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A New Family of Solvable Lattice Models Associated with $A_n^{(1)}$

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Abstract.

Presented is a new family of solvable solid-on-solid models in two-dimensional statistical mechanics. The site variables take the values in a set of not necessarily dominant integral weights of the affine Lie algebra $A_n^{(1)}$. The local state probabilities are obtained and the critical behavior is studied. Our family gives an extension of some recently discovered hierarchies of solvable solid-on-solid models.

§1. Introduction

In this paper we present a new family of solvable lattice models associated with the affine Lie algebra $A_n^{(1)}$. They are unrestricted and restricted solid-on-solid (SOS) models formulated as Interaction Round a Face (IRF) models [1] in two-dimensional statistical mechanics.

Before going into the details we like to devote the section 1.1 to explaining what is meant by the association of Lie algebras with lattice models and what is its significance. The subject is known to involve several topics which are closely interrelated; classification of nondegenerate classical r-matrices, construction of the corresponding quantum R-matrices, an intertwiner relating a class of vertex models to face models, computation of 1 point functions for both kinds of models and so forth. Here we prefer to be selective and aim the elemetary guide and review that lead to a motivation of the present work. We hope that it serves as a glance to the theory of solvable lattice models which is growing to a more and more important subject in mathematical physics. The description of our new model will start from section 1.2.

1.1. A little about the background

Let us begin by demonstrating the connection between solvable lattice models and Lie algebras taking the 8-vertex solid-on-solid (8VSOS)

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by Andrews, Baxter and Forrester [2] as an example. Fix an integer $L \geq 4$. We consider two-dimensional square lattice $\mathcal{L}$ with a site variable a_i attached to each site i. We call a_i *local state* or simply *state* and assume that it belongs to a finite set

$$\mathcal{S} = \{1, 2, \cdots, L-1\}. \tag{1.1a}$$

We impose the admissibility condition on the pair (a, b) of the local state a and its right/lower neighbor b as follows.

$$b = a - 1 \text{ or } a + 1. \tag{1.1b}$$

We call such an ordered pair (a, b) of the states *admissible*. Thus for example $L = 4$, the bond configurations in Fig. 1 are admissible while those in Fig.2 are not.

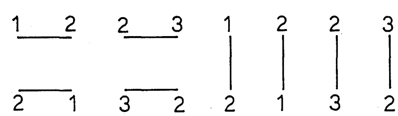


Fig. 1. The allowed bond configurations.

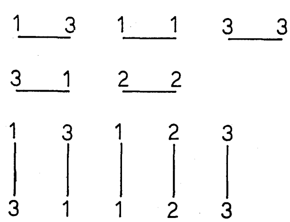


Fig. 2. The forbidden bond configurations.

The elementary interaction is given by the Boltzmann weight $W\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ depending on the state configuration $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ round the face. Here the four states a, b, c, d are ordered clockwise from the NW corner. We set $W\begin{pmatrix} a & b \\ d & c \end{pmatrix} = 0$ unless the four pairs $(a,b),(b,c),(a,d)$ and (d,c) are admissible. The finite set $\mathcal{S}$ of the local states and the admissibility condition on adjacent states are basic data for restricted face models. They are essentially reflected in the form of the Yang-Baxter equation (YBE) for the associated model:

$$\tag{1.2} \begin{aligned} &\sum_g W_{u+v}\begin{pmatrix} a & b \\ f & g \end{pmatrix} W_u\begin{pmatrix} f & g \\ e & d \end{pmatrix} W_v\begin{pmatrix} b & c \\ g & d \end{pmatrix} \\ &= \sum_g W_{u+v}\begin{pmatrix} g & c \\ e & d \end{pmatrix} W_u\begin{pmatrix} a & b \\ g & c \end{pmatrix} W_v\begin{pmatrix} a & g \\ f & e \end{pmatrix}. \end{aligned}$$

Here $a,b,c,d,e,f,g \in \mathcal{S}$ and the Boltzmann weights are assumed to be the functions of the *spectral parameter* $u \in \mathbf{C}$. In our case of the 8VSOS model, the basic data (1.1) has the nice feature that the YBE (1.2) allows the following neat solution in terms of elliptic theta functions:

$$\tag{1.3a} W_u\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{pmatrix} = \frac{H(1+u)}{H(1)},$$

$$\tag{1.3b} W_u\begin{pmatrix} a & a\pm 1 \\ a\mp 1 & a \end{pmatrix} = \frac{\sqrt{H(a+1)H(a-1)}}{H(a)}\frac{H(u)}{H(1)},$$

$$\tag{1.3c} W_u\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a \end{pmatrix} = \frac{H(a\mp u)}{H(a)},$$

$$\tag{1.3d} H(u) = 2|p|^{1/8}\sin\frac{\pi u}{L}\prod_{k=1}^{\infty}(1-2p^k\cos\frac{2\pi u}{L}+p^{2k})(1-p^k),$$

where $|p| < 1$. Using this parametrization one can compute several physical quantities, e.g., the free energy and the 1 point functions, etc. Thus it would be significant to look for an intrinsic meaning of the basic data (1.1), i.e., the set of local states and their admissibility conditions leading to "good" solvability.

Now consider the classical simple Lie algebra $sl(2,\mathbf{C})$ generated by the elements e, f, h under the relations:

$$[e,h] = -2e, \quad [f,h] = 2f, \quad [e,f] = h.$$

For $a \in \mathbf{Z}_{>0}$, we denote by V_a its a-dimensional irreducible module ("spin" $\frac{a-1}{2}$ representation) generated from the normalized highest weight vector v_a $(hv_a = (a-1)v_a, ev_a = 0, |v_a|^2 = (v_a,v_a) = 1)$. Then the following is the most familiar example of the irreducible decomposition of a tensor product:

$$V_a \otimes V_2 = V_{a-1} \oplus V_{a+1}. \tag{1.4}$$

Our viewpoint for the admissibility condition (1.1b) is essentially to interpret the allowed b therein as the indices appearing in the RHS of (1.4). In order to justify such picture, let us reconstruct the solution (1.3) in the *rational limit* $L \to \infty$ where $H(x)$ is simply replaced by x.

We first consider the normalized highest weight vector of V_{a-1} (resp. V_{a+1}) in the RHS of (1.4) which we denote by v_{aa-1} (resp. v_{aa+1}). They are easily determined in terms of the Wigner coefficients by requiring $|v_{aa\pm 1}|^2 = 1$ and $\Delta^{(2)}(e)v_{aa\pm 1} = 0$. (The diagonal action $\Delta^{(N)}(k) \in \mathrm{End}(V_a \otimes V_2^{\otimes N-1})$ of $k \in sl(2,\mathbf{C})$ is defined by $\Delta^{(N)}(k) = \sum_{j=1}^{N} 1 \otimes \cdots \otimes \overbrace{k}^{j-\mathrm{th}} \otimes \cdots \otimes 1$.) Thus we have

$$\begin{aligned} v_{aa+1} &= v_a \otimes v_2, \\ v_{aa-1} &= \frac{1}{\sqrt{a(a-1)}} f v_a \otimes v_2 - \sqrt{\frac{a-1}{a}} v_a \otimes f v_2. \end{aligned} \tag{1.5}$$

Using this twice we get

$$V_a \otimes V_2 \otimes V_2 = \sum_{c=a,a\pm 2} \Omega_{ac} \otimes V_c, \tag{1.6}$$

where Ω_{ac} is the space of highest weight vectors which we now denote by v_{abc} with $(a,b),(b,c)$ admissible. They have the same weight corresponding to c, i.e., $\Delta^{(3)}(h)v_{abc} = (c-1)v_{abc}$. Explicitly, Ω_{ac} is given as follows.

$$\begin{aligned} \Omega_{aa+2} &= \mathbf{C}v_{aa+1a+2}, \\ \Omega_{aa} &= \mathbf{C}v_{aa-1a} \oplus \mathbf{C}v_{aa+1a}, \\ \Omega_{aa-2} &= \mathbf{C}v_{aa-1a-2}, \end{aligned} \tag{1.7a}$$

where the orthonormal highest weight vectors read as

(1.7b)

$$v_{aa+1a+2} = v_a \otimes v_2 \otimes v_2,$$

$$v_{aa-1a} = \frac{1}{\sqrt{a(a-1)}}\big(fv_a \otimes v_2 \otimes v_2 - (a-1)v_a \otimes fv_2 \otimes v_2\big),$$

$$v_{aa+1a} = \frac{1}{\sqrt{a(a+1)}}\big(fv_a \otimes v_2 \otimes v_2 + v_a \otimes fv_2 \otimes v_2 - av_a \otimes v_2 \otimes fv_2\big),$$

$$v_{aa-1a-2} = \frac{1}{\sqrt{a(a-1)^2(a-2)}}\big(f^2 v_a \otimes v_2 \otimes v_2 - (a-2)fv_a \otimes fv_2 \otimes v_2 - (a-2)fv_a \otimes v_2 \otimes fv_2 + (a-2)(a-1)v_a \otimes fv_2 \otimes fv_2\big).$$

Next we introduce an operator $1 \otimes R(u)$ $(u \in \mathbf{C})$ acting on the space (1.6), where $R(u) \in \mathrm{End}_{sl(2,\mathbf{C})}(V_2 \otimes V_2)$ is defined as $R(u) = 1 + uP$ with P being the transposition $P(x \otimes y) = y \otimes x$. In view of the fact that $R(u)$ belongs to the commutant of $sl(2,\mathbf{C})$, we regard $1 \otimes R(u)$ as acting on Ω_{ac}. Then the rational limit $W_u^{(0)}\begin{pmatrix} a & b \\ d & c \end{pmatrix} = \lim_{L\to\infty} W_u \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ of our Boltzmann weights (1.3) can be recovered by considering the $1 \otimes R(u)$ action on the base vectors v_{abc} :

$$1 \otimes R(u)|_{\Omega_{ac}} v_{abc} = \sum_d W_u^{(0)} \begin{pmatrix} a & b \\ d & c \end{pmatrix} v_{adc}, \tag{1.8a}$$

or equivalently

$$W_u^{(0)} \begin{pmatrix} a & b \\ d & c \end{pmatrix} = \big(v_{adc}, 1 \otimes R(u)|_{\Omega_{ac}} v_{abc}\big). \tag{1.8b}$$

Moreover, the YBE for the weights $W_u^{(0)} \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ is a consequence of that for $R(u)$ valid in $\mathrm{End}(V_2^{\otimes 3})$:

(1.9)

$$\big(1\otimes R(u)\big)\big(R(u+v)\otimes 1\big)\big(1\otimes R(v)\big) = \big(R(v)\otimes 1\big)\big(1\otimes R(u+v)\big)\big(R(u)\otimes 1\big).$$

Namely, by letting the identity $1 \otimes (1.9)$ act on $V_a \otimes V_2^{\otimes 3}$ and noting that the identity holds in the commutant of $sl(2,\mathbf{C})$, we get (1.2) for $W_u^{(0)}$'s. We note that this kind of intertwiner relating R-matrices to Boltzmann weights of face models has been described in more general setting in [3] and appendix B of [4]. In particular, the trigonometric case ($p \to 0$) can be treated similarly [3] by employing the representation theory of quantized universal enveloping algebra $U_q(sl(2,\mathbf{C}))$ [5].

So far we have shown that there is at least a way to understand the admissibility condition (1.1b) if one brings the classical simple Lie algebra $sl(2,\mathbf{C})$ into the game together with its irreducible representations. We have explicitly derived the Boltzmann weights (1.3) in the limit $L \to \infty$ by making the vector representation (the one corresponding to V_2 in (1.4), (1.6)) play a distinct role. Through the procedure the local state a appeared as the index labelling the highest weight module V_a of $sl(2,\mathbf{C})$. The meaning of this as well as the remaining data L in (1.1a) becomes clearer in the calculation of the *local state probabilities* (LSPs). In fact, the explanation below implies that the model is associated with the *affine Lie algebra* $A_1^{(1)} = \widehat{sl}(2,\mathbf{C})$.

Let Λ_0, Λ_1 denote the fundamental weights of $A_1^{(1)}$. For $\ell \in \mathbf{Z}_{>0}$, we put $P_+(\ell) = \{(\ell - j)\Lambda_0 + j\Lambda_1 \mid j = 0, 1, \cdots, \ell\}$. This is the set of level ℓ dominant integral weight. As it turns out, the LSP result ($0 < p < 1, -1 < u < 0$) in [2] is most simply stated by identifying a state $a \in \mathcal{S}$ with $(L-1-a)\Lambda_0 + (a-1)\Lambda_1 \in P_+(L-2)$. Let $L(\xi)$, $L(\eta)$ be the irreducible highest weight $A_1^{(1)}$ modules with the highest weights $\xi \in P_+(L-3), \eta \in P_+(1)$. We write the character of $L(\xi)$ as χ_ξ, etc. The following is the character identity describing the affine version of (1.4), i.e., the irreducible decomposion of $A_1^{(1)} \oplus A_1^{(1)}$ module $L(\xi) \otimes L(\eta)$ with respect to the diagonal subalgebra.

$$\chi_\xi \chi_\eta = \sum_{a \in P_+(L-2)} b_{\xi\eta a} \chi_a. \tag{1.10}$$

Here the quantity $b_{\xi\eta a}$ is the branching coefficient for the pair $(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$ with the levels $(L-3)+1 = L-2$. It is the irreducible character of Virasoro algebra having the central charge $c = 1 - 6/L(L-1)$. The LSP P_a itself follows from (1.10) by the principal specialization (denoted by "sp"):

$$P_a = \frac{b_{\xi\eta a}\chi_a^{sp}}{\chi_\xi^{sp}\chi_\eta^{sp}}. \tag{1.11}$$

In the actual computation, the Virasoro character $b_{\xi\eta a}$ arises essentially as the trace of the corner transfer matrix [1] and the choice of ξ, η is in one-to-one correspondence with the boundary conditions (or ground states). Note that the above identification of $\mathcal{S}$ with $P_+(L-2)$ meets the requirement $\sum_{a\in\mathcal{S}} P_a = 1$. Thus the LSP calculation fits the Goddard-Kent-Olive (GKO) construction [6] of Virasoro modules using the pair $(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$. Moreover, the highest weight of $V_a\,(a \in \mathcal{S})$ is equal to the *classical part* $\overline{a}$ of the corresponding dominant integral weight $a \in P_+(L-2)$.

To summarize, we refine the basic data (1.1) for the 8VSOS model to a triad $(A_1^{(1)}, 2, L-2)$: associated affine Lie algebra $A_1^{(1)}$, the vector representation of its classical part (signified here by the dimensionality 2) and the level $L-2$ of the dominant integral weights corresponding to the local states.

So much for the 8VSOS model, let us go to the situation in which the above data is generalized to $(X_n^{(1)}, \pi, \ell)$. Here $X_n^{(1)}$ is an affinization of the classical simple Lie algebra X_n, (π, V_π) is understood as an irreducible representation of the latter and ℓ is a positive integer not too small. The corresponding restricted face model will have the following features [7].

(i) The local states take their values in the finite set of level ℓ dominant integral weights of $X_n^{(1)}$.

(ii) A pair (a, b) of the local states is admissible (b is allowed to occupy the right/lower neighbor of a) if and only if

$$\begin{aligned} &[\,V\big(\overline{\sigma(a)}\big) \otimes V_\pi \,:\, V\big(\overline{\sigma(b)}\big)\,] > 0 \\ &\text{for any Dynkin diagram automorphism } \sigma \text{ of } X_n^{(1)}. \end{aligned} \tag{1.12}$$

Here $V(\overline{a})$ for an example denotes the irreducible X_n module whose highest weight is given by the classical part $\overline{a}$ of a. The symbol [:] stands for the multiplicity of $V\big(\overline{\sigma(b)}\big)$ occuring in the irreducible decomposition of $V\big(\overline{\sigma(a)}\big) \otimes V_\pi$. To be precise, these are not a complete characterization of a model. In some cases there are more than one independent solutions to the YBE, which we do not discuss here. We remark that the condition (1.12) also appeared in [8] to describe the fusion rule of the vertex operators in conformal field theories.

Numerous (elliptic) restricted face models appeared in preceding works are classified according to the above scheme. (For $X_n^{(1)} = A_1^{(1)}$, we specify the irreducible representation π of A_1 by its dimensionality k.) Ising [9] and the hard hexagon [10] models can be viewed as $(A_1^{(1)}, 2, 2)$ and $(A_1^{(1)}, 2, 3)$ models, respectively, which form the first two of the 8VSOS hierarchy $(A_1^{(1)}, 2, L-2)\,(L \geq 4)$. Several works [11–15] were done to extend the models and the LSP calculations to the full $A_1^{(1)}$

family $(A_1^{(1)}, N+1, L-2)\,(N \geq 1, L \geq N+3)$ until the complete result was obtained in [16,17]. The LSP was determined as in (1.11) by using the same GKO pair but with the levels $(L-N-2)+N = L-2$. Application of these calculations to the representation theory of affine Lie algebras [18] is an outcome of further research. In the case $\pi =$ the vector representation, the decomposition (1.12) for $X_n^{(1)} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ is multiplicity free and the associated models $(X_n^{(1)}, \pi, \text{general})$ have been studied in [19] $(X_n^{(1)} = A_n^{(1)})$ and in [7] [20] $(X_n^{(1)} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)})$. Various GKO pairs arise in the description of their LSPs. Construction of the full $A_n^{(1)}$ face model $(A_n^{(1)}, \text{general}, \text{general})$ is attained in [4]. Here the decomposition (1.12) is no longer multiplicity free. The corresponding new degrees of freedom can properly be taken into account by considering an IRF model involving *edge variables* as well as site variables. Their LSP is not known at present.

Now we turn to another aspect of solvable restricted face models which do *not* necessarily fall in the above classification scheme. Below we quote a curious mathematical phenomenon known as Mckay's observation [21] which may have an interplay with such solvable models introduced subsequently.

Consider the Lie group $SU(2)$ and its finite subgroups $\Gamma = \mathcal{C}_n, \mathcal{D}_n, \mathcal{T}, \mathcal{O}$ and $\mathcal{I}$. These are the cyclic group $\cong \mathbf{Z}_n$, the binary dihedral, tetrahedral, octahedral and icosahedral groups, respectively. As the names suggest, the latter four are preimages of the automorphism groups of the regular polyhedra under the double covering map $SU(2) \to SO(3)$. Let ρ denote the 2-dimensional faithful representation of $\Gamma \subset SU(2)$ (which is irreducible except for $\mathcal{C}_n$). Let $\{\rho_j \mid j = 1, 2, \cdots, r\}$ further be a set of finite dimensional irreducible representation of Γ. We consider the irreducible decomposition of the tensor product:

$$\rho_a \otimes \rho = \sum_{b=1}^{r} A_{ab}\rho_b, \tag{1.13}$$

where A is an r by r matrix with non-negative integrer elements. Then what Mckay's observation tells is that the matrices $2I_r - A$ give rise to the generarized Cartan matrices for the affine Lie algebras $A_{r-1}^{(1)}, D_{r-1}^{(1)}, E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$ according as $\Gamma = \mathcal{C}_r, \mathcal{D}_{r-3}, \mathcal{T}, \mathcal{O}$ and $\mathcal{I}$, respectively.

Let us now return to solvable lattice models having this phenomenon in mind. We consider the restricted face models whose basic data, in an analogous sense as before, are given by the tensor product decomposition (1.13). Namely, the IRF models in which

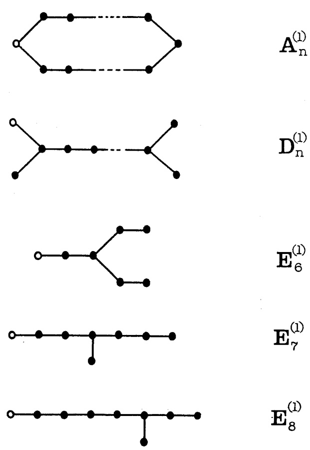


Fig. 3. The extended Dynkin diagrams for $A_n^{(1)}, D_n^{(1)}$, and $E_k^{(1)}$, $(k = 6, 7, 8)$. The classical ones for A_n, D_n, E_k are obtained by deleting the node associated to the trivial representation. Up to the symmetries of the diagram, this has been specified by an open circle.

(i) the local states take their values in a finite set of nodes on the Dynkin diagrams of types $A^{(1)}, D^{(1)}, E^{(1)}$ or A, D, E,
(ii) a pair (a, b) of the local states is admissible (allowed to occupy adjacent lattice sites) if and only if the corresponding nodes are connected on the diagram.
As for extended Dynkin diagrams $A_n^{(1)}$ $(n \geq 2)$ and $D_n^{(1)}$ $(n \geq 5)$, elliptic solutions to the YBE have been found in [22–24]. Note that the meaning of $X_n^{(1)}$ here is different from the preceding ones in the $(X_n^{(1)}, \pi, \ell)$ models. (For $D_n^{(1)}$ case, two other solutions are available as special cases of the models $(B_{\frac{n-3}{2}}^{(1)}, \pi, 2)$ $(n :$ odd $\geq 7)$ and $(D_{2n-1}^{(1)}, \pi, 2)$, $\pi =$ the vector representation.) On the other hand, models corresponding to the Dynkin diagrams of classical types A_n, D_n, E_6, E_7, E_8 are also known to be solvable [25]. Among them the cases A_n, D_n have been shown to admit elliptic parametrizations of the Boltzmann weights. Actually, (1.3) for the 8VSOS model provides such a solution with the former. These models are in a sense relatives of the 8VSOS model. Their LSPs

are evaluated by using theta function identities like (1.10) involving the $A_1^{(1)}$ characters and similar quantities [23–25]. It is not known at present whether such results admit structual understanding by means of the finite subgroups of $SU(2)$ as well as the affine Lie algebra $A_1^{(1)}$, etc. We note that similar structures have been observed in the classification of modular invariant partition functions for $A_1^{(1)}$ systems [26] in conformal field theories.

1.2. Present models

In the preceding section we briefly described a class of restricted face models basically characterized by the data $(X_n^{(1)}, \pi, \ell)$. For $X_n^{(1)} = A_1^{(1)}$, we also observed that there are some other solvable models looking like the relatives of the 8VSOS models $(A_1^{(1)}, 2, \ell)$, $(\ell \geq 2)$. The purpose of this paper is to construct this latter kind of face models for the higher rank case $(A_n^{(1)}, \pi, \text{general})$, $\pi =$ the vector representation. We will build restricted and unrestricted version of the model which contain finite or infinite number of the local states, accordingly. The unrestricted models have the following features (i)–(iii) common with those treated in [19].
(i) The site variables (local states) range over the dual space $\mathcal{H}^*$ of the Cartan subalgebra $\mathcal{H}$ of $A_n^{(1)}$.
(ii) There is a condition that the local states a and its right/lower neighbor b should be *weakly admissible* in the sense that $b - a =$ a weight of the vector representation of the classical Lie algebra A_n.
(iii) As the functions of the spectral parameter u, Boltzmann weights have elliptic parametrization satisfying the YBE.

An intriguing properties of our family emerges in the construction of restricted models from the unrestricted ones. It is possible to restrict the local states to a finite set of *not necessarily dominant integral* weights of $A_n^{(1)}$. For $n = 1$, we find that the resulting family of restricted face models reduces to a relative of the 8VSOS models described in section 1.1 (the one corresponding to the Dynkin diagram of type $A^{(1)}$). Our family contains a continuously varying parameter ζ (see section 3) in the Boltzmann weight parametrization other than u and the elliptic nome p.

Besides the construction, the studies of the case $n = 1$ [24][27] imply a rich structure of the whole family in the calculation of the LSPs. Here we execute it for general $n \geq 1$ retaining the parameter ζ. We exploit theta function identities involving $A_n^{(1)}$ characters and study the behavior of the LSPs in the vicinity of the critical point using the automorphic properties of relevant quantities. The analysis is parallel with [19] where the LSP was determined by using the GKO pair $(A_n^{(1)} \oplus A_n^{(1)}, A_n^{(1)})$ with the levels $(\ell - 1) + 1 = \ell$.

The plan of the paper goes as follows. In the next section, unrestricted models are formulated and the solution to the YBE is given. In section 3, restriction is done to a finite set of not necessarily dominant integral weights. In section 4, the LSP of the restricted model is obtained. The critical behavior is studied and the exponents, e.g. the conformal anomaly and the scaling dimensions are explicitly determined. Proofs of some mathematical identities are given in appendices A and B.

§2. Unrestricted models

Let us begin by fixing the following notations for the affine Kac-Moody algebra $A_n^{(1)}$ [28]. Let $\mathcal{H}^* = \mathbf{C}\Lambda_0 \oplus \cdots \oplus \mathbf{C}\Lambda_n \oplus \mathbf{C}\delta$ be the dual space of the Cartan subalgebra $\mathcal{H}$ of $A_n^{(1)}$ spanned by the null root δ and the fundamental weights $\Lambda_\mu (0 \le \mu \le n)$ having the classical part $\overline{\Lambda}_\mu = \Lambda_\mu - \Lambda_0$. We extend the suffixes to all integers by setting $\Lambda_\mu = \Lambda_{\mu+n+1}$ and put $\rho = \sum_{\mu=0}^n \Lambda_\mu$. The inner product on the space $\mathcal{H}^*$ is defined in terms of the orthonormal vectors ϵ_μ $(0 \le \mu \le n)$, as follows.

$$\overline{\Lambda}_\mu = \sum_{j=0}^{\mu-1} \epsilon_j - \mu\epsilon, \qquad \epsilon = \frac{1}{n+1}\sum_{\mu=0}^{n} \epsilon_\mu, \tag{2.1}$$
$$\langle \Lambda_0, \epsilon_\mu \rangle = \langle \delta, \epsilon_\mu \rangle = 0, \langle \Lambda_0, \Lambda_0 \rangle = \langle \delta, \delta \rangle = 0, \langle \Lambda_0, \delta \rangle = 1.$$

We shall also use the symbol e_μ to mean

$$e_\mu = \Lambda_{\mu+1} - \Lambda_\mu = \epsilon_\mu - \epsilon, \quad (0 \le \mu \le n), \tag{2.2}$$

and identify it with a weight of the vector representation of the classical part A_n. An element of $\mathcal{H}^*$ will be called a *state*. We call an ordered pair of the states (a, b) *weakly admissible* if and only if

$$b = a + e_\mu, \qquad \text{for some} \quad 0 \le \mu \le n. \tag{2.3}$$

For a state $a \in \mathcal{H}^*$, we put

$$\ell ev(a) = \langle a, \delta \rangle, \tag{2.4a}$$
$$a_{\mu\nu} = a_\mu - a_\nu, \tag{2.4b}$$
$$\overline{a} + \overline{\rho} = a_0 e_0 + \cdots + a_n e_n. \tag{2.4c}$$

The quantity (2.4a) is known as the *level* of a.

Now we proceed to the construction of unrestricted face model on a planar square lattice $\mathcal{L}$. To each site i of $\mathcal{L}$ we assign a state $a^{(i)} \in \mathcal{H}^*$ under the condition that the pair (a, b) of the states a and its right/lower neighbor b should be weakly admissible in the above sense. Then the local states are confined to the following subset $\hat{S}_{\ell,n}(\xi) \subset \mathcal{H}^*$ labelled by $\xi \in \mathcal{H}^*$ and $\ell \in \mathbf{C}$.

(2.5)
$$\hat{S}_{\ell,n}(\xi) = \{a + \xi |\, \ell ev(a+\xi) = \ell,\ a \in \mathbf{C}\Lambda_0 \oplus \cdots \oplus \mathbf{C}\Lambda_n,\ a_{\mu\nu} \in \mathbf{Z}, \forall \mu, \nu\}.$$

Consider a state configuration $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ round a face, where a, b, c and d are the states on the NW, NE, SE and SW corner, respectively. For such configuration we assign a non-zero Boltzmann weight $W\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ only if the pairs $(a,b), (b,c), (a,d), (d,c)$ are weakly admissible and set $W\begin{pmatrix} a & b \\ d & c \end{pmatrix} = 0$, otherwise. Our model is the two-dimensional statistical system having the partition function:

$$Z = \sum_{\text{config.}} \prod_{\text{faces}} W\begin{pmatrix} a^{(i)} & a^{(j)} \\ a^{(l)} & a^{(k)} \end{pmatrix}.$$

We introduce a spectral parameter $u \in \mathbf{C}$ and parametrize the Boltzmann weights as follows.

$$\text{(2.6a)}\quad W_u\begin{pmatrix} a & a+e_\mu \\ a+e_\mu & a+2e_\mu \end{pmatrix} = \frac{H(1+u)}{H(1)},$$

$$W_u\begin{pmatrix} a & a+e_\nu \\ a+e_\mu & a+e_\mu+e_\nu \end{pmatrix}$$

$$\text{(2.6b)}\quad = \frac{\sqrt{H(a_{\mu\nu}+1)H(a_{\mu\nu}-1)}}{H(a_{\mu\nu})}\frac{H(u)}{H(1)} \qquad \text{if } \mu-\nu \text{ is even},$$

$$\text{(2.6c)}\quad = \frac{\sqrt{\Theta(a_{\mu\nu}+1)\Theta(a_{\mu\nu}-1)}}{\Theta(a_{\mu\nu})}\frac{H(u)}{H(1)} \qquad \text{if } \mu-\nu \text{ is odd},$$

$$W_u\begin{pmatrix} a & a+e_\mu \\ a+e_\mu & a+e_\mu+e_\nu \end{pmatrix}$$

$$\text{(2.6d)}\quad = \frac{H(a_{\mu\nu}-u)}{H(a_{\mu\nu})} \qquad \text{if } \mu-\nu \text{ is even},$$

$$\text{(2.6e)}\quad = \frac{\Theta(a_{\mu\nu}-u)}{\Theta(a_{\mu\nu})} \qquad \text{if } \mu-\nu \text{ is odd},$$

where $H(u) = H(u,p)$ and $\Theta(u) = \Theta(u,p)$ are Jacobi's theta functions:

$$\text{(2.7a)} \quad H(u,p) = 2|p|^{1/8}\sin\frac{\pi u}{L}\prod_{k=1}^{\infty}(1-2p^k\cos\frac{2\pi u}{L}+p^{2k})(1-p^k),$$

$$\text{(2.7b)} \quad \Theta(u,p) = \prod_{k=1}^{\infty}(1-2p^{k-1/2}\cos\frac{2\pi u}{L}+p^{2k-1})(1-p^k).$$

In (2.6b–e) we have assumed $\mu \neq \nu$. Using the standard theta function identities one can directly show

Theorem 1. *For fixed p ($|p| < 1$) and $L(\neq 0)$, the Boltzmann weights* (2.6–7) *satisfy the Yang-Baxter equation(YBE):*

$$\text{(2.8)} \quad \begin{aligned} &\sum_g W_{u+v}\begin{pmatrix} a & b \\ f & g \end{pmatrix} W_u\begin{pmatrix} f & g \\ e & d \end{pmatrix} W_v\begin{pmatrix} b & c \\ g & d \end{pmatrix} \\ &= \sum_g W_{u+v}\begin{pmatrix} g & c \\ e & d \end{pmatrix} W_u\begin{pmatrix} a & b \\ g & c \end{pmatrix} W_v\begin{pmatrix} a & g \\ f & e \end{pmatrix}. \end{aligned}$$

The solution (2.6–7) can be related to the one obtained in [19] by changing a_μ to $a_\mu + \frac{\sqrt{-1}L\mu}{4\pi}\ln p$ and using the invariance of the YBE (2.8) under the transformation: $W_u\begin{pmatrix} a & b \\ d & c \end{pmatrix} \to e^{\langle\gamma,a+c-b-d\rangle u}W_u\begin{pmatrix} a & b \\ d & c \end{pmatrix}$, $\forall\gamma \in \mathcal{H}^*$. Using the relation one can also verify the following *inversion properties* [1] necessary for the computation of local state probabilities (LSPs).

$$\text{(2.9a)} \sum_g W_u\begin{pmatrix} a & g \\ d & c \end{pmatrix} W_{-u}\begin{pmatrix} a & b \\ g & c \end{pmatrix} = \frac{H(1+u)H(1-u)}{H(1)^2}\delta_{bd},$$

$$\text{(2.9b)} \sum_g \overline{W}_u\begin{pmatrix} a & b \\ d & g \end{pmatrix} \overline{W}_{-n-1-u}\begin{pmatrix} c & d \\ b & g \end{pmatrix} = \frac{H(u)H(-1-n-u)}{H(1)^2}\delta_{ac},$$

where $\overline{W}_u$ is defined by

$$\text{(2.10a)} \quad \overline{W}_u\begin{pmatrix} a & b \\ d & c \end{pmatrix} = \left(\frac{g_a g_c}{g_b g_d}\right)^{1/2} W_u\begin{pmatrix} a & b \\ d & c \end{pmatrix},$$

$$\text{(2.10b)} \quad g_a = \prod_{\mu<\nu,\mu-\nu=\text{even}} H(a_{\mu\nu}) \prod_{\mu<\nu,\mu-\nu=\text{odd}} \Theta(a_{\mu\nu}).$$

§3. Restricted models

From the unrestricted models in the previous section, models having finite numbers of states can be built through the *restriction*, which we shall now explain.

Consider the set of local states $\hat{\mathcal{S}}_{\ell,n}(\xi = \zeta\mathcal{T})$ (see (2.5)) with $\ell \in \mathbf{Z}_{>0}, \zeta \in \mathbf{R} \setminus \mathbf{Z}$ and the vector $\mathcal{T}$ defined by

$$\mathcal{T} = \sum_{\mu:\text{even}} e_\mu = -\sum_{\mu:\text{odd}} e_\mu. \tag{3.1}$$

We choose the parameter L in (2.7) to be an integer

$$L = n + \ell + 1. \tag{3.2}$$

In the sequel we shall consider sets of non-integral weights:

$$\mathcal{S}, \mathcal{S}(a), \mathcal{R}_{\ell,n}(\zeta\mathcal{T}) \subset \mathcal{S}_{\ell,n}(\zeta\mathcal{T}) \subset \hat{\mathcal{S}}_{\ell,n}(\zeta\mathcal{T}) \quad (a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})), \tag{3.3a}$$

$$\mathcal{S}_{\ell,n}(\zeta\mathcal{T}) = \coprod_{a\in\mathcal{S}} \mathcal{S}(a). \tag{3.3b}$$

Here $\mathcal{S}$ and $\mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ form finite sets. The local states of the restricted model will range over the former and the latter will be used in the LSP calculation. In terms of $\hat{\mathcal{S}}_{\ell,n}(\zeta\mathcal{T})(2.5)$, the sets $\mathcal{S}, \mathcal{S}(a), \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ and $\mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ are defined as follows.

$$\mathcal{S} = \{a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T}) |\ \ 0 < a_{01} \leq L\}, \tag{3.4a}$$

$$\mathcal{S}(a) = \{b \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T}) |\ \ b \equiv a \mod \mathbf{Z}L\mathcal{T}\ \}, \tag{3.4b}$$

$$\mathcal{S}_{\ell,n}(\zeta\mathcal{T}) = \{a \in \hat{\mathcal{S}}_{\ell,n}(\zeta\mathcal{T}) |\ \ a \text{ satisfies (R)}\ \}, \tag{3.4c}$$

$$\begin{aligned}\mathcal{R}_{\ell,n}(\zeta\mathcal{T}) &= \{a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T}) | -L < a_{ij} \leq L, i = \text{even}, j = \text{odd}\} \\ &= \{a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T}) | a_{0n''} \leq L, a_{1n'} < L\},\end{aligned} \tag{3.4d}$$

where the condition (R) on $a = (L + a_n - a_0 - 1)\Lambda_0 + (a_0 - a_1 - 1)\Lambda_1 + \cdots + (a_{n-1} - a_n - 1)\Lambda_n \in \hat{\mathcal{S}}_{\ell,n}(\zeta\mathcal{T})$ reads as

$$\text{(R)} \qquad \begin{aligned} L + a_{n'} &> a_0 > a_2 > \cdots > a_{n'}, \\ L + a_{n''} &> a_1 > a_3 > \cdots > a_{n''}, \end{aligned} \tag{3.5}$$

and (n', n'') is specified by

$$(n', n'') = \begin{cases} (n-1, n), & \text{if } n \text{ is odd,} \\ (n.n-1), & \text{if } n \text{ is even.} \end{cases} \tag{3.6}$$

Note that $\mathcal{S}$ is a finite set of level ℓ non-integral weights. In view of (3.3b) and the invariance of the Boltzmann weights:

$$(3.7)\qquad W_u\begin{pmatrix} a+kL\mathcal{T} & b+kL\mathcal{T} \\ d+kL\mathcal{T} & c+kL\mathcal{T}\end{pmatrix} = W_u\begin{pmatrix} a & b \\ d & c\end{pmatrix}\quad \forall k\in\mathbf{Z},$$

we are going to restrict the local states to the set $\mathcal{S}$ by identifying all the states $b\in\mathcal{S}(a)(a\in\mathcal{S})$ with a. This is done as follows.

For two states $a,b\in\mathcal{S}$, we call an ordered pair (a,b) *admissible* if and only if

$$(3.8)\qquad b-a\equiv e_\mu \quad \bmod \mathbf{Z}L\mathcal{T} \quad \text{for some}\quad 0\le\mu\le n.$$

Let $a,b,c,d\in\mathcal{S}$ be four states. In terms of the Boltzmann weights (2.6) for the unrestricted model, we define those for the restricted model $W_u^R\begin{pmatrix} a & b \\ d & c\end{pmatrix}$ to be

$$(3.9)\qquad \begin{cases} W_u\begin{pmatrix} a^\dagger & b^\dagger \\ d^\dagger & c^\dagger\end{pmatrix} & \text{if } (a,b),(b,c),(a,d),(d,c) \text{ are admissible,} \\ 0 & \text{otherwise.}\end{cases}$$

Here $a^\dagger\in\mathcal{S}(a)$, etc. are to be chosen so that the pairs $(a^\dagger,b^\dagger),(b^\dagger,c^\dagger)$, $(a^\dagger,d^\dagger)$ and $(d^\dagger,c^\dagger)$ are weakly admissible. The following is an immediate consequence of Theorem 1 and the above definitions.

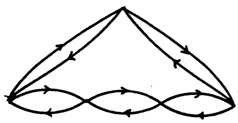
Theorem 2. *The restricted Boltzmann weights* $W_u^R\begin{pmatrix} a & b \\ d & c\end{pmatrix}$ *are finite and satisfy the Yang-Baxter equation* (2.8) *among themselves.*

In the case $n=1$, the above construction reduces to the models studied in [24] as '$A_{L-1}^{(1)}$ ' model and 'cyclic SOS' model in [27].

The set of local states and the admissibility condition are visualized in terms of an *incidence diagram*, the analogue of the Dynkin diagram for the case $n=1$ in Fig.3. It is a connected oriented graph that consists of $|\mathcal{S}|$ nodes each corresponding to a local state $a\in\mathcal{S}$ and arrows pointing from a to b if (a,b) is admissible. In Fig.4 we give some examples for the present model and for the corresponding $(A_n^{(1)}$, vector rep.,$\ell)$ model.

§4. Local state probabilities

Local state probability (LSP) $P(a)$ of an SOS model is by definition the probability that a site (say 1) of the lattice $\mathcal{L}$ is of state a (under



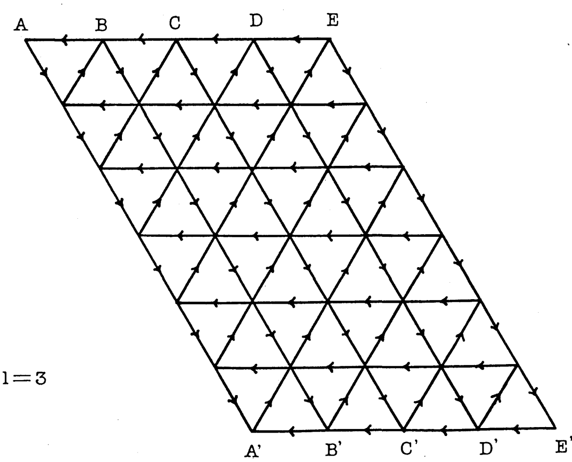
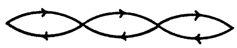


Fig. 4. Examples of the incidence diagrams for the present models. Each node corresponds to a local state. An arrow pointing from a to b is put if and only if the ordered pair (a, b) is admissible. In the diagram for $n = 2, \ell = 3$, the nodes A, B, C, D, E are identified with A', B', C', D', E'.

suitable boundary conditions)

$$P(a) = Z^{-1} \sum_{\text{config.}} \delta_{a^{(1)}=a} \prod_{\text{faces}} W \begin{pmatrix} a^{(i)} & a^{(j)} \\ a^{(l)} & a^{(k)} \end{pmatrix} .$$

It is a physical quantity for an SOS model and considered as an analogue of the spontaneous magnetization. Here we evaluate the LSPs for the restricted face model in section 3 in a regime $-(n+1)/2 < u < 0, 0 <$



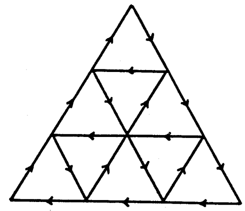


Fig. 5. The incidence diagrams for the $(A_n^{(1)}$, vector rep., $\ell)$ models [19] having the same values of the parameters n and ℓ.

$p^{1/2} < 1$. The main tool is the corner transfer matrix (CTM) method. We refer the readers to [1] and appendix A of [2] for the details of the method.

Let $a, b, b+e_\nu (0 \le \nu \le n)$ be the states in $\mathcal{S}$ among which the latter two refer to the boundary condition. In the working below we shall use a variable x defined by

$$(4.1) \qquad p^{1/2} = \exp(-\varepsilon/2), \qquad x = \exp(-4\pi^2/L\varepsilon).$$

(The parameter ε introduced here should not be confused with the vector ϵ in (2.1).)

In section 3 we have restricted the local states to the set $\mathcal{S}$ by considering $\mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ mod $\mathbf{Z}L\mathcal{T}$. In fact, we find the mathematical description of the LSP becomes simpler by viewing $\mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ as the set of the local states rather than $\mathcal{S}$. Thus we also consider a "probability" $\overline{P}(a,b,\nu)$ satisfying $\sum_{a\in\mathcal{R}_{\ell,n}(\zeta\mathcal{T})} \overline{P}(a,b,\nu) = 1$ in addition to the LSP $P(a,b,\nu)$ itself obeying $\sum_{a\in\mathcal{S}} P(a,b,\nu) = 1$. Actually, $\mathcal{S}$ in (3.4a) and $\mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ in (3.4d) are not so different and the following relation holds.

$$(4.2) \qquad P(a,b,\nu) = \sum_{d\in\mathcal{R}_{\ell,n}(\zeta\mathcal{T}),\mathcal{S}(a)} \overline{P}(d,b,\nu),$$

where the sum consists of at most two terms. Below we formulate the LSP $P(a,b,\nu)$. The result for the $\overline{P}(a,b,\nu)$ will be given in (4.13).

4.1. Multiple sum expressions

Our strategy to evaluate the LSP of the restricted model goes as follows. Firstly we regard all the elements in $\mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ as independent local states. Next we apply the CTM method and derive the "finite size result" $u_d g_m(d, b, b+e_\nu; x^{n+1})$ (see (4.5,6)) proportional to the LSP for $d \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$. Finally we obtain the LSP $P(a,b,\nu)$ for $a \in \mathcal{S}$ by summing up the $u_d g_m$ over $d \in \mathcal{S}(a)$ with proper normalization and taking the limit $m \to \infty$.

Let a, b, c, d be the elements of $\mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ such that the pairs (a,b), $(b,c), (a,d), (d,c)$ are weakly admissible. The eigenvalues of the CTM are deduced from the behavior of the Boltzmann weights $W_u\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (2.6) in the limit $x \to +0, u \to -0$ keeping $w = x^u$ fixed. Noting that $\zeta \in \mathbf{R} \setminus \mathbf{Z}$ and the identity

$$(4.3) \qquad \begin{aligned} \left\{ \begin{matrix} H(u,p) \\ \Theta(u,p) \end{matrix} \right\} &= \sqrt{\frac{2\pi}{\varepsilon}}\, x^{L/8+u(u-L)/2L} E(\pm x^u, x^L), \\ E(z,q) &= \prod_{k=1}^{\infty} (1 - zq^{k-1})(1 - z^{-1}q^k)(1-q^k), \end{aligned}$$

we have the following in the above limit.

$$(4.4a) \qquad \lim W_u \begin{pmatrix} a & b \\ d & c \end{pmatrix} F\, w^{f_b + f_d - f_a - f_c} = \delta_{bd} w^{-H(a,b,c)},$$

$$(4.4b) \qquad F = x^{-u(u+L+\frac{2}{n+1})/2L}, \qquad f_a = \frac{1}{2L}|a+\rho|^2,$$

$$(4.4c) \qquad H(a, a+e_\mu, a+e_\mu+e_\nu) = \big([a_{\mu\nu}] - a_{\mu\nu}\big)/L,$$

where for $y \in \mathbf{R}$, the symbol $[y]$ is uniquely determined by $[y] \equiv y \bmod L$, $0 < [y] \le L$. This is a *low temparature limit* in the sense that the local states along the SW-NE direction are frozen to the same element in $\mathcal{S}_{\ell,n}(\zeta\mathcal{T})$. In view of this, we fix the boundary condition to those configurations invariant under the translation along the SW-NE direction. Now the YBE and the inversion properties (2.9,10) imply that the computation of $P(a,b,\nu)$ is reduced to the study of $m \to \infty$ limit of the quantity $P_m(a,b,\nu)(m \in \mathbf{Z}_{\geq 0})$ defined below.

$$(4.5a)\qquad P_m(a,b,\nu) = Q_m(a,b,\nu)/N_m(b,\nu),$$

$$(4.5b)\qquad Q_m(a,b,\nu) = \sum_{d \in \mathcal{S}(a)} u_d g_m(d,b,b+e_\nu;x^{n+1}),$$

$$(4.5c)\qquad N_m(b,\nu) = \sum_{a \in \mathcal{S}} Q_m(a,b,\nu),$$

$$(4.5d)\qquad = \sum_{a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})} u_a g_m(a,b,b+e_\nu;x^{n+1}),$$

$$(4.5e)\qquad u_a = x^{-\langle a+\rho,\rho\rangle} \prod_{\mu<\nu} E\big((-)^{\mu-\nu}x^{a_{\mu\nu}},x^L\big),$$

wherein (4.5c–d) we have used (3.3b). The quantity $g_m(a,b,b+e_\nu;q)$ is a q-polynomial called *one dimensional configuration sum*:

$$(4.6)\qquad g_m(a,b,b+e_\nu;q) = \sum q^{\sum_{j=1}^m jH(b^{(j)},b^{(j+1)},b^{(j+2)})}.$$

Here the outer sum extends over the states $b^{(2)},\cdots,b^{(m)} \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$, under the constraint that $(b^{(j)},b^{(j+1)})\,(1 \leq j \leq m)$ is weakly admissible. The center state $b^{(1)}$ is fixed to a and b, $b+e_\nu$ specify the boundary condition $b^{(m+1)} = b, b^{(m+2)} = b+e_\nu$.

4.2. One dimensional configuration sums

Let us express the 1D configuration sum (4.6) in a form that is suitable for passing to the thermodynamic limit $m \to \infty$. For the purpose we find it convenient to introduce subgroups W_e and W_o of the Weyl group of $A_n^{(1)}$, each having the following semidirect product structure:

$$(4.7)\qquad \begin{aligned} &W_e = S^e \times Q_e, \quad W_o = S^o \times Q_o,\\ &Q_e = \mathbf{Z}(\epsilon_0-\epsilon_2) \oplus \mathbf{Z}(\epsilon_2-\epsilon_4) \oplus \cdots \oplus \mathbf{Z}(\epsilon_{n'-2}-\epsilon_{n'}),\\ &Q_o = \mathbf{Z}(\epsilon_1-\epsilon_3) \oplus \mathbf{Z}(\epsilon_3-\epsilon_5) \oplus \cdots \oplus \mathbf{Z}(\epsilon_{n''-2}-\epsilon_{n''}). \end{aligned}$$

Here S^e (resp. S^o) is a subgroup of the symmetric group S_{n+1} and acts on $\mathcal{H}^*$ as a permutation of $\{\epsilon_0,\epsilon_2\ldots,\epsilon_{n'}\}$ (resp. $\{\epsilon_1,\epsilon_3\ldots,\epsilon_{n''}\}$). The Q_e and Q_o are sublattices of the root lattice $Q = \mathbf{Z}(\epsilon_0-\epsilon_1) \oplus \cdots \oplus \mathbf{Z}(\epsilon_{n-1}-\epsilon_n)$ of A_n. They act on $\mathcal{H}^*$ as translations t_α $(\alpha \in Q)$

$$(4.8)\qquad t_\alpha(\gamma) = \gamma + \langle\gamma,\delta\rangle\alpha - \big(\frac{1}{2}\langle\gamma,\delta\rangle|\alpha|^2 + \langle\gamma,\alpha\rangle\big)\delta.$$

With these notations the following formula is valid for the 1D configuration sum (4.6).

Theorem 3. *Let $a, b, b+e_\nu \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ and $m \in \mathbf{Z}_{\geq 0}$. Then we have*

$$(4.9\mathrm{a})\quad g_m(a,b,b+e_\nu;q) = \sum_{w \in W_e \cdot W_o} \det w\, f_m(b+\rho-w(a+\rho),b,\nu;q),$$

$$(4.9\mathrm{b})\quad f_m(p,b,\nu;q) = q^{|p-\eta(b,\nu)|^2/2+c(m,b,\nu)} \begin{bmatrix} m \\ p \end{bmatrix},$$

$$(4.9\mathrm{c})\quad c(m,b,\nu) = \frac{1}{2(n+1)}(m+\sum_{\mu=0}^{n} h_\mu)^2 - \frac{1}{2}(m+\sum_{\mu=0}^{n} h_\mu^2),$$

$$(4.9\mathrm{d})\quad \eta(b,\nu) = (1+h_0-h_n)\Lambda_0 + \sum_{\mu=1}^{n}(h_\mu - h_{\mu-1})\Lambda_\mu,$$

$$(4.9\mathrm{e})\quad h_\mu = H(b-e_\mu, b, b+e_\nu),$$

wherein (4.9b) *the symbol* $\left[\begin{smallmatrix} m \\ p \end{smallmatrix}\right]$ *stands for the q-multinomial coefficient* [29] :

$$(4.9\mathrm{f})\qquad \begin{bmatrix} m \\ p \end{bmatrix} = \frac{(q)_m}{(q)_{p_0}\cdots(q)_{p_n}}, \quad (q)_k = \prod_{j=1}^{k}(1-q^j),$$

only when $p \in \mathcal{H}^*$ can be written as $p \equiv \sum_{\mu=0}^{n} p_\mu e_\mu \bmod \mathbf{C}\delta$, $p_\mu \in \mathbf{Z}_{\geq 0}$, $\sum_{\mu=0}^{n} p_\mu = m$. We assume that $\left[\begin{smallmatrix} m \\ p \end{smallmatrix}\right] = 0$, otherwise.

The proof is given in appendix A. In what follows we shall deal with the case $\eta = \eta(b,\nu) \in \{\Lambda_0, \ldots, \Lambda_n\}$, $\xi \stackrel{\text{def}}{=} b - \eta \in \mathcal{S}_{\ell-1,n}(\zeta\mathcal{T})$.

4.3. Theta function identities

A crucial step in the LSP calculation is to find the theta function identity (so called "sums-of-products identity") that simplifies the normalization constant $N_m(b,\nu)$ appearing in (4.5a, c, d). To state this we introduce theta functions $\Theta_{\mu,N}(u,\tau)$, $\Theta^{(-)}_{\mu,N}(u,\tau)$ for $N \in \mathbf{R}_{>0}$ and $u, \mu \in \sum_{k=1}^{n} \mathbf{C}\overline{\Lambda}_k$ as follows.

$$(4.10\mathrm{a})\qquad \Theta^{(-)}_{\mu,N}(u,\tau) = \sum_{w \in S_{n+1}} \det w\, \Theta_{w(\mu),N}(u,\tau),$$

$$(4.10\mathrm{b})\qquad \Theta_{\mu,N}(u,\tau) = \sum_{\gamma \in Q+\mu/N} \exp 2\pi i\Big(\frac{N\tau}{2}\langle\gamma,\gamma\rangle - N\langle\gamma,u\rangle\Big).$$

Let λ be a level $j(\in \mathbf{Z}_{\geq 0})$ dominant integral weight of $A_n^{(1)}$ (i.e., an element of $\hat{\mathcal{S}}_{j,n}(0)$ satisfying $j+n+1+\lambda_n > \lambda_0 > \cdots > \lambda_n$). Then the

Weyl-Kac character formula [28] tells that the $A_n^{(1)}$ character with the highest weight λ is given by the ratio:

$$\chi_\lambda(z_1,\ldots,z_n,q)=\frac{\Theta^{(-)}_{\overline{\lambda}+\overline{\rho},n+j+1}(u,\tau)}{\Theta^{(-)}_{\overline{\rho},n+1}(u,\tau)}, \tag{4.11}$$
$$z_k=e^{2\pi i\langle\epsilon_{k-1}-\epsilon_k,u\rangle},\quad q=e^{2\pi i\tau}.$$

Hereafter we shall formally extend this expression of χ_λ to $\lambda\in\sum_\mu \mathbf{C}\Lambda_\mu$. Now we state the theta function identity relevant for our model.

Theorem 4. *For $\xi\in\hat{S}_{\ell-1,n}(\zeta\mathcal{T})$, $\eta\in\{\Lambda_0,\ldots,\Lambda_n\}$ and $\beta\in\mathbf{C}$, we have*

(4.12a) $\quad\chi_{\xi+\beta\mathcal{T}}(z_1,\ldots,z_n,q)\chi_\eta(z_1,\ldots,z_n,q)$

(4.12b) $\quad=\displaystyle\sum_{a\in\mathcal{S}_{\ell,n}(\zeta\mathcal{T})} b^{(Q_e\oplus Q_o)}_{\xi+\beta\mathcal{T}\,\eta\,a+\beta\mathcal{T}}(q)\chi_{a+\beta\mathcal{T}}(z_1,\ldots,z_n,q)$

(4.12c) $\quad=\displaystyle\sum_{a\in\mathcal{R}_{\ell,n}(\zeta\mathcal{T})} b^{(Q)}_{\xi+\beta\mathcal{T}\,\eta\,a+\beta\mathcal{T}}(q)\chi_{a+\beta\mathcal{T}}(z_1,\ldots,z_n,q),$

where for a lattice M, the function $b^{(M)}_{\xi\eta a}(q)$ is given by

(4.12d)
$$\eta(\tau)^n\, b^{(M)}_{\xi\eta a}(q)$$
$$=\sum_{w\in S^e\cdot S^o}\det w\sum_{\gamma\in M+\frac{w(\overline{a}+\overline{\rho})}{L}-\frac{\overline{\xi}+\overline{\rho}}{L-1}} q^{\frac{L(L-1)}{2}\langle\gamma,\gamma\rangle}\qquad \textit{if } \overline{a}\equiv\overline{\xi}+\overline{\eta}\mod Q,$$
$$=0\qquad\qquad \textit{otherwise,}$$

$$\eta(\tau)=q^{1/24}\varphi(q),\qquad \varphi(q)=(q)_\infty. \tag{4.12e}$$

This has been proved in appendix B. The identity (4.12a–c) formally looks like a character identity describing the irreducible decomposition of the tensor products of $A_n^{(1)}$ modules with the "highest weights" $\xi+\beta\mathcal{T}$ and η. Our LSP calculation is relevant to the identity (4.12) with the special choice $\beta=\beta_0\stackrel{\mathrm{def}}{=}L\varepsilon i/4\pi$ so that $x^{\beta_0}=e^{-\pi i}$. From Theorems 3 and 4 we obtain the result for the $\overline{P}(a,b,\nu)$ as

$$\overline{P}(a,b,\nu)=\frac{b^{(Q)}_{\xi+\beta_0\mathcal{T}\,\eta\,a+\beta_0\mathcal{T}}(x^{n+1})\chi_{a+\beta_0\mathcal{T}}(x,\ldots,x,x^{n+1})}{\chi_{\xi+\beta_0\mathcal{T}}(x,\ldots,x,x^{n+1})\chi_\eta(x,\ldots,x,x^{n+1})}. \tag{4.13}$$

The line of the argument to get (4.13) goes as follows. Straightforward limit $m \to \infty (m \equiv 0 \mod n+1)$ of the 1D configuration sum in (4.9) gives rise to

(4.14)
$$\begin{aligned}&\lim q^{-c(m,b,\nu)-(a+\beta_0\mathcal{T}+\rho)^2/2L+(\xi+\beta_0\mathcal{T}+\rho)^2/2(L-1)} g_m(a,b,b+e_\nu;q)\\ &= b^{(Q_e\oplus Q_o)}_{\xi+\beta_0\mathcal{T}\,\eta a+\beta_0\mathcal{T}}(q).\end{aligned}$$

On the other hand, the principal specialization $z_1 = \cdots = z_n = x, q = x^{n+1}$ of the theta function $\Theta^{(-)}_{\mu,N}(u,\tau)$ is known to become a simple infinite product [28]

$$x^{(n+1)\langle\mu,\mu\rangle/2N-\langle\mu,\overline{\rho}\rangle}\varphi(x^N)^{-n(n-1)/2}\prod_{\kappa<\lambda} E(x^{\langle\mu,\epsilon_\kappa-\epsilon_\lambda\rangle}, x^N). \tag{4.15}$$

Thus, up to a-independent factors, the quantity u_a in (4.5e) is equal to $\chi_{a+\beta_0\mathcal{T}}(x,\ldots,x,x^{n+1})x^{-(n+1)(a+\beta_0\mathcal{T}+\rho)^2/2L}$. From this and (4.14) one finds that the $m \to \infty$ limit of (4.5c, d) is proportional to the theta function identity (4.12a, b) under the principal specialization. This leads us to (4.13) together with (4.2) giving the LSP result of our model.

4.4. Critical behaviors

Our model becomes critical as the elliptic nome p tends to 0. Here we study the critical behaviors of the probability $\overline{P}(a,b,\nu)$ (4.13). Using $x^{\beta_0} = e^{-\pi i}$, we explicitly write it down as follows.

$$\overline{P}(a,b,\nu) = T_{\xi\eta a}\, c_{\xi\eta a}(x^{n+1}), \tag{4.16a}$$

where the function $c_{\xi\eta a}(q) \overset{\text{def}}{=} b^{(Q)}_{\xi+\beta_0\mathcal{T}\,\eta a+\beta_0\mathcal{T}}(q)q^{\frac{|\beta_0\mathcal{T}|^2}{2L(L-1)}+\langle\frac{\overline{a}+\overline{\rho}}{L}-\frac{\overline{\xi}+\overline{\rho}}{L-1},\beta_0\mathcal{T}\rangle}$ takes the form (by the definition $c_{\xi\eta a}(q) = 0$ unless $\overline{a} \equiv \overline{\xi} + \overline{\eta} \mod Q$)

(4.16b)
$$\begin{aligned}&\eta(\tau)^n c_{\xi\eta a}(q = x^{n+1})\\ &= \sum_{w\in S^e\cdot S^o} \det w \sum_{\alpha\in Q} e^{(n+1)\pi i\langle\alpha,\mathcal{T}\rangle} q^{\frac{L(L-1)}{2}|\alpha+w(\overline{a}+\overline{\rho})/L-(\overline{\xi}+\overline{\rho})/(L-1)|^2}.\end{aligned}$$

The remaining part $T_{\xi\eta a}$ is a ratio of infinite products:

$$(4.16c)\ T_{\xi\eta a} = x^{\phi}\left(\frac{\varphi(x^{L-1})\varphi(x^{n+2})}{\varphi(x^L)\varphi(x^{n+1})}\right)^{n(n-1)/2}$$
$$\prod_{\kappa<\lambda}\frac{E\big((-)^{\kappa-\lambda}x^{a_{\kappa\lambda}},x^L\big)E\big(x^{\lambda-\kappa},x^{n+1}\big)}{E\big((-)^{\kappa-\lambda}x^{\xi_{\kappa\lambda}},x^{L-1}\big)E\big(x^{\eta_{\kappa\lambda}},x^{n+2}\big)},$$

$$(4.16d)\ \phi = \langle \xi+\eta-a,\rho\rangle + (n+1)\Big(\frac{|a+\rho|^2}{2L}-\frac{|\xi+\rho|^2}{2(L-1)}+\frac{|\rho|^2}{2(n+1)}-\frac{|\eta+\rho|^2}{2(n+2)}\Big).$$

In deriving (4.16) we have taken advantage of the property $w(\mathcal{T}) = \mathcal{T}$ for $w \in S^e\cdot S^o$. In order to study the small p behavior, we rewrite the expression (4.16) in terms of elliptic functions with conjugate modulus. In the sequel, we shall use the quantity $t = p^{\frac{L}{n+1}}$ as the variable measuring the *deviation from criticality.* For $T_{\xi\eta a}$, we utilize (4.3) to get

(4.17)
$$T_{\xi\eta a} = \sqrt{\frac{(L-1)(n+2)}{L(n+1)}}^{\,n}\left(\frac{H(\pi/3,t^{\frac{n+1}{3(L-1)}})H(\pi/3,t^{\frac{n+1}{3(n+2)}})}{H(\pi/3,t^{\frac{n+1}{3L}})H(\pi/3,t^{\frac{1}{3}})}\right)^{n(n-1)/2}$$
$$\prod_{\kappa<\lambda,\kappa-\lambda=\text{even}}\frac{H(\pi a_{\kappa\lambda}/L,t^{\frac{n+1}{L}})H(\pi(\lambda-\kappa)/(n+1),t)}{H(\pi\xi_{\kappa\lambda}/(L-1),t^{\frac{n+1}{L-1}})H(\pi\eta_{\kappa\lambda}/(n+2),t^{\frac{n+1}{n+2}})}$$
$$\prod_{\kappa<\lambda,\kappa-\lambda=\text{odd}}\frac{\Theta(\pi a_{\kappa\lambda}/L,t^{\frac{n+1}{L}})H(\pi(\lambda-\kappa)/(n+1),t)}{\Theta(\pi\xi_{\kappa\lambda}/(L-1),t^{\frac{n+1}{L-1}})H(\pi\eta_{\kappa\lambda}/(n+2),t^{\frac{n+1}{n+2}})}.$$

From this we find that as $t \to 0$, $T_{\xi\eta a}$ vanishes as

$$(4.18a)\qquad T_{\xi\eta a} = \text{const}\ t^{\frac{c}{24}} + \text{ higher order terms in } t,$$

where c is a positive constant taking the values:

$$(4.18b)\qquad c = \begin{cases} n\big(1-\frac{(n+1)(n+2)}{4L(L-1)}\big) & \text{if } n \text{ is even,}\\ n\big(1-\frac{(n^2-1)(n+3)}{4nL(L-1)}\big) & \text{if } n \text{ is odd.}\end{cases}$$

On the other hand, the necessary formula to rewrite $c_{\xi\eta a}(q)$ (4.16b) reads as follows. ($N \in \mathbf{R}_{>0}$, $\mu \in \sum_{k=1}^{n} \mathbf{R}\overline{\Lambda}_k$, $\mathrm{Im}\tau > 0$)

$$\begin{aligned}
&\sum_{\alpha \in Q} \exp \pi i\big((n+1)\langle \alpha, \mathcal{T}\rangle + N\tau|\alpha + \mu/N|^2\big) \\
(4.19) \qquad &= \frac{1}{\sqrt{n+1}} \sqrt{\frac{1}{-Ni\tau}}^{\,n} \sum_{\lambda} \exp\Big(-\frac{\pi i}{N\tau}|\lambda|^2 - \frac{2\pi i}{N}\langle \mu, \lambda\rangle\Big).
\end{aligned}$$

Here the summation in the RHS extends over $\lambda \in Q^*_+$ or Q^*_- according as n is odd or even with $Q^*_\pm$ defined by

(4.20)

$$\begin{aligned}
Q^*_\pm &= \sum_{k=1}^{n} (\mathbf{Z} + \frac{1 \mp 1}{4})\overline{\Lambda}_k \\
&= \{\sum_{\mu=0}^{n} \lambda_\mu \epsilon_\mu \mid \lambda_i \in \mathbf{R}, \sum_{i=0}^{n} \lambda_i = 0, \lambda_i - \lambda_j \in \mathbf{Z} + (1 \mp 1)\frac{1-(-)^{i-j}}{8}\}.
\end{aligned}$$

This is a direct consequence of Poisson's summation formula and $\langle \alpha, \mathcal{T}\rangle \in \mathbf{Z}$ for $\alpha \in Q$. Applying (4.19) to $c_{\xi\eta a}(x^{n+1})$ (4.16b), we get (see (4.1))

$$\begin{aligned}
&c_{\xi\eta a}(x^{n+1}) = t^{-n/24}\varphi(t)^{-n} \sum_{\lambda} A(\lambda)\, t^{\frac{|\lambda|^2}{2L(L-1)}} \\
(4.21) \qquad &A(\lambda) = \frac{1}{\sqrt{n+1}} \sqrt{\frac{\varepsilon}{2\pi(n+1)(L-1)}}^{\,n} \\
&\qquad \sum_{w, w' \in S^e \cdot S^o} \det w \, \exp\Big(-2\pi i \langle \frac{w(\overline{a}+\overline{\rho})}{L} - \frac{\overline{\xi}+\overline{\rho}}{L-1}, w'(\lambda)\rangle\Big),
\end{aligned}$$

where the λ-sum now has the smaller support in (see (3.5))

$$(4.22) \qquad Q^{*R}_\pm = \{\lambda_0\epsilon_0 + \cdots + \lambda_n\epsilon_n \in Q^*_\pm \mid \lambda_i\text{'s satisfy (R)}\}.$$

Thus (4.21) expresses the function $c_{\xi\eta a}(x^{n+1})$ as the linear combination of

$$(4.23) \qquad A(\lambda)\, t^{-\frac{c}{24}+\Delta(\lambda)}(1 + O(t)),$$

with c given by (4.18b) and the rational power $\Delta(\lambda)(\geq 0)$ having the form:

$$(4.24) \qquad \Delta(\lambda) = \frac{|\lambda|^2 - |\lambda_{\min}|^2}{2L(L-1)}, \qquad \lambda \in \begin{cases} Q^{*R}_- & \text{if n is even,} \\ Q^{*R}_+ & \text{if n is odd.} \end{cases}$$

Here $\lambda_{\min} \in Q_{\pm}^{*R}$ is specified to be

(4.25)
$$\begin{cases} \frac{1}{2}\sum_{i=1}^{n}\overline{\Lambda}_i = \sum_{i=0}^{\frac{n}{2}}(\frac{n}{4}-i)\epsilon_{2i} + \sum_{i=0}^{\frac{n}{2}-1}(\frac{n-2}{4}-i)\epsilon_{2i+1} & \text{if } n \text{ is even,} \\ \sum_{i=1}^{\frac{n-1}{2}}\overline{\Lambda}_{2i} = \sum_{i=0}^{\frac{n-1}{2}}(\frac{n-1}{4}-i)(\epsilon_{2i}+\epsilon_{2i+1}) & \text{if } n \text{ is odd,} \end{cases}$$

and has the squared length $|\lambda_{\min}|^2 = \frac{n(n+1)(n+2)}{48}$ or $\frac{(n^2-1)(n+3)}{48}$ according as n is even or odd. Combining (4.16a), (4.17) and (4.21), we obtain the small t expansion of the probability $\overline{P}(a,b,\nu)$. In the limit $t \to 0$, it converges to a finite value corresponding to the contribution $\lambda = \lambda_{\min}$ in (4.23). This implies that the system undergoes the second order phase transition as t tends to zero. (Recall that the low tempara-ture limit $x \to 0$ corresponds to $t \to 1$.) The deviation of $\overline{P}(a,b,\nu)$ from its critical value consists of the terms $A(\lambda)t^{\Delta(\lambda)}(1+O(t))$ and the order $O(t^{\frac{n+1}{3L}})$ terms coming from $T_{\xi\eta a}$ (4.17). They depend on the parameter ζ through the amplitude such as $A(\lambda)$.

This kind of expansion for the LSPs has appeared repeatedly in the previous studies [15,17,19,23,24], which imply that c(4.18b) and $\Delta(\lambda)$ (4.24) for $A(\lambda) \neq 0$ are the conformal anomaly and the scaling dimensions of the model. We note that these results extend those obtained in [24] for $n=1, \zeta \to 0$.

Appendix A. Proof of Theorem 3

By the definition (4.6) the 1D configuration sum $g_m(a,b,c;q)$ satisfies the linear recurrence relation and the initial condition:

(A.1a)
$$g_m(a,b,c;q) = \sum_d g_{m-1}(a,d,b;q)q^{mH(d,b,c)},$$

(A.1b)
$$g_0(a,b,c;q) = \delta_{ab},$$

wherein (A.1a), the sum is taken over $d \in S_{\ell,n}(\zeta\mathcal{T})$ such that the pair (d,b) is weakly admissible. The function $H(a,b,c)$ has been defined in (4.4c). Conversely, (A.1) uniquely characterizes the quantity $g_m(a,b,c;q)$ for $m \geq 0$. As the first step to prove (A.1) we show

Lemma A.1. *Suppose that $b \in S_{\ell,n}(\zeta\mathcal{T})$, $m \in \mathbf{Z}_{>0}$ and let $f_m(p,b,\nu;q)$ be as in (4.9b–f). Then we have*

(A.2a)
$$f_m(p,b,\nu;q) = \sum_{0\leq\mu\leq n} f_{m-1}(p-e_\mu, b-e_\mu, \mu; q)q^{mH(b-e_\mu,b,b+e_\nu)},$$

(A.2b)
$$f_0(p,b,\nu;q) = \delta_{0p}.$$

Proof. Firstly we write down the function $f_m(p,b,\nu;q)$ (4.9b–f), as

(A.3)

$$f_m(p,b,\nu;q) = q^{\frac{1}{2}\sum_{\mu=0}^{n} p_\mu(p_\mu-1)+\sum_{\mu=0}^{n} H(b-e_\mu,b,b+e_\nu)p_\mu}\begin{bmatrix} m \\ p \end{bmatrix}$$

$$\text{if } p \equiv \sum_{\mu=0}^{n} p_\mu e_\mu \ mod \mathbf{C}\delta,\ p_\mu \in \mathbf{Z}_{\geq 0}, \sum_{\mu=0}^{n} p_\mu = m,$$

$$=0 \text{ otherwise.}$$

The initial condition (A.2b) is obvious in this form of $f_m(p,b,\nu;q)$. On the other hand, equation (A.2a) is reduced to the following.

$$\begin{bmatrix} m \\ p \end{bmatrix} = \sum_{\mu=0}^{n} q^{s(\mu)} \begin{bmatrix} m-1 \\ p-e_\mu \end{bmatrix}, \tag{A.4a}$$

$$s(\mu) = \sum_{j\neq\mu} c(\mu,j)p_j, \tag{A.4b}$$

$$c(\mu,j) = ([b_{j\mu}] - [b_{j\nu}-1] + [b_{\mu\nu}-1])/L, \tag{A.4c}$$

where we have used (4.4c). Define $\hat{b}_j$ $(0 \leq j \leq n)$ by $\hat{b}_j \equiv b_j \bmod L$ and $0 < \hat{b}_j - b_\nu - 1 \leq L$. These are distinct real numbers because $b \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$, $\zeta \in \mathbf{R}\setminus\mathbf{Z}$. Let $\mu_0, \mu_1, \cdots, \mu_n$ be a permutation of $0, 1, \cdots, n$ such that $\hat{b}_{\mu_0} < \hat{b}_{\mu_1} < \cdots < \hat{b}_{\mu_n}$. From these definitions we deduce $c(\mu_k, \mu_j) = 1$ if $j < k$ and $= 0$ if $j > k$. Thus $s(\mu_k) = p_{\mu_0} + \cdots + p_{\mu_{k-1}}$ $(0 \leq k \leq n)$. Now (A.4a) becomes a standard recursion formula for the q-multinomial coefficients if we write the RHS as $\sum_{k=0}^{n} q^{s(\mu_k)} \left[\begin{smallmatrix} m-1 \\ p-e_{\mu_k} \end{smallmatrix}\right]$.

Q.E.D.

Using this lemma and (4.9a), we see that (A.1b) holds. Moreover, the 1D configuration sum $g_m(a,b,c;q)$ satisfies the recursion relation (A.1a) but with the sum taken over $d = b - e_\mu, 0 \leq \mu \leq n$. In this case the pair (d,b) is indeed weakly admissible but d does *not* necessarily belong to $\mathcal{S}_{\ell,n}(\zeta\mathcal{T})$. Thus the remaining task is to show the following which asserts that such contributions vanish.

Lemma A.2. *Assume that $a, b \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ and $b - e_\mu \notin \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$. Then we have*

$$g_{m-1}(a, b-e_\mu, b; q) = 0. \tag{A.5}$$

Proof. The situation $b \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$, $b - e_\mu \notin \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ occurs when one of the following is satisfied. (See (3.5).)

(i) $b_\mu - 1 = b_{\mu+2} \qquad 0 \le \mu \le n-2,$
(ii) $L + b_{n'} - 1 = b_0 \qquad \mu = n',$
(iii) $L + b_{n''} - 1 = b_1 \qquad \mu = n''.$
Here we shall show (A.5) for the case (i). The other cases are verified similarly. For $0 \le \mu \le n-2$, let r_μ be the element of $W_e \cdot W_o$ exchanging ϵ_μ and $\epsilon_{\mu+2}$. The condition (i) implies $r_\mu(b - e_\mu + \rho) = b - e_\mu + \rho$. In view of this and (4.9a), we have (A.5) if the following holds.

$$\text{(A.6)} \qquad f_{m-1}(r_\mu(p), b - e_\mu, \mu; q) = f_{m-1}(p, b - e_\mu, \mu; q).$$

Substituting (A.3) to this we find it is equivalent to

$$\text{(A.7)} \qquad \sum_{i=0}^{n} H(b - e_i - e_\mu, b - e_\mu, b)(p_i - r_\mu(p)_i) = 0.$$

This can be directly checked by using $r_\mu(p)_i = p_i (i \ne \mu, \mu+2)$, $r_\mu(p)_\mu = p_{\mu+2}, r_\mu(p)_{\mu+2} = p_\mu$ and $H(b - 2e_\mu, b - e_\mu, b) = H(b - e_{\mu+2} - e_\mu, b - e_\mu, b) = 1$. Q.E.D.

Appendix B. Proof of Theorem 4

Here we shall prove the identity (4.12). We assume $\xi \in \hat{S}_{\ell-1,n}(\zeta\mathcal{T})$ and $\eta \in \{\Lambda_0, \cdots, \Lambda_n\}$ throughout. Then the quantity χ_η defined by (4.11) is a level 1 character, which is known to have the following expression [28, p217].

$$\text{(B.1)} \qquad \chi_\eta(z_1, \cdots, z_n; q) = \Theta_{\overline{\eta},1}(u, \tau)\eta(\tau)^{-n},$$

where $\Theta_{\mu,N}(u,\tau)$ is defined in (4.10b). Our proof of (4.12) consists of two steps.

Step 1. (4.12a) is equal to (4.12c).
We set

$$\text{(B.2)} \qquad Y = \Theta^{(-)}_{\xi+\beta\mathcal{T}+\overline{\rho},L-1}(u,\tau)\Theta_{\overline{\eta},1}(u,\tau).$$

Because of (4.11) and (B.1), this equals to $\chi_{\xi+\beta\mathcal{T}}\chi_\eta$ in (4.12a) multiplied by $\eta(\tau)^n \Theta^{(-)}_{\overline{\rho},n+1}(u,\tau)$. On the other hand, a product of theta functions can be expanded in terms of the *Theta Null Werte* as [28. p188]

(B.3)
$$\Theta_{\mu_1,N_1}(u,\tau)\Theta_{\mu_2,N_2}(u,\tau) = \sum_{\mu \in Q/(N_1+N_2)Q} \Theta_{N_2\mu+\mu_1+\mu_2,N_1+N_2}(u,\tau) \times \Theta_{N_1N_2\mu - N_2\mu_1 + N_1\mu_2, N_1N_2(N_1+N_2)}(0,\tau).$$

Applying this to (B.2) we get

$$Y = \sum_{w \in S_{n+1}} \det w \sum_{\mu \in Q/LQ} \Theta_{\overline{\eta}+\mu+w(\overline{\xi}+\beta\mathcal{T}+\overline{\rho}),L}(u,\tau) \times \Theta_{(L-1)(\overline{\eta}+\mu)-w(\overline{\xi}+\beta\mathcal{T}+\overline{\rho}),L(L-1)}(0,\tau). \tag{B.4}$$

The forthcoming Lemma B.1 converts the summation over μ in (B.4) to those over $w' \in S^e \cdot S^o$ and $\lambda \in \mathcal{R}'_{\ell,n}(\zeta\mathcal{T})$. Here $\mathcal{R}'_{\ell,n}(\zeta\mathcal{T})$ is defined by (3.4d) with the following condition (R′) instead of (R) :

$$\begin{aligned} (\mathrm{R}') \qquad & L + a_{n'} \geq a_0 \geq a_2 \geq \cdots \geq a_{n'}, \\ & L + a_{n''} \geq a_1 \geq a_3 \geq \cdots \geq a_{n''}. \end{aligned} \tag{B.5}$$

Lemma B.1. *For each $w \in S_{n+1}$ and $\mu \in Q/LQ$, there exist some $w' \in S^e \cdot S^o$ and the unique $\lambda \in \mathcal{R}'_{\ell,n}(\zeta\mathcal{T})$ such that*

$$\overline{\eta} + \mu + w(\overline{\xi} + \beta\mathcal{T} + \overline{\rho}) \equiv ww'(\overline{\lambda} + \beta\mathcal{T} + \overline{\rho}) \qquad \mathrm{mod}\ LQ. \tag{B.6}$$

Proof. Due to the invariance $w'(\mathcal{T}) = \mathcal{T}$ for $w' \in S^e \cdot S^o$, (B.6) is equivalent to the condition

$$\overline{\xi} + \overline{\rho} + w^{-1}(\overline{\eta} + \mu) \equiv w'(\overline{\lambda} + \overline{\rho}) \qquad \mathrm{mod}\ LQ. \tag{B.7}$$

Write the LHS as $r_0 e_0 + \cdots + r_n e_n$. In view of $\mathrm{mod}\ LQ$, r_i's can be reduced to the domain $-L < r_i - r_j \leq L$ for $i - j \in 2\mathbf{Z}_{\geq 0}$ and $i - j \in 2\mathbf{Z} + 1$. We rearrange them in the order:

$$\begin{aligned} & L + r_{i_{n'}} \geq r_{i_0} \geq r_{i_2} \geq \cdots \geq r_{i_{n'}}, \\ & L + r_{i_{n''}} \geq r_{i_1} \geq r_{i_3} \geq \cdots \geq r_{i_{n''}}, \end{aligned} \tag{B.8}$$

where $(i_0, i_2, \cdots, i_{n'})$ (resp. $(i_1, i_3, \cdots, i_{n''})$) is a (not necessarily unique) permutation of $(0, 2, \cdots, n')$ (resp. $(1, 3, \cdots, n'')$). Then the following choice of $w' \in S^e \cdot S^o$, $\lambda \in \mathcal{R}'_{\ell,n}(\zeta\mathcal{T})$ satisfies (B.7).

$$w' = \begin{pmatrix} 0, 2, \cdots, n' \\ i_0, i_2, \cdots, i_{n'} \end{pmatrix} \cdot \begin{pmatrix} 1, 3, \cdots, n'' \\ i_1, i_3, \cdots, i_{n''} \end{pmatrix}, \tag{B.9a}$$

$$\begin{aligned} \lambda = & (L + r_{i_n} - r_{i_0} - 1)\Lambda_0 + (r_{i_0} - r_{i_1} - 1)\Lambda_1 + \cdots \\ & + (r_{i_{n-1}} - r_{i_n} - 1)\Lambda_n. \end{aligned} \tag{B.9b}$$

Here w' (B.9a) stands for the permutation replacing ϵ_0 by ϵ_{i_0}, etc.

Q.E.D.

In the working below, the following properties of theta functions will be utilized.

(B.10a)
$$\begin{aligned}\Theta_{w(\mu),N}(0,\tau) &= \Theta_{\mu,N}(0,\tau)\\ \Theta^{(-)}_{w(\mu),N}(u,\tau) &= \det w\, \Theta^{(-)}_{\mu,N}(u,\tau)\end{aligned} \qquad \text{for } w \in S_{n+1},$$

(B.10b)
$$\begin{aligned}\Theta_{\mu,N}(u,\tau) &= \Theta_{\mu',N}(u,\tau)\\ \Theta^{(-)}_{\mu,N}(u,\tau) &= \Theta^{(-)}_{\mu',N}(u,\tau)\end{aligned} \qquad \text{if } \mu \equiv \mu' \mod NQ,$$

(B.10c)
$$\Theta^{(-)}_{\mu,N}(u,\tau) = 0 \quad \text{if } (\langle \mu, \epsilon_0 - \epsilon_n\rangle - N)\prod_{i=1}^{n}\langle \mu, \epsilon_{i-1} - \epsilon_i\rangle = 0.$$

Now we substitute (B.7) into (B.4). Performing the summation over $w \in S_{n+1}$ by using (4.10) and (B.10a), we get

(B.11)
$$\begin{aligned}Y = \sum_{\lambda \in \mathcal{R}'_{\ell,n}(\zeta\mathcal{T}),\, \overline{\lambda}\equiv\overline{\xi}+\overline{\eta},\ \mathrm{mod}\ Q} \Theta^{(-)}_{\overline{\lambda}+\beta\mathcal{T}+\overline{\rho},L}(u,\tau) \sum_{w\in S^e\cdot S^o} \det w\\ \Theta_{(L-1)w(\overline{\lambda}+\beta\mathcal{T}+\overline{\rho})-L(\overline{\xi}+\beta\mathcal{T}+\overline{\rho}),L(L-1)}(0,\tau)/m(\lambda+\rho).\end{aligned}$$

Here $m(\mu)\,(\mu \in \mathcal{H}^*)$ is the number of elements in $S^e \cdot S^o$ fixing the classical part $\overline{\mu}$ mod LQ. Because of the support property (B.10c), the summand in (B.11) vanishes unless $\lambda \in \mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ in which case one has $m(\lambda+\rho) = 1$. This establishes the step 1.

*Step.*2 (4.12b) is equal to (4.12c).

Note that the identity shown in step 1 can be written as follows.

(B.12a)
$$\begin{aligned}Y = \sum_{a\in\mathcal{R}_{\ell,n}(\zeta\mathcal{T}),\,\overline{a}\equiv\overline{\xi}+\overline{\eta},\ \mathrm{mod}\ Q}\ \sum_{w\in S^e\cdot S^o}\sum_{\alpha\in Q}\Theta^{(-)}_{y,L}(u,\tau)\\ \times q^{\frac{L(L-1)}{2}\left(y/L-(\overline{\xi}+\beta\mathcal{T}+\overline{\rho})/(L-1)\right)^2},\end{aligned}$$

(B.12b)
$$y = L\alpha + w(\overline{a}+\beta\mathcal{T}+\overline{\rho}),$$

where we have used (4.12c, d) and (B.10a, b). Similarly, (4.12b) multiplied by $\eta(\tau)^n\Theta^{(-)}_{\overline{\rho},n+1}(u,\tau)$ is expressed in the same form (B.12) but with the sum over $a \in \mathcal{S}_{\ell,n}(\zeta\mathcal{T})$ and $\alpha \in Q_e \oplus Q_o$ in place of $a \in \mathcal{R}_{\ell,n}(\zeta\mathcal{T})$ and $\alpha \in Q$, respectively. Note that the running variables w, a and α appear in the summand only through the combination y in (B.12b). Moreover it can be shown that these values of y are all distinct for both cases. In view of this and $w(\mathcal{T}) = \mathcal{T}$, it suffices to check the following in order to complete the step 2.

Lemma B.2.

$$\coprod_{w \in S^e \cdot S^o} B_w = \coprod_{w \in S^e \cdot S^o} C_w, \tag{B.13}$$

where the sets B_w and C_w are defined by

$$B_w = \{L\alpha + w(\overline{a} + \overline{\rho}) \mid a \in \mathcal{S}_{\ell,n}(0),\ \alpha \in Q_e \oplus Q_o\}, \tag{B.14a}$$
$$C_w = \{L\alpha + w(\overline{a} + \overline{\rho}) \mid a \in \mathcal{R}_{\ell,n}(0),\ \alpha \in Q\}. \tag{B.14b}$$

Proof. We show $\coprod_{w \in S^e \cdot S^o} B_w \supseteq \coprod_{w \in S^e \cdot S^o} C_w$. The proof of the opposite inclusion is similar. Take an element $x = L\alpha + w(\overline{a} + \overline{\rho}) \in \coprod_w C_w$. Since $x \equiv w(\overline{a} + \overline{\rho}) \bmod LQ$, $a \in \mathcal{R}_{\ell,n}(0)$, there is a unique $\alpha' \in Q_e \oplus Q_o$ such that

$$\begin{aligned} &L\alpha' + r_0 e_0 + \cdots + r_n e_n = x, \\ &|r_i - r_j| \in \{1, 2, \cdots, L-1\} \quad \text{for } i - j = \text{ even}. \end{aligned} \tag{B.15}$$

Let $(i_0, i_2, \cdots, i_{n'})$ (resp. $(i_1, i_3, \cdots, i_{n''})$) be the permutation of $(0, 2, \cdots, n')$(resp. $(1, 3, \cdots, n'')$) uniquely specified by the requirement

$$\begin{aligned} &L + r_{i_{n'}} > r_{i_0} > r_{i_2} > \cdots > r_{i_{n'}}, \\ &L + r_{i_{n''}} > r_{i_1} > r_{i_3} > \cdots > r_{i_{n''}}. \end{aligned} \tag{B.16}$$

Using these i_k 's, define $w' \in S^e \cdot S^o$ and $\lambda \in \mathcal{S}_{\ell,n}(0)$ by (B.9). Then we have $x = w'(\lambda) + L\alpha' \in \coprod_w B_w$. Q.E.D.

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