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Integrable Field Theory from Conformal Field Theory

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Dedicated to Professor I. M. Gelfand on his 75th birthday

Abstract.

Equations of motion for two-dimensional quantum field theory obtained as some relevant perturbation around CFT are analyzed. It is shown that for particular degenerate fields taken as the perturbations, the resulting field theories posseses non-trivial local integrals of motion. The example is the scaling limit of the Ising model at $T = T_c$ but with non-vanishing magnetic field. Implications of the integrals of motion for corresponding particle theory are discussed and the exact S-matrix for the critical Ising model with magnetic field is conjectured.

§1. Introduction

Two directions of mathematical physics experienced a particularly rapid development during the last years. These are conformal field theory (CFT) [1,2] and integrable lattice models (ILMs) of statistical mechanics [3,4]. The subjects seem to be deeply related both in their mathematical contents and physical interpretations. It is not of course very surprising that the ILMs exhibit in their critical limits exactly the same exponents as those predicted by CFT [5]; this agrees with our general expectation based on universality theory [6]. A really amazing features of ILMs are particular forms of corner transfer matrix spectrum and of local height probabilities [4,7] which strongly suggest that these models carry an infinite dimensional algebraic structure characteristic for CFT, even away from critical point. This structure can be explicitly demonstrated in particular case of Ising model [8]. Some progress in unmasking roots of this structure in more general models has been achieved very recently [9,10].

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A lattice model in its scaling limit is interpreted as (euclidean) field theory (FT). This FT is usually massive (the mass being related to correlation length by $m \sim R_c^{-1}$) whereas its short distance limit is described by CFT. For ILM this massive FT is certainly integrable. In particular, one can try to analyze this FT starting from lattice model. On the other hand, this massive FT can be considered as deformation of corresponding CFT, *i.e.* the CFT perturbed by an appropriate relevant operator. Here I will adopt this second point of view on the massive FT. The relation to ILM means that there are particular perturbations around CFT which lead to integrable massive FT. I will show how one can recover these particular directions of deformation using the technique of CFT [11,12].

The structure of CFT is governed by Virasoro algebra (or some extended infinite dimensional symmetry which include Virasoro algebra) which is generated by appropriate component $T(z)$ ($\bar{T}(\bar{z})$ for antiholomorphic part) of stress-energy tensor. A composite fields made up of $T(z)$ (like $:T^2:$, $:T^3:$, $:(\partial_z T)^2:$, etc.) give rise to infinite set of local integrals of motion (IM) in CFT. Amongst these there is a subset of commuting IM. IM in CFT are analized in Section 2. In Sections 3,4 it is shown that some of these commuting operators can survive as IM in perturbed FT

$$H = H_{\mathrm{CFT}} + \lambda \int \Phi(x) d^2x, \tag{1.1}$$

(where H_{CFT} is the action of original CFT) with particular relevant field Φ taken as the perturbation. The IM of the perturbed FT are the integrals of the local densities

$$P_s = \oint [\, T_{s+1} dz + \Theta_{s-1} d\bar{z} \,], \tag{1.2}$$

where $(z, \bar{z})$ are standard complex coordinates of $\mathbf{R}^2$ and T_{s+1} and Θ_{s-1} are some local fields satisfying

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}. \tag{1.3}$$

The integer-valued index s which labels the IM for given FT (1.1) indicate the spin of this operator

$$[\, M, P_s \,] = s P_s, \tag{1.4}$$

where M is euclidean rotation generator. (1.2) is positive-spin part of IM. Of course there is also similar negative-spin part which is obtained from (1.2) by spatial reflection; the reflection symmetry is always implied here. The set of the positive integers taken by s in (1.2) is specific for each particular model.

In Section 4, we discuss three sets of models of FT

$$H_p^{(1,3)} = H_p + \lambda \int \Phi_{(1,3)} d^2x \,; \tag{1.5 a}$$

$$H_p^{(1,2)} = H_p + \lambda \int \Phi_{(1,2)} d^2x \,; \tag{1.5 b}$$

$$H_p^{(2,1)} = H_p + \lambda \int \Phi_{(2,1)} d^2x, \tag{1.5 c}$$

where H_p denotes (an action of) unitary minimal CFT [13] with Virasoro central charge

$$c_p = 1 - \frac{6}{p(p+1)}\,; \qquad p = 3, 4, 5, \ldots \tag{1.6}$$

and the fields $\Phi_{(1,3)}, \Phi_{(1,2)}, \Phi_{(2,1)}$ taken as perturbations in (1.5) are spinless primary fields in standard notations [1]. We show that all these models possess a number of non-trivial local IM of the form (1.2). We prove the existance of IM (1.2) with the following spins

$$s = 1, 3, 5, 7 \quad \text{for} \quad H_p^{(1,3)}; \tag{1.7a}$$

$$s = 1, 5, 7, 11 \quad \text{for} \quad H_p^{(1,2)} \text{ and } H_p^{(2,1)} \text{ with } p \geq 5, \tag{1.7b}$$

and conjecture that these are the first few representatives of the infinite sets

(1.8a)
$$s = 2n - 1; \qquad n = 1, 2, 3, \ldots \quad \text{for} \quad H_p^{(1,3)};$$

(1.8b)
$$s = 1,\ 6n \pm 1; \qquad n = 1, 2, 3, \ldots \quad \text{for} \quad H_p^{(1,2)},\ H_p^{(2,1)} \quad \text{with} \quad p \geq 5.$$

The models (1.5 b,c) with $p < 5$ also possess non-trivial IM (1.2). The operator P_1 is always the complex component of momentum $P^\mu : P_1 = P^1 + iP^2 = \partial_z$ (T_2 and Θ_0 in (1.2) are the corresponding components of stress-energy tensor); this IM is considered to be trivial.

A massive FT (being continued to Minkowski space-time) is equivalent to relativistic scattering theory and therefore it is uniquely characterized by S-matrix. The scattering theory which corresponds to a FT possessing even one non-trivial IM of the form (1.2) (and its spatially reflected partner) is proved to be purely elastic *i.e.* there is no particle production and a set of individual particle momenta is conserved asymptotically [14,15]. In this case, the S-matrix factorizes in terms of two-particle scattering amplitudes. The two-particle S-matrix has to satisfy "factorization (or Yang-Baxter) equations" and certain bootstrap requirements [16]. Obviously, the models (1.5) all possess more than one non-trivial IM (1.2) with required properties and therefore the corresponding scattering theories certainly exhibit the above properties. In Section 5, we discuss a purely elastic scattering theory (PEST) and describe limitations imposed by local IM on the two-particle S-matrix itself. In particular, we show that if a PEST contains two neutral particles A_1, A_2 of different masses $m_1 \neq m_2$ and the particles A_1 and A_2 appear both as the bound state poles of A_1A_1 and A_2A_2 scattering amplitude, then this PEST is not compatible with more than one non-trivial IM (1.2) unless

$$\frac{m_2}{m_1} = \frac{\sqrt{5} \pm 1}{2}. \tag{1.9}$$

If (1.9) is satisfied, this PEST agrees with IM with spins s having no common divisor with 30, *i.e.* s can take the values

$$s = 1, 7, 11, 13, 17, 19, 23, 29, \ldots. \tag{1.10}$$

In Section 6, we consider separately the model (1.5 b) with $p = 3$. This model describes scaling theory of Ising model at $T = T_c$ but with non-vanishing magnetic field. We prove that the model possesses IM (1.2) with

$$s = 1, 7, 11, 13, 17, 19 \tag{1.11}$$

and conjecture the existence of IM with all s given by (1.10). Motivated by this coincidence and some physical reasoning, we further conjecture that the model $H_3^{(1,2)}$ contain particles A_1, A_2 with the mass ratio (1.9). By solving the bootstrap equations, we construct the corresponding minimal S-matrix. The minimal PEST proves to contain eight particles A_1, A_2, ..., A_8 with the masses

$$\begin{aligned}
&m_1 = m; \qquad m_2 = 2m\cos\frac{\pi}{5}; \qquad m_3 = 2m\cos\frac{\pi}{30} \\
&m_4 = 2m_2\cos\frac{7\pi}{30}; \qquad m_5 = 2m_2\cos\frac{2\pi}{15}; \qquad m_6 = 2m_2\cos\frac{\pi}{30}; \\
&m_7 = 4m_2\cos\frac{\pi}{5}\cos\frac{7\pi}{30}; \qquad m_8 = 4m_2\cos\frac{\pi}{5}\cos\frac{2\pi}{15}.
\end{aligned} \tag{1.12}$$

We finally conjecture that this is the S-matrix for $H_3^{(1,2)}$. The possibilities to check this conjecture are also discussed.

§2. Integrals of motion in CFT

Conformal symmetry of CFT is generated by the left and right components $T = T^{zz}$ and $\bar{T} = T^{\bar{z}\bar{z}}$ of symmetric traceless stress-energy tensor $T^{\mu\nu}$. Assuming spatial reflection symmetry, I will explicitly discuss only the left chiral part of the conformal algebra. The component T satisfies the equation

$$\partial_{\bar{z}} T = 0 \tag{2.1}$$

which makes it possible to define infinite set of Virasoro generators L_n, $n = 0, \pm 1, \pm 2, \ldots$ acting in the space $\mathbf{A}$ of local fields of CFT

$$L_n A(z,\bar{z}) = \oint_z d\zeta\, (\zeta - z)^{n+1} T(\zeta) A(z,\bar{z}), \tag{2.2}$$

where $A \in \mathbf{A}$. Because of (2.1), the operators L_n are integrals of motion. They satisfy the Virasoro algebra (VIR)

$$[\, L_n, L_m \,] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\, \delta_{n+m,0}, \tag{2.3}$$

where the central charge c is a numerical constant, characteristic of CFT. The identity operator $\mathbf{I}$ is the particular field in $\mathbf{A}$, satisfying the equations

$$L_n \mathbf{I} = 0 \qquad \text{for} \qquad n \geq -1. \tag{2.4}$$

The application of the operators L_n with $n \leq -2$ to $\mathbf{I}$ gives rise to infinite set of local fields. For instance,

$$(L_{-2}\, \mathbf{I})(z) = \oint_z d\zeta\, T(\zeta)(\zeta - z)^{-1} = T(z) \tag{2.5}$$

and, more generally,

$$(L_{-n}\, \mathbf{I})(z) = \frac{1}{(n-2)!}\, \partial_z^{n-2} T(z). \tag{2.6}$$

The fields obtained by the successive applications of more than one operators L_{-n} with $n \geq 2$ can be identified with the composite fields made of $T(z)$ and its derivatives. For instance, the field

$$T_4(z) \equiv (\, L_{-2} L_{-2}\, \mathbf{I})(z) = \oint_z d\zeta\, (\zeta - z)^{-1} T(\zeta) T(z) \tag{2.7}$$

is naturally interpreted as regularized square of T: $T_4 =: T^2 :(z)$. Let Λ be the infinite dimensional space spanned by these composite fields (including the identity operator $\mathbf{I}$ itself). By definition, the space Λ is an irreducible Virasoro module with the highest weight equal to zero. The space Λ admits the following decomposition

(2.8) $$\Lambda = \bigoplus_{s=0}^{\infty} \Lambda_s$$

in terms of eigen-spaces of the operator L_0

(2.9) $$L_0 \Lambda_s = s\, \Lambda_s; \qquad \bar{L}_0 \Lambda_s = 0.$$

All the fields belong to Λ_s have the conformal dimensions $(s, 0)$ and therefore the spin s. The fields constituing Λ are all analytic, *i.e.* they satisfy the equations like (2.1):

(2.10) $$\partial_{\bar{z}} \Lambda = 0.$$

Every field $T_s^{(\alpha)} \in \Lambda_s$ gives rise to an infinite set of operators

(2.11) $$\oint_z d\zeta\, T_s^{(\alpha)}(\zeta)(\zeta - z)^{n+s-1}; \qquad n = 0, \pm 1, \pm 2, \ldots$$

which are integrals of motion. Clearly, these operators are not all algebraically independent, rather all of them can be expressed in terms of Virasoro generators. For instance,

(2.12) $$\oint_z d\zeta\, T_4(\zeta)(\zeta - z)^{n+3} = \sum_{m=-\infty}^{\infty} :L_m L_{n-m}: + \rho_n L_n,$$

where $T_4 \equiv (\, L_{-2} L_{-2} \mathbf{I}\,)$,

(2.13) $$\rho_{2m} = (m+1)(m+2); \qquad \rho_{2m-1} = (m+1)^2,$$

and the symbol $: \, :$ in the r.h.s. of (2.12) denotes standard normal ordering; the operators with bigger values of index are placed to the right. In fact, the operators (2.11) are not even linearly independent. This is because there are some fields in Λ which are total ∂_z derivatives: the fields (2.6) with $n > 2$ are the examples. In order to separate linearly independent set, one can take the factor space $\hat{\Lambda} = \Lambda / L_{-1} \Lambda$ instead of Λ; here $L_{-1}\Lambda \subset \Lambda$ is the subspace in Λ constituted by the total derivatives. Like Λ itself, the space $\hat{\Lambda}$ enjoys the following decomposition

(2.14) $$\hat{\Lambda} = \bigoplus_{s=0}^{\infty} \hat{\Lambda}_s; \qquad L_0 \hat{\Lambda}_s = s\, \hat{\Lambda}_s.$$

I shall denote basic vectors of $\hat{\Lambda}_s$ as $T_s^{(\kappa)}$. Since Λ is the irreducible highest weight Virasoro module, the dimensionalities of $\hat{\Lambda}_s$ can be deduced from corresponding character formula

$$\sum_{s=0}^{\infty} q^s \dim(\hat{\Lambda}_s) = (1-q)\,\chi_0(q) + q; \tag{2.15}$$

here $\chi_0(q)$ is the character for this module. In particular, if c does not belong to the set

$$c(p,p') = 1 - \frac{6(p-p')^2}{pp'}, \tag{2.16}$$

where p, p' run over all relatively prime positive integers ((2.16) correspond to "strongly degenerate" case [1]), the character formula takes the form

$$\chi_0(q) = (1-q)\prod_{m=2}^{\infty}(1-q^m)^{-1} + q. \tag{2.17}$$

For $c \in \{c(p,p')\}$ the character formula can be found in [1], but even for $c = c(p,p')$ (2.17) gives correct dimensionalities of Λ_s with $s < (p-1)(p'-1)$. The first few dimensionalities together with the corresponding basic vectors are listed in the Table 1.

s	0	1	2	3	4	5	6
$\dim(\hat{\Lambda}_s)$	1	0	1	0	1	0	2
Basic Vectors	$\mathbf{I}$	–	$T_2 = L_{-2}\,\mathbf{I}$	–	$T_4 = L_{-2}^{\,2}\,\mathbf{I}$	–	$T_6^{(1)} = L_{-2}^{\,3}\,\mathbf{I}$ $T_6^{(2)} = L_{-3}^{\,2}\,\mathbf{I}$

Table 1. Dimensionalities and basic vectors of $\hat{\Lambda}_s$ with $s \leq 6$.

The operators

$$\mathcal{L}^{(\kappa)}_{s,n} A(z,\bar{z}) = \oint_z d\zeta\, T^{(\kappa)}_s(\zeta)(\zeta - z)^{n+s-1} A(z,\bar{z}) \tag{2.18}$$

with $n = 0, \pm 1, \pm 2, \ldots$ constitute an infinite set of linearly independent IM in any CFT. In the subsequent sections, I will show how some of these operators (being suitably modified) can survive as IM if the CFT is perturbed by particular relevant field.

In fact, the known models of CFT usually possess an extended symmetry (like superconformal, Kac-Moody or W-algebra symmetry) which is bigger than (and includes) the conformal symmetry itself. This symmetry is generated by some extra basic analytic fields, independent of $T(z)$. The space of analytic fields in this case is bigger than Λ and hence there are more IM than it is described above. Some of these extra IM can also survive under appropriate perturbations (see [17]). Here I will not consider IM of this type and therefore I do not prepare corresponding notations.

§3. Equations of motion in perturbed field theory

The space $\mathbf{A}$ of the local fields of CFT splits into a sum of irreducible highest weight representation spaces of VIR $\times$ $\overline{\text{VIR}}$

$$\mathbf{A} = \bigoplus_\alpha [\varPhi_\alpha]; \tag{3.1}$$

$[\varPhi_\alpha]$ is known as "conformal class" of certain primary field satisfying the equations

$$\begin{aligned} &L_n \varPhi_\alpha = \bar{L}_n \varPhi_\alpha = 0 \qquad \text{for} \qquad n > 0; \\ &L_0 \varPhi_\alpha = \varDelta_\alpha \varPhi_\alpha; \qquad \bar{L}_0 \varPhi_\alpha = \bar{\varDelta}_\alpha \varPhi_\alpha, \end{aligned} \tag{3.2}$$

where $(\varDelta_\alpha,\ \bar{\varDelta}_\alpha)$ are "left" and "right" conformal dimensions of $\varPhi_\alpha$. For the sake of simplicity, I will assume that the CFT under consideration is unitary [13] (although most of the arguments below apply to general case). In a unitary CFT, all the dimensions are non-negative: $\varDelta_\alpha, \bar{\varDelta}_\alpha \geq 0$ and (except for some specific CFT like Gaussian model) it is only the identity field $\mathbf{I}$ which has both left and right dimensions equal to zero. For a spinless primary field $\varDelta_\alpha = \bar{\varDelta}_\alpha$; in this case $d_\alpha = 2\varDelta_\alpha$ is the anomalous scale dimension of $\varPhi$. The spinless primary field $\varPhi_\alpha$ is called relevant provided $\varDelta_\alpha < 1$; I will use index r to label the relevant fields. In the unitary CFT (again except for those specific cases), there are only finitely many relevant fields.

The CFT is associated with the fixed point of renormalization group (RG) [6]. The space $\bigoplus_r \Phi_r$ can be interpreted as a tangent space to corresponding unstable manifold taken just at the fixed point. A shift along the unstable manifold is described by the appropriate perturbation

$$H_\lambda = H_{\mathrm{CFT}} + \sum_r \lambda_r \int \Phi_r(x) d^2x, \tag{3.3}$$

where H_{CFT} is the fixed-point action (hamiltonian in statistical mechanics) and λ_r are the "coupling constants" showing the direction of the shift. The parameters λ_r are the scalars with respect to space rotations but λ_r carry the scale dimension $\lambda_r \sim (m^2)^{1-\Delta_r}$. This can be expressed by ascribing the left and right dimensions $(1-\Delta_r,\ 1-\Delta_r)$ to λ_r. RG transformations move (3.3) away from the fixed point. Therefore, the FT (3.3) which approaches the original CFT in the ultraviolet limit is expected to have completely different behaviour at large distances.

In principle, (3.3) can be used to compute the correlation functions in a shifted FT perturbatively, as a power series in λ_r. However, one has to sum up all orders of the perturbation theory (PT) to get exact information about the long-distance behaviour. In addition, this PT is infrared divergent. We notice however that if one analyze local relations between the fields (*i.e.* equations of motion) only finitely many orders of the PT can contribute. Before turning to this analysis, some remarks about definition of the fields in perturbed FT (3.3) are worth making.

The FT is characterized by infinite dimensional vector space $\mathbf{A}$ of local fields (which admits a countable basis $A_j, j = 0, 1, 2, \ldots$) and the totality of correlation functions. In CFT, the space $\mathbf{A}$ have the structure (3.1). In general the FT treated as perturbed CFT, the fields usually need infinite renormalizations in order to compensate ultraviolet divergencies of PT and to make the correlation functions finite. In the case of "superrenormalizable" PT which arises in (3.3), it is enough to add to each field A_j only finitely many terms containing the fields of less scale dimensions (with cut off-dependent coefficients). This does not change the structure of the space $\mathbf{A}$. Therefore I assume that the space $\mathbf{A}$ in the perturbed FT (3.3) has the same structure (3.1) and I keep the same notation for the fields as in the original CFT. Moreover, the fields Φ_α (and their "descendents") in the perturbed theory (3.3) have exactly the same spins and scale dimensions as in CFT. Of course, the perturbed FT is not scale invariant since (3.3) contains dimensionful constants λ_r.

In what follows I will restrict attention to the cases where only one relevant primary field Φ is taken as perturbation, *i.e.* all λ_r except one equal to zero and (3.3) reduces to (1.1). This makes perfect sence if Φ is the most relevant field in **A** or it is the most relevant field in some subalgebra $\mathbf{A}_1$ of the operator product algebra **A**. In fact, all the examples elaborated in the subsequent sections are of this kind. But here this restriction is made mostly for the sake of notational simplicity. Generalization of the arguments presented below for the case of many couplings in (3.3) is not difficult.

Let us consider the space $\hat{\Lambda}$ in perturbed theory (1.1). At $\lambda \neq 0$ the fields $T_s^{(\kappa)} \in \Lambda_s$ do not of course satisfy (2.10). Rather the $\partial_{\bar{z}}$ derivative of $T_s^{(\kappa)}$ has the form

$$\partial_{\bar{z}} T_s^{(\kappa)} = \lambda\, R_{s-1}^{(\kappa)1} + \cdots + \lambda^n R_{s-1}^{(\kappa)n} + \cdots \tag{3.4}$$

where $R_{s-1}^{(\kappa)n}$ are some local fields belonging to $\mathbf{A}$(or $\mathbf{A}_1$). The dimensions of each term in the r.h.s. of (3.4) must be $(s,\ 1)$ to agree with those in l.h.s. Hence the dimensions of the field $R_{s-1}^{(\kappa)n}$ are $(s - n\epsilon,\ 1 - n\epsilon)$ where $\epsilon = 1 - \Delta$ and (Δ, Δ) are the dimensions of Φ. We see that the right dimension of $R_{s-1}^{(\kappa)n}$ must be negative starting from some sufficiently large n. There are no fields with the negative right dimension in **A**. Consequently, the series (3.4) is actually finite. Moreover, in most cases only the first order term $\lambda R_{s-1}^{(\kappa)1}$ in (3.4) does not vanish. Indeed, the term $\lambda^n R_{s-1}^{(\kappa)n}$ with $n > 1$ must vanish unless the relation

$$1 - n\epsilon = \Delta_r \tag{3.5}$$

is satisfied for some relevant dimension Δ_r, since otherwise one cannot find a field with appropriate dimension in **A**. It can be shown that if (3.5) is satisfied for some $n > 0$ and some $\Delta_r \neq 0$, the action (1.1) is not well defined without adding a counterterm proportional to Φ_r, and one has to analyze more general action (3.3). Let us assume now that Φ is the most relevant field in **A**or in some subalgebra $\mathbf{A}_1 \subset \mathbf{A}$. Then the equation (3.5) with $n > 0$ can be satisfied only if ϵ is an inverse integer *i.e.*

$$\Delta = 1 - \frac{1}{N}; \qquad N > 1 \tag{3.6}$$

(one takes $\Delta_r = 0$). If (3.6) holds, there can be N-th order term in (3.4) with $R_{s-1}^{(\kappa)N} \in \Lambda_{s-1}$. We shall meet this situation in Section 4. Here I assume that (3.6) is not the case and hence

$$\partial_{\bar{z}} T_s^{(\kappa)} = \lambda\, R_{s-1}^{(\kappa)}. \tag{3.7}$$

Now we concentrate attention on this first order contribution.

Let Φ be the irreducible highest weight module over the left Virasoro algebra with the highest weight vector $\varPhi$. This is the space spanned by the vectors

$$L_{-n_1}L_{-n_2}\cdots L_{-n_N}\varPhi \tag{3.8}$$

with $N \geq 0$ and $n_1 \geq n_2 \geq \cdots \geq n_N > 0$. In fact, the conformal class $[\varPhi]$ is the direct product $\Phi \otimes \bar{\Phi}$. Like (2.8), the space Φ enjoys the decomposition

$$\Phi = \bigoplus_{s=0}^{\infty} \Phi_s; \qquad L_0\Phi_s = (\varDelta + s)\Phi_s \quad , \quad \bar{L}_0\Phi_s = \varDelta\Phi_s. \tag{3.9}$$

The dimensional counting explained above shows that $R^{(\kappa)}_{s-1} \in \Phi_{s-1}$. Therefore, the symbol $\partial_{\bar{z}}$ in (3.7) can be considered as linear operator

$$\partial_{\bar{z}}: \qquad \hat{\Lambda}_s \longrightarrow \Phi_{s-1}. \tag{3.10}$$

In fact, the action of $\partial_{\bar{z}}$ extends to the whole space Λ and (3.10) is its (arbitrary) restriction to the factor space $\hat{\Lambda} = \Lambda/L_{-1}\Lambda$. As we shall see later, the operator $\partial_{\bar{z}}$ commutes with L_{-1}. Therefore, the relation (3.10) carries all significant information about $\partial_{\bar{z}}$. Let us show how to compute this operator.

The first order correction to any correlation function involving the field $T^{(\kappa)}_s$ is given by the integral

$$\int d\zeta\, d\bar{\zeta}\, \langle \varPhi(\zeta,\bar{\zeta})\, T^{(\kappa)}_s(z)\cdots\rangle_0 \tag{3.11}$$

where $\langle\cdots\rangle_0$ is the correlation function of unperturbed CFT. Obviously, the contribution to $\partial_{\bar{z}}T^{(\kappa)}_s$ can come only from the vicinity of the singular point $(\zeta,\bar{\zeta}) \to (z,\bar{z})$, where we can use the operator product expansion of CFT

$$T^{(\kappa)}_s(z)\,\varPhi(\zeta,\bar{\zeta}) = \sum_{n=0}^{\infty}(z-\zeta)^{n-s}(\mathcal{L}^{(\kappa)}_{s,-n}\varPhi)(\zeta,\bar{\zeta}); \tag{3.12}$$

here $\mathcal{L}^{(\kappa)}_{s,-n}\varPhi$ are the certain local fields. It is possible to show using (3.12) and the equations

$$\partial_{\bar{z}}(\zeta - z)^{-m-1} = -2\pi i \frac{1}{m!}\partial_z^{m}\delta^{(2)}(z-\zeta) \tag{3.13}$$

that the r.h.s. in (3.7) reduces to the integral

$$\partial_{\bar z} T_s^{(\kappa)}(z,\bar z) = \lambda \oint_z \Phi(\zeta,\bar z)\, T_s^{(\kappa)}(z) \frac{d\zeta}{2\pi i} \tag{3.14}$$

taken over small closed contour surrounding z. Here the operator product expansion of unperturbed CFT is implied in the r.h.s. The contour in (3.14) is closed because in the CFT the field $T_s^{(\kappa)}$ is analytic and local with respect to Φ. Recalling the fact that the contour integral in (3.14) is equivalent to commutator (if one implies radial quantization of CFT) [1], we can rewrite (3.14) in the form

$$\partial_{\bar z} T_s^{(\kappa)}(z,\bar z) = \left[T_s^{(\kappa)}(z,\bar z),\ H_{\mathrm{int}}(\bar z) \right], \tag{3.15}$$

where

$$H_{\mathrm{int}}(\bar z) = \lambda \int d\zeta\, \Phi(\zeta,\bar z), \tag{3.16}$$

which is more conventional in the standard PT.

The expression (3.14) makes evident some general properties of the operator $\partial_{\bar z}$. In particular, the equation

$$\partial_{\bar z} L_{-1}\Lambda = L_{-1}\partial_{\bar z}\Lambda \tag{3.17}$$

immediately follows from (3.14) if one takes into account the commutation relation [1]

$$[\, L_{-1}, \Phi(\zeta,\bar\zeta)\,] = \partial_\zeta \Phi(\zeta,\bar\zeta), \tag{3.18}$$

where the operator L_{-1} is defined with respect to the point z as in (2.2).

The formula (3.14) is not too explicit. It is rather efficient, however for computing particular matrix elements of the operator $\partial_{\bar z}$. Let us introduce the set of operators $D_n : \Lambda \to \Phi, n = 0, \pm 1, \pm 2, \ldots$ defined as

$$D_n\Lambda(z,\bar z) = \oint_z \Phi(\zeta,\bar z)(\zeta - z)^n \Lambda(z) \frac{d\zeta}{2\pi i} \tag{3.19}$$

(again the operator product expansion of the original CFT is implied in the r.h.s.). Evidently

$$\partial_{\bar z} = D_0. \tag{3.20}$$

Using the equations

$$[\, L_n,\ \Phi(\zeta,\bar\zeta)\,] = \left\{ (\zeta - z)^{n+1}\partial_\zeta + \Delta(n+1)(\zeta - z)^n \right\} \Phi(\zeta,\bar\zeta), \tag{3.21}$$

which generalize (3.18), one can easily prove the commutation relations

$$[L_n, D_m] = -\big((1-\Delta)(n+1)+m\big)D_{n+m}. \tag{3.22}$$

These relations together with the obvious equations

$$D_{-n-1}\,\mathbf{I} = \frac{1}{n!}L_{-1}{}^{n}\Phi(z,\bar z) \tag{3.23}$$

makes it easy to compute the action of $\partial_{\bar z}$ on particular fields in Table 1. For example

$$\partial_{\bar z}T = \lambda D_0 L_{-2}\mathbf{I} = \lambda(\Delta-1)D_{-2}\mathbf{I} = \lambda(\Delta-1)L_{-1}\Phi. \tag{3.24}$$

This result can be rewritten in the form

$$\partial_{\bar z}T = \partial_z\Theta, \tag{3.25}$$

where

$$\Theta = \lambda(\Delta-1)\Phi, \tag{3.26}$$

which is nothing other than continuity equation $\partial_\mu T^{\mu\nu}=0$ for the stress energy tensor in the FT (1.1). The equation (3.26) expresses the trace component $\Theta = -T^{z\bar z}$ of $T^{\mu\nu}$ in terms of the perturbation Φ. The continuity equation (3.25) ensures conservation of the momentum ((1.2) with $s=1$) in any FT.

One can notice that according to (3.26), Θ formally vanish for $\Delta=1$. Let us stress however that our above analysis is strongly based on the assumption $\Delta<1$ and it does not apply to a "marginal"perturbation with $\Delta=1$. Eq. (3.5) with $\Delta_r=\Delta=1$ is satisfied by arbitrary n. This means that not only the first but an arbitrary order in λ can in principle contribute to the coefficient β in $\Theta=\beta\Phi$; in fact the Eq. (3.26) shows that the first order contribution to β does vanish for $\Delta=1$.

Let us compute $\partial_{\bar z}T_4$, where $T_4 = L_{-2}{}^2\,\mathbf{I}$, as another exercise in our technique. We obtain

$$\begin{aligned}\partial_{\bar z}T_4 =&\lambda D_0L_{-2}L_{-2}\mathbf{I} = \lambda(\Delta-1)(D_{-2}L_{-2}+L_{-2}D_{-2})\mathbf{I}\\ =&\lambda(\Delta-1)(2L_{-2}L_{-1}+\frac{\Delta-3}{6}L_{-1}{}^3)\Phi.\end{aligned} \tag{3.27}$$

We see that in general case of Φ, the r.h.s. of (3.27) is not a total ∂_z derivative. Contrary to (3.24), the Eq. (3.27) in the general case does not imply conservation of anything. The same result holds for higher-spin fields in $\hat\Lambda$. In the next section, I will show however that for particular degenerate fields Φ some of the fields $\partial_{\bar z}T_s^{(\kappa)}$ do reduce to the total ∂_z derivatives and give rise to non-trivial IM in the FT (1.1).

§4. Degenerate fields and integrals of motion

Let us consider the factor space $\hat{\Phi}_s = \Phi_s / L_{-1}\Phi_{s-1}$; $s = 0, 1, 2, \ldots$. Like (2.15), the dimensionalities of these factor spaces can be obtained from the generating function

$$\sum_{s=0}^{\infty} q^{s+\Delta} \dim(\hat{\Phi}_s) = (1-q)\,\chi_\Delta(q), \tag{4.1}$$

where $\chi_\Delta(q)$ is the character of the irreducible representation of VIR with the highest weight Δ. Let Π_s be the projector $\Phi_s \to \hat{\Phi}_s$. We can introduce the operators

$$B_s = \Pi_s D_{0,s}; \qquad \hat{\Lambda}_s \longrightarrow \hat{\Phi}_{s-1}, \tag{4.2}$$

where $D_{0,s}$ is the operator D_0 (defined by (3.19)) restricted to $\hat{\Lambda}_s$. By the definition, every field $T_{s+1} \in \hat{\Lambda}_{s+1}$ satisfying $B_{s+1}T_{s+1} = 0$ enjoys the relation

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}, \tag{4.3}$$

where Θ_{s-1} is some local field belonging to Φ_{s-1}. So, the IM (1.2) in the FT (1.1) appear every time the operator B_{s+1} possesses non-vanishing kernel. As we have seen in previous section, for general field Φ taken as the perturbation, $\ker B_2 = T$ and $\ker B_s = 0$ for $s \neq 2$.

The situation is better if Φ is a certain degenerate field [1]. The degenerate field $\Phi_{(n,m)}$ are labelled by two positive integers $n, m = 1, 2, \ldots$; their dimensions $\Delta = \Delta_{(n,m)}$ are given by Kac formula [1]. From now on, I assume $c < 1$, in this region the dimensions $\Delta_{(n,m)}$ are real. The field $\Phi_{(n,m)}$ satisfies the null-vector equation of level nm. We shall see soon that in four cases, $\Phi = \Phi_{(1,2)}, \Phi_{(2,1)}, \Phi_{(1,3)}$ and $\Phi_{(3,1)}$, some of the operators B_s with $s > 2$ exhibit non-vanishing kernels. However the field $\Phi_{(3,1)}$ is not relevant for $c < 1$ and the implication of the above statement for (1.1) with $\Phi = \Phi_{(3,1)}$ is not clear. Therefore in what follows I explicitly mention only the first three cases. The fields $\Phi_{(1,3)}, \Phi_{(1,2)}$ and $\Phi_{(2,1)}$ are not only relevant for $c < 1$ (in fact, $\Phi_{(2,1)}$ is relevant only at $-2 < c < 1$) but in addition they are the most relevant fields in the appropriate subalgebras of the operator product algebra of the degenerate fields. Therefore the first-order analysis developed in Section 3 directly applies (with few exceptions which are discussed below) to these perturbations.

The field $\Phi = \Phi_{(1,3)}$ satisfies the third level null-vector equation

$$\left(L_{-3} - \frac{2}{\Delta+2} L_{-1}L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}{}^{3}\right)\Phi = 0, \tag{4.4}$$

where $\Delta \equiv \Delta_{(1,3)}$. Let us turn back to the Eq. (3.27). Now we can use (4.4) to rewrite (3.27) in the form

$$\partial_{\bar{z}} T_4 = \partial_z \Theta_2, \tag{4.5}$$

where

$$\Theta_2 = \frac{\lambda(\Delta-1)}{\Delta+2}\left\{ 2\Delta L_{-2} + \frac{(\Delta-2)(\Delta-1)(\Delta+3)}{6(\Delta+1)} L_{-1}{}^{2} \right\}\Phi. \tag{4.6}$$

Notice that here a non-zero kernel of B_4 appeared for a trivial reason: $\dim \hat{\Phi}_3) = 0$ due to (4.4). The similar cause induces non-vanishing kernels of B_s for the next few even s. If c does not belong to the set (2.15), the corresponding character $\chi_{(1,3)}$ is given by

$$\chi_{(1,3)}(q) = \prod_{n=1, n\neq 3}^{\infty} (1-q^n)^{-1}. \tag{4.7}$$

The dimensionalities of $\hat{\Phi}_s$ (computed from (4.1),(4.7)) are compared versus $\dim(\hat{\Lambda}_{s+1})$ in the Table 2 for $s \leq 11$.

s	1	2	3	4	5	6	7	8	9	10	11
$\dim(\hat{\Lambda}_{s+1})$	1	0	1	0	2	0	3	1	4	2	7
$\dim(\hat{\Phi}_s)$	0	1	0	2	1	3	2	5	4	8	7

Table 2. Dimensionalities of the space $\hat{\Lambda}_{s+1}$ and $\hat{\Phi}_s$ (for $\Phi = \Phi_{(1,3)}$) computed from (2.17) and (4.1), (4.7), respectively.

We see that for $s = 1, 3, 5, 7$ $\dim(\hat{\Lambda}_{s+1})$ exceed by one $\dim(\hat{\Phi}_s)$ and hence there are at least one-dimensional kernels for B_{s+1} with these s. We conclude that there are specific fields $T_{s+1} \in \hat{\Lambda}_{s+1}$ with $s = 1, 3, 5, 7$ which satisfy (4.3) with appropriate $\Theta_{s-1} \in \Phi_{s-1}$. More elaborated computation shows that

$$T_6 = T_6^{(1)} - \frac{c+2}{6} T_6^{(2)}; \tag{4.8}$$

where the fields $T_s^{(\kappa)}$ are defined in Table 1; the expressions for Θ_4 and Θ_6 are rather cumbersome. Hence the perturbed FT (1.1) with $\Phi = \Phi_{(1,3)}$ and $c \notin \{c(p,p')\}$ possess (at least) three non-trivial IM (1.2) with $s = 3, 5, 7$. These IM form the commutative set as one can check by brute-force computation. †

For $s > 7$, the simple dimensional counting does not demonstrate existence of non-vanishing kernel of B_{s+1}; at large s, the dimensionalities of $\hat{\Phi}_s$ enumerate many times the dimensionalities of $\hat{\Lambda}_{s+1}$. Nevertheless I conjecture that for $\Phi = \Phi_{(1,3)}$ the operators B_{s+1} possess one dimensional kernel for all odd s (1.8 a). This is certainly true in the limit $c \to \infty$ as one can learn from the theory of KdV equation.

There are two reasons why one has to be careful in applying the above result to the cases $c \in \{c(p,p')\}$. First, for $c = c(p,p')$ the extra null-vector equations [1] modify the structures of the spaces $\hat{\Lambda}_s$ and $\hat{\Phi}_s$ starting from $s = (p-1)(p'-1)$ and $s = (p-1)(p'-3)$, respectively. Obviously, for the most cases of $c(p,p')$ these modifications are not relevant to our above analysis which concerns the levels $s \leq 8$. In fact, if one restricts attention by the unitary CFT with $c = c(p,\, p+1), p = 3, 4, 5, \ldots$ [13], it is only the CFT with $c = c(3,4) = 1/2$ which needs separate analysis to justify our above conclusion about the IM with $s = 5, 7$. We will return to this theory at the end of this section.

Another reason to be careful with $c(p,p')$ theories is that for these cases $\Delta = \Delta_{(1,3)}$ is rational and there is the danger to have the "resonance" condition (3.6) satisfied. This is exactly the case if (and only if) $c = \{c(p, p+1)\}$ with odd $p > 1$ because for $c = c_p$ (1.6)

$$\Delta_{(1,3)} = 1 - \frac{2}{p+1}. \tag{4.9}$$

Thus for the unitary FT (1.5 a) with $p = 2N - 1, N = 2, 3, 4, \ldots$ the contributions $\sim \lambda^N$ can appear in the expressions for $\partial_{\bar{z}} T_s$ and we have

† In fact, the fields (4.8) are exactly the densities for the first few non-trivial IM for quantum KdV equation [18].

to check that these contributions do not spoil the eq's (4.3) with $s = 3, 5, 7$. This check is not difficult. In the general formula

$$\partial_{\bar{z}} T_{s+1}^{(\kappa)} = \lambda\, R_s^{(\kappa)} + \lambda^N R_s^{(\kappa)N}, \tag{4.10}$$

relevant for these cases, the field $R_s^{(\kappa)N}$ belongs to Λ_s. However for $s = 3, 5, 7$ we have $\hat{\Lambda}_s = 0$ and hence $R_s^{(\kappa)N} \in L_{-1}\Lambda_{s-1}$. This means that the equations (4.3) with $s = 3, 5, 7$ are still valid, the only modification specific for (1.5 a) with $p = 2N - 1$ being the additional contribution $\sim \lambda^N$ to Θ_{s-1}.

Let us turn to the FT (1.1) with $\Phi = \Phi_{(1,2)}$. This field satisfies the null-vector equation

$$\Big(L_{-2} - \frac{3}{2(2\Delta+1)} L_{-1}{}^2\Big)\,\Phi = 0 \tag{4.11}$$

where $\Delta = \Delta_{(1,2)}$. Now $\partial_{\bar{z}} T_4$ cannot be reduced to the total ∂_z derivative because of the presense of the term $L_{-2}L_{-1}\Phi$ in (3.27). Therefore in this case there is no IM (1.2) with the spin $s = 3$. Nevertheless the IM with some $s > 3$ do exist. To see that let us again at first assume that $c \notin \{c(p, p')\}$. Then the dimensionalities of $\hat{\Phi}_s$ can be computed from (4.1) with

$$\chi_\Delta(q) = \chi_{(1,2)}(q) = \prod_{n=1, n\neq 2}^{\infty} (1 - q^n)^{-1}. \tag{4.12}$$

The first few dimensionalities $\dim(\hat{\Phi}_s)$ are compared with $\dim(\hat{\Lambda}_{s+1})$ in Table 3.

s	1	2	3	4	5	6	7	8	9	10	11
$\dim(\hat{\Lambda}_{s+1})$	1	0	1	0	2	0	3	1	4	2	7
$\dim(\hat{\Phi}_s)$	0	0	1	1	1	2	2	3	4	5	6

Table 3. Dimensionalities of $\hat{\Lambda}_{s+1}$ versus $\dim(\hat{\Phi}_s)$ for $\Phi = \Phi_{(1,2)}$. $\dim(\hat{\Phi}_s)$ are computed from (4.1), (4.12).

Inspecting this table, one can conclude that the operators B_{s+1} with $s = 1, 5, 7, 11$ possess at least one-dimensional kernel. Let us present explicit expression for $T_6 = \ker B_6$

$$T_6 = T_6^{(1)} + a\, T_6^{(2)}, \tag{4.13}$$

where $T_6^{(\kappa)}$ are given in Table 1 and

$$a = \frac{18}{2\Delta + 1} + \Delta - 2. \tag{4.14}$$

For $c \notin \{c(p,p')\}$ the dimensionalities of $\hat{\Lambda}_{s+1}$ are less than $\dim(\hat{\Phi}_s)$ starting from $s = 12$. Nevertheless I conjecture that in this case there are one-dimensional $\ker B_{s+1}$ for all integers s having no common divisor with 6. The above results of course apply to the case $\Phi = \Phi_{(2,1)}$ as well.

As in the previous example, if $c = c(p,p')$ we have to think about additional null-vector equations satisfied by $\mathbf{I}$ and Φ about the resonance condition (3.6). Again I will comment only the unitary models $c = c(p, p+1)$, $p = 3, 4, 5, \ldots$. The resonance condition is never satisfied for the theories (1.5 b,c) with the only exception $H_3^{(2,1)}$ which coincide with $H_3^{(1,3)}$. So for $H_p^{(1,2)}$ and for $H_p^{(2,1)}$ with $p \geq 4$ we need not to care about the higher-order contributions to $\partial_{\bar{z}} T_s^{(\kappa)}$. As for the additional null-vectors, these can modify the above analysis of B_s with $s \leq 12$ only for $H_p^{(1,2)}$ and $H_p^{(2,1)}$ with $p = 3, 4$. In fact the dimensional counting (analogous to that performed above) in these cases shows even more non-trivial IM. The model $H_p^{(1,2)}$ with $p = 3$ is considered separately in the Section 6. Here I will briefly discuss only the very specific features of the FT $H_3^{(2,1)}$ (which is equivalent to $H_3^{(1,3)}$).

For $c = c_3 = 1/2$ the field $\Phi_{(2,1)}$ coincides with $\Phi_{(1,3)}$. Therefore the field Φ in $H_3^{(2,1)}$ satisfies the null-vector equations (4.4) and (4.11) both. So, the equation (3.27) reduces to

$$\partial_{\bar{z}} T_4 = -\frac{5}{24}\, \lambda\, \partial_z^{\,3} \Phi + b\, \lambda^2 \partial_z T_2, \tag{4.15}$$

where we took into account that $\Delta = \Delta_{(2,1)} = \Delta_{(1,3)} = 1/2$ satisfies the resonance condition (3.6) with $N = 2; T_2 \equiv T$ and b is a constant whose exact value is not important here. By the shift $T_4 \to T_4 + \frac{5}{12}\, \partial_z^{\,2} T_2$ and appropriate renormalization of T_4 we can transform (4.15) to the form

$$\partial_{\bar{z}} T_4 = \partial_z T_2. \tag{4.16}$$

The similar phenomenon takes place for higher s. It can be shown that in $H_3^{(2,1)}$ there is the series of fields $T_{2n} \in \Lambda_{2n}$, $n = 1, 2, 3, 4, \ldots$ satisfying the chain-type equation

$$\partial_{\bar{z}} T_{2n+2} = \partial_z T_{2n} \tag{4.17}$$

with $n = 0, 1, 2, \ldots$, where by definition $T_0 = \frac{1}{2}\lambda\Phi$. Since there is analogous series $\bar{T}_{2n} \in \bar{\Lambda}_{2n}$, we can extend the chain (4.17) for all integer $n = 0, \pm1, \pm2, \ldots$ by identifying $T_{-2n} \equiv \bar{T}_{2n}$. The equations (4.17) can be considered as the particular form of (1.2). But besides the standard set of local IM

$$P_{2n+1} = \oint [\, T_{2n+2} dz + T_{2n} d\bar{z} \,] \tag{4.18}$$

the Eq. (4.17) lead to enormous set of additional IM. The equations

$$\partial_{\bar{z}} X^{(0)}_{2n+1} = \partial_z X^{(0)}_{2n-1}, \tag{4.19}$$

where

$$X^{(0)}_{2n+1} = z\, T_{2n+2} + \bar{z}\, T_{2n}, \tag{4.20}$$

are simple consequences of (4.17). By applying repeatedly the recurrent relation

$$X^{(k+1)}_{2n+k} = z\, X^{(k)}_{2n+k+1} + \bar{z}\, X^{(k)}_{2n+k-1} \tag{4.21}$$

0 one derives the set of fields $X^{(k)}_{2n+k+1}$; $k = -1, 0, 1, 2, \ldots$; $n = 0, \pm1, \pm2, \ldots$, which satisfy $\partial_{\bar{z}} X^{(k)}_{2n+k+1} = \partial_z X^{(k)}_{2n+k-1}$ and hence give rise to the large set of IM

$$P^{(k)}_{2n+k} = \oint [\, X^{(k)}_{2n+k+1} dz + X^{(k)}_{2n+k-1} d\bar{z} \,], \tag{4.22}$$

where $P^{(-1)}_{2n-1}$ (which coincides with (4.18)) forms the commutative subset.

The appearence of this huge amount of IM in the FT $H_3^{(2,1)}$ is not very surprising. This FT describes the scaling limit of the Ising model away from the critical point [1] which is well known to be the theory of free massive Majorana fermions. The fields T_{2n} in (4.17) are simply the local fermion bilinears

$$\begin{aligned} T_{2n} &= (i\lambda)^{-2n+1} : \partial_z^{\,n-1}\psi\, \partial_z^{\,n}\psi :, \qquad n = 1, 2, 3, \ldots \\ T_0 &= \Theta = i\lambda : \bar{\psi}\psi :, \end{aligned} \tag{4.23}$$

and similarly for $n < 0$; here $\psi, \bar{\psi}$ are the chiral components of the free fermion fields. The free field structure of the Ising model FT makes it possible to derive non-linear differential equations for its correlation functions [19,20]. The model $H_3^{(1,2)}$ describes the scaling limit of the Ising model at $T = T_c$ but with non-zero magnetic field. This FT also exhibits non-trivial local IM but it does not reduce to a free field theory. It will be discussed in Section 6.

As we have shown above, all the unitary FT (1.5) possess a number of non-trivial IM and we conjectured that these FT are integrable. How one can use these IM to analyze the physical contents of these FT ? The subsequent sections are devoted to this problem. Here I only make some remarks about the qualitative properties one expect of these FT in the large-distance region.

As is shown in [21,22], the FT (1.5 a) with $\lambda > 0$ and with sufficiently large p corresponds to RG trajectory which connect the fixed point H_p with the fixed point H_{p-1}. In other words, this FT whose ultraviolet behaviour is governed by CFT H_p exhibits also conformally invariant infrared asymptotic described by CFT H_{p-1}. I expect this statement to be correct for all $p \geq 4$ (the FT $H_3^{(1,3)} = H_3^{(2,1)}$ which was discussed above is the exception). The conformal symmetry is violated at some "intermediate"distances $\sim \lambda^{-(p+1)/2}$. One could say that the long-range correlation in this FT are caused by "massless particles" whereas something like "massive excitations" are responsible for the conformal symmetry breaking at $R \sim \lambda^{-(p+1)/2}$. However there is no good concept of massless particles in two dimensions and in particular it is not clear whether a kind of scattering theory of the massive objects interacting *via* the "massless particles" can be defined. I consider the finding of a good physical principle for characterlization of the states in this situation as important open problem.

For the FT (1.5 a) with $\lambda < 0$ and for (1.5 c) there are no reasons to expect any non-trivial zeros of the β-function and therefore these FTs apparently develop finite correlation length $R_c \sim |\lambda|^{-1/\epsilon}$ and the correlations fall down exponentially at $R \gg R_c$. We expect these FTs to contain only massive particles and hence to be characterlized by the S-matrix. Some properties of the scattering theory governed by non-trivial IM are discussed in the next section.

§5. Integrals of motion and scattering theory

Let us consider relativistic scattering theory containing n sorts of particles $A_a, a = 1, 2, \ldots, n$ with the masses m_a, I will use the symbol

$A_a(p)$ to denote the particle A_a having the 2-momentum p^μ. The 2-momentum satisfies the mass-shell condition

$$p_\mu p^\mu = p\bar{p} = m^2, \tag{5.1}$$

where

$$p = p^0 + p^1; \qquad \bar{p} = p^0 - p^1 \tag{5.2}$$

are the light-cone components of p^μ. The states

$$\mid A_{a_1}(p_1)A_{a_2}(p_2)\cdots A_{a_N}(p_N)\ \rangle_{\mathrm{in(out)}} \tag{5.3}$$

form the basis of the asymptotic in(out)-states which is assumed to be complete in a local FT. S-matrix is the unitary transformation connecting the "in" and "out" basis.

Let us assume that the underlying FT possesses a number of nontrivial commutative IM P_s; $s = s_1, s_2, \ldots$, where the label s indicates the spin of P_s (see (1.4)), which are the integrals of local densities (1.2). The operators P_s are diagonalized by the asymptotic states (5.3)

(5.4)
$$\begin{aligned} &P_s \mid A_{a_1}(p_1)A_{a_2}(p_2)\cdots A_{a_N}(p_N)\ \rangle_{\mathrm{in(out)}} \\ &\qquad = \left(\omega_s^{a_1}(p_1) + \cdots + \omega_s^{a_N}(p_N)\right) \mid A_{a_1}(p_1)\cdots A_{a_N}(p_N)\ \rangle_{\mathrm{in(out)}}. \end{aligned}$$

Here $\omega_s^a(p)$ are the eigen values of P_s for the one-particle states

$$P_s \mid A_a(p)\,\rangle = \omega_s^a(p) \mid A_a(p)\,\rangle. \tag{5.5}$$

It is possible to show that these functions enjoy the symmetry

$$-\omega_s^a(p) = \pm\omega_s^a(-p), \tag{5.6}$$

where the sign in the r.h.s. corresponds to the **C**-parity of P_s; here **C** is the charge conjugation. The Lorentz transformation properties (1.4) dictate the following general form of $\omega_s^a(p)$

$$\omega_s^a(p) = \kappa_s^a p^s, \tag{5.7}$$

where κ_s^p are constants and p is defined in (5.2). The operator P_1 is assumed to be the left-cone component of momentum;

$$\kappa_1^a = 1. \tag{5.8}$$

Comparing (5.7) with (5.6), one conclude that **C**-even (**C**-odd) IM P_s have odd (even) spins s. Being the integrals of motion, the operators

P_s commute with S-matrix. It can be shown [14,15] that under these assumptions, the scattering theory is "purely elastic" *i.e.* the number of particles and the set of their individual momenta are conserved asymptotically. In addition, the N-particle S-matrix factorizes in terms of the two-particle scattering amplitude [23]. The two-particle S-matrix must satisfy the "factorization equations" (also known as the Yang-Baxter relations) [16].

We also assume here that all the particles A_a have different masses, *i.e.* $m_a \neq m_b$ if $a \neq b$. This assumption in particular implies that all the particles A_a are neutral: $\bar{A}_a = A_a$, where $\bar{A}$ denotes the corresponding anti-particle. In this case, the S-matrix is diagonal in the basis (5.3)

(5.9)
$$| A_{a_1}(p_1) \cdots A_{a_N}(p_n) \rangle_{\text{in}} = S_{a_1 \ldots a_N}(p_1, \ldots, p_N) | A_{a_1}(p_1) \cdots A_{a_N}(p_N) \rangle_{\text{out}}$$

whereas the N-particle S-matrix element in (5.9) is simply the product of $N(N-1)/2$ two-particle ones

$$S_{a_1 \ldots a_N}(p_1, \ldots p_N) = \prod_{i<j} S_{a_i a_j}(p_i, p_j). \tag{5.10}$$

The two-particle S-matrix is defined by

$$| A_a(p_1) A_b(p_2) \rangle_{\text{in}} = S_{ab}(p_1, p_2) \, | A_a(p_1) A_b(p_2) \rangle_{\text{out}}. \tag{5.11}$$

The factorization equation [16] are trivially satisfied in this case. In the theory enjoying the spatial-reflection symmetry, the amplitudes $S_{ab}(p_1, p_2)$ satisfy

$$S_{ab}(p_1, p_2) = S_{ba}(p_1, p_2). \tag{5.12}$$

The amplitudes $S_{ab}(p_1, p_2)$ are, in fact, the functions of one variable $\theta = \theta_{12} = \theta_1 - \theta_2$, where θ_k are the rapidities

$$p_k = m_k e^{\theta_k}; \qquad \bar{p}_k = m_k e^{-\theta_k}, \tag{5.13}$$

which can be used instead of momenta p_k^μ to characterize the asymptotic particles. I shall often use the symbol $A_a(\theta)$ for the particle A_a having the rapidity θ. All the two-particle amplitudes $S_{ab}(\theta)$ are meromorphic functions of θ, real at $\operatorname{Re}\theta = 0$. Since we assumed the particles to be neutral, the amplitudes S_{ab} enjoy the crossing-symmetry condition in the form

$$S_{ab}(\theta) = S_{ba}(i\pi - \theta). \tag{5.14}$$

Because of the symmetry (5.12), the order of the indices ab in (5.14) is, in fact, irrelevant. The unitarity condition reduces to the functional equations

$$S_{ab}(\theta)S_{ab}(-\theta) = 1. \tag{5.15}$$

Due to (5.14) and (5.15), the amplitudes $S_{ab}(\theta)$ are $2\pi i$-periodic functions which are completely determined by the positions of their zeros and poles in the "physical strip" $0 \leq \operatorname{Im}\theta \leq \pi$. The poles in this strip are located at $\operatorname{Re}\theta = 0$. The simple poles correspond to "bound-state" particles either of direct channel of $A_a A_b$ scattering or of the cross channel, depending on the sign of the residue.† The basic bootstrap requirement is that these "bound states" must belong to the same set of particles $A_1, A_2, \ldots, A_n$. The pole of $S_{ab}(\theta)$ located at $\theta = iu^c_{ab}$ and having positive residue represents the particle A_c provided u^c_{ab} is related to the particle masses by the equation

$$m_c{}^2 - m_a{}^2 - m_b{}^2 = 2\, m_a m_b \cos u^c_{ab}. \tag{5.16}$$

In this case, according to (5.14), there is the pole of $S_{ab}(\theta)$ at $\theta = i\bar{u}^c_{ab}$; where $\bar{u}^c_{ab} \equiv \pi - u^c_{ab}$ with the negative residue, which represent the same particle A_c as the "bound state" of the cross-channel.

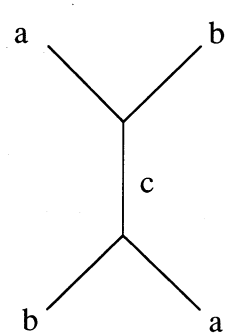

Fig. 1. The diagram associated with the pole singularity (5.17) in $S_{ab}(\theta)$.

The singularity

$$S_{ab}(\theta) \sim \frac{R^c_{ab}}{\theta - iu^c_{ab}} \qquad \text{as} \qquad \theta \to iu^c_{ab} \tag{5.17}$$

†The amplitudes can also possess double or multiple poles associated with the multiple rescattering processes [24].

corresponds to the diagram in Fig. 1, which shows that the residue R^c_{ab} can be represented as the square

$$R^c_{ab} = f_{abc} f_{abc} \tag{5.18}$$

of the "fusion constant" f_{abc} which must be completely symmetric function of the indices a, b, c. In other words, if $S_{ab}(\theta)$ possesses the pole (5.17), the amplitude $S_{ac}(\theta)$ must possess the pole at $\theta = iu^b_{ca}$ with exactly the same residue (5.18). The constant f_{abc} can be associated with the three-leg diagram in Fig. 2, where the angles between the legs are represented by u^c_{ab}.

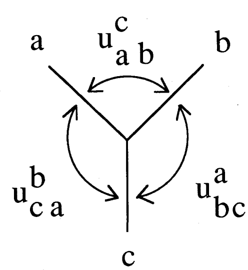

Fig. 2. The "three-particle vertex" associated with f_{abc}.

Note that the relation

$$u^c_{ab} + u^b_{ca} + u^a_{bc} = 2\pi \tag{5.19}$$

is the simple consequence of (5.16).

The "bound state" poles appear, of course, in the multi-particle S-matrix elements, too. The corresponding residues must reduce to the amplitudes of the scattering involving the "bound state" particle. In particular, the residue at $\theta_{12} = iu^c_{ab}$ of the three-particle amplitude

$$S_{abd}(\theta_1, \theta_2, \theta_3) = S_{ab}(\theta_{12}) S_{ad}(\theta_{13}) S_{bd}(\theta_{23}), \tag{5.20}$$

where $\theta_{ij} = \theta_i - \theta_j$, must reproduce the two-particle amplitude $S_{cd}(\theta)$. This requirement leads to the bootstrap equations

$$S_{cd}(\theta) = S_{bd}(\theta - iu^a_{bc}) S_{ad}(\theta + i\bar{u}^b_{ac}) \tag{5.21}$$

which restrict possible form of the two-particle S-matrix.

The "fusion structure" of the S-matrix described above must be compatible with the IM. The two-particle state (5.11) can be analytically continued to the complex values of p_1, p_2. In the vicinity of the bound state pole (5.17) ($\epsilon \to 0$)

$$|\, A_a(\theta + i\bar{u}^b_{ac} - \epsilon) A_b(\theta - i\bar{u}^a_{bc} + \epsilon)\,\rangle_{\text{in}} \sim \frac{1}{\epsilon}\,|\, A_c(\theta)\,\rangle. \tag{5.22}$$

Therefore the equation

$$P_s\,|\, A_a(\theta_1) A_b(\theta_2)\,\rangle_{\text{in}} = (\, \kappa^a_s m_a e^{s\theta_1} + \kappa^b_s m_b e^{s\theta_2}\,)\,|\, A_a(\theta_1) A_b(\theta_2)\,\rangle_{\text{in}} \tag{5.23}$$

being analytically continued to the vicinity of this pole leads to the relation

$$\kappa^a_s m^s_a e^{-is\bar{u}^b_{ac}} + \kappa^b_s m^s_b e^{is\bar{u}^a_{bc}} = \kappa^c_s m^s_c. \tag{5.24}$$

This can be considered as the (usually over-determined) system of linear equations for the constants κ^a_s. Of course, (5.24) always admit a trivial solution $\kappa^a_s = 0$. Let us stress, however, that $\kappa^a_s = 0$ for $a = 1, 2, \ldots, n$ imply $P_s \equiv 0$ as it follows from (5,4), (5.7) and the asymptotic completeness condition. To see that (5.24) indeed provide significant limitations, let us consider some models.

Let us assume that the theory contains the particle A_1 which can be considered as "fundamental", *i.e.* all other particles A_a can be obtained as the "bound states" of some numbers of A_1. This property is assumed here merely to state that $\kappa^1_s \neq 0$ for all IM P_s which the theory possesses. Let us assume in addition that the particle A_1 appears as the "bound state" in $A_1 A_1$ scattering ("φ^3-property"), *i.e.* $f_{111} \neq 0$. Then, setting $a = b = c = 1$ in (5.24) and taking into account

$$u^1_{11} = \frac{2\pi}{3}, \tag{5.25}$$

we obtain

$$e^{-\frac{i\pi s}{3}} + e^{+\frac{i\pi s}{3}} = 1. \tag{5.26}$$

This equation is satisfied if and only if the integer s has no common divisor with 6. So, in any FT which possesses even one IM (1.2) with $s = 0 \pmod 3$, the "fundamental particle" with the "φ^3-property" is forbidden. Compairing this result with (1.7 b), we can guess that the "φ^3-property" is characterlistic for the FT (1.5 b,c).

Another interesting model appears if we assume the existance of two-particles A_1 and A_2 such that $f_{112} \neq 0$ and $f_{122} \neq 0$; *i.e.* A_2 can be interpreted as the "bound state" A_1A_1 and A_1 appears as the bound-state pole of A_2A_2.

Using the variables

$$x_1 = \exp\left(\frac{i}{2}u_{11}^2\right); \qquad x_2 = \exp\left(\frac{i}{2}u_{22}^1\right) \tag{5.27}$$

which satisfy

$$\begin{aligned} x_1 + x_1^{-1} &= \frac{m_2}{m_1}; \\ x_2 + x_2^{-1} &= \frac{m_1}{m_2}, \end{aligned} \tag{5.28}$$

(due to (5.24) with $s = 1$), one can rewrite (5.24) in the form

$$\begin{aligned} x_1^s + x_1^{-s} &= \left(\frac{m_2}{m_1}\right)^s \frac{\kappa_s^2}{\kappa_s^1}; \\ x_2^s + x_2^{-s} &= \left(\frac{m_1}{m_2}\right)^s \frac{\kappa_s^1}{\kappa_s^2}. \end{aligned} \tag{5.29}$$

Excluding the unknown constants in the r.h.s. one obtains the system of equations

$$(x_1 + x_1^{-1})(x_2 + x_2^{-1}) = 1 \tag{5.30}$$

and

$$(x_1^s + x_1^{-s})(x_2^s + x_2^{-s}) = 1 \tag{5.31}$$

where s in (5.31) runs the set of spins of non-trivial IM P_s. If the theory possesses more than one non-trivial IM, the algebraic system turns out to be overdetermined. Nevertheless it admits the solution provided $s \neq 0$ (mod 5) :

$$x_1 = \exp\left(\frac{\pi i}{5}\right), \qquad x_2 = \exp\left(\frac{2\pi i}{5}\right) \tag{5.32}$$

(of course, one can change $x_1 \leftrightarrow x_2$). This solution corresponds to the "golden" mass ratio

$$\frac{m_2}{m_1} = 2\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{2} = 1.6180339\ldots\,. \tag{5.33}$$

Let us in addition assume that one of these particles, say A_1, has the "φ^3-property", *i.e.* $f_{111} \neq 0$. The IM (1.2) compatible with this scattering theory are those having s without common divisor with 30. The first possible s are listed in (1.10). In the next section, we shall elaborate this model of scattering theory in more details.

§6. Integrals of motion and S-matrix in critical Ising model with magnetic field

Let us consider the FT (1.5 b) with $p = 3$. In this case $c = 1/2$ and the characters of the irreducible representations of VIR with $\Delta = 0$ and $\Delta = \Delta_{(1,2)} = 1/16$ are given by [25]

$$\chi_0(q) = \frac{1}{2}\{\prod_{n=0}^{\infty}(1 + q^{n+1/2}) + \prod_{n=0}^{\infty}(1 - q^{n+1/2})\} \; ; \tag{6.1 a}$$

$$q^{-1/16}\chi_{(1,2)}(q) = \prod_{n=1}^{\infty}(1 + q^n) = \prod_{n=0}^{\infty}(1 - q^{2n+1})^{-1}. \tag{6.1 b}$$

Using these expressions, it is not difficult to compute the dimensionalities of the first few spaces $\hat{\Lambda}_{s+1}$ in (2.15) and $\hat{\Phi}_s$ in (4.1). The result is shown in Table 4. We presented only odd s since for even s the dim $\hat{\Lambda}_{s+1}$ is less than $\dim(\hat{\Phi}_s)$, $\dim(\hat{\Lambda}_{s+1}) = \dim(\hat{\Phi}_s) + 1$ for s given by (1.1). Since the "resonance condition" (3.6) is not satisfied by $\Delta = 1/16$, we conclude that the FT $H_3^{(1,2)}$ possesses at least five non-trivial IM (1.2) with the spins (1.11).

s	1	3	5	7	9	11	13	15	17	19	21
$\dim(\hat{\Lambda}_{s+1})$	1	1	1	2	2	3	4	5	7	9	11
$\dim(\hat{\Phi}_s)$	0	1	1	1	2	2	3	5	6	8	12

Table 4. The dimensionalities of $\hat{\Lambda}_{s+1}$ and $\hat{\Phi}_s$ (for odd $s \leq 21$) in the FT $H_3^{(1,2)}$.

The minimal CFT H_3 describes the critical point of Ising model, the field $\sigma \equiv \Phi_{(1,2)}$ being identified with the spin density [1]. So, the FT $H_3^{(1,2)}$ describes the field theory of the Ising model at $T = T_c$ but with non-zero magnetic field $h \sim \lambda$ which breaks the $\mathbf{Z}_2$ symmetry. Probably nobody has a doubt that this system develops finite correlation length and hence the field theory $H_3^{(1,2)}$ is expected to be massive. The above results prove that the corresponding scattering theory is purely elastic.

The qualitative features of the particle spectrum in the Ising model at $T - T_c \to 0$ and $\eta = h^{-8}(T - T_c)^{15} \sim 1$ can be predicted by physical reasoning. Those features are described in [26]. In general case $h \neq 0$ the Fourier-transformed two-point function

$$G(p^2) = \int d^2x\, e^{ipx} \langle\, \sigma(x)\sigma(0)\, \rangle \tag{6.2}$$

is expected to possess the complex singularities shown in Fig. 3.

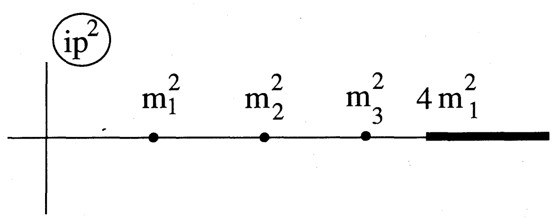

Fig. 3. Complex singularities of the correlator (6.2). The poles are shown by the dots, solid line represents brunch cut.

There are several poles corresponding to stable particles below the continuum represented by the brunch cut. The lightest particle (let us call it A_1) is expected to be "fundamental". Also, since the symmetry is broken at $h \neq 0$, it apparently exhibit "φ^3-property" explained in Section 5. In what follows I will argue that in specific case $\eta = 0$ (*i.e.* $T = T_c$, $h \neq 0$) there are exactly three poles ${m_1}^2$, ${m_2}^2$, ${m_3}^2$ below the threshold $4{m_1}^2$ in Fig. 4, the positions being given by (1.12). In fact, the theory I will describe below contains eight stable particles with the masses (1.12), but five of them lie above the A_1A_1 threshold; the physical reason for their stability is not very clear.

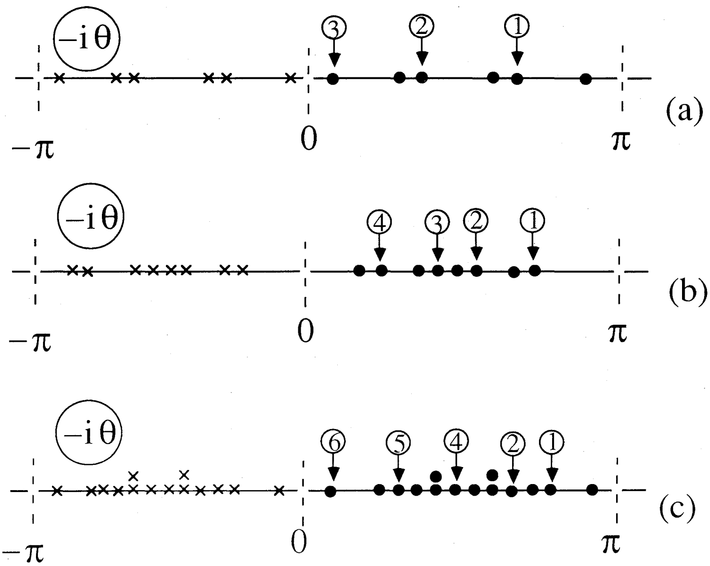

Fig. 4. The locations of zeros (crosses) and poles (dots) of the two-particle amplitudes $S_{11}(\theta)$ (a), $S_{12}(\theta)$ (b) and $S_{22}(\theta)$ (c). In the last case, there are double poles (zeros) which are shown by double dots (crosses). The arrows show the "bound state" poles with positive residues and indicate the corresponding particles.

The set (1.11) of IM (1.2) we have found in the FT $H_3^{(1,2)}$ strongly suggest that the theory contains two particles A_1 and A_2 with the mass ratio (5.31) (see Section 5). Let us assume that these are the lightest particles in the theory. Consider the two-particle amplitude $S_{11}(\theta)$. According to our assumption, it possesses the poles at $\theta = 2\pi i/3$ and $\theta = 2\pi i/5$ with positive residues which correspond to the particles A_1 and A_2, respectively. Due to the crossing symmetry (5.14), there are also the poles $\theta = \pi i/3$ and $3\pi i/5$ with negative residues. The bootstrap equation (5.21) with $a = b = c = d = 1$ takes the form

$$S_{11}(\theta) = S_{11}(\theta - \frac{\pi i}{3})\, S_{11}(\theta + \frac{\pi i}{3}). \tag{6.3}$$

This equation cannot be satisfied without additional poles in $S_{11}(\theta)$. The minimal way to satisfy (6.3) without breaking the above requirements is to add the poles at $\theta = \pi i/15$ (with positive residue) and at $\theta = 14\pi i/15$ (with negative residue). The "minimal" solution of eq. (6.3) takes the form

$$\begin{aligned} S_{11}(\theta) = {} & \operatorname{th}\Big(\frac{\theta}{2}+\frac{\pi i}{6}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{\pi i}{6}\Big)\operatorname{th}\Big(\frac{\theta}{2}+\frac{\pi i}{5}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{\pi i}{5}\Big) \\ & \times\operatorname{th}\Big(\frac{\theta}{2}+\frac{\pi i}{30}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{\pi i}{30}\Big). \end{aligned} \tag{6.4}$$

The locations of the poles and zeros of this amplitude is shown in Fig. 4 (a). The additional pole $\theta = \pi i/15$ represents a new particle A_3 having the mass

$$\frac{m_3}{m_1} = 2\cos\frac{\pi}{30} = 1.9890437... \tag{6.5}$$

which is still below the first threshold $2m_1$. Note that

$$u_{11}^{2} = \frac{2\pi}{5}, \qquad u_{12}^{1} = \frac{4\pi}{5}, \qquad u_{11}^{3} = \frac{\pi}{15}, \qquad u_{13}^{1} = \frac{29\pi}{30}. \tag{6.6}$$

Now we can use the bootstrap equation (5.21) with $a = b = d = 1$ and $c = 2$

$$S_{12}(\theta) = S_{11}(\theta - \frac{\pi i}{5})\, S_{11}(\theta + \frac{\pi i}{5}) \tag{6.7}$$

to compute the amplitude

$$\begin{aligned} S_{12}(\theta) = {} & \\ & \operatorname{th}\Big(\frac{\theta}{2}+\frac{\pi i}{10}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{\pi i}{10}\Big)\operatorname{th}\Big(\frac{\theta}{2}+\frac{4\pi i}{15}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{4\pi i}{15}\Big) \\ & \times\operatorname{th}\Big(\frac{\theta}{2}+\frac{2\pi i}{15}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{2\pi i}{15}\Big)\operatorname{th}\Big(\frac{\theta}{2}+\frac{3\pi i}{10}\Big)\operatorname{cth}\Big(\frac{\theta}{2}-\frac{3\pi i}{10}\Big). \end{aligned} \tag{6.8}$$

Locations of zeros and poles of this amplitude are shown in Fig. 4 (b). Note the poles $\theta = 4\pi i/5 = iu_{12}^{1}$, $\theta = 3\pi i/5$, $\theta = 7\pi i/15$ with positive residues, which represent already known particles A_1, A_2, A_3, respectively, as the "bound states" of $A_1 A_2$. There is one more pole $\theta = 4\pi i/15$ in (6.8). This pole corresponds to new stable particle A_4 with the mass

$$\frac{m_4}{m_1} = 4\cos\frac{\pi}{5}\cos\frac{7\pi}{30} = 2.4048671\dots\,. \tag{6.9}$$

The amplitude $S_{22}(\theta)$, computed by (5.21) with $a = b = 1,\ c = d = 2$

$$S_{22}(\theta) = S_{12}(\theta - \frac{\pi i}{5})\, S_{12}(\theta + \frac{\pi i}{5}) \tag{6.10}$$

exhibits the poles and zeros shown in Fig. 4 (c). The analytic expression is

(6.11)

$$\begin{aligned} S_{22}(\theta) = & \\ & \mathrm{th}\Big(\frac{\theta}{2} + \frac{\pi i}{6}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{\pi i}{6}\Big)\mathrm{th}\Big(\frac{\theta}{2} + \frac{2\pi i}{5}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{2\pi i}{5}\Big) \\ & \times \mathrm{th}\Big(\frac{\theta}{2} + \frac{7\pi i}{30}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{7\pi i}{30}\Big)\mathrm{th}\Big(\frac{\theta}{2} + \frac{11\pi i}{30}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{11\pi i}{30}\Big) \\ & \times \mathrm{th}\Big(\frac{\theta}{2} + \frac{\pi i}{30}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{\pi i}{30}\Big)\Big[\mathrm{th}\Big(\frac{\theta}{2} + \frac{\pi i}{5}\Big)\mathrm{cth}\Big(\frac{\theta}{2} - \frac{\pi i}{5}\Big)\Big]^2 . \end{aligned}$$

One can check that the amplitudes (6.4), (6.8), (6.11) satisfy the bootstrap equations like

(6.12)

$$\begin{aligned} S_{11}(\theta) =& S_{12}(\theta - \frac{\pi i}{3})\, S_{12}(\theta + \frac{\pi i}{3}) = S_{11}(\theta + \frac{3\pi i}{5})\, S_{11}(\theta - \frac{\pi i}{5})\ ; \\ S_{12}(\theta) =& S_{11}(\theta + \frac{2\pi i}{5})\, S_{12}(\theta - \frac{\pi i}{5}) = S_{12}(\theta + \frac{3\pi i}{5})\, S_{22}(\theta - \frac{\pi i}{5})\ ; \\ S_{22}(\theta) =& S_{22}(\theta - \frac{\pi i}{3})\, S_{22}(\theta + \frac{\pi i}{3}) = S_{12}(\theta + \frac{2\pi i}{5})\, S_{22}(\theta - \frac{\pi i}{5}), \end{aligned}$$

where the values

$$u_{12}^2 = \frac{3\pi}{5}; \qquad u_{22}^1 = \frac{4\pi}{5}; \qquad u_{22}^2 = \frac{2\pi}{3} \tag{6.13}$$

are taken into account. Besides the familiar particles A_1, A_2, A_4 which are represented by the poles $\theta = 4\pi i/5$, $\theta = 2\pi i/3$, $\theta = 7\pi i/15$ in (6.11), the amplitude $S_{22}(\theta)$ exhibits the poles at $\theta = 4\pi i/15$ and $\theta = \pi i/15$ which also have the positive residues and hence imply the existance of two more stable particles A_5 and A_6 with the masses

$$\begin{aligned} \frac{m_5}{m_1} &= 4\cos\frac{\pi}{5}\cos\frac{2\pi}{15} = 2.956295\ldots ; \\ \frac{m_6}{m_1} &= 4\cos\frac{\pi}{5}\cos\frac{\pi}{30} = 3.2183404\ldots . \end{aligned} \tag{6.14}$$

To implement the bootstrap program explained in Section 5 one have to construct the two-particle amplitudes involving the particles A_3, A_4, A_5, A_6 and check that they satisfy the bootstrap equations (5.21). In addition, these new "fusions" will create new stable particles which must be involved into the bootstrap equations, *etc.* Here I will not describe explicitly how it goes for the model under consideration. It can be shown that the bootstrap program closes within exactly eight stable particles A_1, $A_2, \ldots, A_8$ with the masses (1.12). I conjecture that this is the S-matrix of the model $H_3^{(1,2)}$, *i.e.* of the FT describing the scaling Ising model with $T = T_c$ and $h \neq 0$.

Note that the numbers (1.10) are exactly the exponents for the Lie algebra E_8, repeated modulo 30. Also, the number of stable particles in the above scattering theory is associated with the rank of E_8. This hidden E_8 structure of the FT $H_3^{(1,2)}$ (as well as the hidden E_7 and E_6 structures in $H_4^{(1,2)}$ and $H_6^{(1,2)}$) has been predicted by V.A. Fateev on the basis of his study of W-algebras associated with the exceptional Lie algebras. Explaining his arguments is beyond the scope of this lecture. This structure strongly suggest that particular IRF integrable lattice model (of "restricted" type) associated with the integral weights of E_8 can be constructed (in the spirit of of [4,7]), whose scaling limit would describe the universality class of the critical Ising model in magnetic field.

The relevance of the above exact S-matrix to the Ising model in magnetic field, being the matter of the conjecture, needs of course the proof or at least some checks. Using the above S-matrix one can in principle construct the Bethe-ansatz states over the physical vacuum. The behaviour of the corresponding spectral density at $E \to \infty$ will then show the ultraviolet central charge ($c = 1/2$ according to our conjecture). A more physical check concerns the long-distance behaviour

$$\begin{aligned} &\langle\, \sigma(R)\sigma(0)\,\rangle \\ &\quad = a_1 K_0(m_1 R) + a_2 K_0(m_2 R) + a_3 K_0(m_3 R) + \mathcal{O}(e^{-2m_1 R}) \end{aligned} \tag{6.15}$$

of the scaled spin-spin correlation function in $T = T_c$, $h \neq 0$ Ising model which follows from our conjecture. Here a_1, a_2, a_3 are (in principle computable) constants. The next-to-leading correlation due to the second term in (6.15) can probably be checked by numerical analysis.

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