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Errata to Vertex Operators in Conformal Field Theory on $\mathbf{P}^1$ and Monodromy Representations of Braid Group in Advanced Studies in Pure Mathematics 16,1988

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Abstract.

We give corrections or comments to the five points in our paper [1].

First fix the notations. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$. Fix a positive integer ℓ and let $\kappa = \ell + 2$ and $q = \exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)$ and introduce the set P_ℓ consisting of all half-integers j with $0 \leq j \leq \ell/2$.

For any $j_k \in P_\ell$ $(1 \leq k \leq 3)$, denote by $\mathcal{W}\left({}_{j_3}{}^{j_2 j_1}\right)$ the space of initial terms of vertex operators of type $\mathbf{v} = \left({}_{j_3}{}^{j_2 j_1}\right)$. Note that in this $\mathrm{A}_1^{(1)}$-case, the space $\mathcal{W}\left({}_{j_3}{}^{j_2 j_1}\right)$ is nothing but the space $\mathcal{V}\left(\begin{smallmatrix} j_2 \\ j_3 j_1 \end{smallmatrix}\right) = \mathrm{Hom}_{\mathfrak{g}}(V_{j_2} \otimes V_{j_1}, V_{j_3})$ in [1] for $j_1 + j_2 + j_3 \leq \ell$, and $\mathcal{W}\left({}_{j_3}{}^{j_2 j_1}\right) = 0$ for $j_1 + j_2 + j_3 > \ell$.

The space $\mathrm{Hom}_{\mathfrak{g}}(V_{j_2} \otimes V_{j_1}, V_{j_3})$ is at most 1-dimensional, and the condition for its nontriviality is the Clebsch-Gordan condition $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$. In such case, we fix the nonzero vector $\phi_{\mathbf{v}}$ as in [1, Appendix 1].

1. Literally is not valid the braid relation in [1, Proposition 4.2], where the notations we used are not appropriate. Now we reformulate Proposition 4.2 ii).

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For any $N \geq 1$ and $s, t, j_k \in P_\ell (1 \leq k \leq N)$, introduce the spaces $\mathcal{W}\left({}_t{}^{j_N \cdots j_1 s}\right)$ and $\mathcal{W}(N;{}_t{}^s)$ defined by

$$\mathcal{W}\left({}_t{}^{j_N \cdots j_1 s}\right) = \sum_{p_1, \ldots, p_{N-1} \in P_\ell} \mathcal{W}\left({}_t{}^{j_N p_{N-1}}\right) \otimes \mathcal{W}\left({}_{p_{N-1}}{}^{j_{N-1} p_{N-2}}\right) \otimes \cdots \otimes \mathcal{W}\left({}_{p_1}{}^{j_1 s}\right)$$

and

$$\mathcal{W}(N;{}_t{}^s) = \sum_{j_1, \ldots, j_N \in P_\ell} \mathcal{W}\left({}_t{}^{j_N \cdots j_1 s}\right).$$

Note

$$\mathcal{W}(M+N;{}_t{}^s) = \sum_{p \in P_\ell} \mathcal{W}(M;{}_t{}^p) \otimes \mathcal{W}(N;{}_p{}^s)$$

Recall that the operator $C(j_4, j_3, j_2, j_1)$ in [1, Section 4.1] acts as

$$C(j_4, j_3, j_2, j_1) : \mathcal{W}\left({}_{j_4}{}^{j_3 j_2 j_1}\right) \longrightarrow \mathcal{W}\left({}_{j_4}{}^{j_2 j_3 j_1}\right),$$

which is defined by the monodromy on the solution space of the four-point functions.

Now define the operators $C_i (1 \leq i \leq N-1)$ on $\mathcal{W}(N;{}_t{}^s)$ as follows:

$$C_i \mathcal{W}\left({}_t{}^{j_N \cdots j_1 s}\right) \subset \mathcal{W}\left({}_t{}^{j_N \cdots j_i j_{i+1} \cdots j_1}\right)$$

and

$$C_i(\phi_N \otimes \cdots \otimes \phi_{i+1} \otimes \phi_i \otimes \cdots \otimes \phi_1) = \phi_N \otimes \cdots \otimes \phi_{i+2} \otimes C(p_{i+1}, j_{i+1}, j_i, p_{i-1})(\phi_{i+1} \otimes \phi_i) \otimes \phi_{i-1} \otimes \cdots \otimes \phi_1$$

for each $\phi_N \in \mathcal{W}\left({}_t{}^{j_N p_{N-1}}\right)$, $\phi_k \in \mathcal{W}\left({}_{p_k}{}^{j_k p_{k-1}}\right)$ $(2 \leq k \leq N-1)$ and $\phi_1 \in \mathcal{W}\left({}_{p_1}{}^{j_1 s}\right)$.

Now Proposition 4.1 ii) should be read as

Proposition 1. i) *As operators on* $\mathcal{W}(3;{}_t{}^s)$*, the relation*

$$C_1 C_2 C_1 = C_2 C_1 C_2$$

holds.

ii) *As operators on* $\mathcal{W}(N;{}_t{}^s)$*, the relation*

$$C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1} \qquad (1 \leq i \leq N-1)$$

holds.

2. There is an error in the definition of the mapping K from the Wenzl's representation $(\pi_\lambda, V_\lambda^{(2,\kappa)})$ to our monodromy representation $(\pi_{N,t}, W(N;t))$ in [1, Proposition 5.3]. We give here the precise definition of the intertwining operator K^{-1} rather than K.

In our notation in this errata the space $W(N;t)$ is $\mathcal{W}\left({}_t{}^{\frac{1}{2}}\cdots{}^{\frac{1}{2}}{}^0\right)$ and has a basis $\{\phi_\mathbf{p};\ \mathbf{p} = (p_N, \dots, p_1) \in \mathcal{P}_\ell(N;t)\}$, where

$$\mathcal{P}_\ell(N;t) = \left\{\mathbf{p} = (p_N, \dots, p_1, 0); p_i \in P_\ell, p_N = t, |p_i - p_{i-1}| = \frac{1}{2}\right\}$$

and

$$\phi_\mathbf{p} = \phi_{\mathbf{v}_N} \otimes \cdots \otimes \phi_{\mathbf{v}_1} \in \mathcal{W}\left({}_t{}^{\frac{1}{2}}{}^{p_{N-1}}\right) \otimes \cdots \otimes \mathcal{W}\left({}_{p_1}{}^{\frac{1}{2}}{}^0\right),$$

$$\mathbf{v}_i = \left({}_{p_i}{}^{\frac{1}{2}}{}^{p_{i-1}}\right).$$

The operators C_i on $\mathcal{W}\left(N;{}_t{}^0\right)$ preserves the subspace $W(N;t)$. By Proposition 1, the braid group B_N generated by C_i $(1 \le i \le N-1)$ acts on the space $W(N;t)$. Denote this representation by $\pi_{N,t}$.

The basis vectors $\phi_\mathbf{p}$ $(\mathbf{p} \in \mathcal{P}_\ell(N;t))$ are eigenvectors w.r.t. the commutative subalgebra $\mathcal{A} = \sum_{i=1}^{N-1} \mathbb{C}(C_i \dots C_1)^i$ of the group algebra $\mathbb{C}[B_N]$.

Let λ be the Young diagram $\lambda(N;t) = [\frac{N}{2} + t, \frac{N}{2} - t]$ (Here we assume that $\frac{N}{2} - t \in \mathbb{Z}_{\ge 0}$, otherwise the space $W(N;t)$ vanishes). The corresponding Wenzl's space $V_\lambda^{(2,\kappa)}$ is generated by the basis $\{\vec{v}_{\vec{p}}; \vec{p} = (\lambda_N, \dots \lambda_1) \in \mathcal{P}_\ell(\lambda)\}$, where $\mathcal{P}_\ell(\lambda)$ is the image of the set $\mathcal{P}_\ell(N;t)$ under the mapping K^{-1}, where K^{-1} is defined as

$$K^{-1}(\mathbf{p}) = (\lambda(N;t), \lambda(N-1, p_{N-1}), \dots, \lambda(1, p_1))$$

for $\mathbf{p} = (t, p_{N-1}, \dots, p_1, 0) \in \mathcal{P}_\ell(N;t)$. The basis vectors $\vec{v}_{\vec{p}}$ $(\vec{p} \in \mathcal{P}_\ell(\lambda))$ are also the eigenvectors w.r.t. the algebra $\mathcal{A}$ of the same eigenvalues as for $K(\vec{p})$.

Thus the mapping $K^{-1} : W(N;t) \to V_\lambda^{(2,\kappa)}$ must have the form

$$K^{-1}(\phi_\mathbf{p}) = \gamma_\mathbf{p} \vec{v}_{K^{-1}(\mathbf{p})},$$

where $\gamma_\mathbf{p}$ is a constant which was given uncorrectly in [1].

Before we give the correct definition of the constants $\gamma_\mathbf{p}$, we need some preliminaries. Introduce an order $<$ in the set $\mathcal{P}_\ell(N;t)$ lexicographically, *i.e.* we call $\mathbf{p} < \mathbf{q}$ for $\mathbf{p} = (t, p_{N-1}, \dots, p_1, 0)$, $\mathbf{q} =$

$(t, q_{N-1}, \ldots, q_1, 0) \in \mathcal{P}_\ell(N;t)$ if $p_j \leq q_j$ for any j. Then it is easily seen that there is the maximal $\mathbf{p}_0$ in $\mathcal{P}_\ell(N;t)$ w.r.t. this order.

We call $\mathbf{p}$ and $\mathbf{q}$ are *adjoining* and denote $\mathbf{p} \sim \mathbf{q}$, if there is an number $k (1 \leq k \leq N-1)$ such that $p_j = q_j \quad (j \neq k)$ and $|p_k - q_k| = 1$. Any $\mathbf{p}$ and $\mathbf{q}$ in $\mathcal{P}_\ell(N;t)$ are connected by a sequence $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ in $\mathcal{P}_\ell(N;t)$ such that $\mathbf{p}_1 = \mathbf{p}$, $\mathbf{p}_n = \mathbf{q}$ and $\mathbf{p}_i \sim \mathbf{p}_{i+1} (1 \leq i \leq n-1)$.

By Proposition 4.8 in [1] and the definition of the Wenzl's representation [3], there hold only the following relations (C) among the constants $\gamma_{\mathbf{p}}$ ($\mathbf{p} \in \mathcal{P}_\ell(N;t)$): let $\mathbf{p} = (t, p_{N-1}, \ldots, p_1, 0)$ and $\mathbf{q}$ be adjoining and $\mathbf{p} < \mathbf{q}$ ($q_k = p_k + 1$). Denote $\mathbf{p}_+ = \mathbf{q}$, $\mathbf{p}_- = \mathbf{p}$ and $p = p_{k-1} = q_{k-1}$. Then there must hold the relation

$$\text{(C)} \qquad \gamma_+(p)\gamma_{\mathbf{p}_+} = \gamma_-(p)\gamma_{\mathbf{p}_-},$$

where

$$\gamma_\pm(p) = \frac{\Gamma\left(\frac{2p+1}{\pm\kappa}\right)}{\Gamma\left(\frac{2p+2}{\pm\kappa}\right)^{1/2}\Gamma\left(\frac{2p}{\pm\kappa}\right)^{1/2}}.$$

Since any $\mathbf{p}$ and $\mathbf{q}$ are connected by an adjoining sequence in $\mathcal{P}_\ell(N;t)$, the constants $\{\gamma_{\mathbf{p}}; \mathbf{p} \in \mathcal{P}_\ell(N;t)\}$ are uniquely determined up to a constant multiple. Normalise them as $\gamma_{\mathbf{p}_0} = 1$ for the maximal $\mathbf{p}_0$. Then $\gamma_{\mathbf{q}} (\mathbf{q} \in \mathcal{P}_\ell(N;t))$ are given as follows: Write $\mathbf{p}_0 = (t, p_{N-1}, \ldots, p_1, 0)$ and $\mathbf{q} = (t, q_{N-1}, \ldots, q_1, 0)$, then $p_j - q_j \in \mathbb{Z}_{\geq 0} (1 \leq j \leq N-1)$. Then

$$\gamma_{\mathbf{q}} = \prod_{j=1}^{N-1} \gamma(p_j, q_j),$$

where for $p \geq q$

$$\gamma(p,q) = \begin{cases} 1, & (p = q) \\ \frac{\gamma_+(p-\frac{1}{2})\cdots\gamma_+(q+\frac{1}{2})}{\gamma_-(p-\frac{1}{2})\cdots\gamma_-(q+\frac{1}{2})}, & (p > q). \end{cases}$$

Then Proposition 5.3 in [1] remains valid.

Proposition 2. *Let $N \in \mathbb{Z}_{>0}$ and $t \in P_\ell$ satisfy $\frac{N}{2} - t \in \mathbb{Z}_{\geq 0}$. Then*

$$K\pi_{\lambda(N,t)} = q^{\frac{3}{4}}\pi_{N,t}K.$$

3. The proof of Proposition 5.4 (The Fusion Rule) in [1] is insufficiently presented. We will develop the theory of the fusions more in detail in our forthcoming paper [2]. In this errata, we only give the meaning of the integral used there to multivalued functions and justify the proof.

Recall that a vertex operator $\Phi(z)$ of type $\mathbf{v} = \binom{ji}{k}$ can be considered as a $\mathrm{Hom}(\mathcal{H}_k^\dagger \otimes V_j \otimes \mathcal{H}_i, \mathbb{C})$ -valued holomorphic function:

$$\Phi(z)(w \otimes v \otimes u) = \langle w|\Phi(v;z)(|u\rangle)\rangle$$

$$(w \in \mathcal{H}_k^\dagger, v \in V_j, u \in \mathcal{H}_i),$$

and Φ is uniquely determined by its initial term $\phi \in \mathcal{W}\binom{ji}{k} \subset \mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_i, V_k) \cong \mathrm{Hom}_{\mathfrak{g}}(V_k^\dagger \otimes V_j \otimes V_i, \mathbb{C})$. The holomorphicity of $\Phi(z)$ is weakly taken, *i.e.* $\Phi(z)$ is holomorphic, if the $\mathbb{C}$-valued function $\langle w|\Phi(v;z)(|u\rangle)\rangle$ is holomorphic in z for any fixed vector $w \otimes v \otimes u$ in $\mathcal{H}_k^\dagger \otimes V_j \otimes \mathcal{H}_i$.

By [1, Theorem 3.4] vertex operators $\Phi_{\mathbf{v}_1}(z)$ of type $\mathbf{v}_1 = \binom{j_2 j_1}{p}$ and $\Phi_{\mathbf{v}_2}(w)$ of type $\mathbf{v}_2 = \binom{j_3 p}{j_4}$ are composable, and the composed operator $\Phi_{\mathbf{v}_2}(w)\Phi_{\mathbf{v}_1}(z)$ is a $\mathrm{Hom}(\mathcal{H}_{j_4}^\dagger \otimes V_{j_3} \otimes V_{j_2} \otimes \mathcal{H}_{j_1}, \mathbb{C})$ -valued multivalued holomorphic function on $M_2 = \{(w,z) \in \mathbb{C}^{*2}; w \neq z\}$ regularized at z=0 and is uniquely determined by the $\mathrm{Hom}_{\mathfrak{g}}(V_{j_4}^\dagger \otimes V_{j_3} \otimes V_{j_2} \otimes V_{j_1}, \mathbb{C})$-valued function

$$\Psi_p(w,z)(u_4 \otimes u_3 \otimes u_2 \otimes u_1) = \langle u_4|\Phi_{\mathbf{v}_2}(u_3;w)\Phi_{\mathbf{v}_1}(u_2;z)(|u_1\rangle)\rangle$$

$$(u_4 \in V_{j_4}^\dagger, u_k \in V_{j_k} (1 \leq k \leq 3)),$$

which satisfies the reduced system of differential equations for 4-point functions(the joint system $E'(\mathrm{J})$ and $B'(\mathrm{J})$, $\mathrm{J} = (j_4, j_3, j_2, j_1)$ in [1, Proposition 4.1]) and they form a basis of its solution space. The holomorphic function $\Psi_p(w,z)$ is known to have the singularity at $w = z$ as

$$\Psi(w,z) = \sum_{r \in P_\ell} (w-z)^{\gamma_r^{(1)}} (\sqrt{2j_4 + 1} F^r U_r^{(1)} + O(w-z))$$

where $U_r^{(1)}$ is the basis vector of $\mathrm{Hom}_{\mathfrak{g}}(V_{j_4}^\dagger \otimes V_{j_3} \otimes V_{j_2} \otimes V_{j_1}, \mathbb{C})$ fixed in $[1, Appendix 1]$, F_p^r is a constant, $O(w-z)$ is a holomorphic function in $(w-z, z)$ near $w = z$ vanishing at $w = z$ and the exponent $\gamma_r^{(1)}$ is given as

$$\gamma_r^{(1)} = \Delta_{j_2} + \Delta_{j_3} - \Delta_r \qquad \left(\Delta_j = \frac{j^2 + j}{\kappa}\right).$$

We can show similarly as Theorems 2.3 and 3.4 in [1] that the function $\Phi_2(w)\Phi_1(z)$ has an expansion as

$$\langle u_4|\Phi_2(u_3;w)\Phi_1(u_2;z)(|u_1\rangle)\rangle = \sum_{r\in P_\ell}(w-z)^{\gamma_r^{(1)}}\Psi_p^r(w,z)(u_4\otimes u_3\otimes u_2\otimes u_1)$$

$$(u_4 \otimes u_3 \otimes u_2 \otimes u_1 \in \mathcal{H}_{j_4}^\dagger \otimes V_{j_3} \otimes V_{j_2} \otimes \mathcal{H}_{j_1})$$

where $\Psi_p^r(w,z)$ is a Laurent series in $w-z$ with coefficients in $\mathbb{C}(z)$.

Assume that a $\mathbb{C}$-valued function $F(w,z)$ is a holomorphic function on some region of M_2 has an expansion as

$$F(w,z) = \sum_{j=0}^{M}(w-z)^{\gamma_j}F_j(w,z),$$

where $\gamma_j \in \mathbb{Q}$, $\gamma_0 = 0, \gamma_j \notin \mathbb{Z}(j \geq 1)$ and $F_j(w,z)$ is a Laurent series in $w-z$. Let C_z is a positively oriented contour around z such that the origin is outside C_z. Then we used the convention that the integral of $F(w,z)$ over C_z means the integral of $F_0(w,z)$ over C_z. Hence the operator

$$\Xi^r(u_3,u_2) = \frac{1}{2\pi\sqrt{-1}}\int_{C_z}(w-z)^{-\gamma_r^{(1)}-1}\Phi_{\mathbf{v}_2}(u_3;w)\Phi_{\mathbf{v}_1}(u_2;z)dw$$

in [1] is nothing but the residue of the above $\Psi_r(w,z)$ at $w=z$.

On the other hand, introduce the space

$$\mathcal{FW}\left({}_{j_4}{}^{j_3j_2j_1}\right) = \sum_{r\in P_\ell}\mathcal{W}\left(\mathbf{w}_2(r)\right)\otimes\mathcal{W}\left(\mathbf{w}_1(r)\right)$$

where

$$\mathbf{w}_2(r) = \left({}_{j_4}{}^{rj_1}\right) \text{ and } \mathbf{w}_1(r) = \left({}_{r}{}^{j_3j_2}\right).$$

Its basis vector $\phi_{\mathbf{w}_2(r)} \otimes \phi_{\mathbf{w}_1(r)}$ determines the $\mathrm{Hom}(V_{j_4}^\dagger \otimes V_{j_3} \otimes V_{j_2} \otimes V_{j_1}, \mathbb{C})$ -valued function by

$$\begin{aligned}&\langle u_4|\Phi_{\mathbf{w}_2(r)}(\Phi_{\mathbf{w}_1(r)}(u_2;z);w-z)(u_1))\rangle\\&= (w-z)^{\gamma_r^{(1)}}\left((\sqrt{2j_4+1}U_r^{(1)}(u_4,u_3,u_2,u_1)+O(w-z)\right)\end{aligned}$$

(regularized at w=z) and these functions furnish also a basis of the solution space of the joint system $E'(\mathbb{J})$ and $B'(\mathbb{J})$ (see [2]).

Thus the analytic continuation gives the isomorphism

$$F : \mathcal{W}\left({}_{j_4}{}^{j_3 j_2 j_1}\right) \longrightarrow \mathcal{FW}\left({}_{j_4}{}^{j_3 j_2 j_1}\right)$$

(the mapping between initial terms).

4. In the proof of [1, Theorem 3.1], we did not take into account the possibility that there may be the solutions of the joint system of $E(\mathbb{J})$ and $B(\mathbb{J})$, $\mathbb{J} = (j_N, \dots, j_1)$ with logarithmic singularities. Any formal solution of the system $\tilde{E}(\mathbb{J})$ in the proof is of the form

$$\Psi(w) = \sum_{a=1}^{R} w^{s^a} \sum_{k \in \mathbb{Z}^N_{\geq 0}} \sum_{m \in \mathbb{Z}^N_{\geq 0}, |m| \leq M} \phi_{a,k,m} w^k (\log w)^m,$$

where M is some bound, $s^a = (s^a_1, \dots, s^a_N)$'s are exponents and $\phi_{a,k,m} \in Hom_{\mathfrak{g}}\left(V^\dagger_{j_N} \otimes V_{j_{N-1}} \otimes \cdots \otimes V_{j_1}, \mathbb{C}\right)$. Apply the arguments in the proof of [1, Theorem3.1], then we get $\phi_{a,0,m} \in \mathcal{W}\left({}_{j_N}{}^{j_{N-1} \cdots j_2 j_1}\right)$. Hence we already know sufficiently many formal solutions in the form without logarithmic terms by means of the products of the vertex operators.

5. The line $\uparrow$ 13 in the page 337 of [1] should be read as

$$\lim_{z \nearrow \infty} z^{2\Delta_{p_N}} \langle vac|(\hat{Y}_q(-m_q) \dots \hat{Y}_1(-m_1)\Phi_{\mathbf{v}_{N+1}}(v_0; z)) = (-1)^q \langle v|.$$

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