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Quantum Groups and Integrable Models

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The term "Quantum Group" and the algebraic constructions associated with it are rather popular nowadays. Different people however, endow this combination of words with different meaning. Here I will present some historical background and a systematic introduction into this rapidly developing theory[1)].

§1. History of the subject

The main source of motivation for quantum groups was the Quantum Inverse Scattering Method (QISM) initiated by L. Faddeev, E. Sklyanin and this author in [1–3]. Their initial aim was to formulate a quantum theory of solitons. Quantum Lie groups and quantum Lie algebras appeared afterwards as abstraction of concrete algebraic constructions constituting the mathematical formalism of QISM. Let us first consider two characteristic examples.

Example 1. In the paper [4] concerning the quantum Liouville model on the lattice, L. Faddeev and the author introduced the $\mathbf{C}$-algebra A_q generated by the elements a, b, c, d with relations

$$\begin{aligned} &ab = qba, \quad ac = qca, \quad bc = cb, \\ &bd = qdb, \quad cd = qdc, \quad ad - da = (q - q^{-1})bc, \\ &q \in \mathbf{C} \setminus \{0\}. \end{aligned} \tag{1}$$

This algebra has the following remarkable property. Consider two commuting copies (a', b', c', d') and (a'', b'', c'', d'') of generators of A_q and form two matrices

$$T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \qquad T'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

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1) This paper is an extended version of the lecture given in Taniguchi symposium at RIMS in October 1988.

Then the set (a, b, c, d), where

$$T = T'T'' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

also generates A_q, *i.e.* satisfies relations (1). In other words, relations (1) are preserved under matrix multiplication. Another observation is that the element

$$\det_q T = ad - qbc$$

— the "quantum determinant"— belongs to the center of A_q.

Setting

$$S(T) = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix},$$

we obtain

$$TS(T) = S(T)T = \det_q T \cdot I,$$

where I is the unit matrix. Thus the quotient of the algebra A_q by the relation $\det_q T = 1$ could be called the "quantum group $SL(2)$" and denoted by $SL_q(2)$. The algebra $SL_q(2)$ with an additional $*$-structure was also introduced by S. Woronowicz [5–6] in his study of "compact matrix pseudogroups". This approach was based on the C^*-algebra point of view.

Example 2. P. Kulish and N. Reshetikhin [7] and E. Sklyanin [8] introduced in their study of concrete problems of QISM the following $\mathbf{C}$-algebra U_h with generators $H, X^{\pm}$ and relations

$$(2) \qquad [H, X^{\pm}] = \pm 2X^{\pm}, \qquad [X^+, X^-] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}.$$

Here the parameter $h \in \mathbf{C}$ plays the role of Planck's constant. As $h \to 0$, relations (2) turn into the commutation relations for the Lie algebra $\mathfrak{sl}(2)$. Therefore the algebra U_h could be considered as a deformation of the universal enveloping algebra $U\mathfrak{sl}(2)$ of the Lie algebra $\mathfrak{sl}(2)$.

V. Drinfeld was the first to make an important observation that main algebraic constructions of QISM are nothing but very special (and very meaningful) examples of bialgebras and Hopf algebras. Using this algebraic language, he gave in [9–10] a natural generalization of Example 2.

Remind (see, for instance, [11]) that a $\mathbf{C}$-algebra A is called a Hopf algebra, if

i) there exists a **C**-algebra homomorphism

$$\Delta : A \longrightarrow A \otimes A$$

called a coproduct, such that the following diagram is commutative:

$$\begin{tikzcd} & A\otimes A \arrow[dr, "\Delta\otimes\mathrm{id}"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A\otimes A\otimes A \\ & A\otimes A \arrow[ur, "\mathrm{id}\otimes\Delta"'] & \end{tikzcd}$$

ii) there exist a **C**-algebra homomorphism $\varepsilon : A \to \mathbf{C}$, called a counit, and a **C**-algebra antihomomorphism $S : A \to A$, called an antipode, such that the following diagrams are commutative:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & A \arrow[d, equal] \\ A\otimes A \arrow[r, shift left, "\mathrm{id}\otimes\varepsilon"] \arrow[r, shift right, "\varepsilon\otimes\mathrm{id}"'] & A\otimes\mathbb{C} = \mathbb{C}\otimes A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[drr, "\varepsilon"'] & A\otimes A \arrow[r, shift left, "\mathrm{id}\otimes S"] \arrow[r, shift right, "S\otimes\mathrm{id}"'] & A\otimes A \arrow[r, "m"] & A \\ & & \mathbb{C} \arrow[ur, "i"'] & \end{tikzcd}$$

Here m is the usual product in the algebra: $m(a \otimes b) = ab$, $a, b \in A$ and i is the natural imbedding of **C** into A: $i(c) = c \cdot 1$, $c \in \mathbf{C}$, where 1 is the unit element in A. If a **C**-algebra satisfies condition i) and has a counit ε it is called a bialgebra.

Let G be a Lie (topological) group. The commutative algebra $\mathrm{Fun}(G)$ of smooth (continuous) functions on G is a typical example of a Hopf algebra and any commutative Hopf algebra is of this form. A typical example of a bialgebra is given by the algebra $\mathbf{C}[t_{ij}]$ of polynomials in n^2 variables t_{ij}, $i, j = 1, \cdots, n$, with coproduct Δ

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj} \tag{3}$$

and counit ε

$$\varepsilon(t_{ij}) = \delta_{ij},\ i,j = 1,\cdots,n,$$

where δ_{ij} is Kronecker's delta. Using the matrix $T = (t_{ij})_{i,j=1}^{n}$ we can rewrite (3) in matrix form

$$\Delta(T) = T \dot{\otimes} T,$$

where the symbol $\dot{\otimes}$ refers to the tensor product of algebras and usual product of matrices. In addition

$$\varepsilon(T) = I,$$

where I is the $n \times n$ unit matrix. Thus the algebra $\mathbf{C}[t_{ij}]$ can be interpreted as the algebra of polynomial functions on the matrix algebra $M_n(\mathbf{C})$ so that the coproduct (3) is induced by the usual matrix product.

In Example 1 we are dealing with the non-commutative deformation of the latter algebra for the case $n = 2$. The main observation shows that A_q is a bialgebra with the same coproduct (3) as in the commutative case. The algebra U_h of Example 2 is also a bialgebra. The coproduct Δ introduced by E. Sklyanin [12] has the form

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(X^{\pm}) &= X^{\pm} \otimes e^{-\frac{hH}{2}} + e^{\frac{hH}{2}} \otimes X^{\pm}. \end{aligned} \tag{4}$$

Moreover, defining the antipode S by

$$S(H) = -H, \quad S(X^{\pm}) = -e^{-\frac{hH}{2}} X^{\pm} e^{\frac{hH}{2}}$$

and the counit ε by

$$\varepsilon(1) = 1, \quad \varepsilon(H) = \varepsilon(X^{\pm}) = 0$$

we make U_h a non-commutative and non-cocommutative Hopf algebra.

It was this particular example that served as a starting point for the work of V. Drinfeld [9–10] and M. Jimbo [13–14] who have generalized the algebra U_h to the general case of simple Lie algebras.

Let us now turn to the QISM. The basic algebraic formulas constituting the essence of the method are

$$RT_1T_2 = T_2T_1R \tag{5}$$

and

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{6}$$

Here $R \in M_{n^2}(\mathbf{C})$ and $T_1 = T \otimes I, T_2 = I \otimes T$, where T is an $n \times n$-matrix with matrix elements belonging to some associative algebra A. The indices 12, 13 and 23 in (6) show the way of imbedding $M_{n^2}(\mathbf{C})$ into $M_{n^3}(\mathbf{C})$ according to the choice of two factors in the triple tensor product $\mathbf{C}^n \otimes \mathbf{C}^n \otimes \mathbf{C}^n$.

Note that in the framework of QISM the matrices T and R depend on an additional complex parameter λ, called the spectral parameter. Hence in (5) one should replace T_1 by $T_1(\lambda)$, T_2 by $T_2(\mu)$ and R by $R(\lambda - \mu)$. Respectively in (6) one should make the replacements $R_{12} \mapsto R_{12}(\lambda - \mu), R_{13} \mapsto R_{13}(\lambda - \nu), R_{23} \mapsto R_{23}(\mu - \nu)$. The matrix $T(\lambda)$ plays the role of the monodromy matrix for the corresponding quantum linear problem:

$$\Phi_{n+1} = Q_n(\lambda)\Phi_n,$$

i.e.

$$T(\lambda) = Q_N(\lambda) \cdots Q_1(\lambda).$$

The main observation equivalent to the existence of a bialgebra structure states that if local matrices $Q_1(\lambda), \cdots, Q_N(\lambda)$ satisfy (5), then the monodromy matrix $T(\lambda)$ also satisfies (5). In this context formula (6), which is nothing but the famous Yang-Baxter equation in QISM; (this name was given to it by Faddeev and myself in [2]) can be considered as a compatibility condition for (5). For certain classes of integrable quantum models there exists a special value of spectral parameter λ, say $\lambda = \infty$, where some simplifications occur. Setting $R = \lim_{\lambda \to \infty} R(\lambda)$ and $T = \lim_{\lambda \to \infty} T(\lambda)$ we arrive to formulas (5) and (6).

Examples 1 and 2 can be constructed by the above procedure using the matrix $Q(\lambda)$ for the quantum Sinh- and Sine-Gordon models (see [4],[7]). In this approach formulas (5) and (6) have been of great help. However Drinfeld [9–10] and Jimbo [13–14], who were generalizing Example 2, did not use the main formulas of QISM (5) and (6) to the full strength. This is why Faddeev, Reshetikhin and the author decided to develop a more systematic approach to quantum Lie groups and quantum Lie algebras based on the exclusive use of formulas (5) and (6). This natural suggestion materialized in our papers [15–16], and my lecture will mostly be based on them.

Before passing to formal definitions I would like to explain the meaning of the word "quantum" in connection with quantum groups. Historically, it points out to their birthplace, QISM. Mathematically it has the same meaning as the term "deformation" as applied to algebraic structures. We will apply this idea to the algebra Fun(G) of polynomial functions — "observables" — on the Lie group G. Its special noncommutative deformation will be called the "algebra of functions on the quantum Lie group G_q" and will be denoted by $\mathrm{Fun}(G_q)$. The quantum group G_q itself should be interpreted as a would-be spectrum of the non-commutative algebra $\mathrm{Fun}(G_q)$ (if such an object exists). Thus the terminology will be as follows: when saying quantum group I will mean the corresponding non-commutative algebra. It is relevant to note that quantum groups should provide a meaningful example for the general program of non-commutative differential geometry of A. Connes [17].

§2. Quantum matrix algebras

Denote by $\mathbf{C}\langle t_{ij}\rangle$ the $\mathbf{C}$-algebra freely generated by t_{ij}, $i,j=1,\cdots,n$. Let $R \in GL(n^2)$ and consider the two-sided ideal $\mathbf{I}_R$ in $\mathbf{C}\langle t_{ij}\rangle$ generated by the relations

$$RT_1T_2 = T_2T_1R.$$

Here $T_1 = T \otimes I, T_2 = I \otimes T \in M_{n^2}(\mathbf{C}\langle t_{ij}\rangle)$, where $T = (t_{ij})_{i,j=1}^n \in M_n(\mathbf{C}\langle t_{ij}\rangle)$ is an $n \times n$ matrix with matrix elements belonging to $\mathbf{C}\langle t_{ij}\rangle$ and I is the unit matrix in $M_n(\mathbf{C})$.

Definition 1. The quotient algebra

$$A_R = \mathbf{C}\langle t_{ij}\rangle \Big/ \mathbf{I}_R$$

is called the algebra of functions on the quantum matrix algebra of rank n associated with the matrix R.

When $R = I \otimes I$, the algebra A_R coincides with the commutative algebra of polynomial functions on $M_n(\mathbf{C})$.

Proposition 1. *The algebra A_R is a bialgebra with coproduct Δ*

$$\Delta(T) = T \dot{\otimes} T$$

and counit ε

$$\varepsilon(T) = I.$$

The proof is evident.

Thus we see that A_R can be considered as a non-commutative deformation of the polynomial algebra on $M_n(\mathbf{C})$ with the same R-independent coproduct (3).

Let now denote by $\mathbf{C}\langle x_1, \cdots, x_n\rangle$ the $\mathbf{C}$-algebra freely generated by $x_1, \cdots, x_n$ and let P be the permutation matrix in $\mathbf{C}^n \otimes \mathbf{C}^n$: $P\, u \otimes v = v \otimes u$ for $u, v \in \mathbf{C}^n$. Set $\hat{R} = PR$ and for any polynomial $f(t) \in \mathbf{C}[t]$

denote by $\mathbf{I}_{f,R}$ the two-sided ideal in $\mathbf{C}\langle x_1, \cdots, x_n\rangle$, generated by the relations

$$f(\hat{R})\, x \otimes x = 0.$$

Here $x \otimes x = (x_i x_j)_{i,j=1}^{n} \in M_n(\mathbf{C}\langle x_1, \cdots, x_n\rangle)$.

Definition 2. The quotient algebra

$$\mathbf{C}^n_{f,R} = \mathbf{C}\langle x_1, \cdots, x_n\rangle \Big/ \mathbf{I}_{f,R}$$

is called the algebra of functions on the quantum n-dimensional vector space, associated with the polynomial $f(t)$ and the matrix R.

Proposition 2. *The map $\delta : \mathbf{C}^n_{f,R} \to A_R \otimes \mathbf{C}^n_{f,R}$ defined by the formula*

$$\delta(x_i) = \sum_{k=1}^{n} t_{ik} \otimes x_k, \quad i = 1, \cdots, n, \tag{7}$$

i.e.

$$\delta(x) = T \dot{\otimes} x,$$

is a $\mathbf{C}$-algebra homomorphism and provide $\mathbf{C}^n_{f,R}$ with the left A_R-comodule structure with coaction δ.

The latter means that the following diagrams are commutative

$$\begin{tikzcd}[row sep=small] & A_R \otimes A_R \otimes \mathbb{C}^n_{f,R} \arrow[dd, equal] \\ A_R \otimes \mathbb{C}^n_{f,R} \arrow[ur, "\mathrm{id}\otimes\delta"] \arrow[dr, "\Delta\otimes\mathrm{id}"'] & \\ & A_R \otimes A_R \otimes \mathbb{C}^n_{f,R} \end{tikzcd}$$

$$\begin{tikzcd} & A_R \otimes \mathbb{C}^n_{f,R} \arrow[dr, "\varepsilon\otimes\mathrm{id}"] & \\ \mathbb{C}^n_{f,R} \arrow[ur, "\delta"] \arrow[rr, equal] & & \mathbb{C} \otimes \mathbb{C}^n_{f,R} = \mathbb{C}^n_{f,R} \end{tikzcd}$$

The proof is clear.

When $\hat{R} = P$ and $f(1) = 0$, the algebra $\mathbf{C}^n_{f,R}$ turns into the commutative algebra $\mathbf{C}[x_1, \cdots, x_n]$ and the coaction δ is induced by the usual action of the matrix algebra $M_n(\mathbf{C})$ on $\mathbf{C}^n$. Thus formula (7) can be interpreted as an R-independent action of the quantum matrix algebra on the quantum vector space.

Several remarks are now in order. The algebras A_R and $\mathbf{C}^n_{f,R}$ naturally inherit the structure of graded algebras from $\mathbf{C}\langle t_{ij}\rangle$ and $\mathbf{C}\langle x_1,\cdots,x_n\rangle$. In this respect they are nothing but special types of finitely generated quadratic algebras. From the functorial point of view the category of quadratic algebras was studied by Y. Manin [18–19]. However from our point of view this approach is rather general. Even the properties of the algebras A_R and $\mathbf{C}^n_{f,R}$ for an arbitrary matrix R can differ drastically from the properties of their commutative analogs $\mathbf{C}[t_{ij}]$ and $\mathbf{C}[x_1,\cdots,x_n]$. For example they can have different Poincaré series — the generating functions for the dimensions of their graded components. In particular, relations (5) for the graded components of degree two imply additional relations for the components of degree three. Equation (6) ensures that these additional relations must be satisfied identically. This is one possible way of incorporating (6) into this algebraic scheme. From now on we will assume that the matrix R satisfies the Yang-Baxter equation (6).

Now, everybody knows the crucial role played by the Yang-Baxter equation in QISM and in related subjects. I will remind here only that in terms of the matrix $\hat{R}$ it reads

$$(\hat{R}\otimes I)(I\otimes\hat{R})(\hat{R}\otimes I)=(I\otimes\hat{R})(\hat{R}\otimes I)(I\otimes\hat{R})$$

and its solutions correspond to the representations $\rho: B_3\to \mathrm{End}(\mathbf{C}^n\otimes \mathbf{C}^n\otimes\mathbf{C}^n)$ of the braid group B_3 satisfying certain locality conditions. By this I mean that

$$\rho(s_1)=\hat{R}\otimes I, \qquad \rho(s_2)=I\otimes\hat{R},$$

where s_1 and s_2 are generators of B_3 satisfying a single relation

$$s_1s_2s_1=s_2s_1s_2.$$

I would like to emphasize here that the problem of complete classification of local representations of the braid group B_3 is not solved even in the case of symmetric group S_3, where $s_1^2=s_2^2=1$. An interesting connection between the Yang-Baxter equation, the braid groups and the monodromy representations was discovered by Kohno (see his lecture in this volume).

However, V. Bazhanov [20] and M. Jimbo [13–14], motivated by QISM, constructed special solutions of the Yang-Baxter equation associated with simple Lie algebras of classical type. The corresponding matrices R act in the tensor square of the vector representation and depend on a complex parameter $q \neq 0$ which is the parameter of deformation; when $q = 1$ R turns into the unit matrix. We will use quantum matrix algebras connected with these R-matrices in defining simple quantum Lie groups by passing to their natural quotient algebras admitting a Hopf structure. This procedure is parallel to the definition of classical Lie groups as algebraic varieties in $M_n(\mathbf{C})$ and was introduced by L. Faddeev, N. Reshetikhin and the author [15]. Contrary to the $q = 1$ case where all simple Lie groups are embedded into the "Universal $M_n(\mathbf{C})$", in the case $q \neq 1$ the algebras A_R attached to the various series of simple Lie algebras are not isomorphic. This illustrates once more the general principle that "quantization removes degeneracy".

§3. Quantum groups $SL_q(n)$ and $GL_q(n)$

The matrix $R = R_q$ associated with the Lie algebra of type A_{n-1}, $n \geq 2$, has the form

$$R_q = q\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} e_{ii} \otimes e_{jj}$$

$$+ (q - q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{n} e_{ij} \otimes e_{ji}, \qquad q \in \mathbf{C} \setminus \{0\},$$

where $e_{ij} \in M_n(\mathbf{C})$, $i, j = 1, \cdots, n$, are matrix units. The corresponding matrix $\hat{R}_q = PR_q$ enters in local representations of the Hecke-Iwahori algebra [13].

Set

$$S(t_{ij}) = (-q)^{i-j}\tilde{t}_{ji},$$

where

$$\tilde{t}_{ij} = \sum_{\sigma\in S_{n-1}} (-q)^{l(\sigma)} t_{1\sigma_1} \cdots t_{i-1\sigma_{i-1}} t_{i+1\sigma_{i+1}} \cdots t_{n\sigma_n}, \quad i, j = 1, \cdots, n.$$

Here $l(\sigma)$ is the length (minimal number of transpositions) of the substitution $\sigma = \sigma(1, \cdots, j-1, j+1, \cdots, n) = (\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$.

We have

Proposition 3. *Let* $S(T) = \big(S(t_{ij})\big)_{i,j=1}^{n}$. *Then*

$$TS(T) = S(T)T = \det_q T \cdot I,$$

where

$$\det_q T = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1\sigma_1} \cdots t_{n\sigma_n}$$

is called the quantum determinant.

The element $\det_q T \in A_q = A_{R_q}$ is central and group-like:

$$\Delta(\det_q T) = \det_q T \otimes \det_q T.$$

Moreover, in the generic case, when q is not a root of 1, the center of A_q is generated by $\det_q T$.

Definition 3. The quotient algebra of A_q by the relation $\det_q T = 1$ is called the algebra of functions on the quantum group $SL_q(n)$ and is denoted by $\mathrm{Fun}(SL_q(n))$.

In a similar way, localizing the algebra A_q with respect to the element $\det_q T$ we obtain the algebra $\mathrm{Fun}(GL_q(n))$.

Theorem 1. *The algebras* $\mathrm{Fun}(SL_q(n))$ *and* $\mathrm{Fun}(GL_q(n))$ *are Hopf algebras with the same coproduct* Δ *and counit* ε *as in* A_q *and with antipodes defined by* $S(T)$ *and* $(\det_q T)^{-1}\, S(T)$ *respectively. In addition*

$$S^2(T) = \mathcal{D} T \mathcal{D}^{-1}$$

where $\mathcal{D} = \mathrm{diag}(1, q^2, \cdots, q^{2(n-1)}) \in M_n(\mathbf{C})$.

In the simplest case $n = 2$ we have

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and the corresponding algebra A_q coincides with the algebra of Example 1.

Now specializing the matrix R in Definition 2 to be R_q and $f(t) = t - q$ we arrive at

Definition 4. The algebra of "q-polynomials" — the algebra $\mathbf{C}_q^n$ with generators $x_1, \cdots, x_n$ and relations

$$x_i x_j = q x_j x_i, \quad 1 \le i < j \le n,$$

is called the algebra of functions on the quantum n-dimensional vector space.

The algebra A_q, $\mathrm{Fun}(SL_q(n))$ and $\mathrm{Fun}(GL_q(n))$ "act" on $\mathbf{C}_q^n$ via formula (7).

Now set $R = R_q$ and $f(t) = t + q^{-1}$. From Definition 2 we obtain for $q^2 \neq -1$

Definition 5. The finite dimensional algebra $\bigwedge \mathbf{C}_q^n$ with generators $x_1, \cdots, x_n$ and relations $x_i^2 = 0,\ x_i x_j = -q^{-1} x_j x_i,\ 1 \leq i < j \leq n$, is called the q-exterior algebra of the quantum vector space $\mathbf{C}_q^n$.

The "action" δ of A_q on $\bigwedge \mathbf{C}_q^n$ leads to the formula $\delta(x_1 \cdots x_n) = \det_q T \otimes x_1 \cdots x_n$ and provides a direct proof of the relation $\Delta(\det_q T) = \det_q T \otimes \det_q T$. This interpretation of the quantum determinant was also given in [19].

In the case $|q| = 1$, the algebra $\mathbf{C}_q^n$ admits a natural completion to the algebra $\mathbf{T}_q^n$ of formal Laurent series $\sum_{k_1, \cdots, k_n = -\infty}^{\infty} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$ with rapidly decreasing coefficients $\{a_{k_1 \cdots k_n}\} \in \mathcal{S}(\mathbf{Z}^n)$.

The algebra $\mathbf{T}_q^n$ is called the algebra of functions on the quantum n-torus. When $n = 2$ it arises in the study of the Kronecker foliation on the 2-torus. The algebra $\mathbf{T}_q^2$ was the main example in Connes's program of non-commutative differential geometry [17]. In particular he calculates the Hochschild (co)homology groups $H_*(A, A)$ ($H^*(A, A^*)$) and the corresponding (co)homology groups $H_*^{\mathrm{DR}}(A)$ ($H_{\mathrm{DR}}^*(A)$) of the non-commutative de Rham complex. These calculations can be generalized to the case $A = \mathbf{T}_q^n$ and, in principle, to the cases $A = \mathrm{Fun}(SL_q(n))$, $\mathrm{Fun}(GL_q(n))$. It is clear that we will have the same dimensions for the de Rham (co)homology groups as in the commutative case $q = 1$. However the spaces $H_*(A, A)$ and $H^*(A, A^*)$ now play the role of the spaces of quantum differential forms and quantum de Rham currents, so a nice geometrical interpretation for them is needed.

In the algebra $\mathrm{Fun}(SL_q(n))$ it is possible to define left coideals corresponding to the algebras of functions on the homogeneous spaces $SL(n)/SL(k) \times SL(n-k), k = 1, \cdots, n-1$. For instance in the $k = 1$ case consider the subalgebra in $\mathrm{Fun}(SL_q(n))$ generated by $t_{in}, S(t_{ni}), i = 1, \cdots, n$. This is a left coideal in $\mathrm{Fun}(SL_q(n))$ and it can be interpreted as the algebra $\mathrm{Fun}(SL_q(n)/SL_q(n-1))$.

Consider now real forms of the quantum group $SL_q(n)$. They are classified by $*$-involutions of the Hopf algebra $\mathrm{Fun}(SL_q(n))$. Here two possibilities occur.

1) *Case* $|q| = 1$

The corresponding $*$-involution has the form $T^* = T$ and leads to the quantum group $SL_q(n, \mathbf{R})$. The analogous involution $x_i^* = x_i, i = 1 \cdots, n$, turns $\mathbf{C}_q^n$ into $\mathbf{R}_q^n$ so that the quantum group $SL_q(n, \mathbf{R})$ "acts" on the quantum n-dimensional real space $\mathbf{R}_q^n$. One can also define a $*$-subalgebra in $\mathrm{Fun}(SL_q(n, \mathbf{R}))$ generated by $\sum_{k=1}^n t_{ik}t_{jk}$, $i, j = 1, \cdots, n$. It is a left coideal and can be interpreted as the algebra of functions on the homogeneous space of rank $n-1$. In the case $n = 2$ we simply obtain the quantum Lobachevsky plane. Natural question is to define quantum Fuchsian groups.

2) *Case* $q \in \mathbf{R}$

The corresponding $*$-involution has the form $T^* = US(T)^t U^{-1}$, where U is a diagonal matrix satisfying $U^2 = I$. Setting $U = I$ we obtain the quantum group $SU_q(n)$ — a compact form of $SL_q(n)$. When $n = 2$ the group $SU_q(2)$ was introduced by S. Woronowicz in [5–6]. In this case there is another choice of matrix $U = \mathrm{diag}(1, -1)$ leading to the quantum group $SU_q(1,1)$. Note that for $q \neq \pm 1$ quantum groups $SL_q(2, \mathbf{R})$ and $SU_q(1,1)$ are defined for different domains of q and are non isomorphic. This illustrates again how quantization removes degeneracy.

§4. Quantum groups $SO_q(n)$ and $Sp_q(n)$

The corresponding matrix $R = R_q$ is of order $N^2 \times N^2$, where $N = 2n+1$ for B_n type and $N = 2n$ for C_n, D_n types and has the following form

$$\begin{aligned} R_q = q \sum_{\substack{i=1 \\ i \neq i'}}^{N} e_{ii} \otimes e_{ii} + e_{\frac{N+1}{2}, \frac{N+1}{2}} \otimes e_{\frac{N+1}{2}, \frac{N+1}{2}} + \sum_{\substack{i,j=1 \\ i \neq j, j'}}^{N} e_{ii} \otimes e_{jj} \\ + q^{-1} \sum_{\substack{i=1 \\ i \neq i'}}^{N} e_{i'i'} \otimes e_{ii} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i > j}}^{N} e_{ij} \otimes e_{ji} \\ - (q - q^{-1}) \sum_{\substack{i,j=1 \\ i > j}}^{N} q^{\rho_i - \rho_j} \varepsilon_i \varepsilon_j e_{ij} \otimes e_{i'j'}, \qquad q \in \mathbf{C} \setminus \{0\}, \end{aligned}$$

where the second term is present only for the type B_n. Here $e_{ij} \in M_N(\mathbf{C})$ are matrix units, $i' = N+1-i$, $j' = N+1-j$, $\varepsilon_i = 1$, $i =$

$1, \cdots, N$, for types B_n, D_n, $\varepsilon_i = 1$, $i = 1, \cdots, \frac{N}{2}$, $\varepsilon_i = -1$, $i = \frac{N}{2} + 1, \cdots, N$, for type C_n and

$$(\rho_1, \cdots, \rho_N) = \begin{cases} (n-\frac{1}{2}, n-\frac{3}{2}, \cdots, \frac{1}{2}, 0, -\frac{1}{2}, \cdots, \frac{1}{2}-n) & \text{for the type } B_n, \\ (n, n-1, \cdots, 1, -1, \cdots, -n) & \text{for the type } C_n, \\ (n-1, n-2, \cdots, 1, 0, 0, -1, \cdots, 1-n) & \text{for the type } D_n. \end{cases}$$

The matrix $\hat{R}_q = PR_q$ enters in local representations of the Birman-Wenzl-Murakami algebra [21].

The matrix R_q satisfies the relations

$$R_q = C_1 (R_q^{t_1})^{-1} C_1^{-1} = C_2 (R_q^{-1})^{t_2} C_2^{-1},$$

where for the matrices in the tensor product $\mathbf{C}^N \otimes \mathbf{C}^N$ the symbols t_1 and t_2 stand respectively for the transposition in the first and second factors. Here $C_1 = C \otimes I, C_2 = I \otimes C$ and $C = q^\rho C_0$, where $\rho = \mathrm{diag}(\rho_1, \cdots, \rho_N)$ and $(C_0)_{ij} = \varepsilon_i \delta_{i'j}$, $i, j = 1, \cdots, N$, so that $C^2 = \varepsilon I$ with $\varepsilon = 1$ for types B_n, D_n and $\varepsilon = -1$ for type C_n.

These properties of the matrix R_q suggest the following

Definition 6. The quotient algebra $\mathrm{Fun}(G_q)$ of the algebra A_q by the relations

$$TCT^tC^{-1} = CT^tC^{-1}T = I$$

is called the algebra of functions either on the quantum group $G_q = SO_q(N)$ if the matrix R_q corresponds to types B_n, D_n or on the quantum group $G_q = Sp_q(n)$ if the matrix R_q corresponds to type C_n.

Theorem 2. *The algebras* $\mathrm{Fun}(SO_q(N))$ *and* $\mathrm{Fun}(Sp_q(n))$ *are Hopf algebras with the standard coproduct* Δ*, counit* ε *and with antipode* S *given by*

$$S(T) = CT^tC^{-1}.$$

It has the property

$$S^2(T) = (CC^t)T(CC^t)^{-1}.$$

The proof is clear.

Consider now the orthogonal case and set in Definition 2 the matrix R to be R_q and $f(t) = t^2 - (q + q^{1-N})t + q^{2-N}$. We arrive at

Definition 7. The algebra $O_q^N(\mathbf{C})$ with generators $x_1, \cdots, x_N$ and relations

$$x_i x_j = q x_j x_i, \quad 1 \le i < j \le N, \ i \ne j',$$

$$x_{i'} x_i = x_i x_{i'} + \frac{q^N - q^{N-2}}{1+q^{N-2}} \sum_{j=1}^{i'-1} q^{\rho_{i'} - \rho_j} x_j x_{j'}$$

$$- \frac{q^2 - 1}{1 + q^{N-2}} \sum_{j=i'}^{N} q^{\rho_{i'} - \rho_j} x_j x_{j'}, \quad 1 \le i < i' \le N,$$

is called the algebra of functions on the quantum n-dimensional Euclidean space.

It is not difficult to see that the Poincaré series for the algebra $O_q^N(\mathbf{C})$ are the same as for the commutative algebra $\mathbf{C}[x_1, \cdots, x_N]$. Moreover the element

$$x^t C x = \sum_{i,j=1}^{N} x_i C_{ij} x_j = \sum_{j=1}^{N} q^{-\rho_j} x_j x_{j'}$$

is central and has the property

$$\delta(x^t C x) = 1 \otimes x^t C x.$$

In other words, the "action" δ preserves the quadratic form $x^t C x$.

In the symplectic case, setting $f(t) = t - q$ we arrive at

Definition 8. The algebra $Sp_q^{2n}(\mathbf{C})$ with generators $x_1, \cdots, x_{2n}$ and relations

$$x_i x_j = q x_j x_i, \quad 1 \le i < j \le N = 2n, \ i \ne j',$$

$$x_{i'} x_i = x_i x_{i'} + (q^2 - 1) \sum_{j=1}^{i'-1} \varepsilon_{i'} \varepsilon_j q^{\rho_{i'} - \rho_j} x_j x_{j'}, \quad 1 \le i < i' \le 2n,$$

is called the algebra of functions on the quantum $2n$-dimensional symplectic space.

In the algebra $Sp_q^{2n}(\mathbf{C})$ the following equality holds:

$$x^t C x = \sum_{i=1}^{2n} q^{-\rho_i} \varepsilon_i x_i x_{i'} = 0$$

and the "action" δ preserves the bilinear form

$$x^t \dot{\otimes} Cx = \sum_{i=1}^{2n} q^{-\rho_i} \varepsilon_i x_i \otimes x_{i'}.$$

By this I mean that

$$m_{13}(\delta \otimes \delta)(x^t \dot{\otimes} Cx) = 1 \otimes x^t \dot{\otimes} Cx,$$

where m_{13} stands for the usual product of the factors with index 1 and 3 in the quadruple tensor product of $Sp_q^{2n}(\mathbf{C})$.

One can also define the quantum exterior algebras of the quantum Euclidean and symplectic spaces and introduce the algebras of functions on the quantum homogeneous spaces like $\mathrm{Fun}(SO_q(N)/SO_q(N-2))$ and $\mathrm{Fun}(Sp_q(n)/Sp_q(n-1))$. Let us describe real forms of these algebras instead.

1) *Case* $|q| = 1$

We have a $*$-involution $T^* = T$ defining the algebras $\mathrm{Fun}(Sp_q(n, \mathbf{R}))$ and $\mathrm{Fun}(SO_q(n,\ n)), \mathrm{Fun}(SO_q(n, n+1))$. However no group of the type $SO(3,1)$ appears, so we are not able to define a quantum Lorentz group. For the case of quantum symplectic space $Sp_q^{2n}(\mathbf{C})$ the involution $x_i^* = x_i$, $i = 1, \cdots, 2n$, turns it into $Sp_q^{2n}(\mathbf{R})$ and the quantum group $Sp_q(n, \mathbf{R})$ "acts" on it via δ.

2) *Case* $q \in \mathbf{R}$

We have $T^* = US(T)^t U^{-1}$, where $U = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_N)$, $\varepsilon_i^2 = 1$, $\varepsilon_i = \varepsilon_{i'}$, $i = 1, \cdots, N$, and $\varepsilon_i = 1$ for $i = i'$. In particular setting $U = I$ we obtain the quantum group $SO_q(N, \mathbf{R})$ — a compact form of $SO_q(N)$. The involution $x_i^* = q^{\rho_i} x_i$, $i = 1, \cdots, N$, turns $O_q^N(\mathbf{C})$ into $O_q^N(\mathbf{R})$ and $SO_q(N, \mathbf{R})$ "acts" on it via δ. This "action" preserves the quadratic form $x^t Cx = x^{*^t} x$ and the quotient algebra of $O_q^N(\mathbf{R})$ by the relation $x^{*^t} x = 1$ is called the quantum $N-1$-sphere S_q^{N-1}. The algebra S_q^2 was introduced in [22].

One can in principle define in analogous manner the quantum groups connected with exceptional Lie groups G_2, F_4, E_6, E_7 and E_8. The main problem here is in constructing the corresponding matrices R_q. For the G_2 case the matrix R_q was found by Reshetikhin [21].

§5. Quantum simple Lie algebras

In the classical case there is a nice way, due to Laurent Schwartz, of introducing the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. Namely, let G be a corresponding Lie group, then

$$U\mathfrak{g} = C_e^{-\infty}(G)$$

where $C_e^{-\infty}(G)$ stands for the space of distributions on G with support in the unit element e. We will introduce quantum universal enveloping algebras by a suitable generalization of this relation.

Denote by $A_R^* = \mathrm{Hom}(A_R, \mathbf{C})$ the algebraic dual of a bialgebra A_R. It naturally has the structure of bialgebra itself and can be considered as a quantum analog of the algebra $C^{-\infty}(G)$. In order to define an analog of $C_e^{-\infty}(G)$ consider the subalgebra $U_R \subset A_R^*$ generated by $l_{ij}^{(\pm)} \in A_R^*$, $i,j = 1,\cdots,n$. These elements are defined as follows. Let $L^{(\pm)} = (l_{ij}^{(\pm)})_{i,j=1}^n \in M_n(A_R^*)$ and define the matrices-functionals $L^{(\pm)}$ by their action on the graded elements of algebra A_R of degree k given by the formula

$$(L^{(\pm)}, T_1 \cdots T_k) = R_1^{(\pm)} \cdots R_k^{(\pm)}. \tag{8}$$

Here $T_i = I \otimes \cdots \otimes \underbrace{T}_{i} \otimes \cdots \otimes I \in M_{n^k}(A_R)$, $i = 1,\cdots,k$, and the matrices $R_i^{(\pm)} \in M_{n^{k+1}}(\mathbf{C})$ act nontrivially only in the factors with indices 0 and i in the tensor product $\underbrace{\mathbf{C}^n \otimes \cdots \otimes \mathbf{C}^n}_{k+1}$ and coincide there with the matrices

$$R^{(+)} = PRP, \qquad R^{(-)} = R^{-1}.$$

When $k = 0$ the RHS of (8) equals to I. Note that due to the Yang-Baxter equation, the action (8) is compatible with relations (5) in the algebra A_R. The subalgebra U_R ia called the algebra of regular functionals on A_R. Thus we see that the Yang-Baxter equation is a necessary ingredient in defining the algebra of regular functionals.

We have

Proposition 4. i) *In the algebra U_R the following relations take place:*

$$R^{(+)} L_1^{(\varepsilon)} L_2^{(\varepsilon)} = L_2^{(\varepsilon)} L_1^{(\varepsilon)} R^{(+)}, \tag{9}$$

and

$$R^{(+)}L_1^{(+)}L_2^{(-)} = L_2^{(-)}L_1^{(+)}R^{(+)}, \tag{10}$$

where $\varepsilon = \pm$, $L_1^{(\varepsilon)} = L^{(\varepsilon)} \otimes I$, $L_2^{(\varepsilon)} = I \otimes L^{(\varepsilon)} \in M_{n^2}(U_R)$.

ii) *The algebra* U_R *is a bialgebra with coproduct* Δ

$$\Delta(L^{(\varepsilon)}) = L^{(\varepsilon)}\dot{\otimes}L^{(\varepsilon)}, \quad \varepsilon = \pm.$$

The proof is clear.

Consider now the case when the matrix R corresponds to the classical types of simple Lie algebras and set $R = cR_q$, where $c = q^{-\frac{1}{n}}$ for type A_{n-1} and $c = 1$ for types B_n, C_n, D_n. In this normalization $\det R = 1$. Since the matrix R_q is lower-triangular it follows from (8) that the matrices $L^{(+)}$ and $L^{(-)}$ are respectively upper and lower triangular. Moreover, we have $l_{ii}^{(+)}l_{ii}^{(-)} = l_{ii}^{(-)}l_{ii}^{(+)} = 1$, $i = 1, \cdots, N$, and from the condition $\det R = 1$ it follows that $l_{11}^{(+)} \cdots l_{NN}^{(+)} = 1$. Let G be a Lie group of classical type. Denote by G_q the corresponding quantum group constructed in Sec.3–4, and let S_q denote the antipode in the Hopf algebra $\mathrm{Fun}(G_q)$. It is not difficult to prove the following

Proposition 5. *In the case* $R = cR_q$ *the algebra* U_R *is a Hopf subalgebra in* $\mathrm{Fun}(G_q)^*$ *with the antipode* S *given by*

$$S(L^{(\pm)}) = S_{q^{-1}}(L^{(\pm)}). \tag{11}$$

The restrictions on elements $l_{ij}^{(\pm)}$ and relations (9),(10) mentioned above completely determine the algebra U_R for the case A_{n-1}. In the case B_n, C_n and D_n one should also add the relations

$$L^{(\pm)}C^tL^{(\pm)^t}(C^{-1})^t = C^tL^{(\pm)^t}(C^{-1})^tL^{(\pm)} = I.$$

Thus in all cases the Hopf algebra U_R has the same number of generators as $\mathrm{Fun}(G_q)$.

Now let $\mathfrak{g}$ be the simple Lie algebra of rank r, corresponding to the Lie group G, $\alpha_1, \cdots, \alpha_r$ be its simple roots and $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, $i,j = 1, \cdots, r$ be its Cartan matrix, where (,) stands for the invariant scalar product. V. Drinfeld [9–10] and M. Jimbo [13–14] introduced the quantum enveloping algebra $U_h\mathfrak{g}$, $h \in \mathbf{C}$, of the Lie algebra $\mathfrak{g}$ as the $\mathbf{C}[[h]]$-

algebra with generators $H_i, X_i^{\pm}$, $i=1,\cdots,r$, and relations

$$(12)\qquad \begin{aligned} &[H_i,H_j]=0, \qquad [H_i,X_j^{\pm}]=\pm(\alpha_i,\alpha_j)X_j^{\pm},\\ &[X_i^+,X_j^-]=\frac{e^{hH_i}-e^{-hH_i}}{e^h-e^{-h}}\delta_{ij}, \qquad i,j=1,\cdots,r, \end{aligned}$$

and

$$(13)\quad \sum_{k=0}^{m}(-1)^k\binom{m}{k}_{q_i} q_i^{-\frac{k(m-k)}{2}}(X_i^{\pm})^k X_j^{\pm}(X_i^{\pm})^{m-k}=0 \qquad \text{if} \quad i\neq j,$$

where $m=1-A_{ij}$, $q_i=e^{h(\alpha_i,\alpha_i)}$ and

$$\binom{m}{k}_q=\frac{(q^m-1)\cdots(q^{m-k+1}-1)}{(q^k-1)\cdots(q-1)}.$$

When $h\to 0$, the algebra $U_h\mathfrak{g}$ goes into the universal enveloping algebra $U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$.

The generators $H_i, X_i^{\pm}$, $i=1,\cdots,r$, play the role of a quantum analog of the Chevalley basis. In the case $\mathfrak{g}=\mathfrak{sl}(2)$ the algebra $U_h\mathfrak{g}$ coincides with the algebra U_h from Example 2.

The algebra $U_h\mathfrak{g}$ is a Hopf algebra with coproduct Δ

$$(14)\qquad \begin{aligned} &\Delta(H_i)=H_i\otimes 1+1\otimes H_i,\\ &\Delta(X_i^{\pm})=X_i^{\pm}\otimes e^{-\frac{hH_i}{2}}+e^{\frac{hH_i}{2}}\otimes X_i^{\pm} \end{aligned}$$

and antipode S

$$(15)\qquad S(H_i)=-H_i, \qquad S(X_i^{\pm})=-e^{-h\rho}X_i^{\pm}e^{h\rho}, \quad i=1,\cdots,r.$$

Here $\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}H_\alpha$, where Δ_+ is the set of positive roots and for $\alpha=\sum_{i=1}^r n_i\alpha_i$ we set $H_\alpha=\sum_{i=1}^r n_iH_i$.

Theorem 3. *Let $R=cR_q$, where the matrix R_q is attached to the simple Lie algebra $\mathfrak{g}$, and set $q=e^h$. Then for a certain completion $\overline{U}_R$ of the algebra U_R we have*

$$\overline{U}_R\cong U_h\mathfrak{g}.$$

Thus we have managed to introduce quantum enveloping algebras of simple Lie algebras using exclusively formulas (5) and (6). In this approach complicated relations (13) for the quantum Chevalley generators and formulas (14),(15) follow from the elementary formulas (9)–(11).

The isomorphism in Theorem 3 can be written down explicitly. For instance, in the simplest case $n = 2$ we have

$$L^{(+)} = \begin{pmatrix} e^{\frac{hH}{2}} & (q - q^{-1})X^+ \\ 0 & e^{-\frac{hH}{2}} \end{pmatrix},$$

$$L^{(-)} = \begin{pmatrix} e^{-\frac{hH}{2}} & 0 \\ -(q - q^{-1})X^- & e^{\frac{hH}{2}} \end{pmatrix},$$

where H and $X^{\pm}$ are generators of the algebra U_h from Example 2.

Summing up we can say that our way of defining quantum universal enveloping algebras is more geometrical than the methods of [9,10,13,14]. In particular, the generators $l_{ij}^{(\pm)}$ play the role of a quantum analog of the Cartan-Weyl basis. Often this basis is more useful than Chevalley basis. Thus for instance in its terms one can write rather simple formulas for the quantum Casimir operators — generators of the center of the algebra U_R. Namely, we have

Theorem 4. *For generic q the center of the algebra U_R is generated by*

$$c_k = \mathrm{tr}(q^{2\rho}(L^{(+)}S(L^{(-)}))^k), \qquad k = 1, \cdots, r.$$

At this point it is appropriate to end this introduction to quantum groups. Before posing some interesting (from my point of view) open problems I would like to mention other subjects which I was unable to cover in this lecture.

a) One can play more with the algebras A_R and U_R for the general matrix R satisfying the Yang-Baxter equation. In particular, there exists a procedure of making them Hopf algebras (see [16]).

b) There exists a natural interpretation (see [16]) of the constructions of Sect.3–5 in terms of the quantum double, a concept introduced by V. Drinfeld [10].

c) In a similar manner one can define quantum loop groups and algebras (see [15]). The problem of defining quantum Kac-Moody algebras is more complicated. V. Drinfeld introduced them in [10] by a rather complicated system of generators and relations. However, recently N. Reshetikhin and M. Semenov-Tian-Shansky found that it is possible to define quantum Kac-Moody algebras along the lines presented in this lecture.

d) Here I have said nothing about the representation theory of quantum groups (by these one should understand corepresentations of $\mathrm{Fun}(G_q)$ or representations of $U_h\mathfrak{g}$; representations of $\mathrm{Fun}(G_q)$ correspond to the problem of classifying quantum Lax operators in the QISM formalism). Surely this subject is very important and rapidly developing. Now we know a lot about representations of quantum groups $SU_q(2)$ and $SU_q(1,1)$ due to the work of S. Woronowicz [5], L. Vaksman and Ya. Soibelman [23], T. Masuda. K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno [24–25], A. Kirillov and N. Reshetikhin [26]. In these papers q-analysis and q-special functions naturally enter the game. Finite dimensional representations of $U_h\mathfrak{g}$ were studied by G. Luztig [27] and M. Rosso [28]. It seems that general considerations presented in the lecture could be useful in the construction of realizations of representations of quantum groups and in treating quantum groups of higher rank.

e) There exists by now well-known connection between quantum groups and braid representations on the one hand and invariants of links on the other hand. I will mention only the work of N. Reshetikhin [21,29] and references therein and Deguchi's talk at this conference.

§6. Problems

1) Intrinsic definition of quantum groups is needed. One can imagine the following analogy. Suppose that one knows nothing about Lie algebras and tries to find all solutions of the Jacobi identity written in terms of the structure constants c_{ij}^k. Then he (or she) eventually discovers intrinsic definition of Lie algebras and classification theorems for them. Since the Yang-Baxter equation plays the role of quantum Jacobi identity, intrinsic definition and classification of quantum groups will give us a list of all solutions of this equation. More seriously we need to define a proper category of quantum groups we are working with.

2) Why do we have a one-parameter continuous family G_q of quantum groups starting from the simple Lie group G ? There should be some cohomology theory for $\mathrm{Fun}(G)$ describing these deformations and having the property that its H^2 group is one-dimensional. It seems there exists an analogy with the approach of A. Lichnerowicz, M. Flato, D. Sternheimer and others to quantization procedure as deformation of symplectic structure. This problem is under consideration now.

3) There are several problems in representation theory. How can one construct models for representations of quantum groups ? What is a quantum method of orbits and a quantum analog of the Borel-Weil-Bott theorem ?

4) Quantum differential geometry. At present we have here only problems. I will mention only one: how can one in addition to the quantum de Rham complex define quantum Dolbeaut complex and an analog of Hodge theory. Certainly this question is also interesting for the general approach of A. Connes to non-commutative differential geometry.

5) Applications to the integrable models of quantum field theory and statistical mechanics. It should be stressed that quantum groups are a rather small selection from the rich structures of QISM. Certainly all constructions of QISM, even the most complicated technically, should have some quantum group meaning; what remains is to reveal it. Another possible application is the "Virasoro puzzle" (called so by J. Cardy in his lecture at Katata), where for certain models away from critical point Virasoro characters nevertheless do appear. Possible explanation is that in this case the model has some deformed Virasoro symmetry; since the characters are a kind of Poincaré series they are not deformed and this is the reason of appearance of true Virasoro characters. Of course in realizing this program one should first define a quantum Virasoro algebra.

6) It seems that this list of problems has a tendency to be infinite. So I will end with the most fantastic possible applications to arithmetic algebraic geometry. Could it be an idea that quantum groups play an interpolations role in the geometry of varieties defined over number fields ?

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