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Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries

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Introduction

Conformal field theory has not only useful application to string theory and two-dimensional critical phenomena but also has beautiful and rich mathematical structure, and it has interested many mathematicians. Conformal field theory is characterized by infinite-dimensional symmetry such as Virasoro algebra. Especially, its correlation functions are characterized by differential equations arising from representations of infinite-dimensional Lie algebras. ([BPZ], [KZ], [EO], [MMS].) Physically, correlation functions should have the properties such as locality, holomorphic factorization and monodromy invariance (duality). To build conformal field theory having such properties, usual approach is to construct holomorphic (chiral) conformal blocks which are the *half*

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of the theory and to study its monodromy. ([TK1], [TK2], [FS], [Va1], [Va2], [Ve], [MS1], [MS2].)

In the present paper, mathematically rigorous formulation of holomorphic (chiral) conformal field theory with gauge symmetry (affine Lie algebra $\widehat{\mathfrak{g}}$) (Wess-Zumino-Witten model) over curves of arbitrary genus is given by means of operator formalism. A curve in our theory may have ordinary double points singularities corresponding to a point of the boundary of the moduli space of curves. The fundamental object in our theory is the *space of vacua*. A vacuum is a linear functional on the direct product of representation spaces of $\widehat{\mathfrak{g}}$ giving vacuum expectation value (correlation function). Our formulation of conformal field theory is a natural generalization of the one developed in [TK1].

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbf{C}$ and $\widehat{\mathfrak{g}}$ the corresponding affine Lie algebra. We fix a positive integer ℓ and consider integrable highest weight representations of $\widehat{\mathfrak{g}}$ with level ℓ. Such representations are parameterized by a finite set of highest weights P_ℓ. Let $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \dots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \dots, t_N^{(\infty)})$ be an N-pointed stable curve with formal neighbourhoods. (For details see Definition 2.1.1 below.) To each point Q_j we associate a representation of $\widehat{\mathfrak{g}}$ corresponding to $\lambda_j \in P_\ell$. Then to $\mathfrak{X}^{(\infty)}$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ we associate the space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ and its dual space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$. The space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ is defined by the gauge condition. (See Definition 2.2.2 below). It will be shown that $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ does only depend on the first order infinitesimal structure $\mathfrak{X}^{(1)}$ of $\mathfrak{X}^{(\infty)}$. (See Remark 4.1.7 below.)

Let $\overline{\mathfrak{M}}^{(\infty)}_{g,N}$ (resp. $\overline{\mathfrak{M}}^{(1)}_{g,N}$) be the moduli space of N-pointed stable curves with formal neighbourhoods (resp. first order infinitesimal structures) and $\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \overline{\mathfrak{M}}^{(\infty)}_{g,N}$ (resp. $\pi^{(1)} : \mathcal{C}^{(1)} \to \overline{\mathfrak{M}}^{(1)}_{g,N}$) be the universal family of N-pointed stable curves on it. Then, the collection of the spaces of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$'s (resp. the dual spaces of vacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$) forms a sheaf $\mathcal{V}^{\dagger(\infty)}_{\vec{\lambda}}$ (resp. $\mathcal{V}^{(\infty)}_{\vec{\lambda}}$) on $\overline{\mathfrak{M}}^{(\infty)}_{g,N}$ and it is the pull back of a sheaf $\mathcal{V}^{\dagger(1)}_{\vec{\lambda}}$ (resp. $\mathcal{V}^{(1)}_{\vec{\lambda}}$) on $\overline{\mathfrak{M}}^{(1)}_{g,N}$.

Precisely speaking, there exist *no* universal families of N-pointed stable curves over the moduli spaces $\overline{\mathfrak{M}}^{(\infty)}_{g,N}$ and $\overline{\mathfrak{M}}^{(1)}_{g,N}$. Therefore, we have to consider local universal families. Namely, we define the sheaves of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ (resp. $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$) attached to local universal families $\mathfrak{F}^{(\infty)} = (\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}\,;\, s_1^{(\infty)}, s_2^{(\infty)}, \dots, s_N^{(\infty)}\,;\, \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \dots, \tilde{t}_N^{(\infty)})$ and $\mathfrak{F}^{(1)} = (\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}\,;\, s_1^{(1)},$

$s_2^{(1)}, \dots, s_N^{(1)}; \tilde{t}_1^{(1)}, \tilde{t}_2^{(1)}, \dots, \tilde{t}_N^{(1)})$, respectively. The sheaves $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ (resp. $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$) are $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-modules (resp. $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules). If a local universal family $\mathfrak{F}'^{(1)}$ is a subfamily of $\mathfrak{F}^{(1)}$ the restriction of the sheaves $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ to the subfamily are the sheaves $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}'^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}'^{(1)})$, respectively.

In the present paper we shall analyze the structure of the sheaves $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$. Though our arguments below often use specific coordinates, they have intrinsic meaning and we could argue as if there were universal family over the moduli space of N-pointed stable curves with infinitesimal structures. Fancy mathematical tool to treat the above situation is the theory of stacks ([DM]). But in the present paper we choose primitive approach described above. Using the idea of Beilinson-Manin-Shechtman [BMS],[BS2], we construct an $\mathcal{O}_{\mathcal{B}^{(1)}}$-module of Lie algebra $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ (the sheaf of twisted first order differential operators) acting on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$, which is the geometric counter part of the Virasoro algebra with central charge c_v defined from the representations as the Sugawara form. (For details see Section 5.)

Main results of the present paper are the following.

1) $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ are coherent $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules. (Theorem 4.2.4.) Hence, the space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ are finite-dimensional. Moreover, $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ are locally free sheaf of finite rank, that is, a vector bundle over $\mathcal{B}^{(1)}$. (Theorem 6.2.1 and Corollary 6.2.3.)

2) The sheaf $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ of twisted first order differential operators acts on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$. (Theorem 5.3.3.) This defines projective flat connections on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ with regular singularities at the locus $D^{(1)} \subset \mathcal{B}^{(1)}$ corresponding to singular curves. The connections are nothing but the Ward-Takahashi identity. Moreover, the solution sheaf of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ gives what physicists call current conformal blocks.

3) $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ has a factorization property [FS]. (Theorem 6.2.6.) Hence the dimension of the space of vacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ does only depend on the genus of the curve C and $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ and can be calculated by a maximally degenerate curve by using the fusion rule. Moreover, the proof in Section 6 shows that we can construct a canonical basis of flat sections of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ from the data on the boundary.

Our result in this paper may be regarded as an infinite-dimensional version of the Beilinson-Bernstein theory [BB], [BK] for representations of finite dimensional simple Lie groups. Here three notions, Virasoro algebra, moduli space, and the braid group and the mapping class group correspond to simple Lie group G , the Flag manifold G/P and the Weyl group of the original theory, respectively.

Let us explain briefly the content of the present paper. In Section 1 we shall give basic results on integrable highest weight representations of an affine Lie algebra $\widehat{\mathfrak{g}}$. The energy-momentum tensor will be defined as the Segal-Sugawara form. Also the automorphism group $\mathcal{D} = \mathrm{Aut}\mathbf{C}((\xi))$ of the field of formal Laurent series $\mathbf{C}((\xi))$ will be introduced and its properties will be studied.

In Section 2 we shall first define the notion of an N-pointed stable curve with n-th infinitesimal neighbourhoods $\mathfrak{X}^{(n)}$ or with formal neighbourhoods $\mathfrak{X}^{(\infty)}$ and define the space of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ and its dual space of vacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ attached to $\mathfrak{X}^{(\infty)}$. The important properties of the space of vacua such as propagation of vacua will be proved. Also we shall define correlation functions of current and study their properties. The propagation of vacua and the properties of correlation functions will play an essential rôle to construct our conformal field theory.

To study the properties of the space of vacua we need to vary the moduli of N-pointed curves with infinitesimal structures. In Section 3 we shall study local universal family of such curves. The content of this section is well-known to the specialists. Since the results in this section are scattered among many references, we shall describe some details of deformation theory of N-pointed curves with infinitesimal structures. We shall use freely the standard technique of the cohomology theory of sheaves which can be found, for example, in [Ha] or [BS1].

In Section 4 we shall define the sheaf of vacua associated with a local universal family of N-pointed stable curves with formal neighbourhoods $(\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}; s_1^{(\infty)}, s_2^{(\infty)}, \dots, s_N^{(\infty)}; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \dots, \tilde{t}_N^{(\infty)})$. We shall show that the sheaf is coherent $\mathcal{O}_{\mathcal{B}^{(1)}}$-module. Here, Gabber's theorem [Ga] plays an essential role.

In Section 5 we shall define the sheaf of twisted first order differential operators $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ acting on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ from the left and on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ from the right. This sheaf defines an integrable connection on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ with regular singularities on the boundary corresponding to singular curves.

Finally in Section 6 we shall show that the sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ are locally free and have the factorization property. Hence the dimension of the space of vacua can be calculated by a maximally degenerate curve by using the fusion rule. Moreover, the proof shows that we can construct a canonical basis of flat sections of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ from the data on the boundary.

The main results of the present paper was announced in [TY]. In [BS2], analogous results have been given for the case of $c < 1$ minimal models. But the analysis of singularity on the boundary was not studied there.

Notations

$\mathfrak{g}$: simple Lie algebra over the complex numbers $\mathbf{C}$.
Δ : set of all non-zero roots of $\mathfrak{g}$.
Δ_+ (Δ_-) : set of all positive (resp. negative) roots of $\mathfrak{g}$.
θ : the maximal root of $\mathfrak{g}$.
$\mu^\dagger := -w(\mu)$ where w is the longest element of Weyl group of $\mathfrak{g}$.
(,) : Cartan-Killing form of $\mathfrak{g}$ normalized as $(\theta, \theta) = 2$.
V_λ ($V_\lambda^\dagger$) : irreducible left (resp. right) $\mathfrak{g}$-module with highest (resp. lowest) weight λ.
P_+ : set of all dominant integral weights.
$\widehat{\mathfrak{g}}$: affine Lie algebra attached to $\mathfrak{g}$. (Definition 1.1.1)
ℓ : level of a representation of $\widehat{\mathfrak{g}}$.
$P_\ell := \{ \lambda \in P_+ \mid 0 \le (\theta, \lambda) \le \ell \}$
$\Delta_\lambda = \dfrac{(\lambda, \lambda) + 2(\lambda, \rho)}{2(g^* + \ell)}$ where $\rho = \dfrac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ and g^* is the dual Coxeter number of $\mathfrak{g}$.
$c_v := \dfrac{\ell \cdot \dim \mathfrak{g}}{g^* + \ell}$
$\mathcal{H}_\lambda$ ($\mathcal{H}_\lambda^\dagger$) : integrable highest weight left (resp. right) $\mathfrak{g}$-module with highest (resp. lowest) weight Λ.
$F_\bullet \mathcal{H}_\lambda$ ($F^\bullet \mathcal{H}_\lambda^\dagger$) : filtration of $\mathcal{H}_\lambda$ (resp. $\mathcal{H}_\lambda^\dagger$). (See 1.3).
$\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1} \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} \mathcal{H}_{\lambda_N}$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_\ell)^N$.
$\mathcal{H}^\dagger_{\vec{\lambda}} := \mathcal{H}^\dagger_{\lambda_1} \widehat{\otimes}_{\mathbf{C}} \cdots \widehat{\otimes}_{\mathbf{C}} \mathcal{H}^\dagger_{\lambda_N}$
$\mathbf{C}((\xi))$: field of all formal Laurent series. That is, the quotient field of the formal power series ring $\mathbf{C}[[\xi]]$.
$X(n) := X \otimes \xi^n$, where $X \in \mathfrak{g}$.
$X(z) := \sum_{n \in \mathbf{Z}} X(n) z^{-n-1}$
$T(z)$: energy-momentum tensor. (Definition 1.2.1)
$X[f] := \mathrm{Res}_{z=0}(X(z) f(z) dz)$ for $f(\xi) \in \mathbf{C}((\xi))$.

$T[\underline{l}] := \mathrm{Res}_{z=0}(T(z)\ell(z)dz)$ for $\underline{l} = \ell(z)\dfrac{d}{dz} \in \mathbf{C}((z))\dfrac{d}{dz}$

$\mathcal{D} := \mathrm{Aut}\,\mathbf{C}((\xi))$

$\mathcal{D}^p := \{\, h \in \mathcal{D} \mid h(\xi) = \xi + a_p\xi^{p+1} + \cdots \,\}$

$(\underline{d}) := \mathbf{C}[[\xi]]\xi\dfrac{d}{d\xi}$

$(\underline{d})^p := \mathbf{C}[[\xi]]\xi^{p+1}\dfrac{d}{d\xi}$

$G[h] := \exp(-T[\underline{l}])$ for $h \in \mathcal{D}^1$ where $h = \exp(\underline{l})$.

$\mathfrak{X}^{(n)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(n)}, t_2^{(n)}, \ldots, t_N^{(n)})$: N-pointed stable curve with n-th infinitesimal neighborhoods. (Definition 2.1.3)

$\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$: N-pointed stable curve with formal neighbourhoods.

$\widehat{\mathfrak{g}}_N := \bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbf{C}((\xi_j)) \oplus \mathbf{C}c$ (Definition 2.2.1)

$\widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)}) := \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$

$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}^{(\infty)})$ ($\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$) : space of vacua (resp. dual space of vacua) associated with $\mathfrak{X}^{(\infty)}$. (Definition 2.2.2)

T_xM (T_x^*M) : tangent (resp. cotangent) space at a point x of a complex manifold M.

Ω_C^1 : sheaf of Kähler differentials of a curve C.

ω_X : dualizing sheaf of a complex space X.

$\Omega_{M/N}^1$: sheaf of relative 1-form for a surjective holomorphic mapping $\pi : M \to N$ of complex manifolds.

$\Theta_{M/N} := \underline{Hom}_{\mathcal{O}_M}(\Omega_{M/N}^1, \mathcal{O}_M)$: sheaf of relative holomorphic vector fields.

$\omega_{M/N}$: relative dualizing sheaf.

$\Theta_M(-\log D)$: sheaf of vector fields on a complex manifold M tangent to an effective divisor D of M.

$\mathfrak{F}^{(n)} = (\,\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}\,;\, s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}\,;\, \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)}\,)$: local universal family of N-pointed stable curves with n-th infinitesimal neighbourhoods. (Definition 3.1.1 and Theorem 3.1.5)

$\mathfrak{F}^{(\infty)} = (\,\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}\,;\, s_1^{(\infty)}, s_2^{(\infty)}, \ldots, s_N^{(\infty)}\,;\, \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \ldots, \tilde{t}_N^{(\infty)})$: local universal family of N-pointed stable curves with formal neighbourhoods.

$\Sigma^{(n)}$: critical locus of $\mathfrak{F}^{(n)}$. ((3.1-8) and Lemma 3.1.6)

$D^{(n)}$: discriminant locus of $\mathfrak{F}^{(n)}$. ((3.1-9) and Lemma 3.1.6)

$\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} := \mathcal{O}_{\mathcal{B}^{(\infty)}} \otimes_{\mathbf{C}} \mathcal{H}_{\vec{\lambda}}$

$\widetilde{\mathcal{H}}_{\vec{\lambda}}^{\dagger(\infty)} := \underline{Hom}_{\mathcal{O}_{\mathcal{B}^{(\infty)}}}(\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}, \mathcal{O}_{\mathcal{B}^{(\infty)}})$

$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$: dual sheaf of vacua attached to a family $\mathfrak{F}^{(\infty)}$. (Definition

4.1.2)
$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$: sheaf of vacua attached to a family $\mathfrak{F}^{(\infty)}$. (Definition 4.1.2)
$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$: dual sheaf of vacua attached to a family $\mathfrak{F}^{(1)}$. (Lemma 4.1.6)
$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$: sheaf of vacua attached to a family $\mathfrak{F}^{(1)}$. (Lemma 4.1.6)
$\mathcal{D}^{1}_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$: sheaf of twisted differential operators.
$\{w; z\}$: Schwarzian derivative. $\{w, z\} = \dfrac{w'''}{w'} - \dfrac{3}{2}\left(\dfrac{w''}{w'}\right)^2$

§1. Integrable highest weight representation of affine Lie algebra

1.1. Affine Lie algebra

In this subsection we recall basic facts on integrable highest weight representations of affine Lie algebras. For the details on integrable highest weight representations of affine Lie algebras we refer the reader to Kac's book [Ka].

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbf{C}$ and $\mathfrak{h}$ its Cartan subalgebra. By Δ we denote the root system of $(\mathfrak{g}, \mathfrak{h})$. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

Fix a lexicographic ordering of $\mathfrak{h}^*_{\mathbf{R}}$ once for all. This gives the decomposition $\Delta = \Delta_+ \sqcup \Delta_-$ of the root system into the positive roots and the negative roots. Let θ be the maximal root. We normalize the Cartan-Killing form

$$(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{C}$$

with the property

$$(1.1\text{-}1) \qquad (\theta, \theta) = 2.$$

Note that the Cartan-Killing form has the following property.

$$(1.1\text{-}2) \qquad ([X, Y], Z) + (Y, [X, Z]) = 0.$$

Let P_+ be the set of dominant integral weights of the Lie algebra $\mathfrak{g}$. There is a one-to-one correspondence between the set of finite dimensional irreducible representations of $\mathfrak{g}$ and the set P_+ of the dominant integral weights of $\mathfrak{g}$.

By $\mathbf{C}[[\xi]]$ and $\mathbf{C}((\xi))$ we mean the ring of formal power series in ξ and the field of formal Laurent power series in ξ, respectively. Namely

$$\mathbf{C}[[\xi]] = \{ \sum_{\nu=0}^{\infty} a_\nu \xi^\nu \mid a_\nu \in \mathbf{C} \},$$
$$\mathbf{C}((\xi)) = \{ \sum_{\nu=m}^{\infty} b_\nu \xi^\nu \mid b_\nu \in \mathbf{C},\, m \in \mathbf{Z} \}.$$

Definition 1.1.1. The affine Lie algebra $\widehat{\mathfrak{g}}$ over $\mathbf{C}((\xi))$ associated with $\mathfrak{g}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c \tag{1.1-3}$$

where c is an element of the center of $\widehat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$\begin{aligned}&[X \otimes f(\xi), Y \otimes g(\xi)] = \\ &\qquad [X,Y] \otimes f(\xi)g(\xi) + c \cdot (X,Y) \operatorname*{Res}_{\xi=0}(g(\xi)df(\xi)),\end{aligned} \tag{1.1-4}$$

for

$$X,\, Y \in \mathfrak{g},\; f(\xi),\, g(\xi) \in \mathbf{C}((\xi)).$$

Note that usually the affine Lie algebra is defined over $\mathbf{C}[\xi, \xi^{-1}]$ but for our theory we need to define it over $\mathbf{C}((\xi))$. Put

$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbf{C}[[\xi]]\xi, \quad \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbf{C}[\xi^{-1}]\xi^{-1}. \tag{1.1-5}$$

We regard $\widehat{\mathfrak{g}}_+$and $\widehat{\mathfrak{g}}_-$ as Lie subalgebras of $\widehat{\mathfrak{g}}$. We have a decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbf{C}c \oplus \widehat{\mathfrak{g}}_-. \tag{1.1-6}$$

Fix a positive integer ℓ (called the *level*) and put

$$P_\ell = \{ \lambda \in P_+ \mid 0 \le (\theta, \lambda) \le \ell \}.$$

Proposition 1.1.2. *For each $\lambda \in P_\ell$ there exists the unique left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ (called the* integrable highest weight $\widehat{\mathfrak{g}}$-module) *satisfying the following properties.*

(1) $V_\lambda = \{ |v\rangle \in \mathcal{H}_\lambda \mid \widehat{\mathfrak{g}}_+ |v\rangle = 0 \}$ *is the irreducible left $\mathfrak{g}$-module with highest weight λ.*
(2) *The central element c acts on $\mathcal{H}_\lambda$ as $\ell \cdot \mathrm{id}$.*

(3) *$\mathcal{H}_\lambda$ is generated by V_λ over $\widehat{\mathfrak{g}}_-$ with only one relation*

$$(X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle = 0 \tag{1.1-7}$$

where $X_\theta \in \mathfrak{g}$ is the element corresponding to the maximal root θ and $|\lambda\rangle \in V_\lambda$ is the highest weight vector.

Similarly we have the integrable highest weight right $\widehat{\mathfrak{g}}$-module $\mathcal{H}^\dagger_\lambda$ which will be discussed in 1.3 below.

1.2. Segal-Sugawara form

In the following we use the following notation freely.

$$X(n) = X \otimes \xi^n, \quad X \in \mathfrak{g}$$
$$X(z) = \sum_{n\in\mathbf{Z}} X(n) z^{-n-1}$$

where z is a variable. Then the normal odering $\overset{\circ}{\circ} \quad \overset{\circ}{\circ}$ is defined by

$$\overset{\circ}{\circ} X(n)Y(m) \overset{\circ}{\circ} = \begin{cases} X(n)Y(m), & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & n = m, \\ Y(m)X(n) & n > m. \end{cases}$$

Definition 1.2.1. The *energy-momentum* tensor $T(z)$ is defined by

$$T(z) = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(z) J^a(z) \overset{\circ}{\circ} \tag{1.2-1}$$

where $\{J^1, J^2, \ldots\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form (,) and g^* is the dual Coxeter number of $\mathfrak{g}$.

Put

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{m\in\mathbf{Z}} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(m) J^a(n-m) \overset{\circ}{\circ}. \tag{1.2-2}$$

Then we have the expansion

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}.$$

The operator L_n is called the Virasoro operator which acts on $\mathcal{H}_\lambda$.

Lemma 1.2.2. *The set $\{L_n\}$ forms a Virasoro algebra and we have*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c_v}{12}(n^3 - n)\delta_{n+m,0}$$
$$[L_n, X(m)] = -mX(n+m), \quad \text{for } X \in \mathfrak{g}$$

where

$$c_v = \frac{\ell \dim \mathfrak{g}}{g^* + \ell}$$

is the central charge of the Virasoro algebra.

For $X \in \mathfrak{g}$, $f = f(z) \in \mathbf{C}((z))$ and $\underline{l} = \ell(z)\frac{d}{dz} \in \mathbf{C}((z))\frac{d}{dz}$ we use the following notation.

$$X[f] = \operatorname*{Res}_{z=0}(X(z)f(z)dz)$$
$$T[\underline{l}] = \operatorname*{Res}_{z=0}(T(z)\ell(z)dz).$$

Lemma 1.2.3. *$X[f]$ and $T[\underline{l}]$ act on $\mathcal{H}_\lambda$ and we have*

$$\begin{aligned} X[f] &= X \otimes f(\xi), \\ [T[\underline{l}], X[f]] &= -X[\underline{l}(f)], \\ [T[\underline{l}_1], T[\underline{l}_2]] &= -T[[\underline{l}_1, \underline{l}_2]] + \frac{c_v}{12}\operatorname*{Res}_{z=0}(\ell_1'''\ell_2 dz). \end{aligned} \tag{1.2-3}$$

1.3. Filtrations and $\mathcal{H}_\lambda^\dagger$

Let us introduce filtrations $\{F_\bullet\}$ on $\mathbf{C}((x))$, $\widehat{\mathfrak{g}}$ and $\mathcal{H}_\lambda$. For any integer p put

$$F_p\mathbf{C}((\xi)) = \xi^{-p}\mathbf{C}[[\xi]], \tag{1.3-1}$$

$$F_p\widehat{\mathfrak{g}} = \begin{cases} \mathfrak{g} \otimes F_p\mathbf{C}((\xi)) & p < 0 \\ \mathfrak{g} \otimes F_p\mathbf{C}((\xi)) \oplus \mathbf{C}c & p \geq 0. \end{cases} \tag{1.3-2}$$

To define a filtration $\{F_\bullet\}$ on $\mathcal{H}_\lambda$, we first define the subspace $\mathcal{H}_\lambda(d)$ of $\mathcal{H}_\lambda$ for a non-negative integer d by

$$\mathcal{H}_\lambda(d) = \{\, |v\rangle \in \mathcal{H}_\lambda \mid L_0|v\rangle = (d + \Delta_\lambda)|v\rangle \,\} \tag{1.3-3}$$

where

$$\Delta_\lambda = \frac{(\lambda,\lambda) + 2(\lambda,\rho)}{2(g^* + \ell)}, \quad \rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha.$$

For a negative integer $-d$ we define

$$\mathcal{H}_\lambda(-d) = \{0\}.$$

Now we define the filtration $\{F_p\mathcal{H}_\lambda\}$ by

$$F_p\mathcal{H}_\lambda = \sum_{d=0}^{p} \mathcal{H}_\lambda(d). \tag{1.3-4}$$

Note that all the filtrations defined above are the increasing ones.

Put

$$\mathcal{H}_\lambda^\dagger(d) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_\lambda(d), \mathbf{C}). \tag{1.3-5}$$

Then the dual space $\mathcal{H}_\lambda^\dagger$ of $\mathcal{H}_\lambda$ is defined to be

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\bar{\lambda}}, \mathbf{C}) = \prod_{d=0}^{\infty} \mathcal{H}_\lambda^\dagger(d). \tag{1.3-6}$$

By definition $\mathcal{H}_\lambda^\dagger$ is a right $\widehat{\mathfrak{g}}$-module. A decreasing filtration $\{F^p\mathcal{H}_\lambda^\dagger\}$ is defined by

$$F^p\mathcal{H}_\lambda^\dagger = \prod_{d \geq p} \mathcal{H}_\lambda^\dagger(d). \tag{1.3-7}$$

There is a canonical complete bilinear pairing

$$\langle \ | \ \rangle : \mathcal{H}_\lambda^\dagger \times \mathcal{H}_\lambda \longrightarrow \mathbf{C}, \tag{1.3-8}$$

which satisfies the following equality for each $a \in \widehat{\mathfrak{g}}$.

$$\langle u|av\rangle = \langle ua|v\rangle, \quad \text{for all } \langle u| \in \mathcal{H}_\lambda^\dagger \text{ and } |v\rangle \in \mathcal{H}_\lambda .$$

Note that the filtrations $\{F_p\}$ and $\{F^p\}$ define the topology on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$, respectively. With respect to this topology $\mathcal{H}_\lambda^\dagger$ is complete and is the integrable highest weight right $\widehat{\mathfrak{g}}$-module with the lowest weight λ. Put

$$V_\lambda^\dagger = \{\, \langle v| \in \mathcal{H}_\lambda^\dagger \mid \ \langle v|\widehat{\mathfrak{g}}_- = 0 \,\}.$$

It is easy to show that $V_\lambda^\dagger = \mathcal{H}_\lambda^\dagger(0)$ and $V_\lambda^\dagger$ is the irreducible right $\mathfrak{g}$-module with lowest weight λ. The integrable highest weight right $\widehat{\mathfrak{g}}$-module with lowest weight λ is generated by $V_\lambda^\dagger$ over $\widehat{\mathfrak{g}}_+$ with only one relation

$$\langle\lambda|(X_{-\theta} \otimes \xi)^{\ell-(\theta,\lambda)+1} = 0.$$

Lemma 1.3.2.

$$X(m)\mathcal{H}_\lambda(d) \subset \mathcal{H}_\lambda(d-m)$$
$$L_m\mathcal{H}_\lambda(d) \subset \mathcal{H}_\lambda(d-m)$$
$$\mathcal{H}^\dagger_\lambda(d)X(m) \subset \mathcal{H}^\dagger_\lambda(d+m)$$
$$\mathcal{H}^\dagger_\lambda(d)L_m \subset \mathcal{H}^\dagger_\lambda(d+m).$$

1.4. $\mathcal{D} = \mathrm{Aut}\mathbf{C}((\xi))$

Let $\mathcal{D}$ be the automorphisms group $\mathrm{Aut}\mathbf{C}((\xi))$ of the field $\mathbf{C}((\xi))$. The group is infinite-dimensional and is regarded as the automorphism group $\mathrm{Aut}\mathbf{C}[[\xi]]$ of the ring $\mathbf{C}[[\xi]]$.

Lemma 1.4.1. *There is an isomorphism*

$$\begin{array}{ccc} \mathcal{D} & \simeq & \{ \sum_{n=0}^{\infty} a_n\xi^{n+1} \mid a_0 \neq 0 \} \\ h & \mapsto & h(\xi) \end{array} \tag{1.4-1}$$

where for h_1, $h_2 \in \mathcal{D}$ the composition $h_1 \circ h_2$ corresponds to a power series $h_2(h_1(\xi))$.

In the following we often identify the group $\mathcal{D}$ with the set of power series given in the right hand side of (1.4.1). For each positive integer p put

$$\mathcal{D}^p = \{ h(\xi) = \xi + a_p\xi^{p+1} + \dots \}. \tag{1.4-2}$$

Then this defines a decreasing filtration

$$\mathcal{D} = \mathcal{D}^0 \supset \mathcal{D}^1 \supset \mathcal{D}^2 \supset \dots .$$

Put

$$\underline{d} = \mathbf{C}[[\xi]]\xi\frac{d}{d\xi} \tag{1.4-3}$$

$$\underline{d}^p = \mathbf{C}[[\xi]]\xi^{p+1}\frac{d}{d\xi} \tag{1.4-4}$$

for each positive integer p. We have a decreasing filtration of ideals

$$\underline{d} = \underline{d}^0 \supset \underline{d}^1 \supset \underline{d}^2 \supset \dots .$$

For any element $\underline{l} \in \underline{d}$ and $f(\xi) \in \mathbf{C}[[\xi]]$ define $\exp(\underline{l})(f(\xi))$ by

$$\exp(\underline{l})(f(\xi)) = \sum_{k=0}^{\infty} \frac{1}{k!}(\underline{l}^k(f(\xi))). \tag{1.4-5}$$

This is well defined and $\exp(\underline{l})$ is an element of $\mathcal{D}$.

Lemma 1.4.2. *The exponential mapping*

$$\begin{array}{rccc} \exp: & \underline{d} & \longrightarrow & \mathcal{D} \\ & \underline{l} & \longmapsto & \exp(\underline{l}) \end{array}$$

is surjective. Moreover, for each positive integer p we have

$$\exp(\underline{d}^p) = \mathcal{D}^p$$

and the exponential mapping is injective on $\underline{d}^p$.

For each positive integer p and an element $\underline{l} \in \underline{d}^p$ define $\exp(T[\underline{l}])$ by

$$\exp(T[\underline{l}]) = \sum_{k=0}^{\infty} \frac{1}{k!} T[\underline{l}]^k. \tag{1.4-6}$$

Lemma 1.4.3. $\exp(T[\underline{l}])$ *is well-defined and is a continuous linear operator on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$. Moreover, it induces the identity operator on $Gr_\bullet^F \mathcal{H}_\lambda$ and $Gr_F^\bullet \mathcal{H}_\lambda^\dagger$.*

Definition 1.4.4. For an automorphism $h \in \mathcal{D}^p$, $p \geq 1$, $G[h]$ is defined by

$$G[h] = \exp(-T[\underline{l}]), \tag{1.4-7}$$

where

$$h = \exp(\underline{l}).$$

Note that by Lemma 1.4.2 $G[h]$ is well-defined.

Theorem 1.4.5. *For $h \in \mathcal{D}^1$ and $f \in \mathbf{C}((\xi))$ we have the following.*

1) $G[h](X \otimes f)G[h^{-1}] = X \otimes h(f)$.
2) $G[h_2]G[h_1] = G[h_2 \circ h_1]$ *for* $h_1, h_2 \in \mathcal{D}$.
3) $G[h]T[\underline{l}]G[h^{-1}] = T[ad(h)(\underline{l})] + \frac{c_v}{12} \operatorname{Res}_{\xi=0}(\{h(\xi); \xi\}\ell(\xi)d\xi)$

where $\{h(\xi);\ \xi\}$ *is the Schwarzian derivative and* $\underline{l} = \ell(\xi)\frac{d}{d\xi} \in \mathbf{C}((\xi))\frac{d}{d\xi}$.

Corollary 1.4.6. *For* $f \in \mathbf{C}((\xi))$ *and* $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$ *the action of* $X_\alpha[f] = X_\alpha \otimes f$ *on* $\mathcal{H}_\lambda$ *and* $\mathcal{H}_\lambda^\dagger$ *are locally nilpotent.*

§2. Pointed stable curves and the associated vacua

2.1. Pointed stable curves

Definition 2.1.1. Data $\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N)$ consisting of a curve C and points $Q_1, \dots, Q_N$ on C are called an *N-pointed stable curve*, if the following conditions are satisfied.

(1) The curve C is a reduced connected complete algebraic curve defined over the complex numbers $\mathbf{C}$. The singularities of the curve C are at worst ordinary double points. That is, C is a semi-stable curve.

(2) $Q_1, Q_2, \dots, Q_N$ are non-singular points of the curve C.

(3) If an irreducible component C_i is a projective line (i.e. Riemann sphere) $\mathbf{P}^1$ (resp. a rational curve with one double point, resp. an elliptic curve), the sum of the number of intersection points of C_i and other components and the number of Q_j's on C_i is at least three (resp. one).

(4) $\dim_{\mathbf{C}} H^1(C, \mathcal{O}_C) = g$.

Note that the above condition (3) is equivalent to saying that $\mathrm{Aut}(\mathfrak{X})$ is a finite group so that $\mathfrak{X}$ has no infinitesimal automorphisms. In the following we often add the following condition (Q) for an N-pointed stable curve $\mathfrak{X}$.

(Q) Each component C_i contains at least one Q_j.

The meaning of the condition (Q) will be clarified in the following Lemma 2.1.4 and Lemma 2.1.5. By virtue of Proposition 2.2.3 below the assumption is not restrictive. (See Remark 2.2.5.)

Definition 2.1.2. Let C be a curve and Q a non-singular point on C. An *n-th infinitesimal neighbourhood* $t^{(n)}$ of C at the point Q is a $\mathbf{C}$-algebra isomorphism

$$t^{(n)} : \mathcal{O}_{C,Q}/\mathfrak{m}_Q^{n+1} \simeq \mathbf{C}[[\xi]]/(\xi^{n+1}) \tag{2.1-1}$$

where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{C,Q}$ consisting of germs of holomorphic functions vanishing at Q.

Taking the limit $n \to \infty$ in the isomorphism (2.1-1), we have an isomorphism

$$t^{(\infty)} : \widehat{\mathcal{O}}_{C,Q} \simeq \mathbf{C}[[\,\xi\,]]. \tag{2.1-2}$$

The isomorphism $t^{(\infty)}$ is called a *formal neighbourhood* of C at Q.

Definition 2.1.3. Data $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(n)}, t_2^{(n)}, \ldots, t_N^{(n)})$ are called an *N-pointed stable curve of genus g with n-th infinitesimal neighbourhoods*, if

(1) $(C\,;\, Q_1, Q_2, \ldots, Q_N)$ is an N-pointed stable curve of genus g.

(2) $t_j^{(n)}$ is an n-th infinitesimal neighbourhood of C at Q_j.

An N-pointed stable curve $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ with formal neighbourhoods is defined similarly.

Lemma 2.1.4. *Assume that an N-pointed stable curve $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ with formal neighbourhoods satisfies the condition (Q). By t_j we denote the Laurent expansions at Q_j with respect to a formal parameter $\xi_j = {t^{(\infty)}}^{-1}(\xi)$. Then, the following homomorphisms are injective.*

(2.1-3)

$$t = \oplus t_j : H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))$$

(2.1-4)

$$t = \oplus t_j : H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))d\xi_j$$

where ω_C is the dualizing sheaf of the curve C.

By this Lemma $H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j))$ (resp. $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$) can be regarded as a subspace of $\oplus_{j=1}^{N}\mathbf{C}((\xi_j))$ (resp. $\oplus_{j=1}^{N}\mathbf{C}((\xi_j))d\xi_j$). There is the residue pairing

(2.1-5)

$$\begin{array}{ccc} \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j)) \times \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))d\xi_j & \rightarrow & \mathbf{C} \\ ((f(\xi_1),..,f(\xi_N), g(\xi_1)d\xi_1,..,g(\xi_N)d\xi_N) & \mapsto & \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}(f(\xi_j)g(\xi_j)d\xi_j). \end{array}$$

The following Lemma is well-known and plays an important role in our theory.

Lemma 2.1.5. *Under the residue pairing $H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j))$ and $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$ are the annihilators to each other.*

2.2. The space of vacua associated with $\mathfrak{X}^{(\infty)}$

First we generalize the notion of an affine Lie algebra to the one over the direct sum of the fields of Laurent series $\oplus_{j=1}^{N}\mathbf{C}((\xi_j))$ and the one over the data $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$.

Definition 2.2.1. Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbf{C}$. The associated affine Lie algebra $\widehat{\mathfrak{g}}_N$ over $\oplus_{j=1}^{N}\mathbf{C}((\xi_j))$ is defined by

$$\widehat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes \mathbf{C}((\xi_j)) \oplus \mathbf{C}c \tag{2.2-1}$$

with the following commutation relations.

(2.2-2)

$$[\oplus_{j=1}^{N} X_j \otimes f_j, \oplus_{j=1}^{N} Y_j \otimes g_j] =$$

$$\oplus_{j=1}^{N}[X_j, Y_j] \otimes f_j g_j + c\sum_{j=1}^{N}(X_j, Y_j) \operatorname*{Res}_{\xi_j=0}(df_j g_j),$$

$$c \in \text{Center}$$

where $\oplus_{j=1}^{N} a_j$ means $(a_1, a_2, \ldots, a_N)$. The Lie subalgebra $\mathfrak{g}(\mathfrak{X}^{(\infty)})$ of $\widehat{\mathfrak{g}}_N$ associated with $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ is defined by

$$\widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)}) = \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)).$$

Here we regard $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$ as a subspace of $\oplus_{j=1}^N \mathbf{C}((\xi))$ by the mapping t given in (2.1-3).

Note that the Lie algebra $\widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)})$ has no centers. By Lemma 1.2.3 we use the notation $X[f_j]$ instead of $X \otimes f_j(\xi_j)$. Also we sometimes use the notation $X[f]$ instead of $X \otimes f$ for a meromorphic function f on the curve C, if there is no danger of confusion.

Let us fix a positive integer ℓ. For $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, a left $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}_{\vec{\lambda}}$ and a right $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}^\dagger_{\vec{\lambda}}$ are defined by

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \dots \otimes \mathcal{H}_{\lambda_N},$$
$$\mathcal{H}^\dagger_{\vec{\lambda}} = \mathcal{H}^\dagger_{\lambda_1} \widehat{\otimes} \dots \widehat{\otimes} \mathcal{H}^\dagger_{\lambda_N},$$

where the left $\widehat{\mathfrak{g}}_N$-action on $\mathcal{H}_{\vec{\lambda}}$ is given by

$$(\oplus_{j=1}^N X_j[f_j])|v_1 \otimes \dots v_N\rangle = \sum_{j=1}^N |v_1 \otimes \dots v_{j-1} \otimes (X_j[f_j])v_j \otimes v_{j+1} \dots \otimes v_N\rangle.$$

The right $\widehat{\mathfrak{g}}_N$-action on $\mathcal{H}^\dagger_{\vec{\lambda}}$ is defined similarly. In what follows we use the following notation.

$$\rho_j(X[f_j])|v_1 \otimes \cdots v_N \otimes\rangle = |v_1 \otimes \cdots \otimes v_{j-1} \otimes (X[f_j])v_j \otimes v_{j+1} \otimes \cdots \otimes v_N\rangle$$
$$\rho_j(X[f]) = \rho_j(X[t_j(f)])$$

for a meromorphic function f on the curve C.

The complete pairing $\langle \ | \ \rangle$ defined in (1.3-8) defines a complete pairing

$$\langle \ | \ \rangle : \mathcal{H}^\dagger_{\vec{\lambda}} \times \mathcal{H}_{\vec{\lambda}} \longrightarrow \mathbf{C} \tag{2.2-3}$$

which is $\widehat{\mathfrak{g}}_N$-invariant:

$$\langle u\rho_j(X[f_j])|v\rangle = \langle u|\rho_j(X[f_j])v\rangle$$

Definition 2.2.2. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) = \mathcal{H}_{\vec{\lambda}}/\widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)})\mathcal{H}_{\vec{\lambda}}$$
$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) = \{\, \langle\Psi| \in \mathcal{H}^\dagger_{\vec{\lambda}} \mid \langle\Psi|a = 0 \quad \text{for any } a \in \widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)}) \,\}.$$

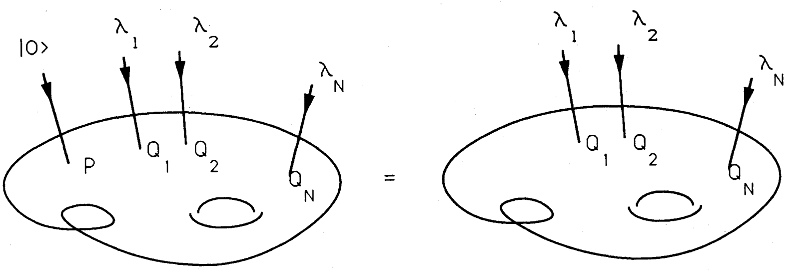

Figure 1.

We call $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ the *space of vacua* associated with $\mathfrak{X}^{(\infty)}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ the dual space of vacua associated with $\mathfrak{X}^{(\infty)}$.

Note that we have an isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \simeq \mathrm{Hom}_{\mathbf{C}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}), \mathbf{C}).$$

The above pairing (2.2-3) $\langle \quad | \quad \rangle$ induces a complete pairing

$$\langle \quad | \quad \rangle : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \longrightarrow \mathbf{C}.$$

For $\mathfrak{X}^{(\infty)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ let P be a non-singular point of the curve C and t a formal parameter of C at P. Put

$$\widetilde{\mathfrak{X}}^{(\infty)} = (C; Q_1, \ldots, Q_N, Q_{N+1}; t_1^{(\infty)}, \ldots, t_N^{(\infty)}, t_{N+1}^{(\infty)})$$

where $Q_{N+1} = P$ and $t_{N+1}^{(\infty)} = t$.

Now let us describe the properties which we call *propagation of vacua*. Since there is a canonical inclusion

$$\begin{aligned} \mathcal{H}_{\vec{\lambda}} &\longrightarrow \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0 \\ |v\rangle &\longrightarrow |v\rangle \otimes |0\rangle \end{aligned}$$

we have a canonical surjection

$$\widehat{\iota}^* \;:\; \mathcal{H}^\dagger_{\vec{\lambda}} \widehat{\otimes} \mathcal{H}^\dagger_0 \longrightarrow \mathcal{H}^\dagger_{\vec{\lambda}}\,.$$

Proposition 2.2.3. *The canonical surjection $\widehat{\iota}^*$ induces a canonical isomorphism*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}^{(\infty)}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})\,.$$

Proof. For an element $\langle\widetilde{\Psi}| \in \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}^{(\infty)})$ put $\langle\Psi| = \widehat{\iota}^*(\langle\widetilde{\Psi}|) \in \mathcal{H}^{\dagger}_{\vec{\lambda}}$. Choose $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$, $X \in \mathfrak{g}$ and $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$. Then by our definition we have

$$\sum_{j=1}^{N}\langle\Psi|\rho_j(X[f])|u\rangle = \sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[f])|u\otimes 0\rangle\,.$$

On the other hand, since f is regular at the point $Q_{N+1} = P$, we have

$$\langle\widetilde{\Psi}|\rho_{N+1}(X[f])|u\otimes 0\rangle = 0\,.$$

Hence we have

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[f])|u\otimes 0\rangle = \sum_{j=1}^{N+1}\langle\widetilde{\Psi}|\rho_j(X[f])|u\otimes 0\rangle = 0\,.$$

Thus we have $\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ and we have a linear mapping

$$\iota^* : \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}^{(\infty)}) \longrightarrow \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})\,.$$

First we shall show that the linear mapping ι^* is injective.

Assume that $\langle\Psi| = \iota^*(\langle\widetilde{\Psi}|) = 0$. By induction on p we show that

$$\text{(2.2-4)}\qquad \langle\widetilde{\Psi}|u\otimes v\rangle = 0\,, \quad \text{for all } u \in \mathcal{H}_{\vec{\lambda}} \text{ and } v \in F_p\mathcal{H}_0\,.$$

By our assumption we have

$$\langle\Psi|u\rangle = \langle\widetilde{\Psi}|u\otimes 0\rangle = 0\,.$$

Hence (2.2-4) is true for $p = 0$. Next assume that (2.2-4) holds for p. Choose an element $X(m)|v\rangle \in F_{p+1}\mathcal{H}_0$, where $|v\rangle \in F_p\mathcal{H}_0$. Choose a meromorphic function $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+1} Q_j))$ and a positive integer M such that

$$\text{(2.2-5)}\qquad f \equiv \eta^m \mod (\eta^M)$$

and that

$$\text{(2.2-6)}\qquad X\otimes\eta^k|v\rangle = 0 \quad \text{for all } k \geq M\,.$$

Then we have

$$\begin{aligned}\langle\widetilde{\Psi}|u\otimes X(m)|v\rangle &= \langle\widetilde{\Psi}|u\otimes(X[f])v\rangle\\ &= -\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[f])u\otimes v\rangle\\ &= 0\end{aligned}$$

since by the induction hypothesis $\langle\widetilde{\Psi}|\rho_j(X[f])u\otimes v\rangle = 0$. Thus (2.2-4) holds for $p+1$. Thus $\langle\widetilde{\Psi}|u\otimes v\rangle = 0$ for any $|u\otimes v\rangle \in \mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_0$. Hence, $\langle\widetilde{\Psi}| = 0$.

Next we shall show that ι^* is surjective. For that purpose, to a given $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ we attach an element $\langle\widetilde{\Psi}| \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_0, \mathbf{C}) = \mathcal{H}^\dagger_{\vec{\lambda}}\hat{\otimes}\mathcal{H}^\dagger_0$. The linear functional $\langle\widetilde{\Psi}|$ is defined inductively as a linear mapping of $\mathcal{H}_{\vec{\lambda}}\otimes F_p\mathcal{H}_0$ to $\mathbf{C}$ as follows. First define

$$\langle\widetilde{\Psi}|u\otimes 0\rangle = \langle\Psi|u\rangle \quad \text{for any } u\in\mathcal{H}_{\vec{\lambda}}.$$

Then we have

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[g])|u\otimes 0\rangle) = \sum_{j=1}^{N}\langle\Psi|\rho_j(X[g])|u\rangle = 0$$

for any element $g\in H^0(C,\mathcal{O}_C(*\sum_{j=1}^N Q_j))$.

Now assume that $\langle\widetilde{\Psi}|$ is defined as a linear mapping of $\mathcal{H}_{\vec{\lambda}}\otimes F_p\mathcal{H}_0$ to $\mathbf{C}$ with

(2.2-7)
$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\rho_j(X[g])|u\otimes v\rangle = 0$$

for any $|u\otimes v\rangle\in\mathcal{H}_{\vec{\lambda}}\otimes F_p\mathcal{H}_0$ and $g\in H^0(C,\mathcal{O}_C(*\sum_{j=1}^N Q_j))$. Then, on $\mathcal{H}_{\vec{\lambda}}\otimes F_{p+1}\mathcal{H}_0$ the linear mapping $\langle\widetilde{\Psi}|$ is defined by

(2.2-8)
$$\langle\widetilde{\Psi}|u\otimes X(m)v\rangle = -\sum_{j=1}^{N}\langle\widetilde{\Psi}|(\rho_j(X[f])u\otimes v\rangle \quad \text{for any } u\in\mathcal{H}_{\vec{\lambda}}, v\in F_p\mathcal{H}_0$$

where a meromorphic function f is chosen in the same way as in (2.2-5) and (2.2-6). It is easy to show that this is well-defined and has the property

$$\sum_{j=1}^{N+1} \langle \widetilde{\Psi} | \rho_j(X[f]) = 0$$

for each element $f \in H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N+1} Q_j))$. A straightforward calculation shows the equality

$$\begin{aligned}\langle \widetilde{\Psi} | u \otimes X(m_1) Y(m_2) v\rangle &- \langle \widetilde{\Psi} | u \otimes Y(m_2) X(m_1) v\rangle \\ &= \langle \widetilde{\Psi} | u \otimes ([X,Y](m_1+m_2) + \ell \cdot (X,Y) m_1 \delta_{m_1+m_2,0}) v\rangle.\end{aligned}$$

This equality shows that the $\langle \widetilde{\Psi}|$ is defined at least as a linear mapping form $\mathcal{H}_{\vec{\lambda}} \otimes M_0$ to $\mathbf{C}$, where M_0 is the Verma module associated to the trivial representation of the affine Lie algebra $\widehat{\mathfrak{g}}$.

To show that $\langle \widetilde{\Psi}|$ is a linear form on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$, it is enough to show the equality

(2.2-9) $$\langle \widetilde{\Psi} | u \otimes X_\theta(-1)^{\ell+1} |0\rangle = 0\,.$$

To prove (2.2-9) we first show

$$\langle \widetilde{\Psi} | u \otimes X_\theta(-1)^n |0\rangle = 0$$

for sufficiently large n depending on $|u\rangle$. Let $f \in H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N+1} Q_j))$ be a meromorphic function on C which satisfies the conditions (2.2-5) and (2.2-6) for $m = -1$. By Corollary 1.4.6 there is a positive integer n depending on $|u\rangle$ such that for any j, $j = 1, \ldots, N$, we have

(2.2-10) $$\rho_j(X_\theta[f])^k |u\rangle = 0, \quad \text{if } k \geq n/N\,.$$

Applying the formula (2.2-8), by (2.2-10) we obtain

$$\begin{aligned}\langle \widetilde{\Psi} | u \otimes X_\theta(-1)^n |0\rangle &= \langle \widetilde{\Psi} | u \otimes (X_\theta[f])^n |0\rangle \\ &= (-1)^n \sum_{n_1+\ldots+n_N=n} \frac{n!}{n_1! n_2! \ldots n_N!} \langle \widetilde{\Psi}| \prod_{j=1}^{N} \rho_j(X_\theta[f])^{n_j} u \otimes 0\rangle \\ &= 0.\end{aligned}$$

Put

$$E = X_{-\theta}(1), \quad F = X_\theta(-1), \quad H = [E, F]\,.$$

Then $\{E, F, H\}$ forms a $\mathfrak{sl}(2, \mathbf{C})$-triplet. Let U_u be a vector subspace of the Verma module M_0 such that $\langle\widetilde{\Psi}|$ is zero on $\mathbf{C}|u\rangle \otimes M_0$ and N_u the $\mathfrak{sl}(2, \mathbf{C})$-module generated by $|0\rangle$. Then the above equality $\langle\widetilde{\Psi}|u \otimes F^n|0\rangle = 0$ means that the $\mathfrak{sl}(2, \mathbf{C})$-module $R_u = N_u + U_u/U_u$ is of finite dimension. Since we have

$$H|0\rangle = \ell|0\rangle,$$

by representation theory of $\mathfrak{sl}(2, \mathbf{C})$ we conclude that $F^{\ell+1}|0\rangle = 0$ in R_u. This means that

$$\langle\widetilde{\Psi}|u \otimes X_\theta(-1)^{\ell+1}|0\rangle = 0\,.$$

Thus we obtain $\langle\widetilde{\Psi}| \in \mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}^{(\infty)})$ such that $\widetilde{\iota}^*(\langle\widetilde{\Psi}|) = \langle\Psi|$. The details of the above argument can be found in [TK1,2.3)]. Q.E.D.

Corollary 2.2.4. *There is a canonical isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \simeq \mathcal{V}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}^{(\infty)})$$

Remark 2.2.5. Proposition 2.2.3 and Corollary 2.2.4 say that in the study of the space of vacua and its dual space attached to an N-pointed stable curve with formal neighbourhoods we can add as many points with formal neighbourhoods as possible we need. Therefore, as we mentioned above, we can always assume that the condition (Q) is satisfied. Below this fact will be often used and play an essential role to prove important theorems.

For an element $\mu \in P_\ell$ put

$$\mu^\dagger = -w(\mu)$$

where w is the longest element of the Weyl group of the simple Lie algebra $\mathfrak{g}$ (in other word, $w(\Delta_+) = \Delta_-$). Note that $\mu^\dagger$ is also characterized by the fact that $-\mu^\dagger$ is the lowest weight of the $\mathfrak{g}$-module V_μ.

For an N-pointed stable curve $\mathfrak{X}^{(\infty)} = (C\,; Q_1, Q_2, \dots, Q_N\,; t_1^{(\infty)}, t_2^{(\infty)}, \dots, t_N^{(\infty)})$ with formal neighbourhoods, assume that the curve C has a double point P. Let $\nu : \widetilde{C} \to C$ be the normalization at the point P. (See, for example, [Se, Chap. IV, Section 1].) Put $\nu^{-1}(P) = \{P', P''\}$. Furthermore we introduce formal neighbourhoods $t'^{(\infty)}$ and $t''^{(\infty)}$ at P' and P'', respectively.

In the proof of the following Proposition 2.2.6 we shall use the results of Theorem 2.4.1. We shall not use Proposition 2.2.6 in the proof of the theorem.

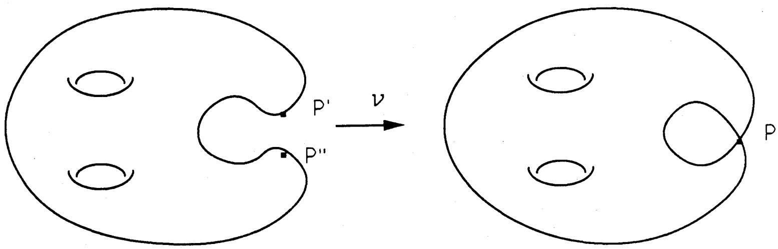

Figure 2.

Proposition 2.2.6. *Under the above notation, for an N-pointed stable curve* $\mathfrak{X}^{(\infty)} = (C\,;\,Q_1, Q_2, \dots, Q_N\,;\,t_1^{(\infty)}, t_2^{(\infty)}, \dots, t_N^{(\infty)})$ *with formal neighbourhoods, put*

$$\widetilde{\mathfrak{X}}^{(\infty)} = (\widetilde{C}; P', P'', Q_1, \dots, Q_N; t'^{(\infty)}, t''^{(\infty)}, t_1^{(\infty)}, \dots, t_N^{(\infty)}).$$

Then there is a canonical isomorphism

$$\bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)}) \xrightarrow{\sim} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}).$$

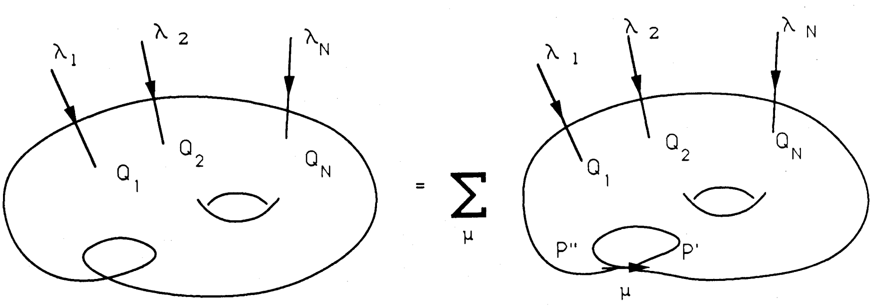

Figure 3.

Proof. The diagonal action of $\mathfrak{g}$ on $V_\mu \otimes V_{\mu^\dagger}$ makes $V_\mu \otimes V_{\mu^\dagger}$ a $\mathfrak{g}$-module and it contains a trivial $\mathfrak{g}$-module with multiplicity one. Let $|0_{\mu,\mu^\dagger}\rangle$ be a basis of the trivial $\mathfrak{g}$-submodule of $V_\mu \otimes V_{\mu^\dagger}$ such that $T(|0_{\mu,\mu^\dagger}\rangle) = |0_{\mu^\dagger,\mu}\rangle$, where T is a canonical isomorphism

$$T : V_\mu \otimes V_{\mu^\dagger} \longrightarrow V_{\mu^\dagger} \otimes V_\mu$$

defined by $T(a \otimes b) = b \otimes a$. Hence $\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$ contains a subspace

$$\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}} \supset |0_{\mu,\mu^\dagger}\rangle \otimes \mathcal{H}_{\vec{\lambda}} \simeq \mathcal{H}_{\vec{\lambda}}.$$

For any element $\langle\widetilde{\Psi}| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})$, define $\langle\Psi| \in \mathcal{H}^\dagger_{\vec{\lambda}}$ by

$$\langle\Psi|\Phi\rangle = \langle\widetilde{\Psi}|0_{\mu,\mu^\dagger} \otimes \Phi\rangle \quad \text{for all } |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}.$$

Then, for any meromorphic function $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$ we have

$$\begin{aligned}\sum_{j=1}^{N}\langle\Psi|\rho_j(X[f])|\Phi\rangle &= \sum_{j=1}^{N}\langle\widetilde{\Psi}|(0_{\mu,\mu^\dagger}) \otimes \rho_j(X[f])\Phi\rangle \\ &= \sum_{j=1}^{N+2}\langle\widetilde{\Psi}|\rho_j(X[f])|0_{\mu,\mu^\dagger} \otimes \Phi\rangle = 0\end{aligned}$$

since if we regard f as a meromorphic function on $\widetilde{C}$, we have $f(P') = f(P'')$ and $\rho_{P'}(X[f])|0_{\mu,\mu^\dagger}\rangle + \rho_{P''}(X[f])|0_{\mu,\mu^\dagger}\rangle = 0$. Hence we have

$$\sum_{j=1}^{N}\langle\Psi|\rho_j(X[f]) = 0 \quad \text{for any } f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)).$$

Thus we have a canonical **C**-linear mapping

$$\iota_\mu : \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)}) \longrightarrow \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}).$$

We shall show that the mapping ι_μ is injective. For that purpose, first we show that for $\langle\Psi| \in \iota_\mu(\langle\widetilde{\Psi}|)$, $\langle\widetilde{\Psi}| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})$ we have

$$\langle\Psi|X(P)|\Phi\rangle dP = \langle\widetilde{\Psi}|X(P)|0_{\mu,\mu^\dagger} \otimes \Phi\rangle dP. \tag{2.2-11}$$

Note that by Claim 3 of the proof of Theorem 2.4.1, the expansion of the left hand side of (2.2-11) at Q_j with respect to the formal parameter ξ_j has the form

$$\sum_{n\in\mathbf{Z}}\langle\Psi|\rho_j(X(n))|\Phi\rangle\xi_j^{-n-1}d\xi_j.$$

Similarly the right hand side of (2.2-11) has the expansion

$$\sum_{n\in\mathbf{Z}}\langle\widetilde{\Psi}|\rho_j(X(n))|0_{\mu,\mu^\dagger}\otimes\Phi\rangle\xi_j^{-n-1}d\xi_j$$
$$=\sum_{n\in\mathbf{Z}}\langle\widetilde{\Psi}|(0_{\mu,\mu^\dagger})\otimes\rho_j(X(n))\Phi\rangle\xi_j^{-n-1}d\xi_j$$
$$=\sum_{n\in\mathbf{Z}}\langle\Psi|\rho_j(X(n))|\Phi\rangle\xi_j^{-n-1}d\xi_j\,.$$

Hence the equality (2.2-11) holds. Similar argument shows the equality

$$\langle\Psi|X_1(P_1)\ldots X_M(P_M)|\Phi\rangle dP_1\ldots dP_M =$$
$$\langle\widetilde{\Psi}|X_1(P_1)\ldots X_M(P_M)|0_{\mu,\mu^\dagger}\otimes\Phi\rangle dP_1\ldots dP_M\,.$$

Now assume that $\langle\Psi|=0$. By Theorem 2.4.1, 3) we have

$$\langle\widetilde{\Psi}|X_2(P_2)\ldots X_M(P_M)|\rho_{P'}(X_1(n))0_{\mu,\mu^\dagger}\otimes\Phi\rangle=0\,.$$

Applying again Theorem 2.4.1, 3), we obtain

$$\langle\widetilde{\Psi}|\rho_{P'}(X_2(n_2)X_1(n_1))0_{\mu,\mu^\dagger}\otimes\Phi\rangle=0$$
$$\langle\widetilde{\Psi}|\rho_{P'}(X_1(n_1))\rho_{P''}(X_2(n_2))0_{\mu,\mu^\dagger}\otimes\Phi\rangle=0$$
$$\langle\widetilde{\Psi}|\rho_{P''}(X_1(n_1)X_2(n_2))0_{\mu,\mu^\dagger}\otimes\Phi\rangle=0\,.$$

Repeating the same process we can show that

$$\langle\widetilde{\Psi}|\widetilde{\Phi}\rangle=0\quad\text{for any}\quad\widetilde{\Phi}\in\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$$

since $\mathcal{H}_\mu\otimes\mathcal{H}_{\mu^\dagger}$ is an irreducible $\widehat{\mathfrak{g}}\times\widehat{\mathfrak{g}}$-module. Hence ι_μ is injective. Let us consider a $\mathbf{C}$-linear homomorphism

$$\iota:\bigoplus_{\mu\in P_\ell}\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})\xrightarrow{\oplus\iota_\mu}\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})\,.$$

We shall show that ι is injective. For that purpose, to the points P' and P'' we associate right $\mathfrak{g}$-modules and integrable right $\widehat{\mathfrak{g}}$-modules.

Fix an element $\langle\Psi|\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$. Let h be a meromorphic function on $\widetilde{C}$ such that

$$\begin{aligned}&h\in H^0(\widetilde{C},\mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^N Q_j))\\ (2.2\text{-}12)\qquad&h(P')=1\\&h(P'')=0.\end{aligned}$$

If h' satisfies also the properties (2.2-12), then $h - h'$ can be regarded as a meromorphic function on C and $h - h' \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$. Hence, for each $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$

$$\sum_{j=1}^{N} \langle \Psi | \rho_j(X[h]) | u \rangle$$

is independent of the choice of a meromorphic function h satisfying (2.2-12). For each element $X \in \mathfrak{g}$ define $\langle \Psi | \rho_{P'}(X) \in \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C})$ by

$$\langle \Psi | \rho_{P'}(X) | u \rangle = -\sum_{j=1}^{N} \langle \Psi | \rho_j(X[h]) | u \rangle, \qquad |u\rangle \in V_{\vec{\lambda}}$$

where h satisfies (2.2-12). This is well-defined.

Next for $X, Y \in \mathfrak{g}$ define $\langle \Psi | \rho_{P'}(X) \rho_{P'}(Y) \in \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C})$ by

$$\langle \Psi | \rho_{P'}(X) \rho_{P'}(Y) | u \rangle = \sum_{j_1=1, j_2=1}^{N} \langle \Psi | \rho_{j_1}(X[h_1]) \rho_{j_2}(Y[h_2]) | u \rangle \qquad |u\rangle \in V_{\vec{\lambda}}$$

where h_1 and h_2 satisfy (2.2-12). The definition is independent of the choice of h_1 by the same reason as above. That the definition is independent of the choice of h_2 is proved as follows. Since $h_2 dh_1$ is a meromorphic one form on $\widetilde{C}$ having poles only at $Q_1, \dots, Q_N$, we have $\sum_{j=1}^N \mathrm{Res}_{Q_j}(h_2 dh_1) = 0$. Therefore, we have the equality

$$\begin{aligned} &\sum_{j_1, j_2}^{N} \langle \Psi | \rho_{j_1}(X[h_1]) \rho_{j_2}(Y[h_2]) | u \rangle \\ &= \sum_{j_1 \neq j_2} \langle \Psi | \rho_{j_2}(Y[h_2]) \rho_{j_1}(X[h_1]) | u \rangle + \sum_{j=1}^{N} \langle \Psi | \rho_j([X, Y][h_1 h_2]) | u \rangle. \end{aligned}$$

The right hand side of the equality shows the independence of the choice of h_2, since $h_1 h_2$ also satisfies the properties (2.2-12). Moreover the above equality shows the equality

$$\langle \Psi | (\rho_{P'}(X) \rho_{P'}(Y) - \rho_{P'}(Y) \rho_{P'}(X)) = \langle \Psi | \rho_{P'}([X, Y]).$$

In this way we can define a right $\mathfrak{g}$-module $U(\langle \Psi |) \subset \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C})$ at the point P'. By the same way we can construct a right $\mathfrak{g}$-module at the point P''.

More generally, we can define an integrable right $\mathfrak{g}$-module $\widehat{U}(\langle\Psi|) \subset \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbf{C})$. For example, $\langle\Psi|\rho_{P'}(X(n))$ is defined as follows. Let g be a meromorphic function on $\widetilde{C}$ such that

$$
\begin{aligned}
&g \in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^{N} Q_j)) \\
&g \equiv \xi'^{n} \mod (\xi'^{M}) \quad \text{at} \quad P' \qquad (2.2\text{-}13)\\
&g(P'') = 0
\end{aligned}
$$

where $\xi' = t'^{-1}(\xi)$ is a formal parameter at the point P'. Then, define $\langle\Psi|\rho_{P'}(X(n))$ by

$$\langle\Psi|\rho_{P'}(X(n))|u\rangle = -\sum_{j=1}^{N}\langle\Psi|\rho_j(X[g])|u\rangle.$$

The definition is independent of the choice of a meromorphic function g satisfying (2.2-13). Similarly we can define $\langle\Psi|\rho_{P'}(X(n))\rho_{P'}(Y(m)) \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbf{C})$ and we have the equality

(2.2-14)
$$
\begin{aligned}
\langle\Psi|(\rho_{P'}(X(n))\rho_{P'}(Y(m)) &- \langle\Psi|\rho_{P'}(Y(m))\rho_{P'}(X(n))) \\
&= \langle\Psi|\rho_{P'}([X,Y](m+n)) + \ell\cdot(X,Y)n\delta_{n+m,0}\langle\Psi|.
\end{aligned}
$$

In this way we can construct a right $\widehat{\mathfrak{g}}$-module $\widehat{U}(\langle\Psi|) \subset \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbf{C})$. Since the action of $\rho_j(X_\alpha[g])$, $X_\alpha \in \mathfrak{g}_\alpha$ is locally nilpotent by Corollary 1.4.6, the action of $\rho_{P'}(X_\alpha(m))$ on $\widehat{U}(\langle\Psi|)$ is locally nilpotent. Hence $\widehat{U}(\langle\Psi|)$ is an integrable right $\widehat{\mathfrak{g}}$-module of level ℓ.

Thus to the point P' we associate a right $\mathfrak{g}$-module

$$U(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) = \bigcup_{\langle\Psi|\in\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})} U(\langle\Psi|)$$

and an integrable right $\widehat{\mathfrak{g}}$-*module*

$$\widehat{U}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) = \bigcup_{\langle\Psi|\in\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})} \widehat{U}(\langle\Psi|)$$

of level ℓ. Since $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ is finite-dimensional, by Theorem 4.2.4, $U(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}))$ is a finite-dimensional right $\mathfrak{g}$-module. By (2.2-14) we have an irreducible decomposition

$$U(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) = \bigoplus_{\mu \in P_\ell} V_\mu^{\dagger \oplus n_\mu}$$

(2.2-15)

$$\widehat{U}(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) = \bigoplus_{\mu \in P_\ell} \mathcal{H}_\mu^{\dagger \oplus n_\mu}.$$

Now we are ready to prove the injectivity of ι. For an element $\langle\widetilde{\Psi}| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})$, put $\langle\Psi| = \iota_\mu(\langle\widetilde{\Psi}|)$ and choose a meromorphic function h on $\widetilde{C}$ satisfying (2.2-12). Then we have

$$\begin{aligned}&\langle\Psi|\rho_{P'}(X_1)\cdots\rho_{P'}(X_k)|u\rangle\\ &\qquad= (-1)^k \sum_{j_1=1,\dots,j_k=1}^{N} \langle\Psi|\rho_{j_1}(X_1[h])\cdots\rho_{j_k}(X_k[h])|u\rangle\\ &\qquad= (-1)^{k+1}\langle\widetilde{\Psi}|\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_k(0))|0_{\mu,\mu^\dagger}\otimes u\rangle.\end{aligned}$$

Since $\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_k(0))|0_{\mu,\mu^\dagger}\rangle$'s generate an irreducible left $\mathfrak{g}$-module isomorphic to $V_\mu^\dagger$, we conclude

$$U(\langle\Psi|) \subset V_\mu^{\dagger\oplus n_\mu}.$$

Hence, for $\langle\widetilde{\Psi}_\mu| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})$ and $\langle\widetilde{\Psi}_\nu| \in \mathcal{V}_{\nu,\nu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)})$, we have

$$U(\langle\widetilde{\Psi}_\mu|) \cap U(\langle\widetilde{\Psi}_\nu|) = \emptyset.$$

This means that ι is injective, since ι_μ is injective.

Finally let us prove that ι is surjective. By (2.2-15) for an element $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ we have a decomposition

$$\langle\Psi| = \sum_{\mu\in P_\ell} \langle\Psi_\mu|, \qquad \langle\Psi_\mu| \in V_\mu^{\dagger\oplus n_\mu}.$$

We construct $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}, \mathbf{C})$ as follows. First note that $V_\mu \otimes V_{\mu^\dagger}$ is generated by elements

$$\begin{gathered}\rho_{P'}(X_1)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\cdots\rho_{P''}(Y_m)|0_{\mu,\mu^\dagger}\rangle\\ X_1,\dots,X_n,Y_1,\dots,Y_m \in \mathfrak{g}.\end{gathered}$$

Moreover, $\langle\widetilde{\Psi}_\mu|$ defines a right $\widehat{\mathfrak{g}}$-module $\widehat{U}(\langle\widetilde{\Psi}_\mu|) \subset \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbf{C})$. For each element $|v\rangle \in \mathcal{H}_{\vec{\lambda}}$ define

$$\langle\widetilde{\Psi}_\mu|0_{\mu,\mu^\dagger} \otimes v\rangle = \langle\Psi|v\rangle.$$

Define

$$\begin{aligned}&\langle\widetilde{\Psi}_\mu|\rho_{P'}(X_1)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\cdots\rho_{P''}(Y_m)0_{\mu,\mu^\dagger} \otimes v\rangle\\ &\quad= (-1)^m\langle\Psi_\mu|\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_n(0))\rho_{P'}(Y_m(0))\cdots\rho_{P'}(Y_1(0))v\rangle.\end{aligned}$$

This is well-defined, since the diagonal action of $\mathfrak{g}$ on $\mathbf{C}|0_{\mu,\mu^\dagger}\rangle$ is trivial. This defines $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(V_\mu \otimes V_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbf{C})$. Now assume that we have already defined $\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(F_p\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbf{C})$ for non-negative integers p and q. Choose an element $\rho_{P'}(X(m))|u \otimes u' \otimes v\rangle \in F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}$ with $|u \otimes u'\rangle \in F_p\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger}$. Choose a meromorphic function f on $\widetilde{C}$ such that

$$\begin{aligned}&f \in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^N Q_j + *P' + *P''))\\ &f \equiv \xi'^m \mod (\xi'^M) \quad \text{at } P'\\ &f \equiv 0 \mod (\xi''^M) \quad \text{at } P''.\end{aligned}$$

Here we choose the positive integer M in such a way that $\rho_{P'}(X(n))|u\rangle = 0$ and $\rho_{P''}(X(n))|u\rangle = 0$ for all $n \geq M$. Then we define

$$\langle\widetilde{\Psi}_\mu|\rho_{P'}(X(m))|u \otimes u' \otimes v\rangle = -\sum_{j=1}^N\langle\widetilde{\Psi}_\mu|\rho_j(X[f])|u \otimes u' \otimes v\rangle.$$

By the similar argument to the proof of Proposition 2.2.4 we can show that the definition is independent of the choice of a meromorphic function f satisfying the above conditions and we have

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbf{C}).$$

Similarly we can define

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(F_p\mathcal{H}_\mu \otimes F_{q+1}\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbf{C}).$$

In this way we can show the existence of

$$\langle\widetilde{\Psi}_\mu| \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbf{C}).$$

Moreover, we can show that

$$\langle \widetilde{\Psi}_\mu | \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)}).$$

By our construction we have $\iota_\mu(\langle \widetilde{\Psi}_\mu |) = \langle \Psi_\mu |$. Q.E.D.

Corollary 2.2.7. *There is a canonical isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \xrightarrow{\sim} \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}^{(\infty)}).$$

Example 2.2.8. Let us consider the space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ with $C = \mathbf{P}^1$. We use the results in 2.4, especially Theorem 2.4.1.

Let z be a global inhomogeneous coordinate of $\mathbf{P}^1$. For N points $a_1, \dots, a_n \in \mathbf{C}$, put

$$u_j = z - a_j, \quad j = 1, \dots, N$$

and

$$\mathfrak{X}^{(\infty)} = (\mathbf{P}^1; a_1, \dots, a_N; u_1, \dots, u_N).$$

Fix $\vec{\lambda} \in (P_\ell)^N$. Let us consider a homomorphism

$$i : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C})$$

defined by

$$i(\langle \Psi |)(|\Phi_0\rangle) = \langle \Psi | \Phi_0 \rangle, \quad |\Phi_0\rangle \in V_{\vec{\lambda}}.$$

Let us show that the homomorphism i defines an injective homomorphism

$$i : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbf{C}). \tag{2.2-16}$$

For that purpose, for an element $X \in \mathfrak{g}$ first consider a meromorphic one form $F = \langle \Psi | X(z) | \Phi_0 \rangle dz$ in Theorem 2.4.1. By Theorem 2.4.1, 5) we have

$$\langle \Psi | X(z) | \Phi_0 \rangle dz = \sum_{j=1}^N \frac{1}{z - a_j} \langle \Psi | \rho_j(X) \Phi_0 \rangle dz \tag{2.2-17}$$

since the left hand side minus the right hand side is a holomorphic one form on $\mathbf{P}^1$, hence zero. By Theorem 2.4.1 3) we have

$$\langle \Psi | \rho_j(X(n)) \Phi_0 \rangle = \operatorname*{Res}_{z=a_j} (u_j^n \langle \Psi | X(z) | \Phi_0 \rangle dz).$$

Since $\langle\Psi|X(z)|\Phi_0\rangle dz$ is a global one form on $\mathbf{P}^1$, we have

$$\sum_{j=1}^{N}\langle\Psi|\rho_j(X)\Phi_0\rangle = \sum_{j=1}^{N} \operatorname*{Res}_{z=a_j}(\langle\Psi|X(z)|\Phi_0\rangle dz) = 0.$$

Hence, $i(\langle\Psi|) \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbf{C})$. By the similar arguments, using Theorem 2.4.1, 4) and 5) we have

$$\begin{aligned}\langle\Psi|X(z)Y(w)|\Phi_0\rangle dzdw =& \frac{\ell\cdot(X,Y)}{(z-w)^2}\langle\Psi|\Phi_0\rangle dzdw \\ &+ \frac{1}{z-w}\langle\Psi|[X,Y](w)|\Phi_0\rangle dzdw \\ &+ \sum_{j=1}^{N}\frac{1}{z-a_j}\langle\Psi|Y(w)|\rho_j(X)\Phi_0\rangle dzdw \\ &+ \sum_{j=1}^{N}\frac{1}{w-a_j}\langle\Psi|X(z)|\rho_j(Y)\Phi_0\rangle dzdw.\end{aligned}$$

The right hand side is uniquely determined by $i(\langle\Psi|)$. In this way we can show that $i(\langle\Psi|)$ determines uniquely the correlation functions of currents

$$\langle\Psi|X_1(z_1)\dots X_A(z_A)|\Phi_0\rangle dz_1\dots dz_A$$

hence, determines uniquely the bilinear pairing

$$\begin{array}{ccc}\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})\times\mathcal{H}_{\vec{\lambda}} & \longrightarrow & \mathbf{C} \\ (\langle\Psi|,|\Phi\rangle) & \longmapsto & \langle\Psi|\Phi\rangle.\end{array}$$

Hence the mapping i is injective.

Finally consider the case $N=3$. In this case the image

$$i(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) \subset \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C})$$

is characterized by the following fusion rule ([GW], [TK1], [TK2]). For $\vec{\lambda} = (\mu,\nu,\lambda) \in P_l^3$, put

(2.2-18) $$W_{\mu,\nu,\lambda} = \left\{\phi \in \mathrm{Hom}_{\mathfrak{g}}(V_\mu\otimes V_\nu\otimes V_\lambda, \mathbf{C}) \mid \text{condition } (*)\right\}$$

where the condition $(*)$ is given as follows. Let $\mathfrak{k}_\theta = \mathbf{C}X_\theta \oplus \mathbf{C}X_{-\theta} \oplus \mathbf{C}[X_\theta, X_{-\theta}]$ be the principal 3-dimensional subalgebra of $\mathfrak{g}$, and let

$$V_\lambda = \bigoplus_{j=0}^{\ell/2} W_{\lambda,j}$$

be the decomposition to the spin-j homogeneous components of $\mathfrak{k}_\theta$-modules. Then the condition $(*)$ is

$(*)$ $\quad \phi|_{W_{\mu,h}\otimes W_{\nu,i}\otimes W_{\lambda,j}} = 0 \quad$ if $\quad h+i+j > l.$

Then, we have

$$i(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})) = W_{\mu,\nu,\lambda}.$$

2.3. Action of $\mathcal{D}$

For an N-pointed stable curve $\mathfrak{X}^{(\infty)} = (C\,;\,Q_1, Q_2, \ldots, Q_N\,;\,t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ of genus g with formal neighbourhoods and an N-tuple $\vec{h} = (h_1, \ldots, h_N) \in \mathcal{D}^{\oplus N}$, let us define $\vec{h}\circ\mathfrak{X}^{(\infty)}$ by

$$\vec{h}\circ\mathfrak{X}^{(\infty)} = (C\,;\,Q_1, \ldots, Q_N\,;\,h_1\circ t_1^{(\infty)}, \ldots, h_N\circ t_N^{(\infty)}). \tag{2.3-1}$$

This defines a left $\mathcal{D}^{\oplus N}$ action on the set of N-pointed stable curves of genus g with formal neighbourhoods.

By Lemma 1.4.2, for an element $h \in \mathcal{D}^1$, there exists the unique derivation $\underline{l} \in \underline{d}^1$ with $h = \exp(\underline{l})$.

Definition 2.3.1. The $(\mathcal{D}^1)^{\oplus N}$-actions on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}^\dagger_{\vec{\lambda}}$ are defined by

$$G[\vec{h}]|\Phi\rangle = \prod_{j=1}^{N} \rho_j(\exp(-T[\underline{l}_j]))|\Phi\rangle, \tag{2.3-2}$$

$$\langle\Psi|G[\vec{h}] = \prod_{j=1}^{N} \langle\Psi|\rho_j(\exp(-T[\underline{l}_j])), \tag{2.3-3}$$

where

$$\vec{h} = (h_1, \ldots, h_N) \in (\mathcal{D}^1)^{\oplus N} \quad h_j = \exp(\underline{l}_j) \quad \underline{l}_j \in \underline{d}^1.$$

Lemma 2.3.2. *For an element $\vec{h} \in (\mathcal{D}^1)^{\oplus N}$, we have*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\vec{h}\circ\mathfrak{X}^{(\infty)}) = \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})G(\vec{h})^{-1}.$$

Remark 2.3.3. The above Lemma says that the space of vacua attached to $\mathfrak{X}^{(\infty)} = (C\,;\,Q_1, Q_2, \ldots, Q_N\,;\,t_1^{(\infty)}, t_2^{(\infty)}, \ldots, t_N^{(\infty)})$ does essentially depend on the first infinitesimal neighbourhoods. This fact will be clarified in Section 4 below.

2.4. Correlation functions

Let C be a semi-stable curve and ω_C its dualizing sheaf. Put $C^M = \overbrace{C \times \ldots \times C}^{M}$. Then C^M has singularities of codimension 1, but still we can define the dualizing sheaf ω_{C^M}, since C^M is locally a complete intersection. (See, for example, [BS1] or [Kl].) Moreover, we can show that

$$\omega_{C^M} = \omega_C^{\boxtimes M}$$

where $\pi_j : C^M \to C$ is the j-th projection and we define

$$\omega_C^{\boxtimes M} = \pi_1^* \omega_C \otimes \pi_2^* \omega_C \otimes \ldots \otimes \pi_M^* \omega_C \, .$$

(See, for example, [Kl].) Since C^M has singularities for a singular semi-stable curve, the (i,j)-th diagonal $\Delta_{ij} = \{(P_1, \ldots, P_N) | P_i = P_j\}$ of C^M is only a Weil divisor and not a Cartier divisor. But it is well-known that $2\Delta_{ij}$ is a Cartier divisor.

Theorem 2.4.1. *Fix $\langle \Psi | \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}^{(\infty)})$. For each non-negative integer M the data*

$$X_1, X_2, \ldots, X_M \in \mathfrak{g}, \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$$

define an element

$$F = \langle \Psi | X_1(P_1) X_2(P_2) \ldots X_M(P_M) | \Phi \rangle dP_1 dP_2 \ldots dP_M$$

of

$$H^0\Big(C^M, \omega_C^{\boxtimes M}\big(\sum_{1 \le i < j \le M} * \Delta_{ij} + \sum_{i=1}^{M} \sum_{j=1}^{N} * \pi_i^{-1}(Q_j)\big)\Big),$$

where $\Delta_{ij} = \{(P_1, \ldots, P_N) | P_i = P_j\}$ is the diagonal. The meromorphic form has the following properties.

0) *For $M = 0$, $F = <\Psi | \Phi>$ is the canonical pairing induced by the pairing (2.2-3).*

1) *F is linear with respect to $|\Phi\rangle$ and multi-linear with respect to X_i's.*

2) *For any permutation $\sigma \in \mathfrak{S}_M$, we have*

$$F = \langle \Psi | X_{\sigma(1)}(P_{\sigma(1)}) X_{\sigma(2)}(P_{\sigma(2)}) \cdots X_{\sigma(M)}(P_{\sigma(M)}) | \Phi \rangle dP_1 dP_2 \ldots dP_M \, .$$

For example, for a transposition $(i, i+1)$ *we have*

$$F = \langle \Psi | X_1(P_1) \cdots X_{i-1}(P_{i-1}) X_{i+1}(P_{i+1}) X_i(P_i) \\ X_{i+2}(P_{i+2}) \ldots X_M(P_M) | \Phi \rangle dP_1 dP_2 \ldots dP_M \, .$$

3) *For* $k = 1, \ldots, N$ *and* $\xi_k = t_k{}^{(\infty)-1}(\xi)$, *if* ξ_k *is a holomorphic coordinate, then we have the equality*

$$\oint_{C_k} \frac{d\xi_k}{2\pi\sqrt{-1}} \xi_k^n \langle \Psi | X(\xi_k) X_1(P_1) X_2(P_2) \ldots X_M(P_M) | \Phi \rangle \\ = \langle \Psi | X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \rho_k(X(n)) \Phi \rangle$$

where C_k *is a contour rounding only* Q_k *and containing no other* Q_j*'s nor* P_i*'s.*

4) *For a local holomorphic coordinate* z *around a nonsingular point we have the following equality.*

$$\langle \Psi | X(P) Y(P') X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \Phi \rangle \\ = \frac{\ell \cdot (X, Y)}{(z(P) - z(P'))^2} \langle \Psi | X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \Phi \rangle \\ + \frac{1}{z(P) - z(P')} \langle \Psi | [X, Y](P') X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \Phi \rangle \\ + \text{regular at } P = P' \, .$$

5) *For a local holomorphic coordinate* z *around* Q_i *and for* $|\Phi\rangle \in V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$, *we have an equality*

$$\langle \Psi | X(P) X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \Phi \rangle \\ = \frac{1}{z(P) - z(Q_i)} \langle \Psi | X_1(P_1) X_2(P_2) \cdots X_M(P_M) | \rho_i(X) \Phi \rangle \\ + \text{regular at } P = Q_i \, .$$

These functions meromorphic one forms F are called *correlation functions of currents*.

Proof. Choose $M+1$ non-singular points $P_1, P_2, \ldots, P_M, P$ of the curve C and their formal neighbourhoods $t_{N+1}^{(\infty)}, t_{N+2}^{(\infty)}, \ldots, t_{N+M+1}^{(\infty)}$. Put

$$\hat{\mathfrak{X}}^{(\infty)} = (C; Q_1, \ldots, Q_N, Q_{N+1}, \ldots, Q_{N+M+1}; t_1^{(\infty)}, \ldots, t_{N+M+1}^{(\infty)})$$
$$\tilde{\mathfrak{X}}^{(\infty)} = (C; Q_1, \ldots, Q_N, Q_{N+1}, \ldots, Q_{N+M}; t_1^{(\infty)}, \ldots, t_{N+M}^{(\infty)})$$

where $Q_{N+i} = P_i$, $i = 1, \dots, M$ and $Q_{N+M+1} = P$. By Proposition 2.2.3 there are canonical isomorphisms

$$\iota_M : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \simeq \mathcal{V}^\dagger_{\vec{\lambda},\vec{0}_M}(\widetilde{\mathfrak{X}}^{(\infty)})$$
$$\iota_{M+1} : \mathcal{V}^\dagger_{\vec{\lambda},\vec{0}_M}(\widetilde{\mathfrak{X}}^{(\infty)}) \simeq \mathcal{V}^\dagger_{\vec{\lambda},\vec{0}_{M+1}}(\widehat{\mathfrak{X}}^{(\infty)})$$

where $\vec{0}_k = (\overbrace{0, \dots, 0}^{k})$. For $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$ put

$$\langle\widetilde{\Psi}| = \iota_M(\langle\Psi|)\,, \quad \langle\widehat{\Psi}| = \iota_{M+1}(\langle\widetilde{\Psi}|)\,.$$

Claim 1. *For any* $|\widetilde{u}\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0}_M}$ *and* $X \in \mathfrak{g}$, $\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle d\eta$ *defines a cotangent vector of the curve* C *at the point* P.

Proof. Choose a meromorphic function $f \in H^0(C, \mathcal{O}_C(*(P + Q_1))$ on C such that

$$f = \eta^{-1} + \text{regular at } P$$
$$f \equiv 0 \mod (\xi_j^{n_j}) \quad \text{at } Q_j,\ j \neq 1$$

where $\eta = t^{(\infty)-1}_{M+1}(\xi)$, $\xi_j = t^{(\infty)-1}_j(\xi)$ and n_j is sufficiently large so that $\rho_j(X[f])|\widetilde{u}\rangle = 0$ and f is holomorphic at Q_j, $j \neq 1$. Then we have

$$\begin{aligned}\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle &= \langle\widehat{\Psi}|\widetilde{u} \otimes (X[f])|0\rangle \\ &= -\langle\widehat{\Psi}|\rho_1(X[f])\widetilde{u} \otimes 0\rangle\,.\end{aligned}$$

Hence, if we change a formal neighbourhood $t^{(\infty)}_{M+1}$ by $\widetilde{t}^{(\infty)}_{M+1}$, we have

$$\widetilde{\eta} = \widetilde{t}^{(\infty)-1}_{M+1}(\xi) = a_1\eta + a_2\eta^2 + \dots, \quad a_1 \neq 0$$

$$\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle_{\widetilde{\eta}} = a_1^{-1}\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle_\eta\,.$$

This implies that $\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle$ depends only on the first order infinitesimal neighbourhood and $\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle_\eta d\eta \in T^*_P C$ is independent of the choice of a formal coordinate.

Claim 2. *Put*

$$\omega_j = \sum_{n \in \mathbf{Z}} \langle\widetilde{\Psi}|\rho_j(X(n))|\widetilde{u}\rangle \xi_j^{-n-1} d\xi_j\,, \quad j = 1, 2, \dots, N + M$$

where $\xi_j = t_j^{(\infty)^{-1}}(\xi)$. *There is a meromorphic* 1*-form*

$$\omega \in H^0(C, \omega_C(*\sum_{j=1}^{N+M} Q_j))$$

on C *such that*

$$t(\omega) = (\omega_1, \omega_2, \dots, \omega_{N+M})$$

where the mapping t *is defined in* (2.1-4).

Proof. For an element $f \in H^0(*\sum_{j=1}^{N+M} Q_j))$ let $f_j(\xi_j) = \sum a_n^{(j)} \xi_j^n$ be the formal Laurent expansion of f at the point Q_j by the formal parameter $\xi_j = t_j^{(\infty)-1}(\xi)$. Hence $t(f) = (f_1(\xi_1), \dots, f_{N+M}(\xi_{N+M}))$. Then we have

$$\begin{aligned}\sum_{j=1}^{N+M} \operatorname*{Res}_{\xi_j=0}(f_j(\xi_j)\omega_j) &= \sum_{j=1}^{N+M} \sum_{n\in\mathbf{Z}} \langle\widetilde{\Psi}|\rho_j(X(n))|\widetilde{u}\rangle a_n^{(j)} \\ &= \langle\widetilde{\Psi}|X \otimes t(f)|\widetilde{u}\rangle = 0\end{aligned}$$

since $\langle\Psi|X \otimes t(f) = 0$ by our assumption. Therefore, by Lemma 2.1.5 there exists an element $\omega \in H^0(C, \omega_C(*\sum_{j=1}^{N+M} Q_j))$ with $t(\omega) = (\omega_1, \dots, \omega_{N+M})$. This proves Claim 2.

Claim 3. *As a cotangent vector at* P *with formal parameter* η, $\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle d\eta$ *and* ω *coincide.*

In the following we express ω by

$$\omega = \langle\widetilde{\Psi}|X(P)|\widetilde{u}\rangle dP\,.$$

Proof. Since $\langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle d\eta$ is a cotangent vector at P, we may assume that η is a local holomorphic coordinate of C at P. Choose a meromorphic function $f \in H^0(C, \mathcal{O}_C(*(P + Q_i))$ on C such that

$$\begin{aligned}&f = \eta^{-1} + \text{regular at } P \\ &f \equiv 0 \mod (\xi_j^{n_j}) \quad \text{at } \ Q_j,\ j \neq i,\ 1 \leq j \leq N+M\end{aligned}$$

where n_j is sufficiently large so that $\rho_j(X[f])|\widetilde{u}\rangle = 0$ and $f\omega$ is holomorphic at Q_j, $j \neq i$, $1 \leq j \leq N + M$. Then we have

$$\begin{aligned}\langle\widehat{\Psi}|\widetilde{u} \otimes X(-1)|0\rangle &= -\sum_{k=1}^{N+M} \langle\widetilde{\Psi}|\rho_k(X[f])|\widetilde{u}\rangle \\ &= -\langle\widetilde{\Psi}|\rho_i(X[f])|\widetilde{u}\rangle.\end{aligned}$$

On the other hand, at the point P we have

$$\begin{aligned}\mathrm{Res}_P(\frac{1}{\eta}\omega) &= \mathrm{Res}_P(f\omega) \\ &= -\sum_{k=1}^{N+M} \mathrm{Res}_{Q_k}(f\omega) \\ &= -\mathrm{Res}_{Q_i}(f\omega) \\ &= -\operatorname*{Res}_{\xi_i=0}\left(f_i(\xi_i)\sum_{n\in\mathbf{Z}}\langle\widetilde{\Psi}|\rho_i(X(n))|\widetilde{u}\rangle\xi_i^{-n-1}d\xi_i\right) \\ &= -\langle\widetilde{\Psi}|\rho_i(X[f])|\widetilde{u}\rangle \\ &= \langle\widehat{\Psi}|\widetilde{u}\otimes X(-1)|0\rangle_\eta\end{aligned}$$

This proves Claim 3.

Now we are ready to prove Theorem 2.4.1. Put

$$|\widetilde{u}\rangle = |u\otimes X_1(-1)0\otimes\cdots\otimes X_M(-1)0\rangle.$$

The above argument shows that

$$\langle\widetilde{\Psi}|\widetilde{u}\rangle = \langle\widetilde{\Psi}|u\otimes X_1(-1)0\otimes\cdots\otimes X_M(-1)0\rangle$$

is regarded as an element of $T^*_{P_1}C\otimes\cdots\otimes T^*_{P_M}C$, if $P_k\neq Q_j$ and $P_j\neq P_k$, $j\neq k$, and depends meromorphically on P_k. Hence, by the Hartogs theorem, it defines an element of $H^0(C^M,\omega_C^{\boxtimes}(\sum_{i<j}*\Delta_{ij}+\sum_{i=1}^M\sum_{j=1}^N *\pi_i^{-1}(Q_j))$. We denote this meromorphic section by

$$\langle\Psi|X_1(P_1)X_2(P_2)\dots X_M(P_M)|u\rangle dP_1dP_2\dots dP_M\,.$$

The assertions 0) and 1) are clear by our definition. For the assertion 2) note that the meromorphic form defined above from the data

$$\widetilde{\mathfrak{X}}^{(\infty)} = (C;Q_1,\dots,Q_N,P_1,\dots,P_M;t_1^{(\infty)},\dots,t_{N+M}^{(\infty)})$$

and the data

$$\widetilde{\mathfrak{X}}^{\infty}_\sigma = (C;Q_1,\dots,Q_N,P_{\sigma(1)},\dots,P_{\sigma(M)};t_1^{(\infty)},\dots,t_N^{(\infty)}, t_{N+\sigma(1)}^{(\infty)},\dots,t_{N+\sigma(M)}^{(\infty)})$$

are the same. This implies the assertion 2).

The assertion 3) follows from Claim 2.

Let us prove the assertion 4). Let the point P' be in a small neighbourhood U of the point P with local coordinate z with center P. Let us choose a meromorphic function $f \in H^0(C, \mathcal{O}_C(*P + \sum_{k=1}^{N+M} *Q_k))$ such that

$$f = z^{-1} + \text{regular at } P.$$

Moreover, changing the local coordinate at P if necessary, we may assume that $f = z^{-1}$. Then $w = z - z(P')$ is a local coordinate of C at P'. As a cotangent vector at each point of $(P, P') \times \overbrace{C \times \ldots \times C}^{M}$,

$$F = \langle \Psi | X(P) Y(P') X_1(P_1) \ldots X_M(P_M) | \Phi \rangle dP dP'$$

is equal to

$$\langle \widehat{\widehat{\Psi}}^* | X(-1) 0_P \otimes Y(-1) 0_{P'} \otimes \widetilde{\Phi} \rangle dz dw$$

where

$$\langle \widehat{\widehat{\Psi}}^* | = \iota(\langle \Psi |), \quad \iota : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \to \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_{M+2}}(\widehat{\widehat{\mathfrak{X}}}^{(\infty)})$$

and

$$|\widetilde{\Phi}\rangle = |\Phi\rangle \otimes X_1(-1)0\rangle \otimes \ldots \otimes X_M(-1)0\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0}_M}.$$

Then we have

(2.4-1)

$$\begin{aligned}\langle \widehat{\widehat{\Psi}}^* | X(-1) 0_P \otimes Y(-1) 0_{P'} \otimes \widetilde{\Phi} \rangle = &- \langle \widehat{\Psi} | (X[f]) Y(-1) 0_{P'} \otimes \widetilde{\Phi} \rangle \\ &- \sum_{k=1}^{N+M} \langle \widehat{\Psi} | Y(-1) 0_{P'} \otimes \rho_k(X[f]) | \widetilde{\Phi} \rangle.\end{aligned}$$

The second term of the right hand side of (2.4-1) is written as

$$- \sum_{k=1}^{N+M} \langle \widetilde{\Psi} | Y(P') | \rho_k(X[f]) \widetilde{\Phi} \rangle dP'$$

hence, it is holomorphic at the point P'. On the other hand, putting $a = z(P')$ we have

$$\begin{aligned}(X[f]) Y(-1) |0_{P'}\rangle &= (X[\frac{1}{w+a}])(Y[w^{-1}])|0_{P'}\rangle \\ &= \left(\frac{[X,Y]}{a}[w^{-1}] - \frac{\ell \cdot (X,Y)}{a^2} \right) |0_{P'}\rangle .\end{aligned}$$

Hence the first term of the right hand side of (2.4-1) has the form

$$\frac{\ell \cdot (X,Y)}{a^2}\langle \Psi | X_1(P_1)\dots X_M(P_M)|\Phi\rangle \\ - \frac{1}{a}\langle \Psi | [X,Y](P')X_1(P_1)\dots X_M(P_M)|\Phi\rangle .$$

Since $-a = z(P) - z(P')$, we have the desired result.

The similar argument proves the assertion 5). Q.E.D.

Furthermore we can show the following Proposition.

Proposition 2.4.2.

1) *For $k = 1, \dots, N$, we have*

$$\oint_{C_k} \frac{d\xi_k}{2\pi\sqrt{-1}}\xi_k^{n+1}\langle \Psi | T(\xi_k)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle \\ = \langle \Psi | X_1(P_1)X_2(P_2)\dots X_M(P_M)|\rho_k(L_n)\Phi\rangle,$$

where the energy-momentum tensor $T(z)$ is defined by

$$T(z) = \frac{1}{2(g^* + l)}\lim_{w\to z}\Big\{\sum_{a=1}^{\dim \mathfrak{g}} J^a(z)J^a(w) - \frac{\ell \dim \mathfrak{g}}{(z-w)^2}\Big\}.$$

Here, $\{J^1, \dots, J^{\dim \mathfrak{g}}\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$.

2) *For a holomorphic coordinate transformation $w = w(z)$ we have*

$$\langle \Psi | T(w)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dw^2 \\ = \langle \Psi | T(z)X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz^2 \\ - \frac{c_v}{12}\{w(z);z\}\langle \Psi | X_1(P_1)X_2(P_2)\dots X_M(P_M)|\Phi\rangle dz^2$$

where $\{w(z);z\}$ is the Schwarzian derivative.

2.5. The space of vacua, general case.

Now we define the space of vacua without assuming the condition (Q) in 2.1. Let $\mathfrak{X}^{(\infty)} = (C; Q_1, \dots, Q_N; t_1^{(\infty)}, \dots, t_N^{(\infty)})$ be an N-pointed stable curve with infinitesimal neighbourhoods. For a sufficiently large integer M, choose M points Q_k with infinitesimal neighbourhood $t_k^{(\infty)}$, $k = N+1, \dots, M+N$ in such a way that $\widetilde{\mathfrak{X}}^{(\infty)} = (C; Q_1, \dots, Q_{M+N}; t_1^{(\infty)}, \dots, t_{M+N}^{(\infty)})$ is an $(M+N)$-pointed stable curve with infinitesimal neighbourhoods and satisfies the condition (Q). Put $\vec{0}_M = (\underbrace{0,\dots,0}_{M})$ and define

$$
\begin{aligned}
\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) &= \mathcal{V}_{\vec{\lambda},\vec{0}_M}(\widetilde{\mathfrak{X}}^{(\infty)}) \\
\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) &= \mathcal{V}_{\vec{\lambda},\vec{0}_M}(\widetilde{\mathfrak{X}}^{(\infty)}).
\end{aligned} \tag{2.5-1}
$$

The definition does not depend on the choice of the data $(Q_{N+1},\dots, Q_{M+N}; t^{(\infty)}_{N+1},\dots,t^{(\infty)}_{N+M})$.

§3. Universal family of pointed stable curves

3.1. Deformations of pointed stable curves

Let C be a compact Riemann surface of genus g. Infinitesimal deformations of the Riemann surfaces are parameterized by the cohomology group $H^1(C,\Theta_C)$, where Θ_C is the sheaf of germs of holomorphic vector fields on C. (See, for example [Ko].) More generally infinitesimal deformations of the data $\mathfrak{X}^{(n)} = (C; Q_1, Q_2,\dots,Q_N; t_1^{(n)}, t_2^{(n)}, \dots, t_N^{(n)})$ of an N-pointed Riemann surface of genus g with n-th infinitesimal neighbourhoods are parameterized by the cohomology group $H^1(C,\Theta_C(-(n+1)\sum_{j=1}^N Q_j)$. If C is a singular stable curve, then the cohomology group should be replaced by the cohomology group $Ext^1_{\mathcal{O}_C}(\Omega^1_C,\mathcal{O}_C)$. (See, for example, [Ar], [DM, Section 1], [SGA7, Exposé VI, 6], [Bin].) Here, Ω^1_C is the sheaf of Kähler differentials of the curve C. (See, for example, [Ha, Chap. II, 8] or [Se]. In our situation, we may regard the exact sequence (3.1-3) as a definition of the sheaf Ω^1_C.) Put $\Theta_C = \underline{Hom}_{\mathcal{O}_C}(\Omega^1_C,\mathcal{O}_C)$. There is an exact sequence

$$
\begin{aligned}
0 &\to H^1(C,\Theta_C(-(n+1)\sum_{j=1}^N Q_j)) \\
&\to Ext^1_{\mathcal{O}_C}(\Omega^1_C,\mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j)) \\
&\to H^0(C,\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C,\mathcal{O}_C)) \to 0.
\end{aligned} \tag{3.1-1}
$$

If the stable curve C has q double points $P_1, P_2,\dots,P_q$, then we have

$$
\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C,\mathcal{O}_C)_Q = \begin{cases} \mathbf{C}, & \text{if } Q = P_j, \quad i=1,2,\dots,q \\ 0, & \text{otherwise.} \end{cases}
$$

Hence we have

$$H^0(C, \underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) \simeq \mathbf{C}^q.$$

Each element of $H^1(C, \Theta_C(-(n+1)\sum_{j=1}^N Q_j))$ corresponds to an infinitesimal deformation of the data $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; t_1^{(n)}, t_2^{(n)}, \ldots, t_N^{(n)})$ *preserving the singularities.*

Definition 3.1.1. Data $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ are called a (holomorphic) family of N-pointed stable curves of genus g with n-th infinitesimal neighbourhoods, if the following conditions are satisfied.

(1) Y and B are connected complex manifolds, $\pi : Y \to B$ is a proper flat holomorphic map and $s_1, s_2, \ldots, s_N$ are holomorphic sections of π.

(2) For each point $b \in B$ the data $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \ldots, s_N(b))$ is an N-pointed stable curve of genus g.

(3) $\tilde{t}_j^{(n)}$ is an $\mathcal{O}_B$-algebra isomorphism

$$\tilde{t}_j^{(n)} : \mathcal{O}_Y / I_{s_j}^{n+1} \simeq \mathcal{O}_B[[\xi]]/(\xi^{n+1}),$$

where I_{s_j} is the defining ideal of $s_j(B)$ in Y.

Similarly we define a family of N-pointed stable curve of genus g with formal neighbourhoods $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \ldots, \tilde{t}_N^{(\infty)})$.

Proposition 3.1.2. *For a family of N-pointed stable curve of genus g with formal neighbourhoods $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ and for each point $b \in B$, there exists a $\mathbf{C}$-linear mapping*

$$\rho_b : T_b B \to Ext^1_{\mathcal{O}_{Y_b}}(\Omega^1_{Y_b}, \mathcal{O}_{Y_b}(-(n+1)\sum_{j=1}^N s_j(b))), \tag{3.1-2}$$

where $Y_b = \pi^{-1}(b)$.

The linear mapping ρ_b is called the *Kodaira-Spencer mapping* of the family $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ at the point b.

Since the proposition plays an important role in our formulation of conformal field theory, we give rather detailed discussions about a proof. For the fundamental properties of the functor *Ext* we refer the reader to [Ha, Chap. III, 6]. Put $C = Y_b$, $Q_j = s_j(b)$. Let I_C be the sheaf of the defining ideal of C in Y. There is an exact sequence

$$0 \to I_C/I_C^2 \to \Omega_Y^1 \otimes \mathcal{O}_C \to \Omega_C^1 \to 0. \tag{3.1-3}$$

This gives a locally free resolution of the sheaf Ω_C^1. The sheaf I_C/I_C^2 is the conormal sheaf of the curve C in Y and we have a canonical isomorphism

$$(T_b^*B) \otimes_{\mathbf{C}} \mathcal{O}_C \simeq I_C/I_C^2 .$$

Hence there are canonical isomorphisms

$$\underline{Hom}_{\mathcal{O}_C}(I_C/I_C^2, \mathcal{O}_C) \simeq T_bB \otimes_{\mathbf{C}} \mathcal{O}_C, \tag{3.1-4}$$

$$\underline{Hom}_{\mathcal{O}_C}(\Omega_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \mathcal{O}_C) \simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C. \tag{3.1-5}$$

Put

$$I^0 = \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \quad I^1 = T_bB \otimes_{\mathcal{O}_C} \mathcal{O}_C.$$

In other words, we have an exact sequence

$$0 \to \Theta_C \to I^0 \to I^1 \to \underline{Ext}^1_{\mathcal{O}_C}(\Omega_C^1, \mathcal{O}_C) \to 0.$$

Then applying $\underline{Hom}_{\mathcal{O}_C}(\quad, \mathcal{O}_C)$ to the exact sequence (3.1-3) and using the canonical isomorphisms (3.1-4) and (3.1-5), we obtain a complex of sheaves

$$0 \to I^0 \overset{\pi_*}{\to} I^1 \to 0. \tag{3.1-6}$$

The cohomology groups of the complex (3.1-6) are $\underline{Ext}^{\bullet}_{\mathcal{O}_C}(\Omega_C^1, \mathcal{O}_C)$. That is, we have

$$\underline{Ext}^0 = \mathrm{Ker}\,\{\pi_* : I^0 \to I^1\} = \Theta_C,$$
$$\underline{Ext}^1 = \mathrm{Coker}\,\{\pi_* : I^0 \to I^1\}.$$

Note that the map π_* in (3.1-6) is surjective outside the double points $P_1, P_2, \ldots, P_q$ of the curve C. The cohomology groups $Ext^{\bullet}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is calculated as follows. Choose an open covering $\mathcal{U} = \{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of the curve C. Let $C^k(\mathcal{U}, I^m)$ be k-th cochains with values in the sheaf I^m. Put

$$K^n = \bigoplus_{k+m=n} C^k(\mathcal{U}, I^m).$$

We define the differentials δ^n of $\{K^n\}$ as follows. For any element $\{\phi_\alpha\} \in C^0(\mathcal{U}, I^0) = K^0$ we define

$$\delta^0\{\phi_\alpha\} = (\{\pi_*(\phi_\alpha)\}, \{\phi_\beta - \phi_\alpha\}) \in C^0(\mathcal{U}, I^1) \oplus C^1(\mathcal{U}, I^0) = K^1.$$

For each element $(\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\}) \in C^0(\mathcal{U}, I^1) \oplus C^1(\mathcal{U}, I^0) = K^1$ we define

$$\begin{aligned}\delta^1(\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\}) =& \{(\varphi_\beta - \varphi_\alpha) - \pi_*(\theta_{\alpha\beta})\}, \{\theta_{\beta\gamma} - \theta_{\alpha\gamma} + \theta_{\alpha\beta}\}) \\ & \in C^1(\mathcal{U}, I^1) \oplus C^2(\mathcal{U}, I^0) = K^2.\end{aligned}$$

Other δ^k's are defined to be the zero map. Then $\{K^\bullet, \delta^\bullet\}$ is a complex and if the covering is good, namely each open set $\mathcal{U}_\lambda$ is different from C, then we have

$$Ext^n_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) = \mathrm{Ker}\,\delta^n / \mathrm{Im}\delta^{n-1}.$$

Assume that the covering $\mathcal{U}$ is good. Assume further that each of the points Q_j's and P_i's is contained in only one open set $\mathcal{U}_\gamma$. For each tangent vector $\theta \in T_bB$ of B at b, there is a vector field $\tilde{\theta}$ on a neighbourhood of b. Then there is a lifting $\tilde{\theta}_\alpha$ on $\widetilde{\mathcal{U}}_\alpha \setminus \Sigma$ of the vector field $\tilde{\theta}$, where $\widetilde{\mathcal{U}}_\alpha$ is an open set in Y with $\mathcal{U}_\alpha = \widetilde{\mathcal{U}}_\alpha \cap C$ and Σ is the locus of double points of fibres of π. Put

$$\begin{aligned}\varphi_\alpha &= \theta \in H^0(\mathcal{U}_\alpha, T_b \otimes \mathcal{O}_C) \\ \theta_{\alpha\beta} &= (\tilde{\theta}_\beta - \tilde{\theta}_\alpha)|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \in H^0(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C).\end{aligned}$$

Then $\Psi(\theta) = (\{\varphi_\alpha\}, \{\theta_{\alpha\beta}\})$ is an element of K^1 and by definition we have $\delta^1(\Psi(\theta)) = 0$, hence defines an element $[\Psi(\theta)]$ of $Ext^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$. Thus we have a $\mathbf{C}$-linear mapping

$$\rho_b : T_bB \to Ext^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C).$$

This is the Kodaira-Spencer mapping of $\pi : Y \to B$ at b.

So far we do not consider the points Q_j and n-th infinitesimal neighbourhoods. To define the Kodaira-Spencer mapping of the family $(\pi : Y \to B; s_1, s_2, \dots, s_N; \tilde{t}^{(n)}_1, \tilde{t}^{(n)}_2, \dots, \tilde{t}^{(n)}_N)$ we need to be careful to choose a lifting $\tilde{\theta}_\alpha$ of $\tilde{\theta}$, namely the lifting should respects the n-th formal neighbourhoods. For simplicity assume that the point Q_j is contained in an open set $\mathcal{U}_j$. Choose local coordinates $(u_1, u_2, \dots, u_m)$ of B with center b. Then we can choose local coordinates of Y with center Q_j as $(u_1, u_2, \dots, u_m, z)$. We may assume that $\mathcal{U}_j$ is contained in the coordinate neighbourhood of Q_j with the above coordinates. By these coordinates the vector field $\tilde{\theta}$ is expressed in a form $\sum a_k(u)\dfrac{\partial}{\partial u_k}$. Then, as $\tilde{\theta}_j$ we choose the same form $\sum a_k(u)\dfrac{\partial}{\partial u_k}$. Other lifting is given by a form

$$\sum a_k(u)\frac{\partial}{\partial u_k} + A(u,z)\frac{\partial}{\partial z}.$$

To preserve the n-th formal neighbourhoods $A(u,z)$ has the zero of order $n+1$ at Q_j. Precisely speaking, if we choose the lifting $\tilde{\theta}_j$ above then we have an element $\Psi(\theta)$ as above. This lifting *does* depend on the choice of the local coordinates. If we choose other local coordinates, $\Psi(\theta)$ changes by adding $\delta^0(\{\phi_\alpha\})$. ϕ_α corresponds to an infinitesimal change of local coordinates of $\mathcal{U}_\alpha$. Hence ϕ_α needs to preserve the n-th formal neighbourhoods. Let I^0_{n+1} and I^1_{n+1} be $\mathcal{O}_C$-submodules of I^0 and I^1, respectively defined by

$$\begin{aligned} I^0_{n+1} &= \underline{Hom}_{\mathcal{O}_C}(\Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C, \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j)) \\ &\simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j) \\ I^1_{n+1} &= \underline{Hom}_{\mathcal{O}_C}(I_C/I_C^2, \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j)) \\ &\simeq T_bB \otimes_{\mathbf{C}} \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j). \end{aligned}$$

Then, again we have a complex of sheaves

$$0 \to I^0_{n+1} \stackrel{\pi_*}{\to} I^1_{n+1} \to 0$$

and $(\{\phi_\alpha\}, \{\theta_{\alpha\beta}\})$ defines a cohomology class $[\Psi(\theta)]$ of the complex $\{K^\bullet_{n+1}, \delta^\bullet\}$, where we define

$$K^p_{n+1} = \bigoplus_{m+\ell=p} C^m(\mathcal{U}, I^\ell_{n+1}).$$

The cohomology group of $\{K^\bullet_{n+1}, \delta^\bullet\}$ is $Ext^\bullet_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum Q_j))$. Hence $[\Psi(\theta)]$ is an element of $Ext^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum Q_j))$. This defines the Kodaira-Spencer mapping of the family $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \vec{t}^{(n)}_1, \vec{t}^{(n)}_2, \ldots, \vec{t}^{(n)}_N)$. We thus prove the Proposition 3.1.2.

A sheaf version of Proposition 3.1.2 is the following.

Corollary 3.1.3. *If* $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \vec{t}^{(n)}_1, \vec{t}^{(n)}_2, \ldots, \vec{t}^{(n)}_N)$ *is a family of* N*-pointed* smooth *curve of genus* g *with* n*-th infinitesimal* *neighbourhoods, the Kodaira-Spencer mapping ρ_s induces an $\mathcal{O}_B$-module homomorphism*

$$\rho : \Theta_Y \to R^1\pi_*\underline{\mathrm{Hom}}(\Omega^1_{Y/B}, \Theta_Y(-(n+1)\sum_{j=1}^{N} s_j(B))).$$

Definition 3.1.4. A family $(\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ of N-pointed stable curves of genus g is called a *local universal family*, if the Kodaira-Spencer mapping

$$\rho_s : T_s \to Ext^1_{\mathcal{O}_{C_s}}(\Omega^1_{\mathcal{O}_{C_s}}, \mathcal{O}_{C_s}(-(n+1)\sum_{j=1}^{N} s_j^{(n)}(s))) \tag{3.1-7}$$

is isomorphic at each point $s \in \mathcal{B}^{(n)}$.

The following theorem plays a crucial role in our conformal field theory.

Theorem 3.1.5. *For each N-pointed stable curve $\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N; t_1^{(n)}, t_2^{(n)}, \ldots, t_N^{(n)})$ of genus g with n-th infinitesimal neighbourhoods, there always exists a local universal family $(\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ with point $x \in \mathcal{B}^{(n)}$ such that $C_x = {\pi^{(n)}}^{-1}(x) \simeq C$ and that with respect to this isomorphism we have*

$$Q_j = s_j^{(n)}(x), \quad t_j^{(n)} = \tilde{t}_j^{(n)}|_{C_x}.$$

Proof. The theorem is a consequence of a deformation theory ([Ar], [Sc], [SGA 7], [Bin]). Since we need an explicit description of a local universal family $\mathfrak{F}^{(1)}$ below, we give a method to construct a local universal family.

By our assumption, the curve C has only ordinary double points. Hence, by a deformation theory, there exists a versal family $\pi : \mathcal{C} \to \mathcal{B}$ with specified point $x \in \mathcal{B}$ such that $C_x = \pi^{-1}(x) \simeq C$. Here, "versal" means that the Kodaira-Spencer mapping

$$\rho_x : T_x\mathcal{B} \to Ext^1_{\mathcal{O}_{C_x}}(\Omega^1_{C_x}, \mathcal{O}_{C_x})$$

is isomorphic. (Since the automorphism group of C may be infinite, the family $\pi : \mathcal{C} \to \mathcal{B}$ may not be universal at the point x but semi-universal.)

Put

$$\mathcal{B}^{(0)} = \mathcal{C}^N \setminus \left(\bigcup_{i<j} \Delta_{ij} \bigcup \{ \text{ singular points of } \mathcal{C}^N \ \} \right)$$

where

$$\Delta_{ij} = \{ \ (x_1, \ldots, x_N) \in \mathcal{C}^N \mid \ x_i = x_j \ \}$$

is the (i,j)-th diagonal. There is a natural holomorphic mapping $p : \mathcal{B}^{(0)} \to \mathcal{B}$. Put also

$$\mathcal{C}^{(0)} = \mathcal{C} \times_{\mathcal{B}} \mathcal{B}^{(0)}$$

and let $\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}$ be the projection to the second factor. By our definition, $(Q_1, \ldots, Q_N) \in p^{-1}(x)$. Put $x_0 = (Q_1, \ldots, Q_N) \in \mathcal{B}^{(0)}$. Then we have ${\pi^{(0)}}^{-1}(x_0) = C_x \times x_0 \simeq C$. Moreover, we can define holomorphic sections

$$s_j^{(0)} : \mathcal{B}^{(0)} \to \mathcal{C}^{(0)}$$

by

$$s_j^{(0)}((P_1, \ldots P_N)) = (P_j, P_1, \ldots, P_N) \in \mathcal{C} \times_{\mathcal{B}} \mathcal{B}^{(0)}.$$

Then we have $s_j^{(0)}(x_0) = (Q_j, x_0)$. It is easy to show that $\mathfrak{F}^{(0)} = (\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}; s_1^{(0)}, \ldots, s_N^{(0)})$ is universal at each point of $\mathcal{B}^{(0)}$.

Next we construct the family $\mathfrak{F}^{(1)}$. For that purpose, put

$$T_{s_j^{(0)}} \mathcal{C}^{(0)} = \bigcup_{y \in \mathcal{B}^{(0)}} T_{s_j^{(0)}(y)} C_y \, .$$

Thus $T_{s_j^{(0)}} \mathcal{C}^{(0)}$ consists of tangent vectors of $\mathcal{C}^{(0)}$ at $s_j^{(0)}(\mathcal{B}^{(0)})$ tangent to the fibres of $\pi^{(0)}$. $T_{s_j^{(0)}} \mathcal{C}^{(0)}$ is a holomorphic line bundle over $\mathcal{B}^{(0)}$. Put further

$$\begin{aligned} T^{\times}_{s_j^{(0)}} \mathcal{C}^{(0)} &= T_{s_j^{(0)}} \mathcal{C}^{(0)} - \text{zero section} \\ \mathcal{B}^{(1)} &= T^{\times}_{s_1^{(0)}} \mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \cdots \times_{\mathcal{B}^{(0)}} T^{\times}_{s_N^{(0)}} \mathcal{C}^{(0)} \\ \mathcal{C}^{(1)} &= \mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{B}^{(1)} \, . \end{aligned}$$

Let $\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}$ be the projection to the second factor and $p_1 : \mathcal{B}^{(1)} \to \mathcal{B}^{(0)}$ the natural holomorphic mapping. The holomorphic sections $s_j^{(0)}$ lifts to holomorphic sections $s_j^{(1)} : \mathcal{B}^{(1)} \to \mathcal{C}^{(1)}$ by

$$y \longmapsto (s_j^{(0)}(p_1(y)), y) \, .$$

Moreover, for each element $y = (v_1, \dots, v_N) \in \mathcal{B}^{(1)}$, by using the canonical isomorphism $\mathcal{O}_{\mathcal{C}^{(1)}}/I_{s_j^{(1)}} \simeq \mathcal{O}_{\mathcal{B}^{(1)}}$, we can define $\mathcal{O}_{\mathcal{B}^{(1)}}$-module homomorphism

$$\begin{array}{rcccc} \tilde{t}_j^{(1)} & : & \mathcal{O}_{\mathcal{C}^{(1)}}/I^2_{s_j^{(1)}} & \to & \mathcal{O}_{\mathcal{B}^{(1)}} \oplus \mathcal{O}_{\mathcal{B}^{(1)}}\xi \\ & & f & \mapsto & (f(s_j^{(1)}(y)), v_j(f)\xi) \end{array}$$

where we regard v_j as a derivation.

Note that the first order infinitesimal neighbourhood $t_j^{(1)}$ of the curve C defines a derivation $v_j \in T^{\times}_{Q_j}C$ by

$$t_j^{(1)}(f) = f(Q) + v_j(f)\xi$$

where f is a holomorphic function at the point Q. Hence the data $\mathfrak{X}^{(1)}$ define a point $x_1 \in \mathcal{B}^{(1)}$ with $p_1(x_1) = x_0$. Moreover, ${\pi^{(1)}}^{-1}(x_1)$ is isomorphic to the curve C and with respect to this isomorphism we have

$$s_j^{(1)}(x_1) = Q_j\,, \quad \tilde{t}_j^{(1)}|_C = t_j^{(1)}\,.$$

It is easy to show that our family $\mathfrak{F}^{(1)} = (\,\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}\,;\, s_1^{(1)}, s_2^{(1)}, \dots, s_N^{(1)}\,;\, \tilde{t}_1^{(1)}, \tilde{t}_2^{(1)}, \dots, \tilde{t}_N^{(1)})$ is universal at each point of $\mathcal{B}^{(1)}$.

Similarly, using the n-th jet bundle, we can construct local universal family of N-pointed stable curves with n-th infinitesimal neighbourhoods. Q.E.D.

Let $(\,\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}\,;\, s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}\,;\, \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \dots, \tilde{t}_N^{(n)})$ be a local universal family of N-pointed stable curve of genus g with n-th infinitesimal neighbourhoods. Put

(3.1-8)
$$\Sigma^{(n)} = \{\, P \in \mathcal{C}^{(n)} \mid d\pi_P^{(n)} : T_P\mathcal{C}^{(n)} \to T_{\pi^{(n)}(P)}\mathcal{B}^{(n)} \text{ is not surjective}\,\}$$

(3.1-9)
$$D^{(n)} = \pi^{(n)}(\Sigma^{(n)}).$$

The set $\Sigma^{(n)}$ is called the *critical locus* of the family and $D^{(n)}$ is called the *discriminant locus* of the family. The following lemma is a consequence of the deformation theory of singular curves with ordinary double points. (See for example [Ar], [DM, Section 1] or [SGA 7, Exposé VI, 6].)

Lemma 3.1.6. *For a local universal family* $(\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \dots, \tilde{t}_N^{(n)})$ *of* N*-pointed stable curve of genus* g *with* n*-th infinitesimal neighbourhoods, assume* $2g - 2 + N \geq 1$.

1) *We have*

$$\dim \mathcal{B}^{(n)} = 3g - 3 + (n+1)N$$
$$\dim \mathcal{C}^{(n)} = 3g - 2 + (n+1)N.$$

2) *The critical locus* $\Sigma^{(n)}$ *is a smooth subvariety of codimension* 2 *in* $\mathcal{C}^{(n)}$.

3) *The discriminant locus* $D^{(n)}$ *is a divisor with normal crossings in* $\mathcal{B}^{(n)}$.

3.2. Kodaira-Spencer mapping

Let us consider a local universal family $\mathfrak{F}^{(n)} = (\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \dots, \tilde{t}_N^{(n)})$ of N-pointed stable curves of genus g with n-th infinitesimal neighbourhoods. In the following we need to consider locally a family $\mathfrak{F}^{(n)}$. For that purpose we introduce the following local coordinates of $\mathcal{C}^{(n)}$.

For a point $P \in \Sigma^{(n)}$ of the critical locus of $\pi^{(n)}$, we can choose local coordinates $(u_1, u_2, \dots, u_{M-1}, z, w)$ of $\mathcal{C}^{(n)}$ with center P and local coordinates $(\tau_1, \tau_2, \dots, \tau_M)$ of $\mathcal{B}^{(n)}$ with center $\pi^{(n)}(P)$ such that the holomorphic mapping $\pi^{(n)}$ is given by

$$(u_1, u_2, \dots, u_{M-1}, z, w) \longmapsto (u_1, u_2, \dots, u_{M-1}, zw) = (\tau_1, \tau_2, \dots, \tau_M).$$

In other word, we have

$$\pi^{(n)^*}\tau_k = \begin{cases} u_k\,, & k = 1, 2, \dots, M-1 \\ zw\,, & k = M\,. \end{cases}$$

For a point $P \in \mathcal{C}^{(n)} \setminus \Sigma^{(n)}$ we can choose local coordinates $(u_1, u_2, \dots, u_M, z)$ of $\mathcal{C}^{(n)}$ with center P and local coordinates $(\tau_1, \tau_2, \dots, \tau_M)$ of $\mathcal{B}^{(n)}$ with center $\pi^{(n)}(P)$ such that the holomorphic mapping is given by

$$(u_1, u_2, \dots, u_M, z) \longmapsto (u_1, u_2, \dots, u_M) = (\tau_1, \tau_2, \dots, \tau_M).$$

An $\mathcal{O}_{\mathcal{C}^{(n)}}$-module $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is defined by the following exact sequence

$$\pi^{(n)^{-1}}\Omega^1_{\mathcal{B}^{(n)}} \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} \mathcal{O}_{\mathcal{C}^{(n)}} \to \Omega^1_{\mathcal{C}^{(n)}} \to \Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to 0.$$

The sheaf $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is called the sheaf of germs of relative 1-forms of the family $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$. Let us describe the sheaf $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$, by using the above local coordinates. In a neighbourhood of a point $P \in \mathcal{C}^{(n)} \setminus \Sigma^{(n)}$, the sheaf $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is locally isomorphic to $\mathcal{O}_{\mathcal{C}^{(n)}} dz$. In a small neighbourhood of $P \in \Sigma^{(n)}$, we have an $\mathcal{O}_{\mathcal{C}^{(n)}}$-module isomorphism

$$\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq (\mathcal{O}_{\mathcal{C}^{(n)}} dz + \mathcal{O}_{\mathcal{C}^{(n)}} dw)/\mathcal{O}_{\mathcal{C}^{(n)}}(wdz + zdw)\,. \tag{3.2-1}$$

Moreover, we have the following lemma.

Lemma 3.2.1. *The following sequence*

$$0 \to {\pi^{(n)}}^{-1}\Omega^1_{\mathcal{B}^{(n)}} \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} \mathcal{O}_{\mathcal{C}^{(n)}} \to \Omega^1_{\mathcal{C}^{(n)}} \to \Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to 0 \tag{3.2-2}$$

is exact and gives a locally free resolution of the sheaf $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$.

Let $\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ be the relative dualizing sheaf of $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$. Since $\mathcal{C}^{(n)}$ and $\mathcal{B}^{(n)}$ are non-singular and $\pi^{(n)}$ is flat, we have an $\mathcal{O}_{\mathcal{C}^{(n)}}$-module isomorphism

$$\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \omega_{\mathcal{C}^{(n)}} \otimes {\pi^{(n)}}^* \omega^{-1}_{\mathcal{B}^{(n)}}$$

where w_Y is the dualizing sheaf (canonical sheaf) of a complex manifold Y. (See, for example, [Kl].) The relative dualizing sheaf $\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is described locally as follows.

In a small neighbourhood of a point $P \in \mathcal{C}^{(n)} \setminus \Sigma^{(n)}$, we have

$$\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} = \Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \mathcal{O}_{\mathcal{C}^{(n)}} dz\,.$$

In a small neighbourhood of a point $P \in \Sigma^{(n)}$, we have

$$\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \mathcal{O}_{\mathcal{C}^{(n)}}(dz \wedge dw) \otimes (d\tau_M)^{-1}\,.$$

In particular, we have

$$\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \begin{cases} \mathcal{O}_{\mathcal{C}^{(n)}} \dfrac{dz}{z} & \text{if } z \neq 0 \\ \mathcal{O}_{\mathcal{C}^{(n)}} \dfrac{dw}{w} & \text{if } w \neq 0 \end{cases}$$

with relation

$$\frac{dz}{z} + \frac{dw}{w} = 0$$

if $zw \neq 0$.

Lemma 3.2.2. *There exits an exact sequence*

$$0 \to \Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \otimes_{\mathcal{O}_{\mathcal{C}^{(n)}}} \mathcal{O}_{\Sigma^{(n)}} \to 0\,.$$

Proof. The mapping $\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is given locally in a neighbourhood of a point $P \in \mathcal{C}^{(n)} \setminus \Sigma^{(n)}$ by

$$dz \longmapsto dz$$

and in a neighbourhood of a point $P \in \Sigma^{(n)}$ by

$$\begin{aligned} dz &\longmapsto z(dz \wedge dw) \otimes (d\tau_M)^{-1} \\ dw &\longmapsto w(dz \wedge dw) \otimes (d\tau_M)^{-1}\,. \end{aligned}$$

In particular, we have

$$\begin{aligned} dz &\longmapsto z\frac{dz}{z} \quad \text{if } z \neq 0 \\ dw &\longmapsto w\frac{dw}{w} \quad \text{if } w \neq 0\,. \end{aligned}$$

This proves Lemma 3.2.2. Q.E.D.

Lemma 3.2.3. *Put*

$$\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}}). \tag{3.2-3}$$

Then $\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is an invertible $\mathcal{O}_{\mathcal{C}^{(n)}}$-module and there is an isomorphism

$$\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \underline{Hom}_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\omega_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}}). \tag{3.2-3a}$$

Hence, $\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}$ is an invertible sheaf.

Proof. By (3.2-2) it is easy to show that in a neighbourhood of a point $P \in \mathcal{C}^{(n)} \setminus \Sigma^{(n)}$ we have an isomorphism

$$\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \mathcal{O}_{\mathcal{C}^{(n)}} \frac{\partial}{\partial z}$$

and in a neighbourhood of a point $P \in \Sigma^{(n)}$ we have an isomorphism

$$\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \simeq \mathcal{O}_{\mathcal{C}^{(n)}}(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}). \tag{3.2-4}$$

By this fact and (3.2-1), we have the desired result. Q.E.D.

From the exact sequence (3.2-2) we obtain the following Corollary.

Corollary 3.2.4. *The following sequence*

$$0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \Theta_{\mathcal{C}^{(n)}} \to {\pi^{(n)}}^{-1}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}}) \to 0$$

of $\mathcal{O}_{\mathcal{C}^{(n)}}$-modules is exact.

Lemma 3.2.5. *There exists an exact sequence*

$$0 \to \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) \to \Theta_{\mathcal{B}^{(n)}} \xrightarrow{t} \pi_*^{(n)}\Theta_{\Sigma^{(n)}} \to 0 \tag{3.2-5}$$

where

$$\Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) = \{\, v \in \Theta_{\mathcal{B}^{(n)}} \mid v(I_{D^{(n)}}) \subset I_{D^{(n)}} \,\}$$

and $I_{D^{(n)}}$ is the sheaf of the defining ideal of $D^{(n)}$ in $\mathcal{B}^{(n)}$.

Proof. First note that the sheaf $\Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)})$ is a sheaf of germs of vector fields on $\mathcal{B}^{(n)}$ tangent to $D^{(n)}$. Since $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$ is a local universal family, using the Kodaira-Spencer mapping and (3.1-1), for each point $s \in \mathcal{B}^{(n)}$ we have an exact sequence

$$0 \to H^1(C_s, \Theta_{C_s}(-(n+1)\sum_{j=1}^{N} s_j(s))) \to T_s\mathcal{B}^{(n)} \to H^0(C_s, \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s})) \to 0.$$

Each element of $H^1(C_s, \Theta_{C_s}(-(n+1)\sum_{j=1}^{N} s_j(s)))$ corresponds to a tangent vector of $\mathcal{B}^{(n)}$ at s preserving the singularities of C_s. Hence the sheaf version of the above exact sequence is the exact sequence (3.2-5).

Q.E.D.

Theorem 3.2.6. *Let $(\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \dots, \tilde{t}_N^{(n)})$ be a local universal family of N-pointed stable curves of genus g with n-th infinitesimal neighbourhoods. Then there exists an $\mathcal{O}_{\mathcal{B}^{(n)}}$-module isomorphism*

$$\rho : \Theta_s(-\log D^{(n)}) \xrightarrow{\sim} R^1\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) \tag{3.2-6}$$

where we put $S_j^{(n)} = s_j(\mathcal{B}^{(n)})$ and $S^{(n)} = \sum_{j=1}^{N} S_j^{(n)}$.

Proof. Applying $\underline{Hom}_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\ \ ,\mathcal{O}_{\mathcal{C}^{(n)}})$ to the exact sequence (3.2-2), we obtain the following exact sequence

(3.2-7)

$$0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \Theta_{\mathcal{C}^{(n)}} \to \pi^{(n)^{-1}}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}}) \to 0.$$

This exact sequence splits into the following short exact sequences.

$$0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}} \to \Theta_{\mathcal{C}^{(n)}} \xrightarrow{\kappa} \mathcal{M} \to 0. \tag{3.2-8}$$

$$0 \to \mathcal{M} \to \pi^{(n)^{-1}}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}}) \to 0. \tag{3.2-9}$$

Let $\mathcal{T}$ be a sheaf of germs of holomorphic vector fields on $\mathcal{C}^{(n)}$ preserving n-th infinitesimal neighbourhoods. The sheaf $\mathcal{T}$ is given by

$$\mathcal{T} = \{ v \in \Theta_{\mathcal{C}^{(n)}} \mid v(I_S) \subset I_S^{n+1} \}, \tag{3.2-10}$$

where we put $S = \sum_{j=1}^{N} S_j$. The sheaf $\mathcal{T}$ is an $\mathcal{O}_{\mathcal{C}^{(n)}}$-submodule of $\Theta_{\mathcal{C}^{(n)}}$ and coincides with $\Theta_{\mathcal{C}^{(n)}}$ outside $\bigcup_{j=1}^{N} S_j$. For a point $P \in S_j$ we let $(u_1, u_2, \ldots, u_M, z)$ be local coordinates of $\mathcal{C}^{(n)}$ with center P such that $(u_1, u_2, \ldots, u_M)$ are the coordinates of $\mathcal{B}^{(n)}$ with center $\pi^{(n)}(P)$ and that S_j is defined by the equation $z = 0$ in a neighbourhood of P. Then, in a neighbourhood of P the sheaf $\mathcal{T}$ is generated by

$$z^{n+1}\frac{\partial}{\partial z}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_M}$$

as an $\mathcal{O}_{\mathcal{C}^{(n)}}$-module. Hence $\mathcal{T}$ is locally free on $\mathcal{C}^{(n)}$.

Let us consider the exact sequences (3.2-8) and (3.2-9). Since the support of $\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}})$ is in $\Sigma^{(n)}$, the sheaf $\mathcal{M}$ is equal to $\pi^{(n)^{-1}}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}}$ on $\mathcal{O}^{(n)} \setminus \Sigma^{(n)}$. By using the above local coordinates of $\mathcal{C}^{(n)}$ with center $P \in S_j$, the restriction of κ to $\mathcal{T}$ in a neighbourhood of P is given by

$$a(u,z)z^{(n+1)}\frac{\partial}{\partial z} + \sum B_j(u,z)\frac{\partial}{\partial u_j} \longmapsto \sum B_j(u,z)\frac{\partial}{\partial u_j}.$$

Hence $\kappa : \mathcal{T} \to \mathcal{M}$ is surjective and its kernel is $\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})$ in a neighbourhood of P. On the other hand, on $\mathcal{B}^{(n)} \setminus \bigcup_{j=1}^{N} S_j$ the sheaf $\mathcal{T}$ is equal to $\Theta_{\mathcal{C}^{(n)}}$. Thus we have an exact sequence

$$0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)}) \to \mathcal{T} \to \mathcal{M} \to 0. \tag{3.2-8a}$$

From the exact sequence (3.2-8a) we obtain a long exact sequence

(3.2-11)
$$\begin{aligned} 0 \to \pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) &\xrightarrow{\tau} \pi_*^{(n)}\mathcal{T} \\ \to \pi_*^{(n)}\mathcal{M} &\xrightarrow{\rho} R^1\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) \\ &\to R^1\pi_*^{(n)}\mathcal{T} \to R^1\pi_*^{(n)}\mathcal{M} \to 0\,. \end{aligned}$$

Put $\mathcal{B}_0 = \mathcal{B}^{(n)} \setminus D^{(n)}$, $\mathcal{C}_0 = {\pi^{(n)}}^{-1}(\mathcal{B}_0)$, $\pi_0 = \pi^{(n)}|\mathcal{C}_0$. Then on $\mathcal{B}_0$, $\pi_{0*}\mathcal{M} = \Theta_{\mathcal{B}^{(n)}}$ and the homomorphism ρ is the Kodaira-Spencer mapping by Corollary 3.1.3. Since our family is a local universal one, ρ is isomorphic on $\mathcal{B}_0$. Therefore, the sheaf homomorphism τ in (3.2-11) is isomorphic on $\mathcal{B}_0$. But on $\mathcal{B}_0$ we have $\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) = 0$. Therefore, $\pi_*^{(n)}\mathcal{T} = 0$ on $\mathcal{B}_0$. As $\mathcal{T}$ is locally free, $\pi_*^{(n)}\mathcal{T}$ is torsion free, hence $\pi_*^{(n)}\mathcal{T} = 0$ on $\mathcal{B}^{(n)}$. This also implies

$$\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) = 0 \tag{3.2-12}$$

on $\mathcal{B}^{(n)}$.

Next we show that ρ in (3.2-11) is isomorphic. For that purpose it is enough to show that $R^1\pi_*^{(n)}\mathcal{T}$ is locally free, because, if $R^1\pi_*^{(n)}\mathcal{T}$ is locally free, as ρ is isomorphic on $\mathcal{B}_0$, $\operatorname{Coker}\rho$ is a torsion subsheaf of $R^1\pi_*^{(n)}\mathcal{T}$, hence zero. By the cohomology theory of coherent sheaves,

$$\chi(\mathcal{T} \otimes \mathcal{O}_{C_s}) = \dim_{\mathbf{C}} H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) - \dim_{\mathbf{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$$

is independent of $s \in \mathcal{B}^{(n)}$, where $C_s = \pi^{n-1}(s)$. (See, for example, [BS1].) Moreover, if $\dim_{\mathbf{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$ is independent of s, say k, since we have $H^2(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$, $R^1\pi_*^{(n)}\mathcal{T}$ is a locally free $\mathcal{O}_{\mathcal{B}^{(n)}}$-module of rank k on $\mathcal{B}^{(n)}$. Therefore, it is enough to show that $H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$ for all $s \in \mathcal{B}^{(n)}$.

Since C_s is a locally complete intersection in $\mathcal{C}^{(n)}$, we have an exact sequence

$$0 \to \Theta_{C_s} \to \Theta_{\mathcal{C}^{(n)}} \otimes \mathcal{O}_{C_s} \to \mathcal{O}_{C_s}(N) \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{\mathcal{O}_{C_s}}, \mathcal{O}_{C_s}) \to 0$$

where N is the normal bundle of C_s in $\mathcal{C}^{(n)}$ which is a trivial bundle of rank $3g-3+(n+1)N$. (See, for example, [Ar].) From this exact sequence we obtain two short exact sequences

$$0 \to \Theta_{C_s} \to \Theta_{\mathcal{C}^{(n)}} \otimes \mathcal{O}_{C_s} \to M_s \to 0,$$

$$0 \to M_s \to \mathcal{O}_{C_s}(N) \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega_{C_s}, \mathcal{O}_{C_s}) \to 0.$$

Similarly as above we have an exact sequence

$$0 \to \Theta_{C_s}(-(n+1)\sum_{j=1}^{N} Q_j) \to \mathcal{T} \otimes \mathcal{O}_{C_s} \to M_s \to 0,$$

where $Q_j = s_j(s)$. This gives a long exact sequence

(3.2-13)

$$\begin{aligned} 0 = H^0(C_s, \Theta_{C_s}(-(n+1)\textstyle\sum Q_j)) & \\ \to H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) & \to H^0(C_s, M_s) \\ \xrightarrow{\rho} H^1(C_s, \Theta_{C_s}(-(n+1)\textstyle\sum Q_j)) & \to H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}). \end{aligned}$$

The cohomology group $H^0(C_s, M_s)$ parameterizes infinitesimal displacements of C_s in $\mathcal{C}^{(n)}$. (For the details see Tsuboi [Ts], where the theory is formulated without n-th infinitesimal neighbourhoods, but the extension of the theory to our situation is immediate.) Since $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$ is a local universal family, infinitesimal displacements of C_s in $\mathcal{C}^{(n)}$ and infinitesimal deformations of C_s coincide. Hence the homomorphism ρ in (3.2-13) is isomorphic. Hence we have $H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$.

Finally we show that $\pi^{(n)}_* \mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)})$. From (3.2-9) we obtain an exact sequence

$$0 \to \pi^{(n)}_* \mathcal{M} \to \Theta_{\mathcal{B}^{(n)}} \xrightarrow{t} \pi^{(n)}_* (\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}^{(n)}}}(\Omega^1_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}, \mathcal{O}_{\mathcal{C}^{(n)}})).$$

The homomorphism t is the same to the one appearing in the exact sequence (3.2-5). Hence t is surjective. Therefore, by Lemma 3.2.5 $\pi^{(n)}_* \mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)})$. Q.E.D.

Remark 3.2.7. The homomorphism ρ in the above Theorem 3.2.6 is also called *Kodaira-Spencer mapping*. The above proof shows that there exists an exact sequence

$$\begin{aligned} 0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)\textstyle\sum S_j) \to \mathcal{T} \to {\pi^{(n)}}^{-1}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}} & \\ \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s}) \to 0 & \end{aligned}$$

where $\mathcal{T}$ is a subsheaf of $\Theta_{\mathcal{C}^{(n)}}$ defined in (3.2-10). Choose a small Stein open set $\mathcal{U} \subset \mathcal{B}^{(n)}$ and a vector field $v \in H^0(\mathcal{U}, \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}))$. Choose also a Stein open covering $\{\mathcal{U}_j\}_{j\in J}$ of $\pi^{(n)-1}(\mathcal{U})$. Then v also defines an element $\pi^{(n)^*}v \in H^0(\mathcal{U}_j, \pi^{(n)-1}\Theta_{\mathcal{B}^{(n)}} \otimes \mathcal{O}_{\mathcal{C}^{(n)}})$, whose image to $\underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega_{C_s}, \mathcal{O}_{C_s})$ is zero, since the tangent vector v is a direction of an infinitesimal deformation preserving singularities. Therefore, if $\mathcal{U}_j$ is small enough, we can find an element $v_j \in H^0(\mathcal{U}_j, \mathcal{T})$ which is mapped to $\pi^{(n)^*}v$. Then, we have

$$v_{ij} = v_j - v_i \in H^0(\mathcal{U}_i \cap \mathcal{U}_j, \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S))$$

and $\{v_{ij}\}$ defines an element

$$[\{v_{ij}\}] \in H^1(\pi^{(n)-1}(\mathcal{U}), \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S)).$$

The mapping

$$v \longmapsto [\{v_{ij}\}]$$

is nothing but the Kodaira-Spencer mapping ρ in Theorem 3.2.6.

3.3. Tower of local universal families

Let $\mathfrak{F}^{(0)} = (\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}; s_1^{(0)}, s_2^{(0)}, \dots, s_N^{(0)})$ be a local universal family of N-pointed stable curves of genus g. The proof of Theorem 3.1.5 says that the family $\mathfrak{F}^{(0)}$ can be constructed from a local versal family $\pi : \mathcal{C} \to \mathcal{B}$ of the semi-stable curve C. The following theorem is an easy consequence of the proof of Theorem 3.1.5 and plays an essential role in our theory.

Theorem 3.3.1. *Let* $(\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}; s_1^{(0)}, s_2^{(0)}, \dots, s_N^{(0)})$ *be a local universal family of N-pointed stable curves of genus g. Then for each non-negative integer n we have a local universal family* $\mathfrak{F}^{(n)} = (\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \dots, \tilde{t}_N^{(n)})$ *of N-pointed stable curves of genus g with n-th infinitesimal neighbourhoods such that the following diagram is commutative.*

(3.3-1)

$$\begin{array}{ccccccccccc}
\mathcal{C} & \leftarrow & \mathcal{C}^{(0)} & \leftarrow & \mathcal{C}^{(1)} & \leftarrow \cdots \leftarrow & \mathcal{C}^{(n)} & \leftarrow & \mathcal{C}^{(n+1)} & \leftarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{B} & \leftarrow & \mathcal{B}^{(0)} & \leftarrow & \mathcal{B}^{(1)} & \leftarrow \cdots \leftarrow & \mathcal{B}^{(n)} & \leftarrow & \mathcal{B}^{(n+1)} & \leftarrow \cdots
\end{array}$$

which is compatible with sections and infinitesimal neighbourhoods. Here $\pi : \mathcal{C} \to \mathcal{B}$ *is a versal family of semi-stable curves associated with the family* $\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}$. *(See the proof of Theorem* 3.1.5.*)*

By the theorem, as a limit, we have a family $(\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}$ $; s_1^{(\infty)}, s_2^{(\infty)}, \dots, s_N^{(\infty)} ; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \dots, \tilde{t}_N^{(\infty)})$ where $\mathcal{C}^{(\infty)}$ and $\mathcal{B}^{(\infty)}$ are regarded as infinite dimensional complex manifolds and each fibre of $\pi^{(\infty)}$ consists of an N-pointed stable curve $\mathfrak{X}^{(\infty)} = (C; Q_1, Q_2, \dots, Q_N;$ $t_1^{(\infty)}, t_2^{(\infty)}, \dots, t_N^{(\infty)})$ of genus g with formal neighbourhoods. Moreover, there exist canonical holomorphic mappings $\varphi^{(n)} : \mathcal{C}^{(\infty)} \to \mathcal{C}^{(n)}$ and $\psi^{(n)} : \mathcal{B}^{(\infty)} \to \mathcal{B}^{(n)}$. The group $\mathcal{D}^{\oplus N}$, $\mathcal{D} = \mathrm{Aut}\mathbf{C}((\xi))$ acts on $\mathcal{B}^{(\infty)}$ from left. (See (2.3-1).)

Remark 3.3.2. More generally, in Theorem 3.3.1 by the proof of Theorem 3.1.5 we can always assume that for $n > p$ the holomorphic mapping $\mathcal{B}^{(n)} \to \mathcal{B}^{(p)}$ is a principal fibre bundle with structure group $(\mathcal{G}_{n,p})^{\oplus N}$, where the group $\mathcal{G}_{n,p}$ is the subgroup of ring automorphisms $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[\xi]/(\xi^n))$ which induce the identity automorphism of the ring $\mathbf{C}[\xi]/(\xi^p)$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{C}^{(n)} & \longrightarrow & \mathcal{C}^{(p)} \\ \downarrow & & \downarrow \\ \mathcal{B}^{(n)} & \longrightarrow & \mathcal{B}^{(p)} \end{array}$$

is cartesian. That is, $\mathcal{C}^{(n)} = \mathcal{C}^{(p)} \times_{\mathcal{B}^{(n)}} \mathcal{B}^{(p)}$. In the following we always assume that the families $\mathfrak{F}^{(n)}$'s have this property.

Corollary 3.3.3. *A* $(\mathcal{D}^p)^{\oplus N}$*-invariant holomorphic function on* $\mathcal{B}^{(\infty)}$ *is the pull-back of a holomorphic function on* $\mathcal{B}^{(p)}$.

In the following we generalize Corollary 3.3.3 to the case of sheaves on $\mathcal{B}^{(\infty)}$.

Lemma 3.3.4. *For any non-negative integer n the following sequence*

(3.3-2)

$$0 \to \Theta_{\mathcal{C}^{(n+1)}/\mathcal{B}^{(n+1)}}(-(n+1)S^{(n+1)})$$
$$\to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)}) \otimes \mathcal{O}_{\mathcal{C}^{(n+1)}}$$
$$\to \Theta_{\mathcal{C}^{(n+1)}/\mathcal{B}^{(n+1)}} \otimes (\oplus_{j=1}^N (I^{n+1}_{S_j^{(n+1)}} / I^{n+2}_{S_j^{(n+1)}})) \to 0$$

of $\mathcal{O}_{\mathcal{C}^{(n+1)}}$-modules is exact, where we put $S_j^{(p)} = s_j^{(p)}(\mathcal{B}^{(p)})$, $S^{(p)} = \sum_{j=1}^N S_j^{(p)}$ and $I_{S_j^{(p)}}$ is the sheaf of defining ideal of $S_j^{(p)}$ in $\mathcal{C}^{(p)}$.

Similarly the following sequence of $\mathcal{O}_{\mathcal{B}^{(n+1)}}$-modules is exact.

$$\begin{aligned} 0 \to \Theta_{\mathcal{B}^{(n+1)}/\mathcal{B}^{(n)}} &\to \Theta_{\mathcal{B}^{(n+1)}}(-\log D^{(n+1)}) \\ &\to \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) \otimes \mathcal{O}_{\mathcal{B}^{(n+1)}} \to 0. \end{aligned} \tag{3.3-3}$$

Let us define a sheaf $\Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)})$ on $\mathcal{B}^{(\infty)}$ by

$$\Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)}) = \varprojlim_n \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} \mathcal{O}_{\mathcal{B}^{(\infty)}}. \tag{3.3-4}$$

By Theorem 3.2.6 we have the following Proposition.

Proposition 3.3.5. *There is a canonical $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module isomorphism*

$$\rho : \Theta_{\mathcal{B}^{(\infty)}} \xrightarrow{\sim} \varprojlim_n (R^1\pi_*^{(n)}\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n+1)}) \otimes \mathcal{O}_{\mathcal{B}^{(\infty)}}).$$

For a local universal family $\mathfrak{F}^{(n)} = (\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$ we let $I_{S_j^{(n)}}$ be the sheaf of the defining ideal of $S_j^{(n)} = s_j^{(n)}(\mathcal{B}^{(n)})$ in $\mathcal{C}^{(n)}$. In the following we use the following notation.

$$\widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}} = \varprojlim_m \mathcal{O}_{\mathcal{C}^{(n)}}/I^{m+1}_{S_j^{(n)}},$$

$$\widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}(p) = \varprojlim_m \mathcal{O}_{\mathcal{C}^{(n)}}(pS_j^{(n)})/I^{m+1}_{S_j^{(n)}} \quad \text{for each positive integer } p,$$

$$K_{\widehat{S}_j^{(n)}} = \varinjlim_p \widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}(p).$$

Also we fix an element $\xi_j \in I_{S_j^{(n)}}$ such that

$$\tilde{t}_j^{(n)}\xi_j \equiv \xi \mod (\xi^{n+1}).$$

Then there is an $\mathcal{O}_{\mathcal{B}^{(n)}}$-module isomorphism

$$\widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}} \simeq \mathcal{O}_{\mathcal{B}^{(n)}}[[\xi_j]] \tag{3.3-5}$$

$$K_{\widehat{S}_j^{(n)}} \simeq \mathcal{O}_{\mathcal{B}^{(n)}}((\xi_j)). \tag{3.3-6}$$

Note that the first isomorphism is canonical up to the order n in ξ_j. Taking the limit, for a local universal family $(\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)} ; s_1^{(\infty)}, s_2^{(\infty)}, \ldots, s_N^{(\infty)} ; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \ldots, \tilde{t}_N^{(\infty)})$ we introduce the similar notation and we have *canonical* $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module isomorphisms

$$\widehat{\mathcal{O}}_{\widehat{S}_j^{(\infty)}} \simeq \mathcal{O}_{\mathcal{B}^{(\infty)}}[[\xi_j]] \tag{3.3-7}$$

$$K_{\widehat{S}_j^{(\infty)}} \simeq \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j)) \tag{3.3-8}$$

where $\xi_j = \tilde{t}_j^{(\infty)-1}(\xi)$. The filtration $\{F_\bullet\}$ on $\widehat{\mathcal{O}}_{\widehat{S}_j^{(\infty)}}$ and $K_{\widehat{S}_j^{(\infty)}}$ are defined by

$$F_p K_{\widehat{S}_j^{(\infty)}} \simeq \mathcal{O}_{\mathcal{B}^{(\infty)}}[[\xi_j]]\xi_j^{-p}. \tag{3.3-9}$$

Define

$$\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}} = \underline{Der}_{\mathcal{O}_{\mathcal{B}^{(n)}}}(\widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}, \widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}) \tag{3.3-10}$$

$$\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(p) = \underline{Der}_{\mathcal{O}_{\mathcal{B}^{(n)}}}(\widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}, \widehat{\mathcal{O}}_{\widehat{S}_j^{(n)}}(p)) \tag{3.3-11}$$

$$\begin{aligned} \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*) &= \varinjlim_p \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(p) \\ &= \underline{Der}_{\mathcal{O}_{\mathcal{B}^{(n)}}}(K_{\widehat{S}_j^{(n)}}, K_{\widehat{S}_j^{(n)}}) \end{aligned} \tag{3.3-12}$$

Also we introduce the filtration on $\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*)$ by

$$F_p\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*) = \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(-(p+1)). \tag{3.3-13}$$

Proposition 3.3.6. *Assume that the condition* (Q) *in* 2.1 *is satisfied for each fibre of a local universal family* $\mathfrak{F}^{(n)} = (\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)} ; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)} ; \tilde{t}_1^{(n)}, \tilde{t}_2^{(n)}, \ldots, \tilde{t}_N^{(n)})$.

1) *There exists an* $\mathcal{O}_{\mathcal{B}^{(n)}}$*-module surjective homomorphism*

$$\theta^{(n)} : \bigoplus_{j=1}^N \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*) \to \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) \to 0. \tag{3.3-14}$$

2) $\operatorname{Ker}\theta^{(n)}$ *is equal to the following sheaf*

$$\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(*S^{(n)})) \oplus \bigoplus_{j=1}^N (\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(-(n+1))). \tag{3.3-15}$$

Proof. By the following exact sequence

$$0 \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)}) \to \Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}((m-n-1)S^{(n)}) \to \oplus_{j=1}^{N} \oplus_{k=1}^{m} \mathcal{O}_{\mathcal{B}^{(n)}} \xi_j^{n-m+k} \frac{d}{d\xi_j} \to 0 .$$

we have a long exact sequence

$$0 \to \pi_*^{(n)}((\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(m-n-1)S^{(n)})) \to \oplus_{j=1}^{N}(\oplus_{k=1}^{m}\mathcal{O}_{\mathcal{B}^{(n)}}\xi_j^{n-m+k}\frac{d}{d\xi_j}) \to R^1\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) \to R^1\pi_*^{(n)}\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}((m-n-1))S^{(n)}).$$

If m is sufficiently large, the last term of the above exact sequence vanishes and we have an $\mathcal{O}_{\mathcal{B}^{(n)}}$-module isomorphism

$$\oplus_{k=1}^{m}\mathcal{O}_{\mathcal{B}^{(n)}}\xi_j^{n-m+k}\frac{d}{d\xi_j} \simeq \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(m-n-1)/\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(-(n+1)).$$

Hence, taking $m \to +\infty$, we obtain the following exact sequence

$$0 \to \pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(*S^{(n)})) \to \oplus_{j=1}^{N}\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*)/\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(-(n+1)) \to R^1\pi_*^{(n)}(\Theta_{\mathcal{C}^{(n)}/\mathcal{B}^{(n)}}(-(n+1)S^{(n)})) \to 0.$$

By Theorem 3.2.6 we have the desired result. Q.E.D.

Remark 3.3.7. The geometric meaning of the above homomorphism $\theta^{(n)}$ is as follows. By (3.3-12) there is an $\mathcal{O}_{\mathcal{B}^{(n)}}$-module isomorphism

$$\Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*) \simeq \mathcal{O}_{\mathcal{B}^{(n)}}((\xi_j))\frac{d}{d\xi_j} .$$

This isomorphism is canonical up to the order n in ξ_j. For $(f_1\frac{d}{d\xi_1}, \ldots, f_N\frac{d}{d\xi_N})$, $f_j\frac{d}{d\xi_j} \in \mathcal{O}_{\mathcal{B}^{(n)}}((\xi_j))\frac{d}{d\xi_j}$ let us consider the first order infinitesimal coordinate change

$$\xi_j \to \xi_j + \epsilon f_j(\xi_j). \tag{3.3-16}$$

This defines a first order infinitesimal deformations of each fibre of our family $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$. Moreover, since we do not change the coordinates around singular points of the fibre, the first order infinitesimal deformation preserves the singularities. Hence, it defines a vector field on $\mathcal{B}^{(n)}$ preserving the discriminant locus. This is nothing but our $\theta^{(n)}((f_1\frac{d}{d\xi_1},\dots,f_N\frac{d}{d\xi_N}))$.

Now let us consider the tower (3.3-1) of local universal families. The proof of Proposition 3.3.6 shows that there exists the following commutative diagram.

$$\begin{array}{ccccc} \oplus_{j=1}^N \Theta_{\widehat{S}_j^{(n+1)}/\mathcal{B}^{(n+1)}}(*) & \rightarrow & \oplus_{j=1}^N \Theta_{\widehat{S}_j^{(n)}/\mathcal{B}^{(n)}}(*) \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} \mathcal{O}_{\mathcal{B}^{(n+1)}} & \rightarrow & 0 \\ \downarrow \theta & & \downarrow \theta & & \\ \Theta_{\mathcal{B}^{(n+1)}}(-\log D^{(n+1)}) & \rightarrow & \Theta_{\mathcal{B}^{(n)}}(-\log D^{(n)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} \mathcal{O}_{\mathcal{B}^{(n+1)}} & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Taking the limit $n \to \infty$, we obtain the following Theorem.

Theorem 3.3.8. *Assume that the condition* (Q) *of Section* 2 *is satisfied.*

1) *There exists a surjective $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism*

$$\theta : \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j} \to \Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)}) \to 0.$$

2) *The restriction of θ to $\bigoplus_{j=1}^N \mathbf{C}((\xi_j))\frac{d}{d\xi_j}$*

$$\theta : \bigoplus_{j=1}^N \mathbf{C}((\xi_j))\frac{d}{d\xi_j} \to \Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)})$$

is a Lie algebra homomorphism.

3) *The further restriction of θ*

$$\theta : \bigoplus_{j=1}^N \mathbf{C}[[\xi_j]]\xi_j\frac{d}{d\xi_j} \to \Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)})$$

coincides with the differential of the action of $(\mathcal{D}^1)^{\oplus N}$ on $\mathcal{B}^{(\infty)}$.

4) *We have*

$$\mathrm{Ker}\,\theta = \pi_*^{(\infty)}(\Theta_{\mathcal{C}^{(\infty)}/\mathcal{B}^{(\infty)}}(*\sum_{j=1}^{N} S_j^{(\infty)})).$$

$\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\dfrac{d}{d\xi_j}$ has a structure of $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-Lie algebra given by a Lie bracket $[\ ,\ ]_0$ defined by

(3.3-16)

$$[f(\xi_j)\frac{d}{d\xi_j}, g(\xi_j)\frac{d}{d\xi_j}]_0 = g(\xi_j)\frac{d}{d\xi_j}(f(\xi_j))\frac{d}{d\xi_j} - f(\xi_j)\frac{d}{d\xi_j}(g(\xi_j))\frac{d}{d\xi_j}$$

where $f(\xi_j)$ and $g(\xi_j)$ are local sections of $\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$. But the $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism θ is *not* a Lie algebra homomorphism. This is because $f(\xi_j)$ is a Laurent series whose coefficients are holomorphic functions of the parameter space $\mathcal{B}^{(\infty)}$. To obtain a Lie algebra homomorphism we need to introduce the following Lie algebra structure on $\mathcal{O}_{\mathcal{B}^{(n)}}((\xi_j))\dfrac{d}{d\xi_j}$.

Definition 3.3.9. A bracket $[\ ,\ ]$ on $\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\dfrac{d}{d\xi_j}$ is defined by

(3.3-17)

$$\begin{aligned}[f\frac{d}{d\xi_j}, g\frac{d}{d\xi_j}] &= [f\frac{d}{d\xi_j}, g\frac{d}{d\xi_j}]_0 \\ &\quad + \theta(f\frac{d}{d\xi_j})(g)\frac{d}{d\xi_j} - \theta(g\frac{d}{d\xi_j})(f)\frac{d}{d\xi_j}.\end{aligned}$$

Proposition 3.3.10. *The bracket defined above induces a Lie algebra structure on* $\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\dfrac{d}{d\xi_j}$*, hence also induces a Lie algebra strucure on* $\oplus_{j=1}^{N}\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\dfrac{d}{d\xi_j}$*. With respect to this Lie algebra structure, the homomorphism* θ *in Theorem* 3.3.7, 1) *is a homomorphism of Lie algebras.*

Proof. As was explained in Remark 3.3.7, $(f_1\dfrac{d}{d\xi_1}, \dots, f_N\dfrac{d}{d\xi_N})$ and $(g_1\dfrac{d}{d\xi_1}, \dots, g_N\dfrac{d}{d\xi_N})$ define the first order infinitesimal deformations of each fibre of the family $\mathfrak{F}^{(n)}$ defined by

$$\begin{aligned} A: &\quad \xi_j \to \xi_j + \epsilon_1 f_j(s,\xi_j) \\ B: &\quad \xi_j \to \xi_j + \epsilon_2 g_j(s,\xi_j) \end{aligned}$$

respectively, where s denotes the parameters of the base space $\mathcal{B}^{(n)}$. If we first deform fibres of $\mathfrak{F}^{(n)}$ infinitesimally by A then deform them by B, we obtain

(3.3-18)

$$\begin{aligned} \xi_j \to \xi_j &+ \epsilon_1 f_j(s,\xi_j) + \epsilon_2 g_j(s,\xi_j) \\ &+ \epsilon_1\epsilon_2 \left(\theta^{(n)}(g_j \frac{d}{d\xi_j})(f_j(s,\xi_j)) + g_j \frac{d}{d\xi_j}(f_j(s,\xi_j))\right). \end{aligned}$$

Because, by applying the infinitesimal deformation B each fibre of $\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}$ deforms infinitesimally, hence it changes the parameter s, and we need to add the effect of this fact, which is nothing but the fourth term of the right hand side of 3.3-18. Reversing the order of infinitesimal deformations, we have

(3.3-19)

$$\begin{aligned} \xi_j \to \xi_j &+ \epsilon_1 f_j(s,\xi_j) + \epsilon_2 g_j(s,\xi_j) \\ &+ \epsilon_1\epsilon_2 \left(\theta^{(n)}(f_j \frac{d}{d\xi_j})(g_j(s,\xi_j)) + f_j \frac{d}{d\xi_j}(g_j(s,\xi_j))\right). \end{aligned}$$

By subtracting 3.3-18 by 3.3-19, the coefficient $[\theta^{(n)}(A), \theta^{(n)}(B)]$of $\epsilon_1\epsilon_2$ is equal to $\theta^{(n)}([A,B])$. Taking the limit $n \to \infty$ we obtain the desired result. Q.E.D.

§4. Sheaf of vacua attached to local universal family

4.1. Sheaf of Vacua

Let $\mathfrak{F}^{(\infty)} = (\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}; s_1^{(\infty)}, s_2^{(\infty)}, \dots, s_N^{(\infty)}; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \dots, \tilde{t}_N^{(\infty)})$ be a local universal family of N-pointed stable curves of genus g with formal neighbourhoods. We assume that each fibre of the family $\mathfrak{F}^{(\infty)}$ satisfies the condition (Q) in 2.1. Main purpose of the present section is to define the sheaf of vacua $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(\infty)})$ of attached to the family.

Definition 4.1.1. The sheaf $\widetilde{\mathfrak{g}}_N$ of affine Lie algebra attached to the family $\mathfrak{F}^{(\infty)}$ is a sheaf of $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module

$$\widetilde{\mathfrak{g}}_N = \mathfrak{g} \otimes_{\mathbf{C}} (\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))) \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}} \cdot c$$

with the following commutation relation.

$$[\oplus_{j=1}^N X_j \otimes f_j, \oplus_{j=1}^N Y_j \otimes g_j] = \oplus_{j=1}^N([X_j, Y_j] \otimes (f_j g_j)) \oplus c \cdot \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j=0}((g_j df_j)$$

$$c \in \text{Center}$$

where

$$X_j, Y_j \in \mathfrak{g}, \quad f_j, g_j \in \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j)).$$

Put

$$\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)}) = \mathfrak{g} \otimes_{\mathbf{C}} \pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(*S^{(\infty)})) \tag{4.1-1}$$

where we define

$$S^{(\infty)} = \sum_{j=1}^{N} s_j^{(\infty)}(\mathcal{S}^{(\infty)})$$

$$\pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(*S^{(\infty)})) = \varinjlim_{k} \pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(kS^{(\infty)})).$$

There is a sheaf version of homomorphism defined in (2.1-3), by using the formal neighbourhoods $t_j^{(\infty)}$.

$$\tilde{t} : \pi_*^{(\infty)}(\mathcal{O}_{\mathcal{B}^{(\infty)}}(*S^{(\infty)})) \to \oplus_{j=1}^N \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$$

and we may regard $\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})$ as a Lie subalgebra of $\widetilde{\mathfrak{g}}_N$.

Fix a non-negative integer ℓ. For any $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_\ell)^N$, we define

$$\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} = \mathcal{O}_{\mathcal{B}^{(\infty)}} \otimes_{\mathbf{C}} \mathcal{H}_{\vec{\lambda}}, \tag{4.1-2}$$

$$\widetilde{\mathcal{H}}_{\vec{\lambda}}^{\dagger(\infty)} = \underline{Hom}_{\mathcal{O}_{\mathcal{B}^{(\infty)}}}(\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}, \mathcal{O}_{\mathcal{B}^{(\infty)}}). \tag{4.1-3}$$

The pairing (2.2-3) induces an $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-bilinear pairing

$$(4.1\text{-}4)\qquad \langle\ |\ \rangle : \widetilde{\mathcal{H}}^{\dagger(\infty)}_{\vec{\lambda}} \otimes \widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}} \to \mathcal{O}_{\mathcal{B}^{(\infty)}}$$

which is complete with respect to the filtration introduced below. The sheaf of affine Lie algebra $\widetilde{\mathfrak{g}}_N$ acts on $\widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}}$ and $\widetilde{\mathcal{H}}^{\dagger(\infty)}_{\vec{\lambda}}$ by

$$(4.1\text{-}5)\quad (\oplus_{j=1}^{N}(X_j\otimes\sum_{n\in\mathbf{Z}} a_n\xi_j^n))(F\otimes|\Psi\rangle) = \sum_{j=1}^{N}\sum_{n\in\mathbf{Z}}(a_nF)\otimes\rho_j(X_j(n))|\Psi\rangle$$

The action of $\widetilde{\mathfrak{g}}_N$ on $\widetilde{\mathcal{H}}^{\dagger(\infty)}_{\vec{\lambda}}$ is the dual action of $\widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}}$, that is,

$$\langle\Psi a|\Phi\rangle = \langle\Psi|a\Phi\rangle \quad \text{for any}\ \ a\in\widetilde{\mathfrak{g}}_N.$$

Definition 4.1.2. Put

$$(4.1\text{-}6)\qquad \begin{aligned} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) &= \widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}}/\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})\widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}} \\ \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) &= \underline{Hom}_{\mathcal{O}_{\mathcal{B}^{(\infty)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}), \mathcal{O}_{\mathcal{B}^{(\infty)}}). \end{aligned}$$

These are sheaves of $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-modules on $\mathcal{B}^{(\infty)}$. The sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ is called the *sheaf of vacua* attached to the family $\mathfrak{F}^{(\infty)}$. Note that we have

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) = \{\,\langle\Psi| \in \widetilde{\mathcal{H}}^{\dagger(\infty)}_{\vec{\lambda}} \mid \langle\Psi|a = 0 \quad \text{for any}\ \ a\in\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})\,\}.$$

The pairing (4.1-4) induces a non-degenerate $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-bilinear pairing

$$\langle\ |\ \rangle : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) \otimes \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) \to \mathcal{O}_{\mathcal{B}^{(\infty)}}.$$

Lemma 4.1.3. *Let $\mathfrak{X}^{(\infty)}$ correspond to a point $s\in\mathcal{B}^{(\infty)}$. By the canonical isomorphism $\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s \simeq \mathbf{C}$, where $\mathfrak{m}_s$ is the maximal ideal of the stalk $\mathcal{O}_{\mathcal{B}^{(\infty)},s}$, we have the following canonical isomorphisms.*

$$(4.1\text{-}7)\qquad \begin{aligned} \widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}} \otimes (\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s) &\simeq \mathcal{H}_{\vec{\lambda}} \\ \widetilde{\mathfrak{g}}_N \otimes (\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s) &\simeq \widehat{\mathfrak{g}}_N \\ \widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)}) \otimes (\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s) &\simeq \widehat{\mathfrak{g}}(\mathfrak{X}^{(\infty)}) \\ \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) \otimes (\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s) &\simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \\ \widetilde{\mathcal{H}}^{\dagger(\infty)}_{\vec{\lambda}} \otimes (\mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s) &\simeq \mathcal{H}^{\dagger}_{\vec{\lambda}}. \end{aligned}$$

Moreover, the action of $\widetilde{\mathfrak{g}}_N$ *on* $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ *defined in* (4.1-7) *and the action of* $\widehat{\mathfrak{g}}_N$ *on* $\mathcal{H}_{\vec{\lambda}}$ *are compatible with respect to the above canonical isomorphisms.*

Proof. The first, second and the fifth isomorphisms are clear from the definition. Note that we have

$$\pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(*S^{(\infty)})) = \varinjlim_k \pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(kS^{(\infty)}))$$

and $\pi_*^{(\infty)}(\mathcal{O}_{\mathcal{C}^{(\infty)}}(kS^{(\infty)}))$ comes from $\pi_*^{(n)}(\mathcal{O}_{\mathcal{C}^{(n)}}(kS^{(n)}))$. If k is sufficiently large, we always have the base change

$$\pi_*^{(n)}(\mathcal{O}_{\mathcal{C}^{(n)}}(kS^{(n)})) \otimes_{\mathcal{O}_{\mathcal{B}^{(n)}}} (\mathcal{O}_{\mathcal{B}^{(n)},s}/\mathfrak{m}_s) \simeq H^0(C_s, \mathcal{O}_{C_s}(k\sum_{j=1}^{N} s_j^{(n)}(s)))$$

since we have

$$H^1(C_s, \mathcal{O}_{C_s}(k\sum_{j=1}^{N} s_j^{(n)}(s))) = 0$$

where $C_s = \pi^{(n)-1}(s)$. (See for example, [Ha, Chap. III, Corollary 12.9] or [BS1, Chap. III Corollary 3.5].) This implies the third isomorphism.

Finally let us consider the following commutative diagram of exact sequences

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}) \otimes \mathbf{C}_s & \stackrel{\beta}{\rightarrow} & (\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)}) \otimes \mathbf{C}_s)(\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \otimes \mathbf{C}_s) \\
\downarrow \alpha & & \downarrow \epsilon \\
\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \otimes \mathbf{C}_s & \stackrel{\gamma}{\rightarrow} & \mathcal{H}_{\vec{\lambda}} \\
\downarrow & & \downarrow \\
\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) \otimes \mathbf{C}_s & \stackrel{\delta}{\rightarrow} & \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where we put $\mathbf{C}_s = \mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s$. The above argument shows that the mapping β is surjective and the mapping γ is isomorphic. Hence, the commutativity of the diagram implies that the mapping γ induces isomorphism between the $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\epsilon)$. Therefore, the mapping δ is isomorphic. Q.E.D.

Define the action of $(\mathcal{D}^1)^{\oplus N}$ on $\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$ by

(4.1-8)
$$\vec{h}(\sum_{n\in\mathbf{Z}} a_n\xi_j^n) = \sum_{n\in\mathbf{Z}} L_{\vec{h}}(a_n)h_j(\xi_j^n) \quad \text{for } \vec{h} = (h_1,\dots,h_N) \in (\mathcal{D}^1)^{\oplus N}$$

where $L_{\vec{h}}$ is defined for any $F \in \mathcal{O}_{\mathcal{B}^{(\infty)}}$ by

$$L_{\vec{h}}(F)(s) = F((\vec{h}^{-1}\circ s)), \quad s \in \mathcal{B}^{(\infty)}.$$

Note that the action of $(\mathcal{D}^1)^{\oplus N}$ on $\mathcal{B}^{(\infty)}$ is defined in (2.3-1). (See also 3.1.)

Define the action π of $(\mathcal{D}^1)^{\oplus N}$ on $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ by

$$\pi(\vec{h})(F\otimes|\Psi\rangle) = L_{\vec{h}}(F)\otimes(\rho(G[\vec{h}]))|\Psi\rangle \tag{4.1-9}$$

for $\vec{h} \in (\mathcal{D}^1)^{\oplus N}$. (See (2.3-1).) Also we define the action π of $(\mathcal{D}^1)^{\oplus N}$ on $\widetilde{\mathfrak{g}}_N$ by

$$\pi(\vec{h})(\oplus_{j=1}^N X_j\otimes f_j \oplus a\cdot c) = \oplus_{j=1}^N (X_j\otimes h_j(f_j)) \oplus L_{\vec{h}}(a)\cdot c$$

for $\vec{h} = (h_1,\dots,h_N) \in (\mathcal{D}^1)^{\oplus N}$, $X_j \in \mathfrak{g}$, $f_j \in \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$ and $a \in \mathcal{O}_{\mathcal{B}^{(\infty)}}$.

The following Lemma is an easy consequence of the definition of the actions of $(\mathcal{D}^1)^{\oplus N}$ and Theorem 1.4.5, 1).

Lemma 4.1.4. *$\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ is stable under the action of $(\mathcal{D}^1)^{\oplus N}$ on $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$.*

Let us consider the tower of local universal family (3.3-1) of N-pointed stable curves of genus g with infinitesimal neighbourhoods. As was explained in 3.3, $\varphi^{(1)} : \mathcal{C}^{(\infty)} \to \mathcal{C}^{(1)}$ and $\psi^{(1)} : \mathcal{B}^{(\infty)} \to \mathcal{B}^{(1)}$ are principal fibre bundles with structure group $(\mathcal{D}^1)^{\oplus N}$. Put

$$\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} = \{\, f \in \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \mid \pi(\vec{h})f = f \text{ for all } \vec{h} \in (\mathcal{D}^1)^{\oplus N}\}$$
$$\widetilde{\mathfrak{g}}^{(1)}(\mathfrak{F}^{(\infty)}) = \{\, f \in \widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)}) \mid \pi(\vec{h})f = f \text{ for all } \vec{h} \in (\mathcal{D}^1)^{\oplus N}\}$$

By Lemma 4.1.4, $(\mathcal{D}^1)^{\oplus N}$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$. Put

$$\tilde{\mathcal{V}}_{\vec{\lambda}}^{(1)} = \{\, g \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)}) \mid \pi(\vec{h})g = g \ \text{ for all } \ \vec{h} \in (\mathcal{D}^1)^{\oplus N} \,\}.$$

Then $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}$, $\tilde{\mathfrak{g}}^{(1)}(\mathfrak{F}^{(\infty)})$ and $\tilde{\mathcal{V}}_{\vec{\lambda}}^{(1)}$ are ${\psi^{(1)}}^{-1}\mathcal{O}_{\mathcal{B}^{(1)}}$-modules and we can show that there are canonical isomorphisms:

$$\begin{aligned}
\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} \otimes_{{\psi^{(1)}}^{-1}\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(\infty)}} &\simeq \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \\
\tilde{\mathfrak{g}}^{(1)}(\mathfrak{F}^{(\infty)}) \otimes_{{\psi^{(1)}}^{-1}\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(\infty)}} &\simeq \widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)}) \\
\tilde{\mathcal{V}}_{\vec{\lambda}}^{(1)} \otimes_{{\psi^{(1)}}^{-1}\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(\infty)}} &\simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})
\end{aligned}$$

Lemma 4.1.5. *There exist sheaves $\mathcal{H}_{\vec{\lambda}}^{(1)}$, $\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ of $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules on $\mathcal{B}^{(1)}$ such that*

$$\begin{aligned}
\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} &= {\psi^{(1)}}^{-1}\mathcal{H}_{\vec{\lambda}}^{(1)} \\
\tilde{\mathfrak{g}}^{(1)}(\mathfrak{F}^{(\infty)}) &= {\psi^{(1)}}^{-1}\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}) \\
\tilde{\mathcal{V}}_{\vec{\lambda}}^{(1)} &= {\psi^{(1)}}^{-1}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}).
\end{aligned}$$

Moreover we have a non-canonical isomorphism

$$\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}) \simeq \mathfrak{g} \otimes_{\mathbf{C}} \pi_*^{(1)}\mathcal{O}_{\mathcal{C}^{(1)}}(*S^{(1)}),$$

where $S^{(1)} = \sum_{j=1}^{N} s^{(1)}(\mathcal{B}^{(1)})$.

Similarly we can define the sheaves $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{\dagger(1)}$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(1)})$ on $\mathcal{B}^{(1)}$.

Lemma 4.1.6.

$$\begin{aligned}
\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) &= \mathcal{H}_{\vec{\lambda}}^{(1)} / \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\mathcal{H}_{\vec{\lambda}}^{(1)}. \\
\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(1)}) &= \underline{Hom}_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}), \mathcal{O}_{\mathcal{B}^{(1)}}) \\
&= \{\, \langle\Psi| \in \widetilde{\mathcal{H}}_{\vec{\lambda}}^{\dagger(1)} \mid \langle\Psi|a = 0 \ \textit{for all} \ a \in \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}) \,\}
\end{aligned}$$

Remark 4.1.7. We define

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(1)}) = \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s)$$

where the point $s \in \mathcal{B}^{(1)}$ corresponds to $\mathfrak{X}^{(1)}$. Then by Lemma 4.1.3 and Lemma 4.1.5 we have a canonical isomorphism

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(1)}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$$

where $\mathfrak{X}^{(\infty)}$ is an N-pointed stable curve with formal neighbourhoods whose restriction to the first order infinitesimal structure is $\mathfrak{X}^{(1)}$.

4.2. Coherency

In this subsection we shall prove the coherency of the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$. First we introduce filtrations $\{F_\bullet\}$ which play an important role in the proof of coherency. The filtration $\{F_\bullet\}$ on $\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$ is defined by

$$F_p\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j)) = \mathcal{O}_{\mathcal{B}^{(\infty)}}[[\xi_j]]\xi_j^{-p}, \quad p \in \mathbf{Z}.$$

The filtration $\{F_\bullet\}$ on $\widetilde{\mathfrak{g}}_N$ is defined by

$$(4.2\text{-}1) \qquad F_p\widetilde{\mathfrak{g}}_N = \begin{cases} \mathfrak{g} \otimes \sum_{j=1}^N F_p\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j)) \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}} \cdot c & p \geq 0 \\ \mathfrak{g} \otimes \sum_{j=1}^N F_p\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j)) & p < 0. \end{cases}$$

The filtration $\{F_\bullet\}$ on $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ is defined by

$$(4.2\text{-}2) \qquad F_p\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} = \mathcal{O}_{\mathcal{B}^{(\infty)}} \otimes F_p\mathcal{H}_{\vec{\lambda}}$$

where $F_p\mathcal{H}_{\vec{\lambda}} = \sum_{d \leq p} H_{\vec{\lambda}}(d)$ and

$$\mathcal{H}_{\vec{\lambda}}(d) = \sum_{d_1+\ldots+d_N=d} \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N).$$

It is easy to see that

$$F_p\widetilde{\mathfrak{g}}_N \cdot F_q\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \subseteq F_{p+q}\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}.$$

The actions of $(\mathcal{D}^1)^{\oplus N}$ on $\widetilde{\mathfrak{g}}_N$ and $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ preserve these filtrations. Hence, $\widetilde{\mathfrak{g}}_N^{(1)}$ and $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}$ have filtrations induced by those of $\widetilde{\mathfrak{g}}_N$ and $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$, where $\widetilde{\mathfrak{g}}_N^{(1)}$ is a sheaf on $\mathcal{B}^{(1)}$ whose pull-back to $\mathcal{B}^{(\infty)}$ is the $(\mathcal{D}^1)^{\oplus N}$-invariant part of $\widetilde{\mathfrak{g}}_N$ and $\widetilde{\mathfrak{g}}_N^{(1)} \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(\infty)}} \simeq \widetilde{\mathfrak{g}}_N$. Then we have

$$(4.2\text{-}3) \qquad F_p\widetilde{\mathfrak{g}}_N^{(1)} \cdot F_q\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} \subseteq F_{p+q}\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}.$$

Since $(\mathcal{D}^1)^{\oplus N}$ acts trivially on $Gr_\bullet \widetilde{\mathfrak{g}}_N$ and $Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ by Lemma 1.4.3, we have the following canonical $\mathcal{O}_{\mathcal{B}^{(1)}}$-module isomorphisms.

$$
\begin{aligned}
&Gr_\bullet \widetilde{\mathfrak{g}}_N^{(1)} \simeq \mathcal{O}_{\mathcal{B}^{(1)}} \otimes_{\mathbf{C}} \sum_{j=1}^{N} \mathfrak{g} \otimes \mathbf{C}[\xi_j, \xi_j^{-1}] \oplus \mathcal{O}_{\mathcal{B}^{(1)}} c \\
&Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} \simeq \mathcal{O}_{\mathcal{B}^{(1)}} \otimes_{\mathbf{C}} \mathcal{H}_{\vec{\lambda}}
\end{aligned} \tag{4.2-4}
$$

where $\deg \xi_j = -1$, $\deg c = 0$, and the degree d part of $\mathcal{H}_{\vec{\lambda}}$ is $\mathcal{H}_{\vec{\lambda}}(d)$.

On $\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})$ and $\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}$ we introduce the induced filtrations from that of $\widetilde{\mathfrak{g}}_N^{(1)}$ and $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}$ respectively. On $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ we introduce the quotient filtrations from that of $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}$. Then we have the following exact sequences of graded $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules.

$$
\begin{aligned}
&0 \longrightarrow Gr_\bullet \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}) \longrightarrow Gr_\bullet \widetilde{\mathfrak{g}}_N^{(1)} \\
&0 \longrightarrow Gr_\bullet(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}) \longrightarrow Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} \longrightarrow Gr_\bullet \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \longrightarrow 0.
\end{aligned} \tag{4.2-5}
$$

For a positive integer M, we consider the graded Lie subalgebra $\mathcal{O}_{\mathcal{B}^{(1)}} \otimes (\mathfrak{g} \otimes \sum_{j=1}^{N} \mathbf{C}[\xi_j^{-1}]\xi_j^{-M})$ of $Gr_\bullet \widetilde{\mathfrak{g}}_N^{(1)}$.

The Riemann-Roch theorem implies the following lemma.

Lemma 4.2.1. *There exists a positive integer M such that*

$$
\mathcal{O}_{\mathcal{B}^{(1)}} \otimes \sum_{j=1}^{N} \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[\xi_j^{-1}]\xi_j^{-M} \hookrightarrow Gr_\bullet \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}). \tag{4.2-6}
$$

Fix a positive integer M satisfying (4.2-6) and define a graded $\mathcal{O}_{\mathcal{B}^{(1)}}$-module $\mathcal{V}$ by

$$
\mathcal{V} = Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)} / (\mathcal{O}_{\mathcal{B}^{(1)}} \otimes (\sum_{j=1}^{N} \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[\xi_j^{-1}]\xi_j^{-M})) Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}. \tag{4.2-7}
$$

On the other hand, by (4.2-3) we have

$$
F_p(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}) = \sum_{p_1+p_2=p} F_{p_1}(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})) \cdot F_{p_2}(\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}).
$$

Hence we have

$$
Gr_\bullet(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}) = Gr_\bullet \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)}) \cdot Gr_\bullet \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(1)}.
$$

Therefore, by Lemma 4.2.1 we have the following lemma.

Lemma 4.2.2. *There is a surjective $\mathcal{O}_{\mathcal{B}^{(1)}}$-module homomorphism*

$$\mathcal{V} \longrightarrow Gr_\bullet \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \longrightarrow 0. \tag{4.2-8}$$

Proposition 4.2.3. *The sheaf $\mathcal{V}$ is a coherent $\mathcal{O}_{\mathcal{B}^{(1)}}$-module.*

Proposition 4.2.3, Lemma 4.2.2 and Lemma 4.1.6 imply the following theorem.

Theorem 4.2.4. *The sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ are coherent $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules.*

The rest of this subsection is devoted to prove Proposition 4.2.3. Put

$$V = \mathcal{H}_{\vec{\lambda}} / (\mathfrak{g} \otimes \sum_{j=1}^{N} \mathbf{C}[\xi_j^{-1}]\xi_j^{-M})\mathcal{H}_{\vec{\lambda}}.$$

Then, as $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules we have

$$\mathcal{V} \simeq \mathcal{O}_{\mathcal{B}^{(1)}} \otimes V.$$

Hence it is sufficient to prove that V is a finite dimensional vector space over $\mathbf{C}$. The module $\mathcal{H}_{\vec{\lambda}}$ is generated by a finite dimensional vector space $\mathcal{H}_{\vec{\lambda}}(0)$ as a $U(\mathfrak{g} \otimes \sum_{j=1}^{N} \mathbf{C}[\xi_j^{-1}])$-module, where $U(\mathfrak{b})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{b}$. Let V_0 be the image of $\mathcal{H}_{\vec{\lambda}}(0)$ on V and $\bar{\mathfrak{g}}$ denotes a finite dimensional Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \sum_{j=1}^{N} \mathbf{C}[\xi_j^{-1}]/(\xi_j^{-M})$. Then V_0 is a finite dimensional vector space and we have

$$V = U(\bar{\mathfrak{g}}) \cdot V_0.$$

We define filtrations $\{G_\bullet\}$ on $U(\bar{\mathfrak{g}})$ and V as follows.

$$G_m U(\bar{\mathfrak{g}}) = \begin{cases} \{0\} & m < 0 \\ \mathbf{C} \cdot 1 & m = 0 \\ G_{m-1}U(\bar{\mathfrak{g}}) + \bar{\mathfrak{g}} \cdot G_{m-1}U(\bar{\mathfrak{g}}) & m \geq 1 \end{cases}$$

$$G_m V = G_m U(\bar{\mathfrak{g}}) \cdot V_0.$$

Then we have

$$G_m U(\bar{\mathfrak{g}}) \cdot G_n U(\bar{\mathfrak{g}}) \subseteq G_{m+n} U(\bar{\mathfrak{g}})$$
$$[G_m U(\bar{\mathfrak{g}}), G_n U(\bar{\mathfrak{g}})] \subseteq G_{m+n-1} U(\bar{\mathfrak{g}})$$
$$G_m U(\bar{\mathfrak{g}}) \cdot G_n V \subseteq G_{m+n} V.$$

Now we consider the associated graded objects $Gr_\bullet U(\bar{\mathfrak{g}})$ and $Gr_\bullet V$. We have a $\mathbf{C}$-algebra isomorphism

$$Gr_\bullet U(\bar{\mathfrak{g}}) \simeq S^*(\bar{\mathfrak{g}})$$

where $S^*(\bar{\mathfrak{g}})$ is the symmetric algebra of $\bar{\mathfrak{g}}$.

Furthermore the commutative algebra $S^*(\bar{\mathfrak{g}})$ has the Poisson bracket $\{\, ,\}$ defined by $\{\bar{P}, \bar{Q}\} = \overline{[P,Q]}$ where $P \in G_m U(\bar{\mathfrak{g}})$, $Q \in G_n U(\bar{\mathfrak{g}})$ and $\bar{P} \in S^m(\bar{\mathfrak{g}})$, $\bar{Q} \in S^n(\bar{\mathfrak{g}})$ are associated elements. And if $a, b \in \bar{\mathfrak{g}}$, then $\{a, b\} = [a, b]$.

Consider the ideal

$$\mathfrak{a} = \{a \in S^*(\bar{\mathfrak{g}}) \mid \quad aGr_\bullet V = 0\}.$$

Then Gabber's theorem [Ga] says that the radical $\sqrt{\mathfrak{a}}$ is closed under the Poisson bracket. On the other hand, for each element $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$, and $n \in \mathbf{Z}$, $X_\alpha \otimes \xi_j^n$ acts locally nilpotently on $\mathcal{H}_{\vec{\lambda}}$, so $X_\alpha \otimes \xi_j^{-n} \in \sqrt{\mathfrak{a}}$ for each $n = 0, 1, \ldots, M-1$. Furthermore $H_\alpha \otimes \xi_j^{-n} = \{X_\alpha \otimes 1, X_{-\alpha} \otimes \xi_j^{-n}\} \in \sqrt{\mathfrak{a}}$. Therefore, the radical $\sqrt{\mathfrak{a}}$ contains a maximal ideal $\bar{\mathfrak{g}} S^*(\bar{\mathfrak{g}})$. Hence, $S^*(\bar{\mathfrak{g}})/\mathfrak{a}$ is an Artin ring over $\mathbf{C}$, that is, a finite dimensional $\mathbf{C}$-algebra. Since $Gr_\bullet V$ is a finite $S^*(\bar{\mathfrak{g}})/\mathfrak{a}$-module, $Gr_\bullet V$ is finite dimensional. This implies that V is a finite dimensional vector space over $\mathbf{C}$. This proves Proposition 4.2.3.

§5. Integrable Connection with Regular Singularity

In this section we shall define a sheaf of twisted first order differential operators $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ acting on the sheaf of vacua and the dual sheaf of vacua. In the following we formulate left action of $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$. The right action on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ is obtained easily by using the canonical pairing $\langle \quad | \quad \rangle$ introduced in §4. That is, for $D \in \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ we have

$$\langle \Psi | D \Phi \rangle = \langle \Psi D | \Phi \rangle$$

In this section we use the same notations as those in §4.

5.1. Sheaf $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$

Let $\mathfrak{F}^{(\infty)} = (\pi^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{B}^{(\infty)}; s_1^{(\infty)}, s_2^{(\infty)}, \ldots, s_N^{(\infty)}; \tilde{t}_1^{(\infty)}, \tilde{t}_2^{(\infty)}, \ldots, \tilde{t}_N^{(\infty)})$ be a local universal family of N-pointed stable curves of genus g with formal neighbourhoods. In §4 we defined the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and

$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ associated with the family $\mathfrak{F}^{(\infty)}$. Let $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ be a sheaf of an $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module on $\mathcal{B}^{(\infty)}$ defined by

$$\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v) = (\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j}) \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}}.$$

Let

$$p : \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j} \tag{5.1-1}$$

be a natural projection. By Theorem 3.3.8 there is a surjective $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism

$$\theta : \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j} \to \Theta_{\mathcal{B}^{(\infty)}}(-\log D^{(\infty)}).$$

Put $\overline{\theta} = \theta \circ p$.

Definition 5.1.1. On $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ we define a Lie algebra structure as follows.

1) The constant subsheaf $\mathbf{C}$ of $\mathcal{O}_{\mathcal{B}^{(\infty)}}$ is the center.

2) Let $[\ ,\ ]$ denote the Lie bracket on $\oplus_{j=1}^{N}\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j}$ defined in Definition 3.3.9. For $\vec{\ell}_1 = (\underline{l}_1^1, \underline{l}_1^2, \dots, \underline{l}_1^N)$, $\vec{\ell}_2 = (\underline{l}_2^1, \underline{l}_2^2, \dots, \underline{l}_2^N) \in \oplus_{j=1}^{N}\mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j}$, define the bracket $[\ ,\ ]_{Vir}$ by

$$[(\vec{\ell}_1, 0), (\vec{\ell}_2, 0)]_{Vir} = [\vec{\ell}_1, \vec{\ell}_2] + \frac{c_v}{12}\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}\left(\frac{d^3\ell_1^j(\xi_j)}{d\xi_j^3}\ell_2^j(\xi_j)d\xi_j\right)$$

where $\underline{l}_i^j = \ell_i^j(\xi_j)\frac{d}{d\xi_j}$.

3) For V_1, $V_2 \in \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ and $f \in \mathcal{O}_{\mathcal{B}^{(\infty)}}$ the bracket $[\ ,\ ]_{Vir}$ has the properties

$$[fV_1, V_2]_{Vir} = f[V_1, V_2]_{Vir} - \overline{\theta}(V_2)(f)V_1$$
$$[V_1, fV_2]_{Vir} = f[V_1, V_2]_{Vir} + \overline{\theta}(V_1)(f)V_2 .$$

It is easy to see that the above definition indeed defines a Lie algebra structure on $\widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v)$. In the following we often use the notation $[\ ,\ ]$ instead of $[\ ,\ ]_{Vir}$.

Lemma 5.1.2. *The following exact sequence of* $\mathcal{O}_{\mathcal{B}(\infty)}$*-modules*

$$0 \to \mathcal{O}_{\mathcal{B}(\infty)} \to \widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}(\infty)}((\xi_j))\frac{d}{d\xi_j} \to 0$$

is an extension of the sheaves of Lie algebras with respect to the Lie algebra structure defined above.

The sheaf $\widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v)$ of Lie algebras acts on $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} = \mathcal{O}_{\mathcal{B}(\infty)} \otimes_{\mathbf{C}} \mathcal{H}_{\vec{\lambda}}$ in the following way.

For $F \in \mathcal{O}_{\mathcal{B}(\infty)}$, $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$ and $V = (\vec{\ell}, r) \in \widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v)$ with $\vec{\ell} = (\underline{l}^1, \underline{l}^2, \ldots, \underline{l}^N) \in \oplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}(\infty)}((\xi_j))\frac{d}{d\xi_j}$, $r \in \mathcal{O}_{\mathcal{B}(\infty)}$, we define

(5.1-2)
$$\begin{aligned} &D(V)(F \otimes |\Phi\rangle) \\ &= \theta(\vec{\ell})(F) \otimes |\Phi\rangle - F \otimes (\sum_{j=1}^{N} \rho_j(T[\underline{l}^j])|\Phi\rangle) + rF \otimes |\Phi\rangle. \end{aligned}$$

Proposition 5.1.3. *For* $V \in \widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v)$ *the above action* $D(V)$ *is well-defined and has the following properties.*

0) *We have*

$$D(fV) = fD(V) \quad \textit{for any} \ \ f \in \mathcal{O}_{\mathcal{B}(\infty)} .$$

1) *For* V_1, $V_2 \in \widetilde{Vir}_{\widehat{\mathcal{S}}(\infty)}(c_v)$ *we have*

$$[D(V_1), D(V_2)] = D([V_1, V_2]_{Vir}) .$$

2) *For* $f \in \mathcal{O}_{\mathcal{B}(\infty)}$ *and* $|\Phi\rangle \in \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$, *we have*

$$D(V)(f|\Phi\rangle) = \overline{\theta}(V)(f)|\Phi\rangle + fD(V)|\Phi\rangle .$$

By the natural inclusion

$$(d^1)^{\oplus N} = \bigoplus_{j=1}^{N} \mathbf{C}[[\xi_j]]\xi_j\frac{d}{d\xi_j} \hookrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}(\infty)}((\xi_j))\frac{d}{d\xi_j}$$

$(d^1)^{\oplus N}$ can be regarded as a Lie subalgebra subsheaf of $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$. By the direct calculations we can prove the following two propositions.

Proposition 5.1.4.

1) *The restriction* $D|(d^1)^{\oplus N}$ *is equal to the differential of the action of* $(\mathcal{D}^1)^{\oplus N}$ *on* $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$.

2) *For an element*

$$\vec{\widetilde{X}} = (\widetilde{X}_1, \dots, \widetilde{X}_N) \in \bigoplus_{j=1}^{N} \mathfrak{g} \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))$$

and an element $V \in \widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$, *we have*

$$[D(V), \vec{\widetilde{X}}] = \overline{\theta}(V)(\vec{\widetilde{X}})$$

as operators on $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$.

Proposition 5.1.5. $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$ *preserves* $\widetilde{\mathfrak{g}}(\mathfrak{F}^{(\infty)})\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$.

Corollary 5.1.6. $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$ *acts on* $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(\infty)})$.

Proposition 5.1.7. *Let* $B_{\widehat{S}^{(\infty)}} = \pi_*^{(\infty)}(\Theta_{\mathcal{C}^{(\infty)}/\mathcal{B}^{(\infty)}}(*S^{(\infty)}))$ *be the kernel of the homomorphism* θ *given in Theorem* 3.3.8. *There exists a unique* $\mathcal{O}_{\mathcal{B}^{(\infty)}}$*-module homomorphism*

$$a : B_{\widehat{S}^{(\infty)}} \to \mathcal{O}_{\mathcal{B}^{(\infty)}}$$

such that for any $\vec{\ell} \in B_{\widehat{S}^{(\infty)}}$ *we have*

$$D((\vec{\ell}, 0)) = a(\vec{\ell}) \cdot \mathrm{id}$$

as a linear operator acting on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(\infty)})$. *Moreover, for any* $\vec{h} \in (\mathcal{D}^1)^{\oplus N}$, *we have*

$$a(\pi(\vec{h})(\vec{\ell})) = L_{\vec{h}}(a(\vec{\ell})) \in \mathcal{O}_{\mathcal{B}^{(\infty)}}.$$

Proof. Let $\mathfrak{X}^{(\infty)}$ be an N-pointed stable curve with formal neighbourhoods corresponding to a point $s \in \mathcal{B}^{(\infty)}$. By Lemma 4.1.3, by taking the tensor product $\otimes \mathcal{O}_{\mathcal{B}^{(\infty)},s}/\mathfrak{m}_s$ there are canonical reduction homomorphisms

$$\iota_s : \widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)} \to \mathcal{H}_{\vec{\lambda}}$$
$$\iota_s : \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(\infty)}) \to \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}^{(\infty)}).$$

The actions of $T(\xi_j) = \sum_{n\in\mathbf{Z}} L_n \xi_j^{-n-2}$ on the j-th component of $\widetilde{\mathcal{H}}_{\vec{\lambda}}^{(\infty)}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ are defined by the same way as those on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}^{(\infty)})$, respectively. Then, for any $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ we have

$$\rho_j(T(\xi_j))\iota_s(|\Phi\rangle) = \iota_s(\rho_j(T(\xi_j))|\Phi\rangle)\,.$$

In what follows, for $\langle\Psi| \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}^{(\infty)})$ and $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ we use the following notation freely.

$$\langle\Psi|\Phi\rangle = \langle\Psi|\iota_s(\Phi)\rangle\,.$$

Then for $\langle\Psi| \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}^{(\infty)})$, $|\Phi_0\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and $\vec{\ell} = (\underline{l}_1, \ldots, \underline{l}_N) \in B_{\widehat{\mathcal{S}}^{(\infty)}}$, by (5.1-2) and Proposition 2.4.2 we have

$$\text{(5.1-3)}\qquad \begin{aligned} \langle\Psi|D((\vec{\ell},0))|\Phi_0\rangle &= -\sum_{j=1}^{N}\langle\Psi|\rho_j(T[\underline{l}_j])|\Phi_0\rangle \\ &= -\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}\left(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|\Phi_0\rangle d\xi_j\right) \end{aligned}$$

where $\underline{l}_j = \ell_j(\xi_j)\dfrac{d}{d\xi_j}$ and

$$\begin{aligned} &\langle\Psi|(T(\xi_j))|\Phi_0\rangle d\xi_j^2 \\ &\quad= \lim_{\xi_j'\to\xi_j}\left\{\frac{1}{2(\ell+g^*)}\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(\xi_j')J^a(\xi_j)|\Phi_0\rangle d\xi_j' d\xi_j \right. \\ &\qquad\qquad \left. - \frac{c_v}{2(\xi_j'-\xi_j)^2}\langle\Psi|\Phi_0\rangle d\xi_j' d\xi_j\right\}. \end{aligned}$$

Let us choose a meromorphic form $\omega \in H^0(\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}^{\boxtimes 2}(2\Delta))$ such that

$$\omega(w,z)dwdz = \frac{dwdz}{(w-z)^2} + \text{regular at the diagonal } \Delta$$

where $\pi : \mathcal{C} \to \mathcal{B}$ is the local universal family of N-pointed curves corresponding to our family $\mathfrak{F}^{(\infty)}$. Existence of such a form will be proved in Lemma 5.1.10, below. Let us define a meromorphic form

$\langle\Psi|\widetilde{T}(z)|\Phi_0\rangle dz^2 \in H^0(C, \omega_C^{\otimes 2}(*\sum_{j=1}^N Q_j))$ by

(5.1-4)

$$\langle\Psi|\widetilde{T}(z)|\Phi_0\rangle dz^2 = \lim_{w\to z}\left\{\frac{1}{2(\ell+g^*)}\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(w)J^a(z)|\Phi_0\rangle dwdz - \frac{c_v}{2}\omega(w,z)\langle\Psi|\Phi_0\rangle dwdz\right\}.$$

Also define $S_\omega(z)dz^2$ by

$$S_\omega(z)dz^2 = -6\lim_{w\to z}\{\omega(w,z)dwdz - \frac{dwdz}{(w-z)^2}\}.$$

$S_\omega(z)dz^2$ is called projection connection, [T]. It depends on the choice of local coordinate and we have

$$S_\omega(w)dw^2 = S_\omega(z)dz^2 + \{w,z\}dz^2.$$

Then we have

(5.1-5) $$\langle\Psi|T(\xi_j)|\Phi_0\rangle d\xi_j^2 = \langle\Psi|\widetilde{T}(\xi_j)|\Phi_0\rangle d\xi_j^2 + \frac{c_v}{12}\langle\Psi|\Phi_0\rangle S_{\omega,j}(\xi_j)d\xi_j^2$$

where $S_{\omega,j}(\xi_j)d\xi_j^2$ is the expression of the projection connection $S_\omega(z)dz^2$ at the point Q_j with respect to the formal parameter ξ_j. By (5.1-3) and (5.1-5) we have

$$\langle\Psi|D((\vec{\ell},0))|\Phi_0\rangle = -\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}\left(\ell_j(\xi_j)\langle\Psi|\widetilde{T}(\xi_j)|\Phi_0\rangle d\xi_j\right) - \frac{c_v}{12}\langle\Psi|\Phi_0\rangle\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}\left(\ell_j(\xi_j)S_{\omega,j}(\xi_j)d\xi_j\right).$$

Since $\ell_j(z)\langle\Psi|\widetilde{T}(z)|\Phi_0\rangle dz$ is a global meromorphic one form on the curve C, the first term of the right hand side vanished. Therefore, if we put

(5.1-6) $$a_\omega(s,\vec{\ell}) = -\frac{c_v}{12}\cdot\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)S_{\omega,j}(\xi_j)d\xi_j)$$

then $a(\vec{\ell}) = a_\omega(s,\vec{\ell})$ satisfies the properties of Proposition 5.1.7.

Q.E.D.

Corollary 5.1.8. *There exists a canonical $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism*

$$a : B_{\widehat{\mathcal{S}}^{(\infty)}} \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}} \to \mathcal{O}_{\mathcal{B}^{(\infty)}}$$

such that for $V \in B_{\widehat{\mathcal{S}}^{(\infty)}} \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}}$ and $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(\infty)})$ we have

$$D(V)|\Phi\rangle = a(V)|\Phi\rangle.$$

Proof. For $V = (\vec{\ell}, r) \in B_{\widehat{\mathcal{S}}^{(\infty)}} \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}}$ put

$$a(V) = a(\vec{\ell}) + r.$$

Then a has the desired properties. Q.E.D.

Remark 5.1.9. We can define a non-canonical $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism

$$a : \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v) \to \mathcal{O}_{\mathcal{B}^{(\infty)}}$$

whose restriction to $B_{\widehat{\mathcal{S}}^{(\infty)}} \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}}$ is the canonical homomorphism a in Corollary 5.1.8. Choose a meromorphic form

$$\omega \in H^0(\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}^{\boxtimes 2}(2\Delta))$$

such that

$$\omega = \frac{dwdz}{(w-z)^2} + \text{regular at the diagonal } \Delta.$$

and let $S_{\omega,j}(\xi_j)d\xi_j^2$ be the expression of the projective connection $S_\omega(z)dz^2$ by the formal parameter ξ_j at Q_j. Then, for an element $V = (\vec{\ell}, r) \in \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ with $\vec{\ell} = (\underline{l}_1, \dots, \underline{l}_N) \in \oplus_{j=1}^N \mathcal{O}_{\mathcal{B}^{(\infty)}}((\xi_j))\frac{d}{d\xi_j}$, $a(V)$ is defined by

$$a(V) = -\frac{c_v}{12} \cdot \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)S_{\omega,j}(\xi_j)d\xi_j) + r \tag{5.1-7}$$

where $\underline{l}_j = \ell_j(\xi_j)\frac{d}{d\xi_j}$. Thus the homomorphism a does depend on the choice of ω.

Let $(\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}; s_1, \dots, s_N)$ be a local universal family of N-pointed stable curves of genus g. If the family $\pi^{(0)}$ contains a singular curve, hence $g \geq 1$, we let D be an irreducible component of the discriminant locus. By Lemma 3.1.6 3), D is smooth. We let $\mathcal{C}_D \to D$ be the restriction of the family $\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)}$ to D and let $\sigma : D \to \mathcal{C}_D$ be the section such that $\sigma(D)$ is a locus of double points of fibres corresponding to the component D. Let $\nu : \widetilde{\mathcal{C}}_D \to \mathcal{C}_D$ be the normalization of the locus of the double points $\sigma(D)$ and we let σ', $\sigma'' : D \to \widetilde{\mathcal{C}}_D$ be the sections corresponding to the inverse image of the locus $\sigma(D)$ by the normalization. Let

$$j : \widetilde{\mathcal{C}}_D \times_D \widetilde{\mathcal{C}}_D \longrightarrow \mathcal{C}^{(0)} \times \mathcal{B}^{(0)} \mathcal{C}^{(0)}$$

be the canonical morphism and $p_i : \widetilde{\mathcal{C}}_D \times_D \widetilde{\mathcal{C}}_D \to \widetilde{\mathcal{C}}_D$, $i = 1, 2$ be the i-th projection.

In the proof of the above Proposition 5.1.7 we used the first part of the following Lemma.

Lemma 5.1.10. *Under the above notations, if we choose $\mathcal{B}^{(0)}$ sufficiently small, then there exists a meromorphic form*

$$\omega \in H^0(\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta))$$

such that

$$\omega(w,z) dw dz = \frac{dw dz}{(w-z)^2} + \text{regular at the diagonal } \Delta. \tag{5.1-8}$$

Moreover, ω cab be chosen to satisfy the following property.

$$j^*(\omega) \in H^0(\widetilde{\mathcal{C}}_D \times_D \widetilde{\mathcal{C}}_D, \omega^{\boxtimes 2}(2\Delta)). \tag{5.1-9}$$

That is, $j^*(\omega)$ is holomorphic at $p_i(\sigma'(D))$ and $p_i(\sigma''(D))$, $i = 1, 2$.

Proof. The proof of Theorem 3.1.5 says that our family $\mathfrak{F}^{(0)}$ is constructed from a versal family $\pi : \mathcal{C} \to \mathcal{B}$ of semi-stable curves and there are holomorphic mappings $\phi : \mathcal{C}^{(0)} \to \mathcal{C}$ and $\psi : \mathcal{B}^{(0)} \to \mathcal{B}$. Moreover, it is known that the family $\pi : \mathcal{C} \to \mathcal{B}$ is obtained from a pull-back of a versal family $\hat{\pi} : \widehat{\mathcal{C}} \to \widehat{\mathcal{B}}$ of *stable* curves ([DM]). Hence we have holomorphic mappings $\hat{\phi} : \mathcal{C}^{(0)} \to \widehat{\mathcal{C}}$ and $\hat{\psi} : \mathcal{B}^{(0)} \to \widehat{\mathcal{B}}$. If the family $\hat{\pi} : \widehat{\mathcal{C}} \to \widehat{\mathcal{B}}$ is a family of smooth curves, the above Lemma is a consequence of the existence of Szegö kernel. If the family $\hat{\pi} : \widehat{\mathcal{C}} \to \widehat{\mathcal{B}}$ contains singular stable curves, then applying the arguments of Fay [Fa, Corollary 3.2, Corollary 3.8], we can find a meromorphic form $\hat{\omega} \in H^0(\widehat{\mathcal{C}} \times_{\widehat{\mathcal{B}}} \widehat{\mathcal{C}}, \omega^{\boxtimes 2}_{\widehat{\mathcal{C}}/\widehat{\mathcal{B}}}(2\Delta))$ with

$$\hat{\omega}(w,z) dw dz = \frac{dw dz}{(w-z)^2} + \text{regular at the diagonal } \Delta\,.$$

Now the pull-back ω of $\hat{\omega}$ to $\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}$ is a desired form satisfying (5.1-8).

Suppose that ω and ω' satisfy (5.1-8). Then we have

$$\omega - \omega' \in H^0(\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}).$$

Hence, choosing $\mathcal{B}^{(0)}$ smaller we can find forms

$$\tau, \tau' \in H^0(\mathcal{C}^{(0)}, \omega_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}})$$

such that $\omega + \tau \boxtimes \tau'$ satisfies the condition (5.1-9). Q.E.D.

Remark 5.1.11. There exists a sheaf homomorphism

$$\mathrm{Res}^2_\Delta : \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta) \to \mathcal{O}_\Delta$$

defined by

$$\tau(w, z, u)dwdz \longmapsto a(u)$$

where

$$\tau(w, z, u)dwdz = a(u)\frac{dwdz}{(w-z)^2} + \text{regular at the diagonal } \Delta$$

and (u) is a system of local coordinates of the base space $\mathcal{B}^{(0)}$. This is independent of the choice of local coordinates (w, z) and is well-defined.

5.2. Descent to $\mathcal{B}^{(1)}$

To define the sheaf of twisted differential operators, first we need to define the action π of $\mathcal{D}^{\oplus N}$ on $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$.

For $\vec{h} = (h_1, \ldots, h_N) \in \mathcal{D}^{\oplus N}$ and $V = (\vec{\ell}, r) \in \widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$, define

$$\pi(\vec{h})(\vec{\ell}, r) = (\pi(\vec{h})(\vec{\ell}), r') \tag{5.2-1}$$

where for $\vec{\ell} = (\underline{l}_1, \ldots, \underline{l}_N)$, $\underline{l}_j = l_j \frac{d}{d\xi_j}$, $l_j = \sum a^\nu_j(s)\xi^\nu_j$, we define

(5.2-2)

$$\pi(\vec{h})(\vec{\ell}) = \sum L_{\vec{h}}(a^\nu_j(s))Ad(h_j)(\xi^\nu_j \frac{d}{d\xi_j}),$$

$$r' = L_{\vec{h}}(r) + \frac{c_v}{12}\sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0} \left(L_{\vec{h}}(l_j)\{h_j(\xi_j); \xi_j\}(\frac{dh_j}{d\xi_j})^{-1} d\xi_j \right).$$

Proposition 5.2.1. *For each $\vec{h} \in \mathcal{D}^{\oplus N}$, $\pi(\vec{h})$ is an automorphism of the sheaf $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ preserving the Lie algebra structure. Moreover, as an $\mathcal{O}_{\mathcal{B}^{(\infty)}}$-module homomorphism, $\pi(\vec{h})$ is compatible with the action of $L_{\vec{h}}$.*

Remark 5.2.2. If we regard $\underline{d}^{\oplus N}$ as a constant subsheaf of Lie subalgebras of $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$, then the differential of the action of $\mathcal{D}^{\oplus N}$ on $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$ coincides with the adjoint action of $\underline{d}^{\oplus N}$ on $\widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$. That is, we have

$$\frac{d}{d\epsilon}\pi(\exp(\epsilon\vec{\ell}))(V)|_{\epsilon=0} = [\vec{\ell}, V]$$

where $\vec{h} = \exp(\vec{\ell})$.

Proposition 5.2.3. *For $\vec{h} \in (\mathcal{D}^1)^{\oplus N}$ and $V \in \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v)$, we have*

$$\pi(\vec{h})D(V)\pi(\vec{h}^{-1}) = D(\pi(\vec{h})(V))$$

as an operator on $\widetilde{\mathcal{H}}^{(\infty)}_{\vec{\lambda}}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$.

Corollary 5.2.4. *For $V \in B_{\widehat{\mathcal{S}}^{(\infty)}} \oplus \mathcal{O}_{\mathcal{B}^{(\infty)}}$ and $s \in \mathcal{B}^{(\infty)}$, we have*

$$a(\vec{h}(s), V) = a(s, \pi(\vec{h})V)$$

where $a(s, V)$ is given in Corollary 5.1.8. *Here, we also write explicitly the dependence of s in the homomorphism a.*

Now we are ready to define the sheaf $Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v)$ on $\mathcal{B}^{(1)}$. Put

$$\widetilde{Vir}^{(1)}(c_v) = \{\, V \in \widetilde{Vir}_{\widehat{\mathcal{S}}^{(\infty)}}(c_v) \mid \pi(\vec{h})(V) = V \text{ for all } \vec{h} \in (\mathcal{D}^1)^{\oplus N} \,\}.$$

Proposition 5.2.5. *There exists a sheaf $Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v)$ of an $\mathcal{O}_{\mathcal{B}^{(1)}}$-module over $\mathcal{B}^{(1)}$ such that $Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v)$ is a sheaf of Lie algebras and we have*

$$\widetilde{Vir}^{(1)}(c_v) \simeq {\psi^{(1)}}^{-1} Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v)$$

where $\psi^{(1)} : \mathcal{B}^{(\infty)} \to \mathcal{B}^{(1)}$ is the canonical holomorphic map. Moreover, there is an exact sequence

$$0 \to \mathcal{O}_{\mathcal{B}^{(1)}} \to Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v) \to \oplus_{j=1}^{N} \Theta_{\widehat{S}_j^{(1)}/\mathcal{B}^{(1)}}(*) \to 0$$

Since the action of $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and the actions of $(\mathcal{D}^1)^{\oplus N}$ on $\widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ are compatible, for each $V \in \widetilde{Vir}_{\widehat{S}^{(\infty)}}(c_v)$, we can define the action $D(V)$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$.

Put

$$\widetilde{B}_{\widehat{S}^{(1)}} = \{ V \in B_{\widehat{S}^{(\infty)}} \mid \pi(\vec{h})V = V \text{ for all } \vec{h} \in (\mathcal{D}^1)^{\oplus N} \}.$$

There exists a sheaf $B_{\widehat{S}^{(1)}}$ on $\mathcal{B}^{(1)}$ such that we have

$$\widetilde{B}_{\widehat{S}^{(1)}} \simeq {\psi^{(1)}}^{-1} B_{\widehat{S}^{(1)}} .$$

Moreover, since on ${\psi^{(1)}}^{-1}\Theta_{\widehat{S}_j^{(1)}/\mathcal{B}^{(1)}}(*)$ the action of $(\mathcal{D}^1)^{\oplus N}$ comes from the adjoint action on $\mathcal{O}_{\mathcal{B}^{(1)}}((\xi_j))\dfrac{d}{d\xi_j}$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{B}^{(1)}} \to Vir_{\widehat{S}^{(1)}}(c_v) \to \oplus_{j=1}^N(\Theta_{\widehat{S}_j^{(1)}/\mathcal{B}^{(1)}}(*)) \to 0 \tag{5.2-3}$$

which is an extension of Lie algebras.

Proposition 5.2.6. *The sheaf $B_{\widehat{S}^{(1)}} \oplus (\oplus_{j=1}^N(\Theta_{\widehat{S}_j^{(1)}/\mathcal{B}^{(1)}}(-2)))$ can be regarded as an ideal of Lie subalgebras of $Vir_{\widehat{S}^{(1)}}(c_v)$ and it acts trivially on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$.*

5.3. Sheaf of twisted differential operators

Let us define a locally free sheaf $V_{\mathcal{C}^{(1)}}(c_v)$ of rank two on $\mathcal{C}^{(1)} \setminus \Sigma^{(1)}$. It is locally a direct sum

$$\Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)}) \oplus \omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}.$$

Let $(u_1, \dots, u_M, z)$ and $(u_1, \dots, u_M)$ be local coordinates of $\mathcal{C}^{(1)} \setminus \Sigma^{(1)}$ and those of $\mathcal{B}^{(1)}$, respectively such that $\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}$ is given by the projection to the first M-factors. Then an element $V \in V_{\mathcal{C}^{(1)}}(c_v)$ is expressed by

$$V = (\ell(u,z)\frac{d}{dz}, \pi(u,z)dz).$$

If $(u_1', \dots, u_M', z')$ are other local coordinates, by definition, V is expressed in the form

$$V = (\ell'(u',z')\frac{d}{dz'}, \pi(u',z')dz')$$

where

$$\ell'(u',z') = \ell(u(u',z'),z(u',z'))(\frac{dz'}{dz})$$

$$\pi'(u',z') = \pi(u(u',z'),z'(u',z'))(\frac{dz'}{dz})^{-1} + \frac{c_v}{12}\{z';z\}\ell(u(u',z'),z(u',z'))\left(\frac{dz'}{dz}\right)^{-1}. \tag{5.3-1}$$

This defines $V_{\mathcal{C}^{(1)}}(c_v)$ as a sheaf of $\mathcal{O}_{\mathcal{C}^{(1)}}$-module over $\mathcal{C}^{(1)} \setminus \Sigma^{(1)}$ and the relations (5.3-1) show that the projection to the first factor induces the following exact sequence.

$$0 \to \omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}} \to V_{\mathcal{C}^{(1)}}(c_v) \to \Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)}) \to 0 \tag{5.3-2}$$

Moreover, since $\omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}$ and $\Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)})$ are invertible on $\mathcal{C}^{(1)}$, and $\Sigma^{(1)}$ is of codimension two in $\mathcal{C}^{(1)}$, the sheaf $V_{\mathcal{C}^{(1)}}(c_v)$ can be extended to a locally free sheaf of rank two on $\mathcal{C}^{(1)}$ by using the above exact sequence (5.3-2). Thus we may regard the exact sequence (5.3-2) as the one of $\mathcal{O}_{\mathcal{C}^{(1)}}$-modules over $\mathcal{C}^{(1)}$. Then, by (5.3-2) we obtain an exact sequence

$$0 \to R^1\pi^{(1)}_*\omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}} \to R^1\pi^{(1)}_*V_{\mathcal{C}^{(1)}}(c_v) \to R^1\pi^{(1)}_*\Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)}) \to 0.$$

Note that there are canonical $\mathcal{O}_{\mathcal{B}^{(1)}}$-module isomorphisms

$$R^1\pi^{(1)}_*\omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}} \simeq \mathcal{O}_{\mathcal{B}^{(1)}}$$

and

$$\theta^{(1)} : R^1\pi^{(1)}_*(\Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)})) \simeq \Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}).$$

Put

$$\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)};c_v) = R^1\pi^{(1)}_*V_{\mathcal{C}^{(1)}}(c_v).$$

Then the above exact sequence is rewritten in the form

$$0 \to \mathcal{O}_{\mathcal{B}^{(1)}} \to \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)};c_v) \to \Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}) \to 0. \tag{5.3-3}$$

If we fix $\omega \in H^0(\mathcal{C}^{(0)} \times_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta))$ with $\mathrm{Res}^2_\Delta(\omega) \equiv 1$, the local splitting of the exact sequence (5.3-2) is given as follows.

(5.3-4)

$$\begin{array}{ccc} \Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-2S^{(1)}) \oplus \omega_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}} & \simeq & V_{\mathcal{C}^{(1)}}(c_v) \\ (\ell\frac{d}{dz}, f(z)dz) & \longmapsto & ((\ell\frac{d}{dz}, (f(z) + \frac{c_v}{12}\cdot \ell(z)S(z))dz) \end{array}$$

where $S(z)dz^2$ is a projective connection defined by

$$S(z)dz^2 = -6\lim_{w\to z}\{\omega(w,z)dwdz - \frac{dwdz}{(w-z)^2}\}.$$

Note that the projective connection does depend on the choice of the coordinate z and we have

$$S(w)dw^2 = S(z)dz^2 + \{w;z\}dz^2.$$

This fact and (5.3-2) imply that the splitting (5.3-4) does depend on the choice of a meromorphic form ω. By taking the first direct images of sheaves in (5.3-4), this splitting induces an $\mathcal{O}_{\mathcal{B}^{(1)}}$-module isomorphism

$$\Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}) \oplus \mathcal{O}_{\mathcal{B}^{(1)}} \simeq \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)\,. \tag{5.3-5}$$

Proposition 5.3.1. *There exists a canonical surjective homomorphism of $\mathcal{O}_{\mathcal{B}^{(1)}}$-modules*

$$\overline{\theta}^{(1)} : Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v) \to \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$$

such that the following diagram is commutative.

$$\begin{array}{ccccc} Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v) & \stackrel{\overline{\theta}^{(1)}}{\to} & \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v) & \to & 0 \\ p\downarrow & & \downarrow & & \\ \bigoplus_{j=1}^{N}\Theta_{\widehat{\mathcal{S}}^{(1)}_j/\mathcal{B}^{(1)}}(*) & \stackrel{\theta^{(1)}}{\to} & \Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}) & \to & 0\,. \end{array}$$

Moreover, we have

$$\operatorname{Ker}\overline{\theta}^{(1)} = B_{\widehat{\mathcal{S}}^{(1)}} \oplus (\bigoplus_{j=1}^{N}(\Theta_{\widehat{\mathcal{S}}^{(1)}_j/\mathcal{B}^{(1)}}(-2)))\,.$$

Proof. By the exact sequence (5.2-3) and by Proposition 3.3.6, we have the following diagram of exact sequences.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathfrak{A} & & \\
 & & & & & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathcal{B}^{(1)}} & \to & Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v) & \xrightarrow{p} & \mathfrak{T} & \to & 0 \\
 & & \| & & & & \downarrow \theta^{(1)} & & \\
0 & \to & \mathcal{O}_{\mathcal{B}^{(1)}} & \to & \mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v) & \to & \mathfrak{C} & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

where

$$\mathfrak{A} = B_{\widehat{\mathcal{S}}^{(1)}} \oplus \bigoplus_{j=1}^{N} (\Theta_{\widehat{\mathcal{S}}_j^{(1)}/\mathcal{B}^{(1)}}(-2))$$

$$\mathfrak{T} = \bigoplus_{j=1}^{N} (\Theta_{\widehat{\mathcal{S}}_j^{(1)}/\mathcal{B}^{(1)}}(*))$$

$$\mathfrak{C} = \Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}).$$

By Remark 5.1.9, if we fix $\omega \in H^0(\mathcal{C}^{(0)} \otimes_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta))$ with $\mathrm{Res}^2_{\Delta}(\omega) \equiv 1$, there is an $\mathcal{O}_{\mathcal{B}^{(1)}}$-module homomorphism

$$a_\omega : Vir_{\widehat{\mathcal{S}}^{(1)}}(c_v) \xrightarrow{a} \mathcal{O}_{\mathcal{B}^{(1)}}.$$

By using the splitting (5.3-4), define $\overline{\theta}^{(1)} = (\theta^{(1)}, a_\omega)$. Then, it is easy to show that $\overline{\theta}^{(1)}$ is well-defined and that we have

$$\mathrm{Ker}\, \overline{\theta}^{(1)} = B_{\widehat{\mathcal{S}}^{(1)}} \oplus (\bigoplus_{j=1}^{N} (\Theta_{\widehat{\mathcal{S}}_j^{(1)}/\mathcal{B}^{(1)}}(-2)).$$

Q.E.D.

Next we introduce a Lie algebra structure on $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$.

Lemma 5.3.2. *The above isomorphism* (5.3-5) *defines a Lie algebra structure of* $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ *and the exact sequence* (5.3-3) *is an extension of the sheaves of Lie algebras.*

By Proposition 5.2.6 and Proposition 5.3.1 we obtain the following Theorem.

Theorem 5.3.3. *On* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ *the sheaf* $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$ *of Lie algebras acts as twisted first order differential operators.*

Corollary 5.3.4. *If* $\mathcal{B}^{(0)}$ *is small enough such that we have a splitting* (5.3-5), *then the sheaves* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ *and* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ *are locally free on* $\mathcal{B}^{(1)} \setminus D^{(1)}$.

Proof. Since we have the splitting (5.3-5), $\Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)})$ defines an integrable connection with regular singularities on $\mathcal{B}^{(1)}$. Hence, on $\mathcal{B}^{(1)} \setminus D^{(1)}$ we have an integrable connection. Therefore, the Corollary is a consequence of the theory of connections on coherent sheaves.

Q.E.D.

By Remark 4.1.7 we have the following Corollary.

Corollary 5.3.5. *Under the same assumption as in Corollary* 5.3.4, *for each point* $s \in \mathcal{B}^{(1)} \setminus D^{(1)}$ *we have the canonical isomorphism*

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}^{(1)}).$$

§6. Locally freeness and factorization

6.1. Family of singular stable curves.

Let $\mathfrak{F}^{(1)} = (\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)};\ s^{(1)}_1, s^{(1)}_2, \dots, s^{(1)}_N;\ \tilde{t}^{(1)}_1, \tilde{t}^{(1)}_2, \dots, \tilde{t}^{(1)}_N)$ be a local universal family of N-pointed stable curves with first order infinitesimal neighbourhoods. Here we study the behavior of the $\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v)$-module $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ near the discriminant locus $D^{(1)}$.

Since the problem is local on $\mathcal{B}^{(1)}$, we take sufficiently small family $\mathfrak{F}^{(0)} = (\pi^{(0)} : \mathcal{C}^{(0)} \to \mathcal{B}^{(0)};\ s^{(0)}_1, s^{(0)}_2, \dots, s^{(0)}_N)$ with local coordinates $(\tau_1, \dots, \tau_{3g-3+N})$ on $\mathcal{B}^{(0)}$ such that the discriminant locus is of the form $D = D_1 \cup D_2 \cup \cdots \cup D_k$, $D_i = \{\ (\tau) \mid \tau_{3g-2+N-i} = 0\ \}$, $i = 1, \dots, k$ and the family $\mathfrak{F}^{(1)}$ is obtained from the family $\mathfrak{F}^{(0)}$. (See the proof of Theorem 3.1.5.) Choosing $\mathcal{B}^{(0)}$ smaller, if necessary, we may assume that

$$\mathcal{B}^{(1)} = (\mathbf{C}^*)^N \times \mathcal{B}^{(0)}.$$

Let $(\eta_1, \dots, \eta_N)$ be global coordinates of $(\mathbf{C}^*)^N$. Moreover, we may assume that there exists a meromorphic form

$$\omega \in H^0(\mathcal{C}^{(0)} \otimes_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta))$$

such that $\mathrm{Res}^2_\Delta(\omega) \equiv 1$ and the condition (5.1-9) is satisfied. Fixing it, we have a trivialization

$$\mathcal{D}^1_{\mathcal{B}^{(1)}}(-\log D^{(1)}; c_v) \simeq \Theta_{\mathcal{B}^{(1)}}(-\log D^{(1)}) \oplus \mathcal{O}_{\mathcal{B}^{(1)}}.$$

Let $D_i^{(1)} \subset \mathcal{B}^{(1)}$ be the pull back of $D_i \subset \mathcal{B}^{(0)}$, and put

$$E = \bigcap_{1 \le i \le k} D_i, \quad E^{(1)} = \bigcap_{1 \le i \le k} D_i^{(1)}.$$

Denote by $\pi_E : \mathcal{C}_E \to E$ the restriction of $\mathcal{C}^{(0)}$ to E. Let $\tilde{\pi}_E : \tilde{\mathcal{C}}_E \to E$ be obtained by the simultaneous normalization of $\pi_E : \mathcal{C}_E \to E$ and $\sigma'_p, \sigma''_p : E \to \tilde{\mathcal{C}}_E$ $(p = 1, \dots, k)$ the cross-sections corresponding to the normalized double points.

$$\begin{array}{ccccc} \tilde{\mathcal{C}}_E & \to & \mathcal{C}_E & \hookrightarrow & \mathcal{C}^{(0)} \\ & \searrow \tilde{\pi}_E & \pi_E \downarrow \quad s, \sigma & & \downarrow \\ s, \sigma', \sigma'' & & E & \hookrightarrow & \mathcal{B}^{(0)}. \end{array}$$

We also denote by $\pi_{E^{(1)}} : \mathcal{C}_{E^{(1)}} \to E^{(1)}$ (resp. $\tilde{\pi}_{E^{(1)}} : \tilde{\mathcal{C}}_{E^{(1)}} \to E^{(1)}$) the pull back of $\pi_E : \mathcal{C}_E \to E$ (resp. $\tilde{\pi}_E : \tilde{\mathcal{C}}_E \to E$) to $E^{(1)}$. For simplicity, we use the notation s_j instead of $s_j^{(0)}$ and $s_j^{(1)}$. Also by σ' and σ'' we denote the sections of $\tilde{\pi}_{E^{(1)}} : \tilde{\mathcal{C}}_{E^{(1)}} \to E^{(1)}$ induced from the sections σ' and σ'' of $\pi_E : \mathcal{C}_E \to E$.

Proposition 6.1.1. *The family*

$$(\tilde{\pi}_E : \tilde{\mathcal{C}}_E \to E; \sigma'_p, \sigma''_p, (p = 1, \dots, k), s_1^{(0)}, \dots, s_N^{(0)})$$

is a local universal family of $(N+2k)$-pointed (not necessarily connected) stable curves.

For the preparation of the next subsection we study the relation between the family $\tilde{\pi}_{E^{(1)}} : \tilde{\mathcal{C}}_{E^{(1)}} \to E^{(1)}$ and $\mathfrak{F}^{(1)}$.

For simplicity of notation let us assume that $k = 1$. Hence, $E = D_1$ and $E^{(1)} = D_1^{(1)}$. Put $M = 3g - 4 + 2N$, $\tau = \tau_{3g-g+N}$ and

$$u_i = \begin{cases} \eta_i & i = 1, \dots, N \\ \tau_{i-N} & i = N+1, \dots, M. \end{cases}$$

Hence $(u_1, u_2, \dots, u_M, \tau)$ are coordinates of $\mathcal{B}^{(1)}$ and $E^{(1)}$ is defined by the equation $\tau = 0$.

Lemma 6.1.2. *If we choose $\mathcal{B}^{(0)}$ sufficiently small, then there exist local coordinates $(u_1, \dots, u_M, z)$ (resp. $(u_1, \dots, u_M, w)$) of a neighbourhood X (resp. Y) of $\sigma'(E^{(1)})$ (resp. $\sigma''(E^{(1)})$) in $\widetilde{\mathcal{C}}_{E^{(1)}}$ and a relative vector field $\widetilde{\ell} \in H^0(\widetilde{\mathcal{C}}_{E^{(1)}}, \Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(* \sum_{j=1}^N s_j(E^{(1)})))$ which satisfy the following conditions.*

1) *The sections σ' and σ'' are given by the mappings*

$$\begin{aligned} \sigma' &: (u_1, \dots, u_M) \mapsto (u_1, \dots, u_M, 0) = (u_1, \dots, u_M, z) \\ \sigma'' &: (u_1, \dots, u_M) \mapsto (u_1, \dots, u_M, 0) = (u_1, \dots, u_M, w). \end{aligned}$$

2) $\displaystyle \widetilde{\ell}|_X = \frac{1}{2} z \frac{\partial}{\partial z}, \quad \widetilde{\ell}|_Y = \frac{1}{2} w \frac{\partial}{\partial w}.$

Proof. Let $\nu : \widetilde{\mathcal{C}}_{E^{(1)}} \to \mathcal{C}_{E^{(1)}}$ be the simultaneous normalization. Let $(u_1, \dots, u_M, x)$ (resp. $(u_1, \dots, u_M, y)$) be local coordinates of X (resp. Y) satisfying the condition 1). Since ν is isomorphic (the identity mapping) on $\widetilde{\mathcal{C}}_{E^{(1)}} \backslash (\sigma'(E^{(1)}) \cup \sigma''(E^{(1)})) = \mathcal{C}_{E^{(1)}} \backslash \sigma(E^{(1)})$, by the proof of Lemma 3.2.3, especially by (3.2-4) we have the following exact sequence.

$$0 \to \Theta_{\mathcal{C}_{E^{(1)}}/E^{(1)}} \to \nu_*(\Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(-\sigma'(E^{(1)}) - \sigma''(E^{(1)}))) \xrightarrow{\alpha} \mathcal{O}_{E^{(1)}} \to 0$$

where the $\mathcal{O}_{E^{(1)}}$-module homomorphism α is given by

$$(a(u,x)\frac{\partial}{\partial x}, b(u,y)\frac{\partial}{\partial y}) \mapsto \frac{\partial a(u,0)}{\partial x} + \frac{\partial b(u,0)}{\partial y}.$$

Note that the stalk of $\nu_*(\Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(-\sigma'(E^{(1)}) - \sigma''(E^{(1)})))$ at a point $\sigma(u)$, $(u) \in E^{(1)}$ consists of a pair of local holomorphic vector field $(a(u,x)\frac{\partial}{\partial x}, b(u,y)\frac{\partial}{\partial y})$ with $a(t,0) = 0$, $b(y,0) = 0$ and the definition of α is independent of the choice of local coordinates. The exact sequence induces an exact sequence

$$\begin{aligned}0 \to H^0(\mathcal{C}_{E^{(1)}}, \Theta_{\mathcal{C}_{E^{(1)}}/E^{(1)}}(kS^{(1)})) \\ \to H^0(\mathcal{C}_{E^{(1)}}, \nu_*(\Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(\nu^*(kS^{(1)}) - \sigma'(E^{(1)}) - \sigma''(E^{(1)})))) \\ \stackrel{\alpha}{\to} H^0(E^{(1)}, \mathcal{O}_{E^{(1)}}) \to H^1(\mathcal{C}_{E^{(1)}}, \Theta_{\mathcal{C}_{E^{(1)}}/E^{(1)}}(kS^{(1)}))\end{aligned}$$

for every integer k, where

$$S^{(1)} = \sum_{j=1}^{N} s_j(E^{(1)}).$$

If k is sufficiently large, we have

$$H^1(\mathcal{C}_{E^{(1)}}, \Theta_{\mathcal{C}_{E^{(1)}}/E^{(1)}}(kS^{(1)})) = 0.$$

Hence, by the above exact sequence there exists a relative vector field

$$\begin{aligned}\tilde{\ell} \in H^0(\widetilde{\mathcal{C}}_{E^{(1)}}, \Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(\nu^*(kS^{(1)}) - \sigma'(E^{(1)}) - \sigma''(E^{(1)}))) \\ = H^0(\mathcal{C}_{E^{(1)}}, \nu_*(\Theta_{\widetilde{\mathcal{C}}_{E^{(1)}}/E^{(1)}}(\nu^*(kS^{(1)}) - \sigma'(E^{(1)}) - \sigma''(E^{(1)})))\end{aligned}$$

such that

$$\alpha(\tilde{\ell}) \equiv 1.$$

By the local coordinates given above, $\tilde{\ell}$ has the form

$$\tilde{\ell} = a(u,x)\frac{\partial}{\partial x} \quad \text{on} \quad X$$
$$\tilde{\ell} = b(u,y)\frac{\partial}{\partial y} \quad \text{on} \quad Y$$

with

$$\frac{\partial a(u,0)}{\partial x} + \frac{\partial b(u,0)}{\partial y} \equiv 1.$$

Adding an element coming from $H^0(\mathcal{C}_{E^{(1)}}, \Theta_{\mathcal{C}_{E^{(1)}}/E^{(1)}}(*S^{(1)}))$ if necessary, and choosing $\mathcal{B}^{(0)}$, X and Y smaller, we may assume that $\dfrac{\partial a(u,x)}{\partial x}$ (resp. $\dfrac{\partial b(u,y)}{\partial y}$) has no zero on X (resp. Y). Now define $z = z(u,x)$

and $w = w(u, y)$ by

$$a(u,x)\frac{\partial z}{\partial x} = \frac{1}{2}z, \quad z(u,0) = 0$$
$$b(u,y)\frac{\partial w}{\partial y} = \frac{1}{2}w, \quad w(u,0) = 0.$$

Then, by choosing X and Y smaller, $(u_1, \ldots, u_M, z)$ and $(u_1, \ldots, u_M, w)$ satisfy the above conditions 1) and 2). Q.E.D.

We let $\hat{\pi}_{\tilde{E}^{(1)}} : \widehat{\mathcal{C}}_{\tilde{E}^{(1)}} \to \tilde{E}^{(1)}$ be a local universal family obtained by adding the first order infinitesimal neighbourhoods at σ' and σ''. Lemma 6.1.2 says that at σ' and σ'' we can choose special coordinates z and w. These coordinates induce the first order infinitesimal neighbourhoods of σ' and σ'', hence, we have a holomorphic section

(6.1-1) $$j : E^{(1)} \to \widetilde{E}^{(1)}.$$

Let ξ_j be a formal coordinate at $s_j(E^{(1)})$ such that

$$\tilde{t}_j^{(1)}(\xi_j \bmod I^2_{s_j(E^{(1)})}) = \xi.$$

Let $\ell_j(\xi_j)\dfrac{d}{d\xi_j}$ be the formal Laurent expansion of $\widetilde{\ell}$ with respect to the formal coordinate ξ_j. Thus we have $\ell_j(\xi_j) \in \mathcal{O}_{E^{(1)}}((\xi_j))$. Put

(6.1-2) $$\vec{\underline{l}} = (\ell_1(\xi_1)\frac{d}{d\xi_1}, \ldots, \ell_N(\xi_N)\frac{d}{d\xi_N}).$$

Next we construct the family $\mathfrak{F}^{(1)}$ from the family $(\widetilde{\pi}_{E^{(1)}} : \widetilde{\mathcal{C}}_{E^{(1)}} \to E^{(1)}; s_1^{(1)}, \ldots, s_N^{(1)}; \tilde{t}_1^{(1)}, \ldots, \tilde{t}_N^{(1)})$. Using the notation of Lemma 6.1.2, we may assume that

$$X = \{\, P \in \widetilde{\mathcal{C}}_{E^{(1)}} \mid |z(P)| < 1 \,\}$$
$$Y = \{\, P \in \widetilde{\mathcal{C}}_{E^{(1)}} \mid |w(P)| < 1 \,\}.$$

For a positive number $\varepsilon < 1$ put

$$X_\varepsilon = \{\, P \in \widetilde{\mathcal{C}}_{E^{(1)}} \mid |z(P)| < \varepsilon \,\}$$
$$Y_\varepsilon = \{\, P \in \widetilde{\mathcal{C}}_{E^{(1)}} \mid |w(P)| < \varepsilon \,\}.$$

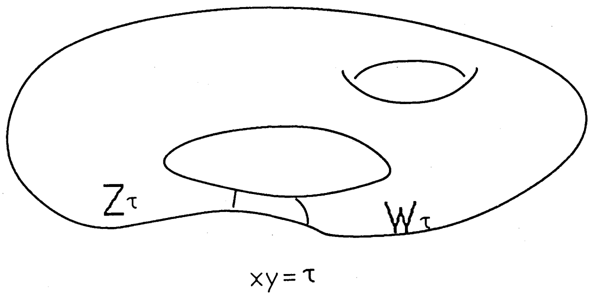

Figure 4.

Fix positive numbers $\varepsilon_1 < \varepsilon_2 < 1$ and let $\{U_\alpha\}_{3 \le \alpha \le A}$ be a finite open covering of $\widetilde{\mathcal{C}}_{E^{(1)}} \setminus (X_{\varepsilon_2} \cup Y_{\varepsilon_2})$ such that

$$U_\alpha \cap X_{\varepsilon_1} = \emptyset, \quad U_\alpha \cap Y_{\varepsilon_1} = \emptyset$$

for any $\alpha = 3, \dots, A$.

Now put

$$\begin{aligned} D &= \{\, \tau \in \mathbf{C} \mid |\tau| < 1 \,\} \\ S_0 &= \{\, (x, y, \tau) \in \mathbf{C}^3 \mid xy = \tau,\ |x| < 1,\ |y| < 1,\ |\tau| < 1 \,\} \\ S &= S_0 \times E \\ Z &= \{\, (P, \tau) \in \widetilde{\mathcal{C}}_{E^{(1)}} \times D \mid P \in \widetilde{\mathcal{C}}_{E^{(1)}} \setminus (X \cup Y) \\ &\qquad\qquad\qquad\qquad \text{or } \ P \in X \text{ and } \ |z(P)| > |\tau| \} \\ W &= \{\, (P, \tau) \in \widetilde{\mathcal{C}}_{E^{(1)}} \times D \mid P \in \widetilde{\mathcal{C}}_{E^{(1)}} \setminus (X \cup Y) \\ &\qquad\qquad\qquad\qquad \text{or } \ P \in Y \text{ and } \ |w(P)| > |\tau| \}. \end{aligned}$$

On $Z \sqcup S \sqcup W$ we introduce an equivalence relation $\sim$ as follows.

1) A point $(P, \tau) \in Z \cap (X \times D)$ and a point $(x, y, \tau', u) \in S$ are equivalent if and only if

$$(x, y, \tau', u) = (z(P), \frac{\tau}{z(P)}, \tau, \widetilde{\pi}_E^{(1)}(P)).$$

2) A point $(P, \tau) \in W \cap (Y \times D)$ and a point $(x, y, \tau', u) \in S$ are equivalent if and only if

$$(x, y, \tau', u) = (\frac{\tau}{w(P)}, w(P), \tau, \widetilde{\pi}_E^{(1)}(P)).$$

3) A point $(P,\tau) \in Z$ and a point $(Q,\tau') \in W$ if and only if

$$(P,\tau) = (Q,\tau').$$

Now put $\mathcal{C}^{(1)} = Z \sqcup S \sqcup W/\sim$. Then it is easy to show that $\mathcal{C}^{(1)}$ is a complex manifold and there is a natural holomorphic mapping $\pi^{(1)} : \mathcal{C}^{(1)} \to E^{(1)} \times D$. Moreover, since we can assume that $s_j(E^{(1)})$'s are contained in $\widetilde{\mathcal{C}}_{E^{(1)}} \setminus (X \cup Y)$, we can define holomorphic sections s_j's by

$$\begin{array}{rccc} s_j : & E^{(1)} \times D & \to & \mathcal{C}^{(1)} \\ & (t,\tau) & \mapsto & (s_j(t),\tau) \in Z. \end{array}$$

By the same way we can define the first order infinitesimal neighbourhoods $\widetilde{t}_j$. It is easy to show that $(\pi^{(1)} : \mathcal{C}^{(1)} \to E^{(1)} \times D; s_1, \dots, s_N; \widetilde{t}_1, \dots, \widetilde{t}_N)$ is isomorphic to our original family $\mathfrak{F}^{(1)}$.

By the same method we can construct a family $(\pi : \mathcal{C} \to E \times D; s_1, \dots, s_N)$ isomorphic to $\mathfrak{F}^{(0)}$. Hence, in the following we identify $\mathfrak{F}^{(0)}$ and $\mathfrak{F}^{(1)}$ with the families constructed above.

For each point $(u,\tau) \in E^{(1)} \times D$ put

$$\begin{aligned} \mathcal{C}_{(u,\tau)} &= \pi^{-1}((u,\tau)) \\ U_\alpha(u,\tau) &= U_\alpha \cap \mathcal{C}_{(u,\tau)}, \quad 3 \le \alpha \le A \\ U_1(u,\tau) &= S \cap Z \cap \mathcal{C}_{(u,\tau)} \\ U_2(u,\tau) &= S \cap W \cap \mathcal{C}_{(u,\tau)}. \end{aligned}$$

Then, for each $\tau \neq 0$, $\mathcal{U}(u,\tau) = \{U_\alpha(u,\tau)\}_{1 \le \alpha \le A}$ is an open covering of the curve $\mathcal{C}_{(u,\tau)}$.

Lemma 6.1.3. *For each point $(u,\tau) \in E^{(1)} \times D$ with $\tau \neq 0$, the image $\rho(\tau\frac{\partial}{\partial\tau})$ of a vector field $\tau\frac{\partial}{\partial\tau}$ by the Kodaira-Spencer mapping*

$$\rho : T_{(u,\tau)}(E^{(1)} \times D) \to H^1(\mathcal{C}_{(u,\tau)}, \Theta_{\mathcal{C}_{(u,\tau)}})$$

is given by a Cech cohomology class $\{\theta_{\alpha\beta}(u,\tau)\} \in \check{H}^1(\mathcal{U}(u,\tau), \Theta_{\mathcal{C}_{(u,\tau)}})$ with respect to the covering $\mathcal{U}(u,\tau)$ given above, where

$$\begin{aligned} \theta_{12}(u,\tau) &= z\frac{\partial}{\partial z} \\ \theta_{21}(u,\tau) &= -\theta_{12}(u,\tau) \\ \theta_{\alpha\beta}(u,\tau) &= 0 \quad \textit{if } (\alpha,\beta) \neq (1,2) \textit{ or } (2,1). \end{aligned}$$

Proof. By the above equivalence relation, on $U_1(u,\tau)\cap U_2(u,\tau)$ we have

$$z = \frac{\tau}{w}.$$

If $U_\alpha(u,\tau)\cap U_\beta(u,\tau) \neq \emptyset$ and $(\alpha,\beta) \neq (1,2)$ nor $(2,1)$, then the relation between local coordinates of $U_\alpha(u,\tau)$ and $U_\beta(u,\tau)$ does not depend on τ. Hence, by the definition of Kodaira-Spencer mapping (see, for example, Kodaira [Ko, §4.2]) we have

$$\rho(\tau\frac{\partial}{\partial\tau})_{12} = \frac{\tau}{w}\frac{\partial}{\partial\tau} = z\frac{\partial}{\partial z}$$
$$\rho(\tau\frac{\partial}{\partial\tau})_{21} = w\frac{\partial}{\partial w} = -z\frac{\partial}{\partial z}$$
$$\rho(\tau\frac{\partial}{\partial\tau})_{\alpha\beta} = 0 \quad \text{if } (\alpha,\beta) \neq (1,2) \text{ nor } (2,1)$$

Q.E.D.

Let us consider the N-tuple of formal vector fields

$$\vec{\underline{l}} = (\ell_1(\xi_1)\frac{d}{d\xi_1}, \ldots, \ell_N(\xi_N)\frac{d}{d\xi_N})$$

defined in (6.1-2). Since we have $\ell_j(\xi_j)\frac{d}{d\xi_j} \in \mathcal{O}_{E^{(1)}}((\xi_j))$, we may regard $\vec{\underline{l}}$ as an N-tuple of formal vector fields on $\mathfrak{F}^{(1)}$, that is, $\ell_j(\xi_j)\frac{d}{d\xi_j} \in \mathcal{O}_{E^{(1)}\times D}((\xi_j))$.

Corollary 6.1.4. *On $\mathcal{B}^{(1)} = E^{(1)} \times D$ we have*

$$\theta^{(1)}(\vec{\underline{l}}) = \tau\frac{\partial}{\partial\tau}$$

where the mapping $\theta^{(1)}$ is defined in Proposition 3.3.6.

Proof. Since both sides of the above equality in the corollary define holomorphic vector fields on $\mathcal{B}^{(1)}$, it is enough to prove the equality for $\tau \neq 0$.

Let us consider an exact sequence

$$0 \to \Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-S^{(1)}) \to \Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}((m-1)S^{(1)}) \to \oplus_{j=1}^{N} \oplus_{k=1}^{m} \mathcal{O}_{\mathcal{B}^{(1)}}\xi_j^{-m+k}\frac{d}{d\xi_j} \to 0$$

for a sufficiently large positive integer m. $\vec{\underline{l}}$ defines an element $\vec{\underline{l}}'$ of the third term of the exact sequence. On the other hand, for each $(u,\tau) \in E^{(1)} \times D$, $\tau \neq 0$, the meromorphic vector field $\widetilde{\ell}$ on $\widetilde{\mathcal{C}}_{E^{(1)}}$ defines meromorphic vector fields $\widetilde{\ell}_{u,\tau}$ on $C_{u,\tau} \setminus \{U_2(u,\tau) \setminus (U_1(u,\tau) \cap U_2(u,\tau))\}$ and $\widetilde{\ell}'_{u,\tau} = \dfrac{1}{2} w \dfrac{\partial}{\partial w}$ on $U_2(u,\tau)$ such that both vector fields have the same image $\vec{\underline{l}}'$ in the above exact sequence. Hence, the image of $\vec{\underline{l}}'$ by the mapping

$$\oplus_{j=1}^{N} \oplus_{k=1}^{m} \mathcal{O}_{\mathcal{B}^{(1)}} \xi_j^{-m+k} \frac{d}{d\xi_j} \to R^1 \pi_*^{(1)}(\Theta_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(-S^{(1)})$$

is given at a point (u,τ) by an element

$$\{\theta_{\alpha,\beta}(u,\tau)\} \in H^1(C_{u,\tau}, \Theta_{C_{u,\tau}})$$

where on $U_1(u,\tau) \cap U_2(u,\tau)$ we have

$$\begin{aligned} \theta_{12}(u,\tau) &= \widetilde{\ell}_{u,\tau}|_{U_1(u,\tau)} - \widetilde{\ell}'_{u,\tau}|_{U_2(u,\tau)} \\ &= \frac{1}{2} z \frac{\partial}{\partial z} - \frac{1}{2} w \frac{\partial}{\partial w} \\ &= z \frac{\partial}{\partial z} \\ \theta_{21}(u,\tau) &= -\theta_{12}(u,\tau) \end{aligned}$$

and on $U_\alpha(u,\tau) \cap U_\beta(u,\tau)$ with $(\alpha,\beta) \neq (1,2)$, $(2,1)$ we have

$$\theta_{\alpha\beta}(u,\tau) = 0.$$

Thus $\vec{\underline{l}}$ defines the cohomology class given in Lemma 6.1.3. Hence we have the equality for $\tau \neq 0$. Q.E.D.

6.2. Locally freeness and factorization

The main purpose of the present subsection is to prove the locally freeness and factorization properties of the sheaf of vacua $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}^{(1)})$ for a local universal family $\mathfrak{F}^{(1)} = (\pi^{(1)} : \mathcal{C}^{(1)} \to \mathcal{B}^{(1)}\,;\, s_1^{(1)}, s_2^{(1)}, \dots, s_N^{(1)}\,;\, \tilde{t}_1^{(1)}, \tilde{t}_2^{(1)}, \dots, \tilde{t}_N^{(1)})$. We use freely the notation and convention in the previous subsection.

Theorem 6.2.1. *The sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ is locally free.*

Proof. By Corollary 5.3.4 the theorem is true for a local universal family of smooth curves. Therefore, we assume that the local universal family $\mathfrak{F}^{(1)}$ contains singular curves. For simplicity we only consider the case $k = 1$, that is, each singular curve has only one double point. General case is reduced to this case by the induction on the number k of the double points of a singular curve.

First fix an element $\mu \in P_\ell$.

Claim 1. *There exists a bilinear pairing*

$$(\ | \ ) : \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger} \to \mathbf{C}$$

unique up to the constant multiple such that we have

$$(X(n)u|v) + (u|X(-n)v) = 0$$

for any $X \in \mathfrak{g}$, $n \in \mathbf{Z}$, $|u\rangle \in \mathcal{H}_\mu$, $|v\rangle \in \mathcal{H}_{\mu^\dagger}$ and $(\ | \ )$ is zero on $\mathcal{H}_\mu(d) \otimes \mathcal{H}_{\mu^\dagger}(d')$, if $d \neq d'$.

Proof. Since $V_\mu \otimes V_{\mu^\dagger}$, considered as a $\mathfrak{g}$-module by the diagonal action, contains only the one-dimensional trivial $\mathfrak{g}$-module $\mathbf{C}|0_{\mu,\mu^\dagger}\rangle$, we have a bilinear form $(\ | \ ) \in \mathrm{Hom}_{\mathfrak{g}}(V_\mu \otimes V_{\mu^\dagger}, \mathbf{C})$ unique up to the constant multiple. Assume that we have a bilinear form $(\ | \ ) \in \mathrm{Hom}(F_p\mathcal{H}_\mu \otimes F_p\mathcal{H}_{\mu^\dagger}, \mathbf{C})$ with desired properties. For an element

$$X(-m)|u\rangle \in F_{p+1}\mathcal{H}_\mu, \quad |u\rangle \in F_p\mathcal{H}_\mu, \quad m > 0$$

and an element $|v\rangle \in F_{p+1}\mathcal{H}_{\mu^\dagger}$ define

$$(X(-m)u|v) = -(u|X(m)v).$$

Note that since $X(m)|v\rangle \in F_{p+1-m}\mathcal{H}_{\mu^\dagger}$, the right hand side is defined already. It is easy to show that in this way we can define the bilinear form $(\ | \ )$ satisfying the conditions of Claim 1. This proves Claim 1.

Now let us choose a basis $\{v_1(d), \dots, v_{m_d}(d)\}$ of $\mathcal{H}_\mu(d)$ and the dual basis $\{v^1(d), \dots, v^{m_d}(d)\}$ of $\mathcal{H}_{\mu^\dagger}(d)$ with respect to the above bilinear form $(\ | \ )$.

Using the holomorphic section $j : E^{(1)} \to \widetilde{E}^{(1)}$ defined in (6.1-1),we put

$$\mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}} = j^*\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}^{(1)}_{\widetilde{E}^{(1)}})$$
$$\mathcal{H}^{\dagger(1)}_{\vec{\lambda},E^{(1)}} = \widetilde{\mathcal{H}}^{\dagger(1)}_{\vec{\lambda}} \otimes_{\mathcal{O}_{B^{(1)}}} \mathcal{O}_{E^{(1)}}.$$

Then, $\mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}}$ is locally free and by Theorem 5.3.3 the sheaf of holomorphic vector fields $\Theta_{E^{(1)}}$ operates on it from right as the integrable connection. Let $\langle\Psi|$ be a section of $\mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}}$. Let us define an element $\langle\widetilde{\Psi}| \in \mathcal{H}^{\dagger(1)}_{\vec{\lambda},E^{(1)}}[[\tau]]$ associated with $\langle\Psi|$. For that purpose first define $\langle\Psi_d| \in \mathcal{H}^{\dagger(1)}_{\vec{\lambda},E^{(1)}}$ by

$$\langle\Psi_d|\Phi\rangle = \sum_{i=1}^{m_d}\langle\Psi|v_i(d)\otimes v^i(d)\otimes\Phi\rangle, \qquad |\Phi\rangle \in \mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}. \tag{6.2-1}$$

Now define $\langle\widetilde{\Psi}| \in \mathcal{H}^{\dagger(1)}_{\vec{\lambda},E^{(1)}}[[\tau]]$ by

$$\langle\widetilde{\Psi}|\Phi\rangle = \sum_{d=0}^{\infty}\langle\Psi_d|\Phi\rangle\tau^d. \tag{6.2-2}$$

This construction of $\tilde{\Psi}$ is known as sewing procedure by phisisists [So]. Now we shall show that $\langle\widetilde{\Psi}|$ satisfies the formal gauge condition. To give the precise meaning of this statement, first we prove the following Claim.

Claim 2. *There is an $\mathcal{O}_{E^{(1)}}$-module injection*

$$\begin{array}{ccc}\pi^{(1)}_*\mathcal{O}_{\mathcal{C}^{(1)}}(*S^{(1)}) & \hookrightarrow & \widetilde{\pi}_{E^{(1)}*}\mathcal{O}_{\widetilde{\mathcal{C}}^{(1)}_{E^{(1)}}}(*(\sigma'+\sigma''+S^{(1)}))[[\tau]] \\ f & \to & \sum_{k=0}^{\infty} f_k\tau^k\end{array}$$

where

$$f_k \in \widetilde{\pi}_{E^{(1)}*}\mathcal{O}_{\widetilde{\mathcal{C}}^{(1)}_{E^{(1)}}}(*S^{(1)}+k(\sigma'+\sigma'')).$$

Proof. Choose a point $P \in \mathcal{C}_{E^{(1)}}$ which is a double point of a fibre of $\pi_{E^{(1)}}$. Then we can choose local coordinates $(u_1,\dots,u_{M-1},z,w)$ of $\mathcal{C}^{(1)}$ with center P and those $(u_1,\dots,u_{M-1},\tau)$ of $\mathcal{B}^{(1)}$ with center $\pi^{(1)}(P)$ such that $\pi^{(1)}$ is given by

$$(u_1,\dots,u_{M-1},z,w) \to (u_1,\dots,u_{M-1},zw).$$

(See the beginning of 3.2.) Since f is holomorphic at P, we have an expansion

$$f = f(u_1,\dots,u_{M-1},z,w) = \sum_{m\geq 0,n\geq 0} f_{m,n}(u)z^m w^n.$$

Define $g_{P'}(u,\tau,z)$ by

$$g_{P'}(u,\tau,z) = f(u,z,\frac{\tau}{z}) = \sum_{k=0}^{\infty} g_k(u,z)\tau^k$$

where

$$g_k(u,z) = \sum_{m=0}^{\infty} f_{m,k}(u)z^{m-k}. \tag{6.2-3}$$

Define also $h_{P''}(u,\tau,w)$ by

$$h_{P'}(u,\tau,w) = f(u,\frac{\tau}{w},w) = \sum_{k=0}^{\infty} h_k(u,w)\tau^k$$

where

$$h_k(u,w) = \sum_{n=0}^{\infty} f_{k,n}(u)w^{n-k}. \tag{6.2-4}$$

For a point $Q \in \mathcal{C}_{E^{(1)}}$ which is not a double point of a fibre, we can choose local coordinates $(u_1,\ldots,u_{M-1},\tau,z)$ of $\mathcal{C}^{(1)}$ with center Q such that $\pi^{(1)}$ is given by the projection to the first M factors. Then we have an expansion

$$f(u_1,\ldots,u_{M-1},\tau,z) = \sum_{k=0}^{\infty} f_{Q,k}(u,z)\tau^k.$$

It is easy to see that $\{g_k(u,z), h_k(u,w), f_{Q,k}(u,z)\}$ defines a local holomorphic section of the sheaf $\tilde{\pi}_{E^{(1)}*}\mathcal{O}_{\tilde{\mathcal{C}}^{(1)}_{E^{(1)}}}(*S^{(1)}+k(\sigma'+\sigma''))$. This proves Claim 2.

Claim 3. *For an element $f \in \pi_*^{(1)}\mathcal{O}_{\mathcal{C}^{(1)}}(*S^{(1)})$ let $\sum_{k=0}^{\infty} f_k\tau^k$ be the expansion defined in Calim 2. Then we have*

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\sum_{k}\rho_j(X\otimes f_k)\tau^k = 0.$$

That is, $\langle\widetilde{\Psi}|$ satisfies the formal gauge condition.

Proof. By definition, for any $|\Phi\rangle \in \mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}$ we have

$$\sum_{j=1}^{N}\langle\widetilde{\Psi}|\sum_{k=0}^{\infty}\rho_j(X\otimes f_k)\tau^k|\Phi\rangle$$

$$= \tau^{\Delta_\mu}\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}\tau^{k+d}\sum_{j=1}^{N}\langle\Psi|\rho_j(X\otimes f_k)|v_i(d)\otimes v^i(d)\otimes\Phi\rangle$$

$$= -\tau^{\Delta_\mu}\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}\tau^{k+d}\langle\Psi|\rho_{\sigma'}(X\otimes g_k)$$

$$+ \rho_{\sigma''}(X\otimes h_k)|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.$$

By (6.2-3) and (6.2-4) we have

$$\rho_{\sigma'}(X\otimes g_k) = \sum_{m=0}^{\infty} f_{m,k}(t)\rho_{\sigma'}(X(m-k))$$

$$\rho_{\sigma''}(X\otimes h_k) = \sum_{n=0}^{\infty} f_{k,n}(t)\rho_{\sigma''}(X(n-k)).$$

Since we have

$$(X(m-k)v_i(d)|v^j(d-m+k)) + (v_i(d)|X(k-m)v^j(d-m+k)) = 0,$$

we have

$$\sum_{i=1}^{m_d}\rho_{\sigma'}(X(m-k))|v_i(d)\otimes v^i(d)\otimes\Phi\rangle$$

$$+ \sum_{j=1}^{m_{d-m+k}}\rho_{\sigma''}(X(-(m-k)))|v_j(d-m+k)\otimes v^j(d-m+k)\otimes\Phi\rangle$$

$$= 0.$$

This proves Claim 3.

Let $\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}$ be the completion along $E^{(1)}$, that is

$$\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}} \simeq \varprojlim_n \mathcal{O}_{\mathcal{B}^{(1)}}/I^n_{E^{(1)}}$$

where $I_{E^{(1)}}$ is the ideal sheaf of $E^{(1)}$. Then, there is an $\mathcal{O}_{E^{(1)}}$-algebra isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}} \simeq \mathcal{O}_{E^{(1)}}[[\,\tau\,]].$$

In the following we identify $\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}$ with $\mathcal{O}_{E^{(1)}}[[\,\tau\,]]$.

Claim 4. *Let* $\widehat{\mathcal{V}}^{\dagger(1)}_{\vec{\lambda}/E^{(1)}}$ *be a sheaf of* $\mathcal{O}_{E^{(1)}}[[\,\tau\,]]$*-module defined by*

$$\widehat{\mathcal{V}}^{\dagger(1)}_{\vec{\lambda}/E^{(1)}} = \{\langle\Phi| \in \mathcal{H}^{\dagger(1)}_{\vec{\lambda},E^{(1)}}[[\,\tau\,]] \mid \sum_{j=1}^{N}\langle\Phi|\sum_{k}\rho_j(X\otimes f_k)\tau^k = 0$$

$$\text{for all } f \in \pi^{(1)}_*\mathcal{O}_{\mathcal{C}^{(1)}}(*S^{(1)})\,\}.$$

Then, there is an $\mathcal{O}_{E^{(1)}}[[\,\tau\,]]$*-module isomorphism*

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}} \simeq \widehat{\mathcal{V}}^{\dagger(1)}_{\vec{\lambda}/E^{(1)}}.$$

Proof. Since the tensor product with $\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}$ is faithfully flat, we have an isomorphism

$$\begin{aligned}\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})&\otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}}\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}\\ &= \underline{Hom}_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}), \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}\\ &\simeq \underline{Hom}_{\widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}, \widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}).\end{aligned}$$

Note that we have an $\mathcal{O}_{E^{(1)}}$-module isomorphism

$$\mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}} \simeq \mathcal{H}^{(1)}_{\vec{\lambda}} \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{B}^{(1)}/E^{(1)}}$$

By Lemma 4.1.6 and faithful flatness we have

$$\begin{aligned}\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) &\otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}}\\ &\simeq \mathcal{H}^{(1)}_{\vec{\lambda}} \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}}/(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\mathcal{H}^{(1)}_{\vec{\lambda}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}(\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\mathcal{H}^{(1)}_{\vec{\lambda}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}} &= \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})(\mathcal{H}^{(1)}_{\vec{\lambda}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}}\\ &\simeq \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}[[\tau]]\end{aligned}$$

where the action of $X \otimes f \in \widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})$ on $\mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}[[\tau]]$ is given by

$$\sum_{j=1}^{N}\rho_j(X\otimes f_k)\tau^k$$

where $\sum f_k \tau^k$ is the element corresponding to f defined in Claim 2. Hence we have

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \widehat{\mathcal{O}}_{\mathcal{O}_{\mathcal{B}^{(1)}}/E^{(1)}} \simeq \mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}[[\,\tau\,]]/\widetilde{\mathfrak{g}}(\mathfrak{F}^{(1)})\mathcal{H}^{(1)}_{\vec{\lambda},E^{(1)}}[[\,\tau\,]].$$

This proves Claim 4.

Now we are ready to prove Theorem 6.2.1. Choose $\mathcal{B}^{(1)}$ small enough so that $\oplus_{\mu \in P_\ell} \mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}}$ is $\mathcal{O}_{E^{(1)}}$-free. Let $\{\langle\Psi_1|,\dots,\langle\Psi_n|\}$ be an $\mathcal{O}_{E^{(1)}}$ free basis of $\oplus_{\mu \in P_\ell} \mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}}$. Let $\{\langle\widetilde{\Psi}_1|,\dots,\langle\widetilde{\Psi}_n|\}$ be elements of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ constructed in (6.2-2) from $\{\langle\Psi_1|,\dots,\langle\Psi_n|\}$. The correspondence $\langle\Psi_i| \mapsto \langle\widetilde{\Psi}_i|$, $i = 1,\dots,n$ defines an $\mathcal{O}_{E^{(1)}}$-module homomorphism

$$\iota \, : \, \bigoplus_{\mu \in P_\ell} \mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}} \longrightarrow \widehat{\mathcal{V}}^{\dagger(1)}_{\vec{\lambda}/E^{(1)}}.$$

First we show that $\langle\widetilde{\Psi}_1|,\dots,\langle\widetilde{\Psi}_n|$ are $\mathcal{O}_{E^{(1)}}[[\,\tau\,]]$-linearly independent. Suppose we have a relation

$$\sum_{i=1}^{n} a_i(\tau)\langle\widetilde{\Psi}_i| = 0, \quad a_i(\tau) \in \mathcal{O}_{E^{(1)}}[[\,\tau\,]].$$

We may assume that one of $a_i(\tau)$'s, say $a_k(\tau)$ satisfies the condition that $a_k(0) \neq 0$. If we put $\tau = 0$ in the above relation, we have

$$\sum_{i=1}^{n} a_i(0)\langle\Psi_i| = 0.$$

Hence $a_i(0) = 0$ for all i. This is a contradiction. Hence, $\langle\widetilde{\Psi}_1|,\dots,\langle\widetilde{\Psi}_n|$ generate $\mathcal{O}_{E^{(1)}}[[\,\tau\,]]$-free submodule of $\widehat{\mathcal{V}}^{\dagger(1)}_{\vec{\lambda}/E^{(1)}}$.

Now choose a point $x \in E^{(1)}$ and $s \in \mathcal{B}^{(1)} \setminus E^{(1)}$. Then, the above argument and Corollary 5.3.5 show that

$$\begin{aligned}\dim_{\mathbf{C}} & \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) \\ &= \dim_{\mathbf{C}} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) \\ &\geq n = \sum_{\mu \in P_\ell} \mathrm{rank} \mathcal{V}^{\dagger(1)}_{\mu,\mu^\dagger,\vec{\lambda},E^{(1)}}.\end{aligned}$$

By Lemma 4.1.3 and Corollary 2.2.6 we have

$$\dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},x}/\mathfrak{m}_x) = \sum_{\mu \in P_\ell} \dim_{\mathbf{C}} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\mathfrak{X}^{(1)}_x).$$

Hence we have

$$\dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) \geq \dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},x}/\mathfrak{m}_x).$$

On the other hand, since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ is coherent and locally free on $\mathcal{B}^{(1)} \setminus E^{(1)}$, we have the inequality

$$\dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) \leq \dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},x}/\mathfrak{m}_x).$$

Hence we have the equality

$$\dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s) = \dim_{\mathbf{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes (\mathcal{O}_{\mathcal{B}^{(1)},x}/\mathfrak{m}_x).$$

Hence $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ is locally free. Q.E.D.

Let $\tilde{\ell}$ be the meromorphic vector field given in Lemma 6.1.2 and $\underline{l}_j$ be the formal Laurent expansion of $\tilde{\ell}$ at Q_j. Put $\vec{\underline{l}} = (\underline{l}_1, \ldots, \underline{l}_N)$. (See (6.1-2).) Under these notations we have the following theorem. In the following theorem we shall only prove that $\tau^{\Delta_\mu}\langle\widetilde{\Psi}|$ is a formal solution but actually we can show that the formal solution converges. To show this we need rather long discussions and we shall give a complete proof in the forthcoming paper.

Theorem 6.2.2. *Let $\langle\widetilde{\Psi}|$ be the formal power series defined in (6.2-2). Then $\tau^{\Delta_\mu}\langle\widetilde{\Psi}|$ is a formal solution of a differential equation of the Fuchsian type*

$$\tau^{\Delta_\mu}\langle\widetilde{\Psi}|(\tau\frac{d}{d\tau} - T[\vec{\underline{l}}] + a(\vec{\underline{l}})) = 0.$$

Proof. Let us fix an element $\omega \in H^0(\mathcal{C}^{(0)} \otimes_{\mathcal{B}^{(0)}} \mathcal{C}^{(0)}, \omega^{\boxtimes 2}_{\mathcal{C}^{(0)}/\mathcal{B}^{(0)}}(2\Delta))$ such that ω satisfies (5.1-8) and (5.1-9). By (5.1-4) we have

$$\langle\widetilde{\Psi}|\widetilde{T}(u)|\Phi\rangle du^2 \in H^0(\mathcal{C}^{(1)}, \omega^{\otimes 2}_{\mathcal{C}^{(1)}/\mathcal{B}^{(1)}}(*S^{(1)})).$$

Let $\widetilde{\ell} = \ell(z)\dfrac{\partial}{\partial z}$ be the meromorphic vector field given in Lemma 6.1.2. Then, for $(u, \tau) \in E^{(1)} \times D$, $\tau \neq 0$,

$$\ell(z)\langle\widetilde{\Psi}|\widetilde{T}(z)|\Phi\rangle dz$$

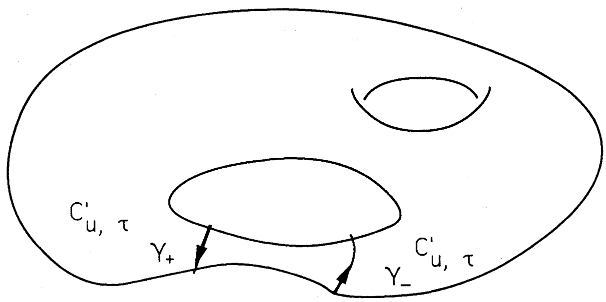

Figure 5.

is a meromorphic form on $C'_{u,\tau} = C_{u,\tau} \setminus \{ (x, y, \tau) \in S_0 \,|\, |x| \le \epsilon \text{ or } |y| \le \epsilon \}$ for a sufficiently small positive number $\epsilon < 1$.

The boundary of $C'_{u,\tau}$ consists of two disjoint simple closed curves γ_+, γ_-. We choose the orientation of $\gamma_\pm$ in such a way that $C'_{u,\tau}$ lies in a right side of $\gamma_\pm$. Then by Proposition 2.4.2,(5.1-5) and (5.1-6) we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_+} \ell(z)\langle\widetilde{\Psi}|\widetilde{T}(z)|\Phi\rangle dz + \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_-} \ell(w)\langle\widetilde{\Psi}|\widetilde{T}(w)|\Phi\rangle dw \\
&\quad = \sum_{j=1}^{N} \mathrm{Res}_{Q_j}(\ell(u)\langle\widetilde{\Psi}|\widetilde{T}(u)|\Phi\rangle du) \\
&\quad = \sum_{j=1}^{N} \langle\widetilde{\Psi}|\rho_j(\mathop{\mathrm{Res}}_{\xi_j=0}(\ell_j(\xi_j)T(\xi_j)d\xi_j))|\Phi\rangle \\
&\qquad\qquad - c_v \sum_{j=1}^{N} \mathop{\mathrm{Res}}_{\xi_j=0}(\ell_j(\xi_j)S_{\omega,j}(\xi_j))\langle\widetilde{\Psi}|\Phi\rangle \\
&\quad = \sum_{j=1}^{N} \langle\widetilde{\Psi}|\rho_j(T[\ell_j])|\Phi\rangle - a(\vec{\underline{\ell}})\langle\widetilde{\Psi}|\Psi\rangle.
\end{aligned}$$

On the other hand, on γ_+ we have $\ell(z)\dfrac{d}{dz} = \dfrac{1}{2}z\dfrac{d}{dz}$. Hence, by (5.1.4) we have

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}} & \int_{\gamma+} \ell(z)\langle\widetilde{\Psi}|\widetilde{T}(z)|\Phi\rangle dz \\
&= \frac{1}{4\pi\sqrt{-1}} \int_{\gamma+} \left(z\langle\widetilde{\Psi}|T(z)|\Phi\rangle - c_v z S_\omega(z)\langle\widetilde{\Psi}|\Phi\rangle\right) dz \\
&= \frac{1}{4\pi\sqrt{-1}} \int_{\gamma+} z\langle\widetilde{\Psi}|T(z)|\Phi\rangle dz,
\end{aligned}$$

since $S_\omega(z)dz^2$ is holomorphic at $z = 0$. Hence, by (6.2-1) and (6.2-2) we have

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}} & \int_{\gamma+} \ell(z)\langle\widetilde{\Psi}|\widetilde{T}(z)|\Phi\rangle dz \\
&= \frac{1}{2}\sum_{d=0}^{\infty} \tau^{\Delta_\mu+d} \sum_{i=1}^{m_d} \int_{\gamma+} z\langle\Psi_d|T(z)|v_i(d)\otimes v^i(d)\otimes\Phi\rangle dz \\
&= \frac{1}{2}\sum_{d=0}^{\infty} \tau^{\Delta_\mu+d} \sum_{i=1}^{m_d} \langle\Psi_d|L_0(v_i(d))\otimes v^i(d)\otimes\Phi\rangle \\
&= \frac{1}{2}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d} (\Delta_\mu + d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}} & \int_{\gamma-} \ell(w)\langle\widetilde{\Psi}|\widetilde{T}(w)|\Phi\rangle dw \\
&= \frac{1}{2}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d} (\Delta_{\mu^\dagger} + d)\tau^{\Delta_{\mu^\dagger}+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

Since we have $\Delta_\mu = \Delta_{\mu^\dagger}$, we obtain

$$\begin{aligned}
\sum_{j=1}^{N} & \langle\widetilde{\Psi}|\rho_j(T[\ell_j])|\Phi\rangle - a(\vec{\underline{\ell}})\langle\widetilde{\Psi}|\Psi\rangle \\
&= \sum_{d=0}^{\infty}\sum_{i=1}^{m_d} (\Delta_\mu + d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\langle\widetilde{\Psi}|\tau\frac{d}{d\tau}|\Phi\rangle\\&=\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(\Delta_\mu+d)\tau^{\Delta_\mu+d}\langle\Psi_d|v_i(d)\otimes v^i(d)\otimes\Phi\rangle.\end{aligned}$$

Hence, $\langle\widetilde{\Psi}|$ is a formal solution of the differential equation

$$\langle\widetilde{\Psi}|(\tau\frac{d}{d\tau}-T[\vec{l}]+a(\vec{l}))=0.$$

This proves Theorem 6.2.2.

Corollary 6.2.3. *$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ is locally free. Moreover, for each point $s\in\mathcal{B}^{(1)}$ we have*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})\otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{O}_{\mathcal{B}^{(1)},s}/\mathfrak{m}_s)\simeq\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}_s).$$

Remark 6.2.4. Similar to Remark 4.1.7 we can define $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(1)})$ by the left hand side of the above isomorphism for $\mathfrak{X}^{(1)}=\mathfrak{X}^{(1)}_s$. Then we have the canonical isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(1)})\simeq\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}^{(\infty)}).$$

Corollary 6.2.5. *The rank of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ can be calculated combinatorically from the fusion rules.*

In this case, the fusion rules, which count the numbers of independent solutions of type $(g,N)=(0,3)$, are given in Example 2.2.8. We use the notation there. The number of the independent solutions is given by $N_{\mu,\nu,\lambda}=\dim W_{\mu,\nu,\lambda}$. By using $N_{\mu,\nu,\lambda}$, the explicit formula for the rank is given in the case of maximally degenerate curves (the corresponding dual diagram is the ϕ^3-diagram) with g loops and N external lines, which has $3g-3+N$ internal lines and $2g-2+N$ vertices, that is,

$$\operatorname{rank}\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})=\sum_{\vec{\mu}:\text{internal}}\ \prod_{(\alpha,\beta,\gamma):\text{vertices}}N_{\alpha,\beta,\gamma}.$$

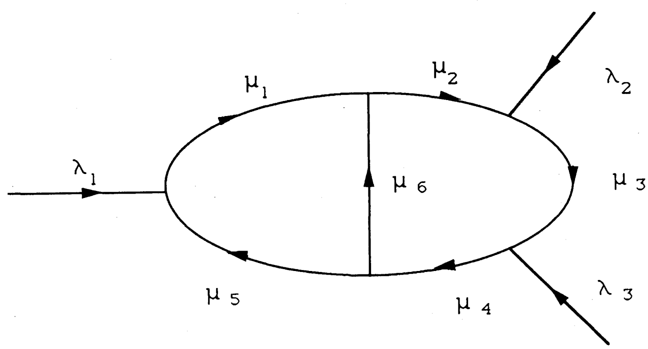

Figure 6.

(See [Ve].)

For each ϕ^3-diagram the above proof (see also the factorization property, Theorem 6.2.6 below) gives a canonical basis of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})$, with which the monodromy around the vanishing cycles are diagonalized. The relation between the bases corresponding to two different diagrams is described by a connection matrix. The matrix provides us the monodromy representation of the braid group, the mapping class group or some generalization of them ([TK1], [TK2], [F], [Va1]).

The sheaf version of Proposition 2.2.5 is the following *factorization property*.

Theorem 6.2.6. *There exists an $\mathcal{O}_{\widetilde{E}(1)}$-module isomorphism.*

$$\bigoplus_{\mu\in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}^{(1)}) \xrightarrow{\sim} (\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{B(1)}} \mathcal{O}_{E(1)}) \otimes_{\mathcal{O}_{E(1)}} \mathcal{O}_{\widetilde{E}(1)}.$$

Proof. We use the notation in the proof of Proposition 2.2.6 freely. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}(1)} = (\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{B(1)}} \mathcal{O}_{E(1)}) \otimes_{\mathcal{O}_{E(1)}} \mathcal{O}_{\widetilde{E}(1)}$$
$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}(1)} = (\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)}) \otimes_{\mathcal{O}_{B(1)}} \mathcal{O}_{E(1)}) \otimes_{\mathcal{O}_{E(1)}} \mathcal{O}_{\widetilde{E}(1)}.$$

Then we have a canonical identification

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}(1)} = \underline{Hom}_{\mathcal{O}_{\widetilde{E}(1)}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}(1)}, \mathcal{O}_{\widetilde{E}(1)}).$$

For an element $\langle\widetilde{\Psi}| \in \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}^{(1)}_{\widetilde{E}^{(1)}})$ and an element $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}^{(1)}}$ define $\iota_{\mu}(\langle\widetilde{\Psi}|) \in \underline{Hom}_{\mathcal{O}_{\widetilde{E}^{(1)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}^{(1)}}, \mathcal{O}_{\widetilde{E}^{(1)}})$ by

$$\iota_{\mu}(\langle\widetilde{\Psi}|)(|\Phi\rangle) = \langle\widetilde{\Psi}|0_{\mu,\mu^{\dagger}} \otimes \Phi\rangle.$$

This is well-defined and induces an $\mathcal{O}_{\widetilde{E}^{(1)}}$-module homomorphism

$$\text{(6.2-5)} \qquad \iota : \bigoplus_{\mu\in P_{\ell}} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}^{(1)}_{\widetilde{E}^{(1)}}) \to \underline{Hom}_{\mathcal{O}_{\widetilde{E}^{(1)}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}^{(1)}}, \mathcal{O}_{\widetilde{E}^{(1)}}).$$

For each point $s \in \widetilde{E}^{(1)}$, put

$$\mathbf{C}_s = \mathcal{O}_{\widetilde{E}^{(1)},s}/\mathfrak{m}_s.$$

By tensoring $\mathbf{C}_s$ to (6.2-5), we have a $\mathbf{C}$-linear mapping

$$\iota_s : \bigoplus_{\mu\in P_{\ell}} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}^{(1)}_{\widetilde{E}^{(1)}}) \otimes \mathbf{C}_s \to \mathrm{Hom}_{\mathbf{C}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}^{(1)})_{\widetilde{E}^{(1)}} \otimes \mathbf{C}_s, \mathbf{C}).$$

By Remark 4.1.7 and Corollary 6.2.2, the mapping ι_s is nothing but the mapping in Proposition 2.2.6. Hence, ι_s is isomorphic. Therefore, ι is an $\mathcal{O}_{\widetilde{E}^{(1)}}$-module isomorphism. Q.E.D.

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