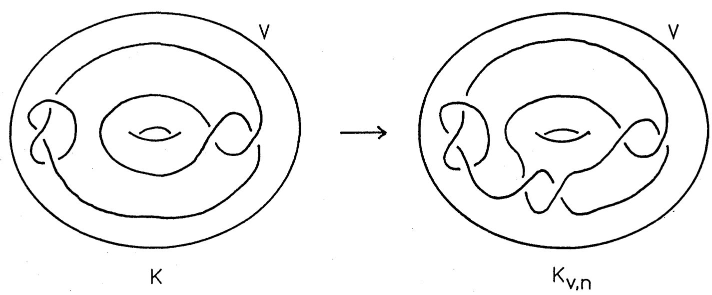

Fig. 8.

In this example K has a locally knotted arc in V (*i.e.* there is a 3-ball $B \subset V$ such that $(B,\ B \cap K)$ is a knotted ball pair). If K does not have a locally knotted arc in V, then we get the following.

Theorem 4.3. *Let V be an unknotted solid torus containing K without a locally knotted arc. Then $K_{V,n}$ is prime for all but at most finitely many integers n.*

Proof. Consider the torus decomposition of $V - \operatorname{int} N(K)$, and let P be a piece containing $\partial N(K)$ and P_0 a piece containing ∂V.

Sublemma. *Suppose that $K \subset V$ does not have a local knot. Then P can not be a composing space.*

Proof of Sublemma. Suppose that P is a composing space. Let T be a component of ∂P which does not separate ∂V and $\partial N(K)$. Note that T

bounds a nontrivial knot exterior E, and a regular fibre of P coincides a boundary of a meridian disk of $N(K)$. Hence we have a saturated annulus A' which joins T and $\partial N(K)$. Then $D' = A' \cup D$ becomes a meridian disk of $W = S^3 - \operatorname{int} E$. Since $K \cap D'$ and $K \cap D$ consist of one point, K has a locally knotted arc in V. This is a contradiction.

Q.E.D.

If P_0 is a cable space, $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is a (nontrivial) torus knot exterior except for at most only two integers n by Lemma 2.4. If P_0 is a k-fold composing space, then $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is a $(k-1)$-fold composing space for any integer n. Finally we consider the case when P_0 is hyperbolic. By Lemma 2.5, we see that $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is also hyperbolic except for at most finitely many integers n. It follows that in any case, $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is boundary irreducible Haken manifold. Now we divide into two cases depending upon whether $P = P_0$ or not. If $P = P_0$, then $V_J \bigcup_{m_J=\ell m^{-n}} P = V_J \bigcup_{m_J=\ell m^{-n}} P_0$ can not be a composing space by Sublemma and the above, and it becomes a decomposing piece in $E(K_{V,n})$. Thus $K_{V,n}$ is prime except for at most finitely many integers n. If $P \neq P_0$, then it turns out that P is still a decomposing piece in $E(K_{V,n})$. Since P is not a composing space, $K_{V,n}$ is prime except for at most finitely many integers n. Q.E.D.

Remark 4.4. Even if K does not have a locally knotted arc in V, there is an example such that $K_{V,n}$ is a composite knot for some integer n (see Figure 9).

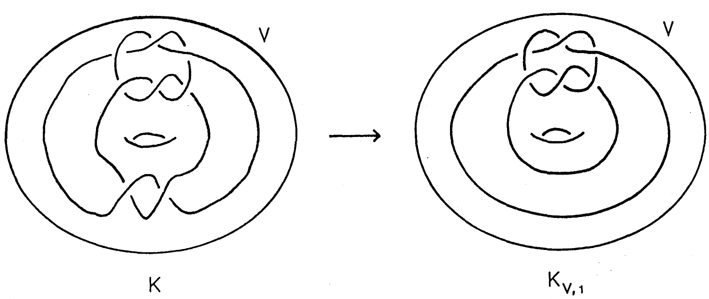

Fig. 9.

When an original knot is trivial, Scharlemann-Thompson [11], Eudave-Munoz and Gordon have shown the following result, which is a generalization of the theorem — "Unknotting number one knots are prime [10]".

Theorem 4.5 ([11]). *Let V be a solid torus containing the unknot O with $w_V(O) \leq 2$. Then $O_{V,n}$ is prime for any integer n.*

Since the unknot can not have a locally knotted arc, as an application of Theorem 4.3, we have the following.

Corollary 4.6. *Let V be a solid torus containing the unknot O. Then $O_{V,n}$ is prime for all but at most finitely many integers n.*

We conclude this paper with the following question.

Question. *Is the result of twisting of the unknot always prime?*

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