Advanced Studies in Pure Mathematics 19, 1989
Integrable Systems in Quantum Field Theory and Statistical Mechanics
pp. 287–346

From the Harmonic Oscillator to the A-D-E Classification of Conformal Models

C. Itzykson

§1. Introduction

There is a growing feeling that arithmetics will play a significant role in understanding various physical models. This is not a recent discovery. Arithmetic questions arise in a large array of problems ranging from crystallography to complex dynamical systems, from the quantization conditions to applications of group theory and, since this is the subject of the present meeting, in the beautiful interplay between solvable two dimensional statistical models and conformal field theory. This goes perhaps against the opinion that the "queen of mathematics" is to be safeguarded from any application. On the other hand, some physicists have recently coined the name "recreational mathematics" to describe these matters. Let the reader decide if this is an appropriate characterization.

In this contribution I would like to exhibit a few simple examples drawn from work with several collaborators, J. M. Luck and E. Aurell for the first two, A. Cappelli and J.-B. Zuber for the next one and more recently M. Bauer. It is a nice opportunity to thank them warmly again, as well as the organizers of the Taniguchi meeting in Kyoto and my colleagues at the Physics Department of the University of Tokyo for their kind hospitality.

§2. The harmonic oscillator

The harmonic oscillator is one of the simplest integrable systems with energy levels

$$\epsilon_n = \epsilon_0(2n+1) \qquad n \text{ non negative integer} \tag{2.1}$$

in an appropriate scale. Consider a gas of non interacting fermions allowed to occupy these levels, at equilibrium at temperature T and

Received January 9, 1989.

chemical potential μ. If E and N denote the total energy and number of particles, the grand canonical partition function is

$$z = \sum \exp \frac{\mu N - E}{kT} = \prod_{n=0}^{\infty} (1 + yq^{2n+1}) \tag{2.2}$$

provided we set

$$y = \exp \frac{\mu}{kT} \qquad q = \exp - \frac{\epsilon_0}{kT}.$$

The magnitude of q is smaller than unity, since ϵ_0 is positive, hence z extends as an entire function in the variable y. The coefficients of z in a power series expansion in y are the canonical partition functions corresponding to a fixed number of particles

$$z(y,q) = \sum_{N=0}^{\infty} y^N \sum_{0 \le n_1 < n_2 < \cdots < n_N} q^{\sum_{s=1}^{N}(2n_s+1)}.$$

The N values of s label the occupied levels arranged according to increasing energy. Each configuration can be considered as an excitation over a ground state, where the first N single particle levels are occupied, with minimal energy $E_0 = \epsilon_0 \sum_{n=0}^{N-1}(2n+1) = \epsilon_0 N^2$. This means that $2n_s + 1 = 2s - 1 + 2l_s$, $0 \le l_1 \le l_2 \cdots \le l_N$, and

$$z(y,q) = \sum_{N=0}^{\infty} y^N q^{N^2} \sum_{0 \le l_1 \le l_2 \cdots \le l_N} q^{2\sum_{s=1}^{N} l_s}.$$

The last sum is readily performed, first on l_N,then on $l_{N-1}, \ldots$, resulting in an identity due to Euler

$$z(y,q) = \prod_{n=0}^{\infty} (1+yq^{2n+1}) = 1 + \sum_{N=1}^{\infty} \frac{y^N q^{N^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2N})}. \tag{2.3}$$

Let us generalize the problem to a gas of fermions and antifermions with opposite chemical potential so that the new conserved quantity, again denoted N, is the charge, equal to the number of fermions minus the number of antifermions. The partition function reads

$$Z(y,q) = \sum_{N=-\infty}^{\infty} y^N Z_N(q) = \prod_{n=0}^{\infty} (1 + yq^{2n+1})(1 + y^{-1}q^{2n+1}). \tag{2.4}$$

Since $Z_N(q) = Z_{-N}(q)$ it is sufficient to compute this canonical partition function for $N \ge 0$. As usual, when dealing with particles and antiparticles it is convenient to introduce negative single particle energy states. In the neutral vacuum all negative one particle states are occupied. Any other Fock state can then be described by enumerating some filled positive energy states (particles) and holes in the negative energy sea (anti-particles). In a sector with fixed positive charge N, the lowest possible energy is such that the first N particle states are occupied ($E_{min} = \epsilon_0 N^2$) and excitations above that level are described in identical terms as those above the ground state, showing that

$$Z_N(q) = Z_{-N}(q) = q^{N^2} Z_0(q).$$

We can relate any neutral excited state to the vacuum by setting a correspondence between occupied levels of increasing energy as indicated in Figure 1. Neutrality and finite excitation energy ensure that far enough down the ladder, the corresponding levels have equal energy, and

$$E = 2\epsilon_0(r_1 + r_2 + \cdots)$$

with $r_1 \geq r_2 \geq \cdots \geq 0$. To ensure that E is finite, only a finite number of r's can be different from zero. Hence each partition of the integer $E/2\epsilon_0$ is in one to one correspondence with a neutral state of energy E. Denoting by α_k the number of r's equal to k, we have

$$Z_0(q) = \prod_{k=1}^{\infty} \sum_{\alpha_k=0}^{\infty} q^{2k\alpha_k} = \frac{1}{\prod_{k=1}^{\infty}(1-q^{2k})} \tag{2.5}$$

which is Euler's generating function for the number of partitions.

Collecting the above results, we find Jacobi's triple product identity in the form

$$\begin{aligned} \theta(y,q) = yq\theta(yq^2,q) &= \sum_{N=-\infty}^{+\infty} y^N q^{N^2} \\ &= \prod_{k=1}^{\infty}(1-q^{2k})(1+yq^{2k-1})(1+y^{-1}q^{2k-1}). \end{aligned} \tag{2.6}$$

Professor Andrews called my attention to the fact that he already met this combinatorial proof (Andrews–1984) which might possibly occur in much earlier work.

The next step is to introduce, besides fermions and antifermions, bosons and antibosons with equal chemical potentials in absolute value.

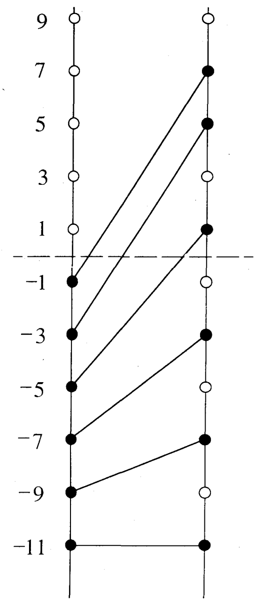


Fig. 1. Correspondence between occupied states in the neutral sector

The complete partition function
(2.7)

$$\Xi(y,q) = \frac{\theta(y,q)}{\theta(-y,q)} = \prod_{k=0}^{\infty} \frac{(1+yq^{2k+1})(1+y^{-1}q^{2k+1})}{(1-yq^{2k+1})(1-y^{-1}q^{2k+1})} = \sum_{N=-\infty}^{+\infty} \Xi_N \cdot y^N$$

is meromorphic in y in the complex plane with the origin deleted. The last series is only valid for $|q| < |y| < |q|^{-1}$. Taking into account the relation

$$\Xi(q^2 y, q) = -\Xi(y,q).$$

Cauchy's residue theorem applied to $\Xi(y,q)y^{-N}$ in an annulus bounded by the circles $|y| = 1$ and $|y| = |q^2|$ which encircle once the simple pole at $y = q$, yields

$$
\begin{aligned}
\Xi(y,q) &= \prod_{k=0}^{\infty} \frac{(1+yq^{2k+1})(1+y^{-1}q^{2k+1})}{(1-yq^{2k+1})(1-y^{-1}q^{2k+1})} \\
&= 2\prod_{k=1}^{\infty}\left(\frac{1+q^{2k}}{1-q^{2k}}\right)^2 \sum_{N=-\infty}^{+\infty} \frac{q^N}{1+q^{2N}}\, y^N
\end{aligned}
\tag{2.8}
$$

an identity again due to Jacobi valid in the annulus $|q| < |y| < |q|^{-1}$. Setting $y = 1$ and using Euler's identity

$$
\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n-1}) = 1 \tag{2.9}
$$

we obtain the remarkable relation

$$
\theta(1,q)^2 = \sum_{n_1,n_2=-\infty}^{+\infty} q^{n_1^2+n_2^2} = 1 + 4\sum_{N=1}^{\infty}\frac{q^N}{1+q^{2N}} \tag{2.10}
$$

which will be used in the next section to study energy levels in a square box.

Returning to the partition function (2.8), we identify the coefficient of y^0 in

$$
\Xi(y,q^{1/2})\,\Xi(-y,q^{1/2}) = 1
$$

make use of the triple product identity as well as (2.9), and finally change q into $-q$, to get a relation analogous to (2.10) but valid in four dimensions

$$
\begin{aligned}
\theta(1,q)^4 &= \sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{n_1^2+n_2^2+n_3^2+n_4^2} = 1 + 8\sum_{N=1}^{\infty}\frac{q^N}{[1+(-q)^N]^2} \\
&= 1 + 8\sum_{\substack{N=1 \\ N\not\equiv 0 \bmod 4}}^{\infty} \frac{Nq^N}{1-q^N}
\end{aligned}
\tag{2.11}
$$

where the last expression results from the previous one by expanding the denominators and resuming in the opposite order. The reader will note that none of the above techniques apply to sums of an odd number of squares. For these consult (Weil–1974).

Let me mention two other identities which express similar facts as (2.10) and (2.11) when squares are replaced by equilateral triangles

$$(2.12)\quad \triangle(q) = \theta(1,q)\theta(1,q^3) + q\theta(q,q)\theta(q^3,q^3)$$
$$= \sum_{n_1,n_2=-\infty}^{+\infty} q^{n_1^2+n_2^2+n_1n_2} = 1 + 6\sum_{N=1}^{\infty} \frac{q^N}{1+q^N+q^{2N}},$$

$$(2.13)\quad \triangle(q)^2 = \sum_{n_1,n_2,n_3,n_4=-\infty}^{+\infty} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1n_2+n_3n_4}$$
$$= 1 + 12 \sum_{\substack{N=1\\ N\not\equiv 0 \bmod 3}}^{\infty} \frac{Nq^N}{1-q^N}.$$

Of course the above results translate into statements on representations of integers by quadratic forms, which go back to Fermat and Lagrange.

§3. Free particle in a box

There are a few cases in which one can find exact expressions for the quantum energy levels of a particle enclosed in a compact domain of Euclidean space bounded by finitely many (hyper-)planes, but otherwise free. The wave functions are required to vanish on the boundary. If we look for the absence of "diffraction", meaning that eigenfunctions can be written as superpositions of finitely many plane waves, the Schwarz reflection principle implies that the group generated by the reflections in the walls of the box must tile the space. These Weyl-Coxeter groups (Coxeter–1973) are in one to one correspondence with complex semi-simple algebras, one of the themes of this discussion. In this section we shall confine ourselves to rank two algebras, meaning that we deal with two dimensional (integrable) problems. These are of four types, A_2 an equilateral triangle, $B_2 \sim C_2$ a triangle with angles $\pi/2$, $\pi/4$, $\pi/4$ obtained by cutting a square along a diagonal, $D_2 \sim A_1 \times A_1$ a rectangle, and G_2 half of an equilateral triangle with angles $(\pi/2, \pi/3, \pi/6)$. In convenient units the energies are simply the integral eigenvalues of the quadratic Casimir operator. It seems that Lamé was the first around 1850 to obtain the spectrum in the equilateral case.

Berry drew attention to the fact that "accidental" degeneracies in such two dimensional cases have a very intriguing pattern (Berry–1981) first observed by the mathematician Landau. For definiteness let us look in parallel at the cases of a square or an equilateral triangle with energy levels

$$E_{\square} = n_1^2 + n_2^2 \qquad n_1, n_2 > 0, \tag{3.1}$$

$$E_{\triangle} = n_1^2 + n_2^2 + n_1 n_2 \qquad n_1, n_2 > 0. \tag{3.2}$$

We have dropped inessential factors, so that with n_1, n_2 integers, the energies are themselves integers. To recover the spectrum of the triangles one should impose $n_1 > n_2$ for instance. Interchanging n_1, n_2 for $n_1 \neq n_2$ leads to an obvious degeneracy. The D_2 case corresponds to a rectangle which does not exhibit unexpected degeneracies when the ratio of its sides is irrational; when this ratio is rational, this is a (difficult) generalization of (3.1).

One might not suspect at first such spectra as (3.1) to exhibit anything spectacular. Since a large circle with area $\pi E_{\max}$ contains approximately $\pi/4 \cdot E_{\max}$ points of the spectrum (3.1) with integral energies up to $E_{\max}$, the asymptotic density of levels, $\pi/4$, slightly smaller than one, would seem to indicate a smooth behavior. It comes therefore as a surprise to discover that the fraction of integers belonging to the spectrum up to the maximal value $E_{\max}$, or equivalently the probability of an integer up to $E_{\max}$, to belong to the spectrum, vanishes like $(\ln E_{\max})^{-1/2}$ for large $E_{\max}$! The above estimate would have rather suggested that it tends to a value $\pi/4$. In fact both spectra (3.1) and (3.2) are extremely wild as we proceed to show, following work done in collaboration with J. M. Luck.

Write $\overline{D}(E)$ for the degeneracy (multiplicity) of level E and define $D(E) = \overline{D}(E)$ if E is not a square, $D(E) = \overline{D}(E) + 1$ if E is a square. It is easier to work analytically with $D(E)$ while , due to the scarcity of squares, this does not affect the asymptotic estimates to be obtained below. Corrections to return from $D(E)$ to $\overline{D}(E)$ could be easily supplied. By $EF(E, d)$ we shall understand the number of positive integers $E' \leq E$ such that $D(E)$ takes the value d. Let us study the moments of the distribution of degeneracies

$$\mu_k(E) = \sum_d d^k F(d, E). \tag{3.3}$$

If $k = 0$, the sum ranges over $d > 0$. The reason for introducing $D(E)$ rather than $\overline{D}(E)$ appears when we expand both sides of (2.10) as double series

$$\sum_{n=1}^{\infty} q^{n^2} + \sum_{n_1,n_2=1}^{\infty} q^{n_1^2+n_2^2} = \sum_{E=1}^{\infty} D(E) q^E \tag{3.4}$$
$$= \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(q^{k(1+4l)} - q^{k(3+4l)})$$

meaning that each divisor of E of the form $4l \pm 1$ contributes ± 1 to $D(E)$. Denote generically by p and r the odd primes equal to $+1$ and -1 mod 4 respectively. If the prime decomposition of $E_\Box$ reads

$$E_\Box = 2^\sigma p_1^{\alpha_1} p_2^{\alpha_2} \cdots r_1^{\beta_1} r_2^{\beta_2} \cdots \qquad p_i \equiv 1\ (4) \quad r_j \equiv -1\ (4) \tag{3.5}$$

then

$$D_\Box(E) = \sum_{\substack{0\le a_i\le\alpha_i \\ 0\le b_j\le\beta_j}} (-1)^{\sum_j b_j} = \prod_i (1+\alpha_i) \prod_j \left(\frac{1+(-1)^{\beta_j}}{2}\right). \tag{3.6}$$

This states that odd primes equal to -1 mod 4 must occur in even power for E to belong to the spectrum, in which case $D_\Box(E)$ is equal to the number of its odd divisors factorizable into primes equal to 1 mod 4. Conversely from (3.6) one can derive the identity (2.10) showing how arithmetics and elliptic functions come naturally into contact. The study of the degeneracies is thus a variant of the study of the number of divisors, both share the same "unpredictability " in the distribution of primes.

Using the same convention in the equilateral triangle case, it follows from (2.12) that a similar formula holds for $D_\triangle(E)$ provided we replace p and r by primes of the form $+1$ and -1 mod 3 and expand $E_\triangle$ as

$$E_\triangle = 3^\sigma p_1^{\alpha_1} p_2^{\alpha_2} \cdots r_1^{\beta_1} r_2^{\beta_2} \cdots \qquad p_i \equiv 1\ (3) \quad r_j \equiv -1\ (3). \tag{3.7}$$

In spite of being explicit, the formula for $D(E)$ is not as informative as one might hope. Following Dirichlet it is useful to introduce the series

$$\eta_k(s) = \sum_{n=1}^{\infty} \frac{D^k(n)}{n^s} \tag{3.8}$$

which allow to obtain the asymptotic behavior of the moments $\mu_k(E)$ defined above. The explicit expression for $D(E)$ enables one to express $\eta_k(s)$ as Eulerian products, the underlying reason being that the quadratic forms under consideration are norms of integers in simple quadratic fields. Explicitly

$$\eta_{k,\square}(s) = \frac{1}{1-2^{-s}} \prod_{\substack{p \text{ prime} \\ p\equiv 1(4)}} \frac{P_k(p^{-s})}{(1-p^{-s})^{k+1}} \times \prod_{\substack{r \text{ prime} \\ r\equiv -1(4)}} \frac{1}{(1-r^{-2s})}, \tag{3.9a}$$

$$\eta_{k,\triangle}(s) = \frac{1}{1-3^{-s}} \prod_{\substack{p \text{ prime} \\ p\equiv 1(3)}} \frac{P_k(p^{-s})}{(1-p^{-s})^{k+1}} \times \prod_{\substack{r \text{ prime} \\ r\equiv -1(3)}} \frac{1}{(1-r^{-2s})}. \tag{3.9b}$$

These expressions involve polynomials $P_k(x)$ such that

(3.10)
$$P_0(x) = P_1(x) = 1 \qquad P_2(x) = 1 + x \qquad P_3(x) = 1 + 4x + x^2 \qquad \cdots$$

and in general

$$P_k(x) = (1-x)^{k+1} \left(\frac{d}{dx} x\right)^k \frac{1}{(1-x)}. \tag{3.11}$$

To use the Dirichlet series $\eta_k(s)$, obviously analytic for $\operatorname{Re} s$ large enough, in the estimate of moments, one may proceed in the classical fashion as follows. First, using a representation of the step function, we express $\mu_k(E)$ as

$$\mu_k(E) = \int_{\operatorname{Re} s = c} \frac{d\operatorname{Im} s}{2\pi} E^{s-1} \eta_k(s) \tag{3.12}$$

with the line $\operatorname{Re} s = c$ in the analyticity domain. The contour is then displaced to the left until we hit a singularity in $\eta_k(s)$. The latter occurs at $s = 1$, as a pole for $k \geq 1$ or a branch point for $k = 0$, as follows from (3.9). The contribution from this singularity yields the dominant behavior of $\mu_k(E)$ for large E. Alternatively, and more sloppily, a behavior $\mu_k(E) \sim a(\ln E)^r$ corresponds in $\eta_k(s)$ to a singularity of the type

$$a\int_1^\infty dn (\ln n)^\gamma n^{-s} = a\frac{\Gamma(\gamma+1)}{(s-1)^{\gamma+1}}.$$

Whichever way, we need to characterize the rightmost singularity of $\eta_k(s)$.

Consider first $\eta_1(s)$. Using the notation $\zeta(s)$ for Riemann's function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{(1 - p^{-s})} \tag{3.13}$$

we derive from (3.9)

$$\eta_1(s) = \zeta(s) L(s) \tag{3.14}$$

with

$$\begin{aligned} L_{\square}(s) &= \prod_{\substack{p \text{ prime} \\ p \equiv 1(4)}} \frac{1}{(1 - p^{-s})} \prod_{\substack{r \text{ prime} \\ r \equiv -1(4)}} \frac{1}{(1 + r^{-s})} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \end{aligned} \tag{3.15a}$$

$$\begin{aligned} L_{\triangle}(s) &= \prod_{\substack{p \text{ prime} \\ p \equiv 1(3)}} \frac{1}{(1 - p^{-s})} \prod_{\substack{r \text{ prime} \\ r \equiv -1(3)}} \frac{1}{(1 + r^{-s})} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right]. \end{aligned} \tag{3.15b}$$

In both cases $L(s)$ is analytic for $\operatorname{Re} s > 0$ and

$$L_{\square}(1) = \frac{\pi}{4} \qquad L_{\triangle}(1) = \frac{\pi}{3^{3/2}}. \tag{3.16}$$

From the known behavior of $\zeta(s)$ —a pole at $s = 1$ with unit residue— we conclude that $\eta_1(s)$ is meromorphic for $\operatorname{Re} s > 0$, with a pole at $s = 1$ of known residue. This leads to

$$E \to \infty \qquad \mu_{1,\square}(E) \sim \frac{\pi}{4} \qquad \mu_{1,\triangle}(E) \sim \frac{\pi}{3^{3/2}} \tag{3.17}$$

which of course agrees with the Weyl estimates for the leading behavior of the integrated density of levels.

We turn to $\mu_0(E)$, the quantity discussed by Berry. Comparison between $\eta_0(s)$ and $\eta_1(s)$ given by (3.9) yields

$$[\eta_{0,\square}(s)]^2 = \eta_{1,\square}(s)\frac{1}{1-2^{-s}} \prod_{\substack{r \text{ prime} \\ r\equiv -1(4)}} \frac{1}{1-r^{-2s}}, \tag{3.18a}$$

$$[\eta_{0,\triangle}(s)]^2 = \eta_{1,\square}(s)\frac{1}{1-3^{-s}} \prod_{\substack{r \text{ prime} \\ r\equiv -1(3)}} \frac{1}{1-r^{-2s}}. \tag{3.18b}$$

The products converge for $\operatorname{Re} s > \frac{1}{2}$, hence cannot vanish in this region, so that one can take their square root (positive on the real axis). From the expression (3.14) we deduce that $\eta_0(s)$ has a branch point at $s = 1$, and

(3.19)

$$E \to \infty \qquad \mu_{0,\square}(E) \sim (2A_\square \ln E)^{-\frac{1}{2}} \qquad \mu_{0,\triangle} \sim (2\sqrt{3}A_\triangle \ln E)^{-\frac{1}{2}}$$

where $A_\square$ and $A_\triangle$ stand for the finite constants

$$A_\square = \prod_{\substack{r \text{ prime} \\ r\equiv -1\ (4)}} (1-r^{-2}) \qquad A_\triangle = \prod_{\substack{r \text{ prime} \\ r\equiv -1\ (3)}} (1-r^{-2}). \tag{3.20}$$

As previously mentioned, these results mean that, far from spreading uniformly over the integers, the levels are highly degenerate and leave huge unoccupied holes, so that asymptotically a particular integer has a vanishing probability of belonging to the spectrum. From the almost constant "density" (3.17), one would be tempted to conclude that $D(E)$ scales like $(\ln E)^{1/2}$. This is supported by the computation of $\mu_2(E)$ but not true for higher moments. Indeed, with the notation introduced in (3.15)

$$\eta_{2,\square}(s) = \frac{1}{1+2^{-s}}\,\zeta^2(s)\zeta(2s)^{-1}L_\square^2(s) \tag{3.21a}$$

$$\eta_{2,\triangle}(s) = \frac{1}{1+3^{-s}}\,\zeta^2(s)\zeta(2s)^{-1}L_\triangle^2(s) \tag{3.21b}$$

we obtain

$$E \to \infty \qquad \mu_{2,\square}(E) \sim \frac{1}{4}\ln E \qquad \mu_{2,\triangle}(E) \sim \frac{1}{6}\ln E. \tag{3.22}$$

Note the rational coefficients. For larger values of k, comes however the surprise. It is advantageous to replace the polynomial $P_k(x)$ of degree

$k-1$ by

$$Q_k(x) = (1-x)^{2^k-k-1} P_k(x) \tag{3.23}$$

where the prefactor insures that $Q_k(x) - 1$ vanishes like x^2 for small x. One then finds

$$E \to \infty \quad \mu_k(E) \sim a_k (\ln E)^{2^{k-1}-1} \qquad k \geq 0 \tag{3.24}$$

with constants a_k given by

$$a_{k,\square} = \frac{\pi}{4}\left(\frac{\pi A_\square}{8}\right)^{2^{k-1}-1} [\Gamma(2^{k-1})]^{-1} \prod_{\substack{p \text{ prime} \\ p\equiv 1\ (4)}} Q_k(p^{-1}), \tag{3.25a}$$

$$a_{k,\triangle} = \frac{\pi}{3^{3/2}}\left(\frac{\pi A_\triangle}{3^{5/2}}\right)^{2^{k-1}-1} [\Gamma(2^{k-1})]^{-1} \prod_{\substack{p \text{ prime} \\ p\equiv 1\ (3)}} Q_k(p^{-1}). \tag{3.25b}$$

The rapid growth of the moments $\mu_k(E)$ as $(\ln E)^{2^{k-1}-1}$ is quite remarkable and perhaps unexpected. We refer to our paper (Itzykson and Luck–1986) for a study of corrections and similar properties in higher dimension.

From the point of view of integrable systems, we understand that degeneracies are related to the existence of higher conservation laws. For instance in the case of the equilateral triangle we have a cubic invariant and for the square a quartic one. It would be interesting to investigate these properties for higher rank Lie algebras and understand in more detail the arithmetic behavior of the Casimir invariants.

§4. Rational billiards

Let us divert for a while from integrable systems to see yet another connection with arithmetics. The simplest system to study as close as possible to an integrable one is a "quantum billiard", which we take again two-dimensional, in the shape of a triangle but such that the group generated by the reflections in the boundaries does not necessarily tile the plane. We assume that the group is "locally finite", by which we mean that the internal angles of the triangle measured in units of π are rational fractions. When some of these fractions in irreducible terms have a numerator larger than unity, we loose the tiling property. It was observed by Richens and Berry that the classical phase space of a free particle bouncing in such a billiard is foliated by compact orientable two dimensional surfaces in a four dimensional phase space (Richens and Berry–1981). The genus of g and Euler characteristics χ of these surfaces are given by

$$\chi = 2 - 2g = \sum_i \frac{Q}{q_i}(1 - p_i) \tag{4.1}$$

where the index i takes three values, we choose $i = (0, 1, \infty)$, the inner angles of the triangle are written $\phi_i = \pi p_i/q_i$ with p_i and q_i coprimes, and Q stands for the lowest common multiple of the denominators q_i, in fact of two of them, since

$$\sum_i \frac{p_i}{q_i} = 1. \tag{4.2}$$

In general we write (p, q) for the greatest common divisor of p and q and we use indifferently the notations $a \equiv 0 \bmod b$, $a \equiv 0(b)$ or $b|a$.

If we require the numerators p_i to be unity, we recover the three integrable triangles (genus 1, tori) discussed in the previous section. To derive formula (4.1) one repeats in essence the reasoning of Riemann and Hurwitz by gluing together triangles corresponding to the finitely many possible directions of the particle velocity (generically $2Q$).

It follows from the Riemann mapping theorem that, no matter which values the angles φ_i may take (possibly φ_i/π irrational) such a triangle can be mapped conformally one to one on the upper half plane or inside the unit circle, with its edges on the boundary and vertices at three preassigned points of the latter. This gives access to a Green function, the inverse of the Laplacian with Dirichlet boundary conditions on the sides of the triangle. As a result one can write for any integer $n \geq 2$ an explicit integral representation for the sum of inverse n-th powers of the energy levels. With J. M. Luck and P. Moussa we investigated these integrals which have some similarity with field correlations in conformal or string field theory.

Let us concentrate on the case where the angles φ_i/π are rational, and show that one can attach to such a triangle an algebraic curve such that triangles and curves come into families with interesting automorphisms and/or analytic maps (Aurell and Itzykson–1988).

I collect first some data in table I using the following notation. Let $l_i = Q/q_i$. We can then write the angles as $\varphi_i = \pi P_i/Q$, with $P_i = l_i p_i$, so that it is sufficient to list P_0, P_1, P_∞ with sum Q such that these integers have no common factor. Up to genus 5 we find the possibilities listed in table I, disregarding a permutation of P_0, P_1, P_∞. The table suggests that the sum $Q = P_0 + P_1 + P_\infty$ ranges between $2g+1$ and $2(2g+1)$.

Proposition 4.1. *Let Q and g be defined as above, then*

$$2g+1 \leq Q \leq 2(2g+1). \tag{4.3}$$

The following proof is due to J.-M. Luck. The first inequality follows from the definition of g written as

$$Q = 2g+1+\sum_i (l_i - 1). \tag{4.4}$$

It is only saturated when all l_i are to one, so that Q is then necessarily odd, and we can take as an example the triangle $\{P_0, P_1, P_\infty\} = \{1, g, g\}$. Henceforth we denote a triangle by the symbol $\{P_0, P_1, P_\infty\}$.

Since the P_i's are without common factor, so are the $l_i = (P_i, Q)$. For every prime p, there exists at least one l_i prime to p, and therefore at least a pair Q, $q_i = Q/l_i$ which admits the same maximal power of p as a common divisor. This same power of p divides the product $q_j q_k$ since it divides the lowest common multiple Q of q_j and q_k. It follows that Q^2 divides $q_0 q_1 q_\infty$ and hence

$$Q \geq \frac{Q}{q_0}\frac{Q}{q_1}\frac{Q}{q_\infty} = l_0 l_1 l_\infty.$$

Let us order the three integers l_i, for instance $l_0 \geq l_1 \geq l_\infty$. Let $q_0 \equiv Q/l_0 = m \geq 2$, where the last inequality says that each angle is smaller than π. It follows from the above that $m \geq l_1 l_\infty$ and therefore that $l_1 + l_\infty \leq m+1$. Consequently

$$l_0 + l_1 + l_\infty \leq \frac{Q}{m} + m + 1$$

for m a divisor of Q larger or equal to 2. If $m = Q$, $l_0 = 1$. Since by assumption $l_0 \geq l_1 \geq l_\infty$, it follows that $l_0 = l_1 = l_\infty$ and we recover the lower bound. Let us therefore assume $2 \leq m \leq Q/2$. In this interval the function $\frac{Q}{m} + m + 1$ assumes its maximum $Q/2+3$ at both ends and $l_0 + l_1 + l_\infty \leq Q/2 + 3$ or equivalently $\sum(l_i - 1) \leq Q/2$. Inserting this in equation (4.4) yields the second inequality in the proposition.

g	P_0	P_1	P_∞	
$g=1$	1	1	1	A_2
	2	1	1	B_2
	3	2	1	G_2
$g=2$	2	2	1	
	3	1	1	
	4	1	1	
	4	3	1	
	5	3	2	
	5	4	1	
$g=3$	3	2	2	
	3	3	1	
	4	2	1	
	5	1	1	
	3	3	2	
	5	2	1	
	6	1	1	
	4	3	2	
	5	3	1	
	6	2	1	
	5	4	3	
	6	5	1	
	8	3	1	
	7	4	3	
	7	5	2	
	7	6	1	
$g=4$	4	4	1	
	5	2	2	
	7	1	1	
	4	3	3	
	6	3	1	
	7	2	1	
	8	1	1	
	7	3	2	
	7	4	1	
	9	2	1	
	6	5	4	
	7	5	3	
	9	5	1	
	10	3	2	
	8	5	3	
	8	7	1	
	9	5	4	
	9	7	2	
	9	8	1	
$g=5$	4	4	3	
	5	3	2	
	5	4	2	
	5	5	1	
	6	3	2	
	6	4	1	
	7	2	2	
	7	3	1	
	8	2	1	
	9	1	1	
	5	5	2	
	10	1	1	
	8	5	2	
	10	4	1	
	10	7	3	
	10	9	1	
	11	6	5	
	11	7	4	
	11	8	3	
	11	9	2	
	11	10	1	

Table I. **Rational triangles classified according to genus up to $g = 5$.**

It can only be saturated for Q even, in which case an example is the triangle $\{1, 2g, 2g+1\}$, which completes the proof.

It follows obviously that the number of triangles of fixed genus is finite.

Let t denote a complex coordinate in the plane of the triangle and x in the upper half plane $\operatorname{Im} x > 0$. Let the vertices of the triangle be mapped respectively on $x = 0, 1, \infty$ and choose a length scale such that

$$|t_1 - t_0| = \frac{\Gamma\left(\frac{P_0}{Q}\right)\Gamma\left(\frac{P_1}{Q}\right)}{\Gamma\left(\frac{P_0+P_1}{Q}\right)}. \tag{4.5}$$

The Schwarz map from the triangle to the upper half plane is then

$$t - t_0 = \int_0^x \frac{dx'}{x'^{1-(P_0/Q)}(1-x')^{1-(P_1/Q)}} \qquad \operatorname{Im} x > 0. \tag{4.6}$$

For definiteness we assume $t_1 - t_0$ real positive, so that the roots in the integrand are such that it tends to a positive value for x' tending to a point on the segment $[0, 1]$. The area of the triangle is then a Veneziano like amplitude

$$A = \frac{\pi}{2} \cdot \frac{\Gamma\left(\frac{P_0}{Q}\right)\Gamma\left(\frac{P_1}{Q}\right)\Gamma\left(\frac{P_\infty}{Q}\right)}{\Gamma\left(\frac{P_1+P_\infty}{Q}\right)\Gamma\left(\frac{P_\infty+P_0}{Q}\right)\Gamma\left(\frac{P_0+P_1}{Q}\right)}. \tag{4.7}$$

When we choose a rational triangle, the derivative

$$y = \frac{dx}{dt} \tag{4.8}$$

is related to x by an algebraic equation

$$y^Q = x^{P_1+P_\infty}(1-x)^{P_\infty+P_0} \tag{4.9}$$

univocally associated to the triangle up to permutations of the vertices. The latter are generated by transpositions whose squares are automorphisms of the corresponding Riemann surface. It is readily checked that the curve (4.9) has a genus g given by formula (4.1) or (4.4) and that the Riemann surface is paved by $2Q$ triangles, each one the preimage of the upper or lower half x-plane with the points 0, 1, ∞ marked on the boundary. The desingularized triangulated surface is a model (even a conformal one) of the classical allowed region of phase space.

Each point 0, 1, ∞ is a singular point of the presentation (4.9) in the vicinity of which one can introduce a regularizing parameter u_i for each of the l_i families of $q_i = Q/l_i$ covering sheets which meet at this point over the x plane. From this one sees that

$$\omega^{(1)} = dt = \frac{dx}{y} \tag{4.10}$$

is a holomorphic differential on the curve with $2(g-1)$ zeroes.

For $g = 1$, the integrable case, the holomorphic differential is unique up to a constant factor and the corresponding abelian integral, the variable t, plays the role of global uniformization parameter. The correspondence between integrable triangles and elliptic curves of modular ratio τ reads

(4.11)

$$\begin{array}{llll} A_2 & \{1,1,1\} & y^3 = [x(1-x)]^2 & \tau = e^{2i\pi/3} \\ B_2 & \{1,1,2\} & y^4 = [x(1-x)]^3 & \tau = e^{2i\pi/4} \\ G_2 & \{1,2,3\} & y^6 = x^5(1-x)^4 & \tau = e^{2i\pi/6} \sim e^{2i\pi/3} \end{array}$$

These curves can be desingularized, being birationally equivalent (over $\mathbf{Q}$) to

$$\begin{array}{ll} A_2 & Y^2 + 4X^3 = 1 \\ B_2 & Y^2 + 4X^4 = 1 \\ G_2 & Y^2 - X^3 = 1. \end{array} \tag{4.12}$$

When the genus g is larger than one, a (compact) Riemann surface admits g linearly independent holomorphic differentials and $\omega^{(1)}$ is only one of them. It is noteworthy that one can choose a basis of differentials in such a way that each of their integrals maps the upper half x plane on a rational triangle. We call such triangles associates of the original triangle.

Let as before P_0, P_1, P_∞ be three integers with no common factor, Q their sum, $\mathcal{T}$ the corresponding triangle and $\mathcal{C}$ the algebraic curve. It will sometimes be convenient to label a triangle with three integers admitting a common factor. Then, reduction by this factor will be understood to define the triangle.

To find the associated triangles we proceed as follows. Let v be a positive integer smaller than Q, and consider representatives of the integers vP_0, vP_1 and vP_∞ mod Q in the range 0 to $Q-1$. We call these representatives $P_i^{(v)}$ and declare the integer v admissible if

$$P_i^{(v)} > 0 \qquad P_0^{(v)} + P_1^{(v)} + P_\infty^{(v)} = Q \tag{4.13}$$

which means of course that the set $\{P_i^{(v)}\}$ defines a triangle $\mathcal{T}^{(v)}$ with angles $\pi P_i^{(v)}/Q$. Thus $\mathcal{T}^{(1)} \equiv \mathcal{T}$ and for v admissible $\mathcal{T}^{(v)}$ is an associated triangle. It is not guaranteed that the $P_i^{(v)}$ have no common factor. When this happens one observes interesting phenomena as we shall see later.

Proposition 4.2.
(i) *The number of admissible values of v, hence the number of associated triangles is precisely g.*
(ii) *For v admissible let $y^{(v)}$ and $w^{(v)}$ be defined as*

$$\left(y^{(v)}\right)^Q = x^{P_1^{(v)}+P_\infty^{(v)}}(1-x)^{P_\infty^{(v)}+P_0^{(v)}} \tag{4.14}$$

$$\omega^{(v)} = \frac{dx}{y^{(v)}} \tag{4.15}$$

then $y^{(v)}$ is a meromorphic function on the Riemann surface of the curve $\mathcal{C}$, more precisely a rational function $y^{(v)}$ of x and y can be defined satisfying (4.14)*. Furthermore $\omega^{(v)}$ is holomorphic on this surface, its integral maps the upper half plane on the associated triangle and as v ranges over admissible values the g holomorphic differentials are linearly independent.*

To prove (i) we note that two distinct values of v in the interval $[0, Q-1]$ cannot lead to the same ordered triplets. Indeed if for $v_2 > v_1$ $v_2 - v_1 \leq Q-2 \quad (v_2 - v_1)P_i = k_i Q$ we would have $P_i/Q = k_i/(v_2 - v_1)$ in contradiction with the fact that Q is the smallest possible common denominator. If for each i, $P_i^{(v)}$ is positive so is $P_i^{(Q-v)} = Q - P_i^{(v)}$ and since

$$\sum P_i^{(v)} + \sum_i P_i^{(Q-v)} = 3Q$$

one of the sums is Q the other $2Q$, showing that of the pair of values v, $Q-v$, only one at most is admissible and v and $Q-v$ are distinct if all $P_i^{(v)}$ are positive. If one $P_i^{(v)}$ vanishes, we have $vP_i = k_i Q$ so that dividing by $l_i = (P_i, Q)$ we conclude that q_i has to divide v. To count the admissible values, it is therefore sufficient to suppress from the $Q-1$ integers from 1 to $Q-1$ the multiples of q_0, q_1 and q_∞, none of which coincide in this range since Q is the lowest common multiple of

q_0, q_1, q_∞, and observe that the remaining set contains pairs v, $Q-v$ of distinct integers (since their residues mod q_i are opposite) out of which only one is admissible. Since we erase $\sum_i (l_i - 1)$ integers, the number of admissible v's is

$$\frac{1}{2}\left\{Q - 1 - \sum_i (l_i - 1)\right\} = g$$

according to (4.4), thus proving the first part of the proposition.

To prove that the functions $y^{(v)}$ are meromorphic and the differentials $\omega^{(v)}$ holomorphic, it is sufficient to use a local desingularizing parameter at $x = 0,\ 1,\ \infty$. If we write $P_i^{(v)} = vP_i - r_iQ$ for v admissible, we readily obtain $y^{(v)}$ as

$$y^{(v)} = y^v x^{r_0+1-v}(1-x)^{r_1+1-v}. \tag{4.16}$$

The irreducibility of the curve (4.9) entails that no nontrivial polynomial in y of degree smaller than Q with rational coefficients in x vanishes; as a consequence one obtains the independence of the differentials $\omega^{(v)}$ using the expression of $y^{(v)}$. Finally the integral of $\omega^{(v)}$ maps the upper half x-plane on the triangle $\mathcal{T}^{(v)}$.

The Riemann surfaces obtained in this process carry a natural triangulation and form a class with interesting algebraic properties. We conclude this section with a few examples.

Consider the triangle $\{1,1,3\}$ of genus 2 and corresponding curve

$$y^5 = [x(1-x)]^4 \tag{4.17a}$$

birationally equivalent to the one of its associated triangle $\{2,2,1\}$ namely

$$y^5 = [x(1-x)]^3 \tag{4.17b}$$

as well as its desingularized model

$$Y^2 + X^5 = 1 \tag{4.17c}$$

all of which exhibit a group of automorphisms of order 10, $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$. Each triangle, completed by a reflected image in one of its sides, appears as an elementary tile in the famous Penrose tilings of the plane.

The Riemann surface with the largest automorphism group in genus 2 occurs for the associated triangles $\{4,3,1\}$ and $\{4,1,3\}$ with equivalent curves

$$y^8 = x^4(1-x)^5 \qquad y^8 = x^4(1-x)^7 \tag{4.18a}$$

both birationally equivalent to

$$y^2 = x(1-x^4) \tag{4.18b}$$

with automorphism group of order 48, a double covering of the octahedral group. Using the form (4.18b) its triangulation in 16 triangles is a double cover of the one of the Riemann sphere of the variable x when projecting the faces of a regular octahedron with vertices at 0, ∞ and the four fourth roots of unity.

As a last example consider the triangle $\{1,4,16\}$ of genus 10 with Q=21. The admissible values of v are classified as follows.

(4.19)

$v=1$	$\{1,4,16\}$	$v=2$	$\{2,8,11\}$	$v=3$	$\{3,12,6\}$
4	$\{4,16,1\}$	8	$\{8,11,2\}$	6	$\{6,3,12\}$
16	$\{16,1,4\}$	11	$\{11,2,8\}$	12	$\{12,6,3\}$
$v=7$	$\{7,7,7\}$				

In three groups the associated triangles only differ by a cyclic permutation of the vertices, while the last one is invariant under any permutation.

For $v=3,6,12$, the integers $P_i^{(v)}$ have a common factor 3, and, up to permutation, the corresponding triangle is $\{1,4,3\}$ of genus 3 (and not 10) while for $v=7$ one obtains in irreducible terms the equilateral triangle $\{1,1,1\}$ of genus 1. This is an example of rational correspondence between algebraic curves of different genus larger than zero. Starting from the irreducible curve

$$g=10 \qquad y^{21} = x^{20}(1-x)^{17} \tag{4.20}$$

if we set

$$Y = \frac{y^7}{x^6(1-x)^5} \qquad X = x \tag{4.21}$$

we obtain the elliptic curve

$$g=1 \qquad Y^3 = [X(1-X)]^2. \tag{4.22}$$

Thus instead of considering the curve (4.20) as ramified above the Riemann sphere of the variable $X = x$, we can also consider it as 7-tuple covering of the torus (4.22). It can also be considered as a triple covering of the curve of genus 3

(4.23)

$$g = 3 \qquad Y'^7 = X'^6(1 - X')^3 \qquad X' = x \quad Y' = \frac{y^3}{[x(1-x)]^2}.$$

According to a theorem of de Franchis, these are at most finitely many holomorphic maps between algebraic curves of genus larger than one. The triangular curves offer a rich zoo of examples. It would of course be great if the above properties afforded a mean to find algebraic relations between the spectra of the associated triangles.

§5. The A-D-E classification of conformal models

As a last and most significant example, I will describe in this section the classification of minimal conformal models and Wess-Zumino-Witten models based on the simplest affine algebra $A_1^{(1)}$ (Cappelli, Itzykson, Zuber–1987). In the concluding remark of the next section I shall touch on extensions to other rational cases.

It is not necessary to recall here the framework of two dimensional conformal field theory, with its Virasoro, Kac-Moody or extended algebras, Ward identities and partition functions. This is covered in other contributions to this volume. Instead I will concentrate on questions of modular invariance, an aspect of the various monodromy problems which have led to amazing connections with many branches of mathematics.

In statistical mechanics, one expects the partition function on a torus, expressed as a sesquilinear form in the characters pertaining to whichever extended algebra is relevant for the description of the finitely many primary operators in a rational conformal field theory, to be modular invariant (Cardy–1986). Such constraints are extremely powerful and reveal some unexpected relations, as examplified by our finding of an A-D-E classification for the simplest cases, suggesting ties with finite group theory, algebraic surfaces, rings of invariant polynomials in enveloping algebra of simple Lie algebras .... To carry over this program to a larger class of rational models is a challenge which might reveal new features. We will emphasize here some arithmetical aspects. The mathematical classification has its physical counterpart in a better understanding of universality classes of critical theories and could perhaps shed some light on the behavior in the vicinity of critical points.

5.1.

The modular group $\Gamma = \mathrm{PSL}(2,\mathbf{Z})$ is the group of transformations

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \tag{5.1}$$

with integral coefficients and unit determinant $ad - bc = 1$. A given element stems therefore from a pair of matrices $\pm A$ in $\overline{\Gamma} = \mathrm{SL}(2,\mathbf{Z})$. The group Γ leaves the real projective line and the upper half τ plane invariant. Henceforth we assume τ to have a positive imaginary part.

The modular group describes the effect of a change in basis on the ratio τ between two generators of a lattice $\mathcal{L}$ in the complex plane, with $\mathbf{C}/\mathcal{L}$ identified with a torus (or an elliptic curve). It is generated by two elements

$$T: \quad \tau \to \tau + 1 \qquad S: \quad \tau \to -\tau^{-1} \tag{5.2a}$$

satisfying

$$S^2 = (ST)^3 = 1. \tag{5.2b}$$

For any $k > 1$, let Γ_k be the invariant subgroup of level k such that $A \equiv \pm 1 \bmod k$, and $\Gamma^k = \Gamma/\Gamma_k$, the factor group with a number of elements $|\Gamma^k|$ given by

$$|\Gamma^k| = \frac{k^3}{2} \prod_{\substack{p\,\mathrm{prime} \\ p|k}} (1 - p^{-2}). \tag{5.3}$$

The factor $\frac{1}{2}$ is omitted when $k = 2$, so that $|\Gamma^2| = 6$. For a pair of coprime integers k_1, k_2, we have $\overline{\Gamma^{k_1 k_2}} \sim \overline{\Gamma^{k_1}} \times \overline{\Gamma^{k_2}}$. Hence the nontrivial part of representation theory corresponds to the case where k is a power of a prime. We note that the center of $\overline{\Gamma^k}$ is made of involutions of the form γI, $\gamma^2 \equiv 1 \bmod k$. If r denotes the number of odd prime divisors of k and $a = 0, 1, 2$ according respectively to $k \not\equiv 0, 4$, $k \equiv 4$ or $k \equiv 0 \bmod 8$, the number of elements in the center is 2^{a+r}. In any representation these elements will belong to the commutant.

5.2.

We will need Dedekind's function already encountered in Section 2

$$q = \exp 2\pi i\tau \quad \mathrm{Im}\tau > 0$$

$$\eta(\tau) = q^{1/24} \prod_{l=1}^{\infty} (1 - q^l) = \sum_{t=-\infty}^{+\infty} (-1)^t q^{\frac{(6t+1)^2}{24}} \tag{5.4}$$

which transforms under the modular group as

$$(5.5)\qquad \begin{aligned} T: &\quad \eta(\tau+1) = \exp\frac{2\pi i}{24}\,\eta(\tau)\\ S: &\quad \eta(-\tau^{-1}) = \left(\frac{\tau}{i}\right)^{1/2}\eta(\tau)\end{aligned}$$

as follows from its definition and Poisson's formula. The product representation shows that $\eta(\tau)$ never vanishes for $\mathrm{Im}\tau > 0$.

Let us introduce the characters corresponding to the "degenerate" series of Virasoro highest weight representations (Feigin and Fuchs–1983, Rocha-Caridi–1985) with central charge

$$(5.6)\qquad c = 1 - \frac{6(p-p')^2}{pp'}$$

such that p and p' are coprime positive integers. The highest (or conformal) weights are given in terms of two integers r and s in the range $0 < r < p'$, $0 < s < p$ (and to avoid duplication, $sp' < rp$ if we assume for instance that $p' < p$) as

$$(5.7)\qquad h_{(r,s)} = \frac{(rp-sp')^2 - (p-p')^2}{4pp'}.$$

When p and p' are successive integers, $p' = m, p = m+1, m \geq 2$ one obtains unitary representations (Friedan, Qiu and Shenker 1985–1986). With L_0 standing for the generator of dilatations in the Virasoro algebra we have the following formula for the character

$$(5.8)\qquad \begin{aligned}\chi_{c,h}(\tau) &= \mathrm{Tr}_{c,h}\, q^{L_0 - \frac{c}{24}}\\ &= \frac{1}{\eta(\tau)}\sum_{t=-\infty}^{+\infty}\left[q^{\frac{(2tpp'+rp-sp')^2}{4pp'}} - q^{\frac{(2tpp'+rp+sp')^2}{4pp'}}\right].\end{aligned}$$

Set

$$(5.9)\qquad n = pp' \quad N = 2n = 2pp' \quad \lambda = rp - sp' \bmod N.$$

If r and s are chosen as above, a representative λ lies in the range $0 < \lambda < n$, multiples of p and p' being excluded. Consequently, the number of possibilities, *i.e.* of independent characters is $\frac{1}{2}(p-1)(p'-1)$.

Since p and p' are coprimes, there exist integers r_0 and s_0 such that $r_0p - s_0p' = 1$ with $h_{(r_0,s_0)}$ negative unless the representation is unitary (a poor man's way of understanding the condition $(p-p')^2 = 1$ as necessary for unitarity, sufficiency then follows from the coset construction described by Olive in this volume). The combination $h-c/24$ inserted in (5.8) will insure "good " modular properties and arises physically from the Casimir effect in a bounded torus. It reads

$$h_{(r_0,s_0)} - \frac{c}{24} = \frac{1}{4pp'} - \frac{1}{24}$$

and is negative in all cases of interest where $pp' \geq 6$ thus could be written $-c'/24$ with $c' \geq 0$.

Define

$$\omega_0 \equiv r_0 p + s_0 p' \quad \bmod N \quad \omega_0 \equiv 1 \quad \bmod 2N \tag{5.10}$$

such that

$$\omega_0 \lambda \equiv rp + sp' \quad \bmod N. \tag{5.11}$$

Among positive integers p and p' are the smallest such that $\omega_0\lambda \equiv \lambda \bmod N$ is equivalent to $\lambda \equiv 0 \bmod p$ and $\omega_0\lambda \equiv -\lambda \bmod N$ is equivalent to $\lambda \equiv 0 \bmod p'$ respectively. The requirement on Virasoro representations is that $\omega_0 \not\equiv \pm 1 \bmod N$. With these notations the characters read

$$\chi_\lambda(\tau) = \frac{1}{\eta(\tau)} \sum_{t=-\infty}^{+\infty} \left[q^{\frac{(tN+\lambda)^2}{2N}} - q^{\frac{(tN+\omega_0\lambda)^2}{2N}} \right] \tag{5.12}$$

and satisfy

$$\chi_\lambda(\tau) = \chi_{-\lambda}(\tau) = \chi_{\lambda+N}(\tau) = -\chi_{\omega_0\lambda}(\tau). \tag{5.13}$$

One observes that Euler's pentagonal identity (5.4) expresses the fact that for $p' = 2,\ p = 3,\ N = 12$, the Virasoro character reduces to unity.

The modular transformation properties follow from Poisson's formula and the behavior of $\eta(\tau)$ given in (5.5)

$$\begin{aligned} T: &\quad \chi_\lambda(\tau+1) = \exp 2\pi i \left(\frac{\lambda^2}{2N} - \frac{1}{24} \right) \chi_\lambda(\tau) \\ S: &\quad \chi_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}} \sum_{\lambda' \bmod N} \exp \frac{2i\pi\lambda\lambda'}{N}\, \chi_{\lambda'}(\tau). \end{aligned} \tag{5.14}$$

These transformations are compatible with the symmetry properties (5.13). Using Gauss's formula giving the trace of the finite Fourier transform

$$\frac{1}{\sqrt{N}}\sum_{\lambda \bmod N}\exp\frac{2i\pi\lambda^2}{N}=\begin{cases}1 & N\equiv 1 \bmod 4\\ 0 & N\equiv 2 \bmod 4\\ i & N\equiv 3 \bmod 4\\ 1+i & N\equiv 0 \bmod 4\end{cases} \tag{5.15}$$

as well as the symmetry $\chi_\lambda = \chi_{-\lambda}$ which insures that S operates in the subspace with eigenvalues ± 1, it can be checked that $S^2 = (ST)^3 = 1$. That the finite Fourier transform makes its appearence among the unitary transformations induced by the modular group, ties nicely the subject with quantum mechanics on systems with finitely many states as we shall see below.

From the preceding it follows that

$$Z_{A_{p'-1},A_{p-1}}(\tau,\overline{\tau}) = \sum_{\lambda=1}^{n-1}|\chi_\lambda(\tau)|^2 \tag{5.16}$$

is a modular invariant partition function, which we call the principal invariant, and defines a consistent symmetric theory (symmetric between holomorphic and antiholomorphic parts) involving only scalar primary operators. The subscript notation will be justified in the following. The choices $[p',p] = [3,4]$ and $[4,5]$ correspond to the Ising model (free Majorana fermions) and its tricritical version. Higher unitary models in this series can be understood as higher multicritical versions which from the point of view of a Landau Lagrangian would have an effective potential of the form ϕ^4, ϕ^6, $\dots$ $\phi^{2(m-1)}$, ... Also included in this series are non unitary minimal models, the typical example being $[p',p] = [2,5]$, interpreted by Cardy as describing the Lee-Yang edge singularity of an Ising model in a pure imaginary magnetic field.

As a side remark, let us note that the two characters involved in this first minimal non unitary model (corresponding respectively to the identity operator and the only non trivial scalar field) have a special arithmetical flavour, being related to the celebrated Rogers-Ramanujan identities. Indeed the value of n in (5.9) is $n = 10$ and $\omega_0 \equiv 11 \bmod 20$, so that using Jacobi's identity the characters read

$$
\begin{aligned}
\chi_1(\tau) &= \frac{1}{\eta(\tau)}\left[\sum_{t\equiv 1(20)} - \sum_{t\equiv 11(20)}\right] q^{\frac{t^2}{40}} \\
&= q^{-\frac{1}{60}} \frac{\sum_m (-1)^m q^{\frac{5m^2+m}{2}}}{\prod_1^\infty (1-q^n)} \\
&= \left[q^{\frac{1}{60}} \prod_{m=0}^{\infty} (1-q^{5m+1})(1-q^{5m+4})\right]^{-1} \\
\chi_3(\tau) &= \frac{1}{\eta(\tau)}\left[\sum_{t\equiv 3(20)} - \sum_{t\equiv 13(20)}\right] q^{\frac{t^2}{40}} \\
&= q^{\frac{11}{60}} \frac{\sum_m (-1)^m q^{\frac{5m^2+3m}{2}}}{\prod_1^\infty (1-q^n)} \\
&= q^{\frac{1}{5}} \left[q^{\frac{1}{60}} \prod_{m=0}^{\infty} (1-q^{5m+2})(1-q^{5m+3})\right]^{-1}
\end{aligned}
\tag{5.17}
$$

where one recognizes in the product forms the quantities which occur in the identities referred above. The ratio

$$
z(\tau) = \frac{\chi_3(\tau)}{\chi_1(\tau)} \tag{5.18}
$$

transforms under T and S as

$$
z(\tau+1) = \exp\frac{2i\pi}{5}\, z(\tau) \qquad z(-\tau^{-1}) = \frac{1 - \frac{1+\sqrt{5}}{2} z(\tau)}{z(\tau) + \frac{1+\sqrt{5}}{2}} \tag{5.19}
$$

which means that, projectively, the action of the modular group is represented by the icosahedral group, revealing the isomorphism of the latter with PSL(2,$\mathbf{Z}/5\mathbf{Z}$). We shall encounter further similar instances later on.

5.3.

In parallel with minimal conformal models it is instructive to study Wess-Zumino-Witten models, abbreviated as W-Z-W, (Gepner and Witten–1986) based on the group SU_2, or its affine Lie algebra version $A_1^{(1)}$. Whenever necessary we use superscripts, conformal or affine, to distinguish the corresponding characters. In the affine case the models are characterized by a level k, a positive integer such that the central charge is

$$c = \frac{3k}{k+2} \tag{5.20}$$

and the highest weights can be specified by a lowest integral or half integral angular momentum l, such that $0 \leq l \leq \frac{k}{2}$. To stress the analogy with minimal models, we set

$$N = 2(k+2) \qquad n = k+2 \geq 2 \qquad \lambda = 2l+1 \tag{5.21}$$

and write the Kac characters (Kac–1985)

$$\chi_\lambda^{aff}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{t=-\infty}^{+\infty} (Nt+\lambda)\, q^{\frac{(Nt+\lambda)^2}{2N}} \tag{5.22}$$

satisfying

$$\chi_\lambda^{aff}(\tau) = \chi_{\lambda+N}^{aff}(\tau) = -\chi_{-\lambda}^{aff}(\tau). \tag{5.23}$$

Here the role of the odd involution is played by the reflection $\lambda \to -\lambda$. The $k+1$ independent characters can be chosen as those with index $\lambda = 1,\ 2,\ \ldots, k+1$. Under modular transformations

$$T:\ \chi_\lambda^{aff}(\tau+1) = \exp 2i\pi(\frac{\lambda^2}{2N} - \frac{1}{8})\chi_\lambda^{aff}(\tau) \tag{5.24a}$$

$$S:\ \chi_\lambda^{aff}(-\tau^{-1}) = \frac{-i}{\sqrt{N}} \sum_{\lambda' \bmod N} \exp 2i\pi\frac{\lambda\lambda'}{N}\ \chi_{\lambda'}^{aff}(\tau) \tag{5.24b}$$

and when $N = 4$, $\lambda = 1$, χ_1^{aff} reduces to unity, thanks to Jacobi's identity. In (5.24b) the factor $-i$ is necessary since χ_λ^{aff} is odd in the interchange $\lambda \leftrightarrow -\lambda$.

In both the conformal as well as the affine case, the modular group Γ^{2N} of level $2N$ is represented by multiples of the identity. As a consequence, both (5.14) and (5.24) generate projective representations (in general reducible) of Γ^{2N}.

5.4.

The classification problem consists in finding all partition function of the form

$$Z(\tau,\overline{\tau}) = \sum_{\lambda,\lambda' \in \mathcal{B}} \overline{\chi_\lambda(\tau)}\ Z_{\lambda,\lambda'}\ \chi_{\lambda'}(\tau) \tag{5.25}$$

where the indices λ, λ' range in a fundamental domain of the symmetry properties within integers mod N, *i.e.* $1 \leq \lambda \leq n-1$ in the affine case, and if $p' \leq p$, $\lambda = rp - sp'$, $1 \leq r \leq p'-1$, $1 \leq s \leq p-1$, $rp - sp' > 0$, in the conformal one. The coefficients $Z_{\lambda,\lambda'}$ should satisfy the following requirements

A — $Z(\tau,\overline{\tau})$ is modular invariant

B — $Z_{\lambda,\lambda'}$ are non negative integers for λ and λ' in $\mathcal{B}$

with a normalization in the affine case $Z_{1,1} = 1$ and in the conformal one $Z_{p-p',\, p-p'} = 1$ expressing the unicity of the vacuum state.

Correspondingly, the investigation divides itself into two parts, which we call problem A, the easiest, and problem B, the real arithmetical one. It will be convenient in the sequel to extend the summation in (5.25) to the full range λ, λ' mod N with due attention paid to (anti-)symmetry properties. We can now identify $Z_{\lambda,\lambda'}$ with a matrix in an N-dimensional vector space with a basis labelled by λ mod N. The overall phases in (5.14) and (5.24) are immaterial when we write the condition for modular invariance

$$\mathcal{T}^\dagger\, Z\, \mathcal{T} = \mathcal{S}^\dagger\, Z\, \mathcal{S} = Z$$

$$\mathcal{T}_{\lambda,\lambda'} = \delta_{\lambda,\lambda' \bmod N} \exp 2i\pi \frac{\lambda^2}{2N} \tag{5.26}$$

$$\mathcal{S}_{\lambda,\lambda'} = \frac{1}{\sqrt{N}} \exp 2i\pi \frac{\lambda\lambda'}{N}.$$

Due to the unitarity of the matrices $\mathcal{T}$ and $\mathcal{S}$ this means that Z belongs to their commutant, so that problem A is nothing but characterizing this commutant, and if possible finding of a basis spanned by matrices with integral coefficients.

5.5.

Throughout this section $N = 2n$ is an even positive integer. Let δ be a positive divisor of n and put $\overline{\delta} = n/\delta$, $\alpha = (\delta, \overline{\delta})$ the greatest common divisor of δ and $\overline{\delta}$ so that α^2 divides n and N. Since δ/α and $\overline{\delta}/\alpha$ are coprime, we can find integers ρ and σ such that

$$\rho\overline{\delta} - \sigma\delta = \alpha \tag{5.27a}$$

and define ω mod N/α^2 through

$$\omega \equiv \frac{\rho\overline{\delta} + \sigma\delta}{\alpha} \bmod N/\alpha^2. \tag{5.27b}$$

It satisfies

$$\begin{aligned}\omega + 1 &\equiv 2\rho\frac{\bar{\delta}}{\alpha} \bmod N/\alpha^2 \qquad \omega - 1 \equiv 2\sigma\frac{\delta}{\alpha} \quad \bmod N/\alpha^2 \\ \omega^2 - 1 &\equiv 0 \quad \bmod 2N/\alpha^2.\end{aligned} \tag{5.28}$$

Therefore if $\alpha|\lambda$, $(\omega\lambda)^2 \equiv \lambda^2 \bmod 2N$. This shows that the matrix

$$(\Omega_\delta)_{\lambda,\lambda'} = \begin{cases} \sum_{\xi \bmod \alpha} \delta_{\lambda,\omega\lambda+\xi N/\alpha \bmod N} & \text{if } \alpha|\lambda \text{ and } \alpha|\lambda' \\ 0 & \text{otherwise} \end{cases} \tag{5.29}$$

commutes with $\mathcal{T}$ and, as an immediate computation confirms, also with $\mathcal{S}$. Interchanging the role of δ and $\bar{\delta}$ amounts to replace ω by $-\omega$; Ω_n corresponds to $\alpha = 1$, $\omega = 1$, *i.e.* $\Omega_n = \mathrm{I}$, while Ω_1 corresponds to $\alpha = 1$, $\omega = -1$, *i.e.* $(\Omega_1)_{\lambda,\lambda'} = \delta_{\lambda+\lambda',0}$. More generally Ω_δ and $\Omega_{n/\delta}$ are linearly relating when operating on the even or odd subspace under $\lambda \leftrightarrow -\lambda$.

Let us call $\sigma(n)$ the number of divisors of n (equal to 2 if n is a prime) we have (Gepner and Qiu–1987)

Proposition 5.1. *The commutant of the matrices $\mathcal{S}$ and $\mathcal{T}$ is spanned by the $\sigma(n)$ linearly independent operators Ω_δ.*

This proposition was stated as a conjecture in an earlier work with A. Cappelli and J.B. Zuber. To prove it we introduce a technique useful in a broader context, namely, we give an explicit realization of the adjoint action of the modular group of level $2N$ on a suitable basis of $N \times N$ matrices related to the Heisenberg group modulo N. We call this realization "finite quantum mechanics". It is presumably familiar in a mathematical context under the name of finite metaplectic group.

Let $\{|\lambda\rangle\}$, $\lambda \bmod N$, be an orthonormal basis in an N-dimensional Hilbert space $\mathcal{H}$. Define the operators Q and P through

$$\begin{aligned} Q|\lambda\rangle &= \exp\frac{2i\pi\lambda}{N}\,|\lambda\rangle \\ P|\lambda\rangle &= |\lambda+1\rangle \\ Q^N = P^N &= I \qquad QP = \exp\frac{2i\pi}{N}\,PQ. \end{aligned} \tag{5.30}$$

Obviously Q and P are related by a Fourier transformation. The polynomials in Q and P generate the full algebra $\mathcal{H}\otimes\mathcal{H}^*$ of operators on $\mathcal{H}$ since we can write any operator M in the form

$$M = \sum_{k,l \bmod N} P^k Q^l \frac{1}{N}\mathrm{Tr}(Q^{-l}P^{-k}M). \tag{5.31}$$

This follows by linearity from a check on elements of the form $|\lambda\rangle\langle\lambda'|$. In view of (5.30) we can "normal order" any monomial in P's and Q's by pushing the P's to the left of the Q's.

The operators $\mathcal{S}$ and $\mathcal{T}$ generate through their adjoint action canonical transformations preserving the commutation relation between P and Q. If we introduce the convenient symbol

$$\{k,l\} = \exp 2i\pi \frac{kl}{2N} P^k Q^l \tag{5.32}$$

it is readily checked that

$$\mathcal{T}^\dagger \, \{k,l\} \, \mathcal{T} = \{k, l-k\} \quad \mathcal{S}^\dagger \, \{k,l\} \, \mathcal{S} = \{l, -k\}. \tag{5.33}$$

The monomials $\{k,l\}$ are periodic in k and $l \bmod 2N$ but are linearly independent only $\bmod N$, since

$$\{k+aN, l+bN\} = (-1)^{kb-al}\{k,l\}. \tag{5.34}$$

We can recast (5.33) as an action of the modular group generated by $\mathcal{S}$ and $\mathcal{T}$ as

$$\begin{aligned} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma} \to \mathcal{A} \qquad & \mathcal{A}^\dagger\{k,l\}\mathcal{A} = \{k',l'\} \\ [k',l'] = [k,l] \begin{pmatrix} a & b \\ c & d \end{pmatrix}. & \end{aligned} \tag{5.35}$$

In fact, since $\overline{\Gamma_{2N}}$ is represented by the identity we obtain effectively an action of $\overline{\Gamma^{2N}}$.

Using (5.29) and (5.31) we find on the one hand

$$\Omega_\delta = \frac{\delta}{N} \sum_{y,z \bmod \frac{n}{\delta}} P^{2\delta y} Q^{2\delta z} \exp 2i\pi \frac{\delta^2 yz}{n}. \tag{5.36}$$

On the other hand to obtain a general element of the commutant one can average the adjoint action of $\overline{\Gamma^{2N}}$ on an arbitrary element of $\mathcal{H}\otimes\mathcal{H}^*$, a familiar idea in finite (or compact) group theory. From the decomposition (5.31) it is sufficient to choose as an arbitrary element a monomial

P^kQ^l. Thus we set

$$\begin{aligned} M'_{k,l} &= \frac{1}{|\overline{\Gamma^{2N}}|} \sum_{\mathcal{A}\in\overline{\Gamma^{2N}}} \mathcal{A}^\dagger\, P^kQ^l\mathcal{A} \\ &= \frac{1}{|\overline{\Gamma^{2N}}|} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\overline{\Gamma^{2N}}} \exp\frac{2i\pi}{N}\left(\frac{abk^2+cdl^2+2bckl}{2N}\right) \\ &\qquad\qquad \times P^{ak+cl}\, Q^{bk+dl}. \end{aligned} \tag{5.37}$$

The operators $M'_{k,l}$ with k and l mod N $(M'_{k+\xi N,l+\xi' N} = M'_{k,l})$ span the commutant, but they are far from being linearly independent. First we observe that $M'_{k,l}$ vanishes unless k and l are both even. To see this, consider the kernel of the maps $\overline{\Gamma^{2N}} \to \overline{\Gamma^N}$ given by the eight matrices

$$\begin{pmatrix} 1+\alpha N & \beta N \\ \gamma N & 1+\alpha N \end{pmatrix} \qquad \alpha,\beta,\gamma = 0,1.$$

Averaging (5.37) over this invariant subgroup should leave $M'_{k,l}$ invariant and multiplies the coefficient of $P^{ak+cl}\, Q^{bk+dl}$ by

$$\frac{1}{8}\sum_{\alpha,\beta,\gamma \bmod 2} \exp\left(2i\pi\left(\frac{\beta}{2}(ak+cl)^2+\frac{\gamma}{2}(bk+dl)^2\right)\right).$$

This is non vanishing if and only if $k' = ak+cl$ and $l' = bk+dl$ are both even. The linear transformation is invertible $\bmod 2$, hence k and l have also to be even. Thus in (5.37) we may assume both indices even which allows us to write

(5.38)

$$M_{k,l} = M'_{k,l}.$$

$$M_{k,l} = \frac{1}{|\overline{\Gamma^n}|}\sum_{\overline{\Gamma^n}} \exp\left(2i\pi\frac{abk^2+cdl^2+2bckl}{n}\right) P^{2(ak+cl)}Q^{2(bk+dl)}.$$

The structure of the formula has in fact reduced the average to $\overline{\Gamma^n}$ and k and l are defined $\bmod n$. Furthermore for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma^n}$ we find $M_{k,l}$ proportional to $M_{ak+cl,bk+dl}$ so that we can restrict ourselves to pick any representative point of the orbit of $[k,l]$ under $\overline{\Gamma^n}$ to label M. This is done as follows. Pick first representatives k,l, in the range 1 to

n and let $d = (k, l)$. Then we find m_1 and m_2 such that $m_1 k + m_2 l = d$ and

$$
\begin{aligned}
&[k,l]\begin{pmatrix} l/d & m_1 \\ -k/d & m_2 \end{pmatrix} = [0,d] \\
&M_{k,l} = \exp(-2i\pi\frac{kl}{n})M_{0,d}.
\end{aligned} \tag{5.39}
$$

The process can be repeated by identifying $M_{0,d}$ with $M_{n,d}$ and introducing $\delta = (n, d)$. We conclude that the $\sigma(n)$ operators

$$
M_\delta \equiv M_{0,\delta} = \frac{1}{|\overline{\Gamma^N}|}\sum_{\overline{\Gamma^n}} \exp 2i\pi\frac{\delta^2 cd}{n} P^{2\delta c}Q^{2\delta d} \tag{5.40}
$$

labelled by divisors of n, generate the commutant. They are linearly independent, since disjoint orbits are labelled by distinct divisors. The relation

$$
\Omega_\delta = \sum_{\delta|\delta'} \frac{\delta\delta'}{n'}|\overline{\Gamma}^{n/\delta'}|M_{\delta'} \tag{5.41}
$$

relates the two sets $\{\Omega_\delta\}$ and $\{M_\delta\}$ through a triangular matrix which can be inverted to yield the M's in terms of the Ω's showing that the latter also generate the commutant and are therefore also linearly independent. This proves the proposition. Of course the advantage of using the basis $\{\Omega_\delta\}$ is that the matrices have in the $|\lambda\rangle$ basis integral coefficients.

5.6.

The heart of the matter is problem B. For the sake of completeness I will repeat here the argument of our paper. The difficulty stems in the conformal or affine case from oddness property of the characters.

We look first at the affine case. From Proposition 5.1 we can write a modular invariant partition function

$$
\begin{aligned}
&Z(\tau,\overline{\tau}) = \frac{1}{2}\sum_{\lambda,\lambda' \bmod N} \overline{\chi_\lambda}(\tau)\; Z_{\lambda,\lambda'}\; \chi'_\lambda(\tau) \\
&Z_{\lambda,\lambda'} = \left[\sum_{\delta|n} c_\delta\Omega_\delta\right]_{\lambda,\lambda'}
\end{aligned} \tag{5.42}
$$

Recall that $\chi_{-\lambda} = -\chi_\lambda$ so that $\chi_{\xi n} = 0$. Let $U \equiv \{1, 2, \cdots, n-1\}$ and $L \equiv \{n+1, \cdots, 2n-1\}$ denote a partition of the integers from 1 to N distinct from 0 and $n \bmod N$ with $L = -U \bmod N$. A fundamental domain $\mathcal{B}$ coincides with U. Since $(\Omega_\delta \chi)_\lambda = -(\Omega_{n/\delta}\chi)_\lambda$ and $(\Omega_\delta)_{\lambda,\lambda'} = (\Omega_\lambda)_{-\lambda,-\lambda'}$, for each factorization $n = \delta\overline{\delta}$ we can replace in (5.42) the combination $c_\delta \Omega_\delta + c_{\overline{\delta}}\Omega_{\overline{\delta}}$ by $(c_\delta - c_{\overline{\delta}})\Omega_{\overline{\delta}}$ or $(c_{\overline{\delta}} - c_\delta)\Omega_{\overline{\delta}}$, we choose whichever has a non negative coefficient. Thus

$$Z(\tau,\tau') = \sum_{\lambda,\lambda' \in U} \overline{\chi_\lambda(\tau)}(Z_{\lambda,\lambda'} - Z_{\lambda,-\lambda'})\chi'_\lambda(\tau) \tag{5.43}$$

with $Z_{\lambda,\lambda'}$ in (5.42) such that $c_\delta \geq 0$, and $c_\delta > 0$ implying $c_{n/\delta} = 0$. Only Ω_n and Ω_1 contribute to the coefficient of $\overline{\chi_1}\chi_1$ which has to be equal to unity. With the above conventions, this implies $c_n = 1, c_1 = 0$. We have chosen matrices Ω_δ with integral non negative entries, we want to find the coefficient c_δ so that this is also true for $Z_{\lambda,\lambda'} - Z_{\lambda,-\lambda'}$. In the following proposition we exhibit all possible solutions for $Z_{\lambda,\lambda'}$.

Proposition 5.2 (A-D-E classification). *Modular invariant affine $(A_1^{(1)})$ partition functions with normalized non negative integral coefficients are of the following types*

$$\begin{array}{llll} & A_{n-1} & n \geq 2 & \Omega_n \\ & D_{n/2+1} & n \text{ even} \geq 6 & \Omega_n + \Omega_2 \\ (5.44) & E_6 & n = 12 & \Omega_{12} + \Omega_3 + \Omega_2 \\ & E_7 & n = 18 & \Omega_{18} + \Omega_3 + \Omega_2 \\ & E_8 & n = 30 & \Omega_{30} + \Omega_5 + \Omega_3 + \Omega_2. \end{array}$$

We have two infinite series A and D and three exceptional E cases. The index on A, D, E is the rank of the corresponding simple and simply laced Lie algebra. Simply laced means that all fundamental roots are of equal length, or equivalently that the corresponding Dynkin diagram has no multiple link. This correspondence is illustrated in table II which gives the partition functions in expanded form : the coefficient of each diagonal term $\overline{\chi}_\lambda\chi_\lambda$, $1 \leq \lambda \leq n-1$ is the multiplicity of λ in the list of Coxeter exponents for the corresponding Lie algebra and n is its Coxeter number.

$n \geq 2$	$\sum_{\lambda=1}^{n-1} \lvert\chi_\lambda\rvert^2$	A_{n-1}	Ω_n
$n = 4\rho + 2$, $\rho \geq 1$	$\sum_{\substack{\lambda\ \text{odd}=1 \\ \lambda\neq 2\rho+1}}^{4\rho+1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$		
	$+ \sum_{\lambda\ \text{odd}=1}^{2\rho-1} (\chi_\lambda\chi^*_{4\rho+2-\lambda} + \text{c.c.})$	$D_{2\rho+2}$	$\Omega_n + \Omega_2$
	$= \sum_{\lambda\ \text{odd}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2$		
	$+2\lvert\chi_{2\rho+1}\rvert^2$		
$n = 4\rho$, $\rho \geq 2$	$\sum_{\lambda\ \text{odd}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho}\rvert^2$		
	$+ \sum_{\lambda\ \text{even}=2}^{2\rho-2} (\chi_\lambda\chi^*_{4\rho-\lambda} + \text{c.c.})$	$D_{2\rho+1}$	$\Omega_n + \Omega_2$
$n = 12$	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2$		
	$+\lvert\chi_5 + \chi_{11}\rvert^2$	E_6	$\Omega_{12} + \Omega_3 + \Omega_2$
$n = 18$	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2$		
	$+\lvert\chi_7 + \chi_{11}\rvert^2 + \lvert\chi_9\rvert^2$		
	$+[(\chi_3 + \chi_{15})\chi_9^* + \text{c.c.}]$	E_7	$\Omega_{18} + \Omega_3 + \Omega_2$
$n = 30$	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2$		
	$+\lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$	E_8	$\Omega_{30} + \Omega_5$ $+\Omega_3 + \Omega_2$

Table II List of modular invariant affine partition functions for $A_1^{(1)}$.

It must be mentioned that an independent proof was found by Kato (Kato–1987). Ours proceeds in three steps, Lemmas 5.1, 2 and 3.

We set aside the case of the obvious invariant (A_{n-1}) with $c_\delta = 0$ except $c_n = 1$. For each divisor $\delta | n$ we define $\alpha(\delta) = (\delta, n/\delta)$ and $\omega(\delta)$ as before, such that $\omega^2(\delta) \equiv 1 \bmod 2N/\alpha^2(\delta)$. Consider in (5.43) the factor of $\overline{\chi}_1(\tau)$, it reads

$$\chi_1(\tau) + \sum_{\delta, \alpha(\delta)=1} c_\delta \chi_{\omega(\delta)}(\tau)$$

arising from those δ's such that $\alpha(\delta) = 1$ and $\omega(\delta)$ prime to N, which for $1 < \delta < n$ are distinct from ± 1. It follows that $\omega(\delta) \in U$. If on the contrary $\omega(\delta) \in L, c_\delta \chi_{\omega(\delta)} = -c_\delta \chi_{-\omega(\delta)}$ would be a negative contribution, which requires that there exists a δ' with $\alpha(\delta') = 1$ and $\omega(\delta') = -\omega(\delta) = \omega$, such that $(c_{\delta'} - c_\delta)\chi_\omega$ is a positive contribution with $c_{\delta'} > c_\delta$. But $\alpha = 1, \omega' = -\omega$ means $\delta' = \delta/n$ and this possibility is excluded by our conventions. We conclude that all $\omega(\delta)$'s such that $c_\delta \neq 0$ and $\alpha(\delta) = 1$, have to belong to U, the corresponding coefficients being positive integers.

If $\alpha^2 | n$, the identity

$$\sum_{\xi \bmod \alpha} \chi_{\alpha\lambda + \xi N/\alpha}(\tau, N) = \alpha \chi_\lambda(\tau, N/\alpha^2) \quad \lambda \bmod N/\alpha^2 \tag{5.45}$$

relating characters at level $k = n - 2$ and $k' = n/\alpha^2 - 2$, shows that both sides vanish when $n = \alpha^2$, since $\chi_\lambda(\tau, N = 2) = 0$. Hence when $n = \alpha^2$, $\sum_{\lambda'} (\Omega_\alpha)_{\lambda,\lambda'} \chi_{\lambda'} = 0$, and the corresponding term may be disregarded in (5.42–43). Define α_{min} to be the smallest $\alpha(\delta)$ for $\delta \neq n$ such that $c_\delta > 0$. Then

Lemma 5.1. (i) $\alpha_{min} = 1$ *or* 2 (ii) *if* $\alpha_{\min} = 2$ *(therefore* $4|n$*) the unique solution is* $Z = \Omega_n + \Omega_2$

It is useful to represent the integers $\bmod N$ on a circle of radius $N/2\pi$ as regularly spaced at distance 1. The upper (lower) semi circle represents U (L). For $\alpha(\delta) = \alpha > 0$ the points $\lambda' = \omega\lambda + \xi N/\alpha$ as ξ varies are the vertices of a regular α-gon. If $\alpha \geq 4$, at least two vertices belong to $L \cup \{0,\ n\}$, one of them being certainly in L. These negative contributions have to be compensated by positive ones. Consider the factor of $\overline{\chi}_{\alpha_{\min}}(\tau)$. By definition of $\alpha_{\min}$ only Ω_δ's such that $c_\delta > 0$ and $\alpha = \alpha_{\min}$ contribute. This factor is

$$\chi_{\alpha_{\min}} + \sum_{\delta,\ \alpha(\delta)=\alpha_{\min}} c_\delta \sum_{\xi \bmod \alpha_{\min}} \chi_{\omega\alpha_{\min} + \xi N/\alpha_{\min}}.$$

Suppose first $\alpha_{\min} \geq 4$. Such α-gon involves at least two terms in L. The case when only one term is in L would require $\alpha_{\min} = 4$ and a set of indices $\omega\alpha_{\min} + \xi N/\alpha_{\min}$ ranging over $0, n/2, n, 3n/2 (16|n)$. Since ω is invertible $\bmod n/4 = (2N/{\alpha_{\min}}^2)$ and $\omega\alpha_{\min} = 0 \bmod n/2$ and a fortiori $\bmod n/4$, it follows that ${\alpha_{\min}}^2 = 16$ is a multiple of n, the only possibility being $n = {\alpha_{\min}}^2$, the case discarded above. Thus each $\alpha_{\min}$-gon has at least two terms in L, and since the coefficient of $\chi_{\alpha_{\min}}$ is unity, so that at best it can only compensate only one of the negative terms, two possibilities of cancellation are open: either the negative terms are compensated by positive ones of the same α-gon or a negative term pertaining to δ is compensated by a positive one from a $\delta' \neq \delta$ contribution. In the first case $2\omega \equiv 0 \bmod N/{\alpha_{\min}}^2$, hence $2 \equiv 0 \bmod N/{\alpha_{\min}}^2$ which implies $n = {\alpha_{\min}}^2$, $\alpha(\delta) = \alpha_{\min}$, an excluded possibility. In the second hypothesis, the corresponding multipliers ω and ω' have to satisfy $\omega' \equiv -\omega \bmod N/{\alpha_{\min}}^2$ which is also excluded since it would imply $\delta\delta' = n$. We conclude that $\alpha_{\min} \leq 3$. If $\alpha_{\min} = 3$, beyond the two previous possibilities excluded by a reasoning similar to the above one, we find a third possibility with an equilateral triangle having a unique point in L compensated by $\chi_{\alpha_{\min}} = \chi_3$ with a coefficient $c_\delta = 1$. But then $\omega \equiv -1 \bmod N/9$, so that the corresponding contribution is $\chi_{-3} + \chi_{-3+N/3} + \chi_{-3-N/3}$ and $-3 \pm N/3$ are in $U \cup \{0,\ n\}$. Thus $3+2n/3 \geq n$, meaning $9 \geq n$ and since $9|n$ we find again an excluded case $n = {\alpha_{\min}}^2$, $\alpha(\delta) = \alpha_{\min}$. This proves part (i) of the lemma: $\alpha_{\min} = 1$ or 2.

Assume now $\alpha_{\min} = 2$ and consider the coefficient of $\overline{\chi}_2$

$$\chi_2 + \sum_{\delta,\, \alpha(\delta)=2} c_\delta(\chi_{2\omega} + \chi_{2\omega+n}).$$

If the two points 2ω and $2\omega + n$ coincide with $\pm n/2 \bmod N$, we have $4\omega = n \bmod N$ and this is the excluded case $n = 4 = 2^2$. The possibility $2\omega = 0 \bmod n$ cannot arise since $\omega^2 \equiv 1 \bmod n$. To obtain a positive contribution, the only possibility left is either that terms arising from two distinct δ and δ' compensate each other, this being excluded as above, of when there exists a unique δ, $\alpha(\delta) = 2$, $c_\delta = 1$ such that, by choosing the representative mod $N/4$, we have $2\omega = -2 \bmod N$ and $2\omega + n \in U$, meaning that the negative term from $\chi_{2\omega}$ is compensated by χ_2. Hence $\delta = 2$, $n = 4k$, $\omega \equiv -1 \bmod 2k$. The partition function contains then $\Omega_n + \Omega_2$ plus possibly a sum over δ's such that $\alpha(\delta') \geq 3$. For $\lambda \in U$, $\sum_{\lambda'}(\Omega_n + \Omega_2)_{\lambda\lambda'}\chi_{\lambda'}$ is equal to χ_λ if λ is odd or to $\chi_{n-\lambda}$ if λ is even. To discuss the possible occurence of other Ω's we can retrace the steps of part (i), replacing the contribution of Ω_n by the one of $\Omega_n + \Omega_2$, which amounts to replace χ_λ by $\chi_{n-\lambda}$ if λ is even in the first contribution to the factor of $\overline{\chi}_{\alpha'_{\min}}$. The same arguments exclude any $\alpha(\delta) \geq 3$ and prove part (ii) of the lemma.

Lemma 5.2. *If n is odd the only possibility is Ω_n.*

Assume on the contrary that there exist additional possibilities. Since n is odd, from the previous lemma, $\alpha_{\min} = 1$. Let γ be such that $2^\gamma < n$ and consider the coefficient of $\overline{\chi}_{2^\gamma}$. Since n is odd the only contributions are from δ's such that $\alpha(\delta) = 1$ (indeed $\alpha(\delta)$ has to divide both n and 2^γ). The coefficient reads

$$\chi_{2^\gamma} + \sum_{\alpha(\delta)=1} c_\delta \chi_{\omega 2^\gamma}.$$

From the beginning of the discussion, all ω's have to belong to U. Choose representatives in the range $0 < \omega < n$. Let us then show that $2^\gamma\omega$ has also to belong to U for any γ such that $2^\gamma < n$. If for some δ, $2^\gamma\omega \in L$, the corresponding negative contribution has to be compensated by some other arising from $\omega' 2^\gamma$ (ω' being possibly one). This requires $2^\gamma(\omega + \omega') \equiv 0 \bmod N$. Let first $\gamma = 1$. Then $\omega + \omega'$ is smaller than N and $2(\omega + \omega')$ being smaller than $2N$ is equal to N, thus $\omega + \omega' = n$. But ω and ω' are odd by $\omega^2 \equiv 1 \bmod 2N$, hence a contradiction. Thus $\omega < n/2$ and the previous argument can be iterated. If $4 < n$ we cannot have $4\omega \in L$. Again and ω' would be needed for compensation and $4(\omega + \omega') \equiv 0 \bmod N$. By the previous bound applied to ω and ω', $0 < 4(\omega + \omega') < 2N$, so that $2(\omega + \omega') = n$ leading again to a contradiction, hence $\omega < n/4$ and so on. Thus for any γ such that $2^\gamma < n$ we have $2^\gamma\omega \in U$ and a positive representative can be chosen smaller than $n2^{-\gamma}$. Take the maximal γ such that $2^\gamma < n < 2^{\gamma+1}$ and ω such that $0 < \omega < n2^{-\gamma}$. Recall that $\omega n/\delta \equiv n/\delta \bmod N$. But γ, a nontrivial divisor of the odd number n is larger than 2. Hence $n/\delta < 2^\gamma(2/\delta) < 2^\gamma$ and $\omega\frac{n}{\delta} < \frac{n}{2^\gamma} \cdot 2^\gamma = n$, Thus $\omega\frac{n}{\delta} = \frac{n}{\delta}$ or $\omega = 1$, contrary to the assumption that it corresponds to a divisor $\delta > 2$. The lemma is proved.

The search for further nontrivial solutions is thus restricted to n even and $\alpha_{\min} = 1$. We henceforth assume n even and study the effect of multiplying invertible integers $\bmod N$ belonging to U by $\omega \neq 1$, $\omega^2 \equiv 1 \bmod 2N$ $(\alpha = 1)$, $\omega \in U$. Invertible integers $\bmod N$ belong to $(\mathbf{Z}/N\mathbf{Z})^*$, and we set $U^* = L \cap (\mathbf{Z}/N\mathbf{Z})^*$, $L^* = L \cap (\mathbf{Z}/N\mathbf{Z})^*$. The following is the crucial observation.

Lemma 5.3. *Let n be even, distinct from* 12 *and* 30, *and* $N = 2n$, *for each $\omega \in U^*$ such that $\omega^2 \equiv 1 \bmod 2N$, excluding $\omega = 1$ and, if $n \equiv 2 \bmod 4$, excluding also $n - 1$, there exists $\lambda \in U^*$ such that $\omega\lambda \in L^*$.*

For each such ω we associate a factorization of n into $\delta\overline{\delta}$ where $(\delta, \overline{\delta}) = 1$ and δ $(\overline{\delta})$ is the smallest positive integer such that $\omega\delta = -\delta \bmod N$ $(\omega\overline{\delta} \equiv \overline{\delta} \bmod N)$. The following pairs are excluded by hypothesis (i) $\{n,\ 1\}$ and $\{1,\ n\}$ corresponding to $\omega = 1$ and $\omega = -1 \notin U^*$

(ii) $\{2,\ \frac{n}{2}\}$ for $n \equiv 2 \bmod 4$ ($\alpha = 1$, $\omega = n-1$) (iii) $\{3,\ 4\}$ for $n = 12$ (iv) $\{3,\ 10\}$, $\{5,\ 6\}$ for $n = 30$.

We search a representative $\lambda \in U^*$ in the range $0 < \lambda < n$, such that a representative λ' of $\omega\lambda$ is in the range $-n < \omega\lambda < 0$. Since $(\delta,\ \overline{\delta}) = 1$ we look for λ and λ' in the form

$$\lambda = \mu\overline{\delta} + \rho\delta \qquad \lambda' = \mu\overline{\delta} - \rho\delta$$

with $0 < \mu < \delta$, $0 < \rho < \overline{\delta}$, μ and ρ prime respectively to δ and $\overline{\delta}$. Since n is even one in the pair δ, $\overline{\delta}$ is even the other odd. The above conditions imply λ prime to δ and $\overline{\delta}$ hence to $n = \delta\overline{\delta}$, hence to $N = 2n$ since n is even. Requiring $0 < \lambda < n$ and $-n < \lambda' < 0$ yields

$$0 < \frac{\mu}{\delta} + \frac{\rho}{\overline{\delta}} < 1 \qquad -1 < \frac{\mu}{\delta} - \frac{\rho}{\overline{\delta}} < 0.$$

As μ/δ and $\rho/\overline{\delta}$ should be positive irreducible fractions smaller than one, the lower bounds are irrelevant. If a solution μ, ρ exists, so does a fortiori a solution 1, ρ. Thus it is sufficient to look for ρ in the range $0 < \rho < \overline{\delta}$ prime to $\overline{\delta}$ such that

$$\frac{1}{\delta} < \frac{\rho}{\overline{\delta}} < 1 - \frac{1}{\delta}. \tag{5.46}$$

It is easy to convince oneself that no such ρ exists in the cases excluded from the lemma. In all other cases we exhibit a solution. We distinguish several cases.

(i) $2 < \overline{\delta} < \delta < n = \delta\overline{\delta}$ Set $\rho = 1$. The inequalities are satisfied since $\frac{1}{\delta} < \frac{1}{\overline{\delta}} < \frac{1}{2} < 1 - \frac{1}{\delta}$.

(ii) $2 < \delta < \overline{\delta} < n$, δ even, hence $\overline{\delta}$ odd of the form $\overline{\delta} = 2k+1$, $k > 1$. Set $\rho = k = \frac{\overline{\delta}-1}{2}$. Clearly $(\rho,\ \overline{\delta}) = 1$ and since $\overline{\delta} > \delta > 2$, the inequalities are satisfied.

(iii) $2 < \delta < \overline{\delta} < n$, δ odd, $\overline{\delta} \equiv 0 \bmod 4$. Choose $\rho = \frac{\overline{\delta}}{2} - 1$ for $\overline{\delta} > 4$. The case $\overline{\delta} = 4$, $\delta = 3$ is one of the excluded ones. Hence $\overline{\delta} = 4k$, $k > 1$, and $\rho = 2k - 1 > 1$. Any common factor of ρ and $\overline{\delta}$ would divide $\overline{\delta} - 2\rho = 2$, hence ρ being odd $(\rho,\ \overline{\delta}) = 1$ and $\rho/\overline{\delta} < 1$. Since δ odd is larger than 2 we have $\delta \geq 3$, so $1/\delta \leq 1/3$ and $1 - 1/\delta \geq 2/3$ while $\rho/\overline{\delta} = \frac{1}{2} - \frac{1}{4k}$ is bounded by $\frac{1}{2} - \frac{1}{8} \leq \rho/\overline{\delta} < \frac{1}{2}$, *i.e.* $1/\delta \leq 1/3 < 3/8 \leq \rho/\overline{\delta} < 1/2 < 2/3 \leq 1 - 1/\delta$.

(iv) $2 < \delta < \overline{\delta} < n$, δ odd, $\overline{\delta} \equiv 2 \bmod 4$. The case $\overline{\delta} = 6$ and δ odd prime to $\overline{\delta}$ in the interval $2 < \delta < \overline{\delta}$ requires $\overline{\delta} = 5$ excluded by hypothesis. Thus $\overline{\delta} > 6$. If $\overline{\delta} = 10$, the possible δ's are 3, 7, 9. The pair

$\delta = 3$, $\overline{\delta} = 10$ is again excluded by hypothesis. For $\delta = 7$ or 9 $\rho = 3$ is a solution. We can now assume $\overline{\delta} = 2k$, k odd ≥ 7. Take $\rho = k - 2$ odd. Any common divisor of ρ and $\overline{\delta}$ divides 4, and ρ being odd has to be 1. Thus $(\rho, \overline{\delta}) = 1$ and $0 < \rho/\overline{\delta} < 1$. Now $1/\delta \leq 1/3$, $1 - 1/\delta \geq 2/3$. On the other hand $\rho/\overline{\delta} < 1/2 < 1-1/\delta$, while the condition $\rho/\overline{\delta} > 1/3 \geq 1/\delta$ means $3k - 6 > 2k$, *i.e.* $k > 6$, which is the case. The lemma is proved.

We can now complete the proof of the proposition. We assume $n > 2$ since $n = 2$ is a trivial case with a unique $\chi_1 = 1$. The choice Ω_n leads to the A-invariant and is the unique one for n odd (Lemma 5.2). We look then for additional terms involving Ω's such that $\omega \neq \pm 1$ when n is even larger or equal to 4. By Lemma 5.1 these additional terms are such that $\alpha_{\text{min}} = 1$ or 2. If $\alpha_{\text{min}} = 2$ the only possibility in $\Omega_n + \Omega_2$ (and $4|n$). Thus we are left with n even and $\alpha_{\text{min}} = 1$.

Consider the coefficient of $\overline{\chi}_\lambda$ for $\lambda \in U^*$. Only the Ω_δ's such that $\alpha(\delta) = 1$ contribute and, if we have a further invariant, some do occur with a positive coefficient (in which case $\overline{\delta} = n/\delta$ does not occur). Let us show that the corresponding multiplier ω must be such that $\omega U^* = U^*$ (we may include $\omega = 1$ corresponding to Ω_n). Indeed if $\lambda \in U^*$ is such that $\omega\lambda \in L^*$ then for positivity another ω' must occur such that $\omega'\lambda \in U^*$ and $(\omega + \omega')\lambda \equiv 0 \bmod N$. Since λ is invertible $\bmod N$ this entails $\omega + \omega' \equiv 0 \bmod N$ or $\delta\delta' = n$ which is excluded by hypothesis. Thus we are in a position to apply Lemma 5.3. We consider in turn the cases $n \equiv 0$ or $2 \bmod 4$.

(i) $n \equiv 0 \bmod 4$. If $n \neq 12$ all Ω_δ, $\delta \neq n$ with $\alpha_{\text{min}} = 1$ are excluded. Then $\alpha_{\text{min}} = 2$, $\Omega_n + \Omega_2$ is the only nontrivial possibility (D-type) besides Ω_n (A-type). For $n = 12$ an additional solution $\Omega_{12} + \Omega_3 + \Omega_2$ is found by inspection (E_6-type).

(ii) $n \equiv 2 \bmod 4$. This implies α_{min} odd therefore equal to one. According to Lemma 5.3 if $n \neq 30$, the only possible additional term with $\alpha(\delta) = 1$ is Ω_2 $(\omega = n - 1)$. By looking at the coefficient of $\overline{\chi}_2$ (*i.e.* $\sum_{\lambda'}(\Omega_n + c_2\Omega_2)_{2\lambda'}\chi_{\lambda'} = (1 - c_2)\chi_2$) it follows that c_2 is 0 or 1. Thus besides Ω_n (A-type), $\Omega_n + \Omega_2$ (D-type) we have in this case for $n \neq 30$ as only further possibility $\Omega_n + \Omega_2 + \sum_{\alpha(\delta)\geq 3} c_\delta \Omega_\delta$ where all the α's are odd. Let us show that except when $n = 18$ and $n = 30$ all coefficients c_δ's have to vanish. The proof parallels the one of Lemma 5.1. Let $\alpha'_{\text{min}} \geq 3$ be the lowest possible value among those occuring in the additional terms. Look at the coefficient of $\overline{\chi}_{\alpha'_{\text{min}}}$ where the first term contributes $\chi_{\alpha'_{\text{min}}} + \chi_{n-\alpha'_{\text{min}}}$ and only those Ω'_δ's such that $\alpha(\delta) = \alpha'_{\text{min}}$ yield further contributions. For each such term the corresponding indices of χ range over the vertices of a regular polygon with α'_{min} vertices. As in Lemma 5.1, indices belonging to L from these polygons cannot be compensated by those belonging to U from the same or another polygon with the same number of vertices $\alpha'_{\min}$. This leaves as unique possibility a single polygon (and a single Ω_δ) with coefficient $c_\delta = 1$, the negative terms of which being compensated by $\chi_{\alpha'_{\min}}$ and $\chi_{n-\alpha'_{\min}}$ with $\alpha'_{\min} = 3$ or 5, since the polygon has at most two vertices in L. Let ω be the corresponding multiplier. If the polygon has two vertices in L, we must have

$$\omega\alpha'_{\min} + \xi N/\alpha'_{\min} \equiv -\alpha'_{\min} \bmod N$$
$$\omega\alpha'_{\min} + (\xi - 1)N/\alpha'_{\min} \equiv -(n - \alpha'_{\min}) \bmod N$$

for some $\xi \bmod \alpha'_{\min}$. Set $n = 2(\alpha'_{\min})^2 q;\ n \geq 18$. Subtract both equations, obtaining $N/\alpha'_{\min} = n - 2\alpha'_{\min} + \rho N$ for some integer ρ, *i.e.* $q[(2\rho+1)\alpha'_{\min} - 2] = 1$. Thus $q = 1$ and $\alpha'_{\min} = 3,\ n = 18$ excluding $\alpha'_{\min} = 5$. If $\alpha'_{\min} = 3$ it is not possible for the equilateral triangle to have a single term in L compensated by $\chi_{\alpha'_{\min}}$ or $\chi_{n-\alpha'_{\min}}$ as in the proof of Lemma 5.1. To summarize the only exceptional cases left correspond to $N = 18$ and $n = 30$, which are readily studied separately with the result quoted in the Proposition 5.2 and this concludes its proof.

This proposition which settles the case of W-Z-W models based on $SU(2)$ is the building block for the classification of various other rational conformal theories. I will quote the main application to minimal models but skip some details.

In the case of minimal models it is simpler to return to the original notation, introduced at the beginning of this section, where the Virasoro characters are labelled by two integers r and $s \bmod 2p'$ and $2p$ respectively, $(p,\ p') = 1$, with appropriate (anti-)symmetries. Then the role of $N = 2n$ in the affine case is played now by the pair $\{2p,\ 2p'\}$.

Proposition 5.3. (i) *Acting on Virasoro characters of the minimal series, the commutant of the modular transformations is spanned by tensor products $\Omega_{\delta'} \otimes \Omega_\delta$ in an obvious notation where $\delta|p'$, $\delta|p$.*
(ii) *As a result, the conformal invariant partition functions of minimal models are specified by a pair of elements in table* II, *where the role of n is played by p' and p. Since p and p' are coprime one of them at least is odd the corresponding invariant being of A-type.*

The partition functions are displayed in table III, assuming for definiteness that p is odd (not necessarily greater than p'). In the unitary case, where we have successive integers m and $m+1$, p is m for m odd or $m+1$ for m even. As a result we have two infinite series of invariants and three pairs of exceptional models.

Proposition 5.3 has been applied to $N=1$ minimal superconformal models, parafermionic theories $\cdots$ and can be extended to coset models involving the algebra $A_1^{(1)}$.

$$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1}|\chi_{rs}|^2 \qquad (A_{p-1}, A_{p-1})$$

$$p'=4\rho+2,\ \rho\geq 1 \qquad \frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{\substack{r\ \mathrm{odd}\ =1\\ r\neq 2\rho+1}}^{4\rho+1}|\chi_{rs}|^2+2|\chi_{2\rho+1\ s}|^2 + \sum_{r\ \mathrm{odd}\ =2}^{2\rho r-1}(\chi_{rs}\chi^*_{p'-r\ s}+\mathrm{c.c.})\right\} \qquad (D_{2\rho+2}, A_{p-1})$$

$$p'=4\rho\ \ \rho\geq 2 \qquad \frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}\ =1}^{4\rho-1}|\chi_{rs}|^2+2|\chi_{2\rho\ s}|^2 + \sum_{r\ \mathrm{even}\ =2}^{2\rho-2}(\chi_{rs}\chi^*_{p'-r\ s}+\mathrm{c.c.})\right\} \qquad (D_{2\rho+1}, A_{p-1})$$

$$p'=12 \qquad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{7s}|^2+|\chi_{4s}+\chi_{8s}|^2+|\chi_{5s}+\chi_{11s}|^2\right\} \qquad (E_6, A_{p-1})$$

$$p'=18 \qquad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{17s}|^2+|\chi_{5s}+\chi_{13s}|^2+|\chi_{7s}+\chi_{11s}|^2 + |\chi_{9s}|^2+\left[(\chi_{3s}+\chi_{15s})\chi^*_{9s}+\mathrm{c.c.}\right]\right\} \qquad (E_7, A_{p-1})$$

$$p'=30 \qquad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}|^2 + |\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}|^2\right\} \qquad (E_8, A_{p-1})$$

Table III. **List of invariant partition functions in terms of conformal characters with p assumed odd. In the unitary case $p=m$, $p'=m+1$ for m odd and $p=m+1$, $p'=m$ for m even.**

5.7.

Attempts have been made to interpret and justify the appearance of an A-D-E classification of $A_1^{(1)}$ W-Z-W models, in particular by Nahm, relating it to quaternionic spaces (Nahm-1987) and/or to link it with other classifications such as solutions of Yang-Baxter equations of lattice models, finite subgroups of SU_2 with the McKay correspondence between their irreducible representations and simply laced affine Dynkin diagrams, isolated singularities of algebraic surfaces. For a review of some of these connections see (Slodowy–1983). We shall give some flavour of these interrelations without much elaboration but observe that any real insight should at least have the virtue to give a hint on how to extend this classification to W-Z-W models based on higher rank Lie groups. This does not seem to be the case when these lines are written.

We shall use as an illustration the E_8 affine invariant ($n = 30$) in table II which we rewrite

$$(5.47) \qquad \begin{aligned} &Z_{E_8} = |y_1|^2 + |y_2|^2 \\ &y_1 = \chi_1 + \chi_{11} + \chi_{19} + \chi_{29} \qquad y_2 = \chi_7 + \chi_{13} + \chi_{17} + \chi_{23}. \end{aligned}$$

The indices run over the E_8 exponents split into two disjoint sets; taken together there are all integers primes to 30, in other words representatives of $(\mathbf{Z}/30\mathbf{Z})^*$. Under the operation T we have

$$(5.48) \qquad \begin{aligned} y_1(\tau+1) &= \exp 2i\pi\left(\frac{1}{120} - \frac{1}{8}\right) y_1(\tau) \\ y_2(\tau+1) &= \exp 2i\pi\left(\frac{49}{120} - \frac{1}{8}\right) y_2(\tau). \end{aligned}$$

Since the fifth power T^5 acts as the identity multiplied by $\exp 2i\pi/12$, it is a theorem that for such a low power the group of level five Γ_5 acts trivially on the ratio $z(\tau) = \frac{y_2(\tau)}{y_1(\tau)}$ and the image of the modular group is realized as $\Gamma^5 = PSL(2;\ \mathbf{Z}/5\mathbf{Z})$ acting on $z(\tau)$ through homographic transformations. This group is isomorphic to the group A_5 of even permutations on five objects and equivalently to the icosahedral group already encountered, with $60 = \frac{1}{2}5!$ elements. Thus we find a "natural" correspondence between the E_8 invariant and this finite subgroup of three dimensional rotations. With the notations

$$z(\tau) = \frac{y_2(\tau)}{y_1(\tau)} \qquad \omega = \exp\frac{2i\pi}{5}$$

(5.49)
$$u = \omega + \omega^{-1} = \frac{\sqrt{5}-1}{2} = 2\cos 2\pi/5$$
$$v = \omega^2 + \omega^{-2} = -\frac{\sqrt{5}+1}{2} = 2\cos 4\pi/5$$

we find

(5.50)
$$z(\tau+1) = \omega^2 z(\tau)$$
$$z(-\tau^{-1}) = \frac{1 - uz(\tau)}{z(\tau)+u}.$$

If following Klein one introduces isobaric polynomials in z *i.e.* polynomials which under modular transformations are multiplied by a power of the denominator (Klein–1956)

(5.51)
$$V = z(z^{10} - 11z^5 - 1)$$
$$E = z^{30} + 1 - 522(z^{25} - z^5) - 10005(z^{20} + z^{10})$$
$$F = z^{20} + 1 + 228(z^{15} - z^5) + 494z^{10}$$

of respective degrees 12 (adding ∞ as a root of V), 30 and 20, their zeros on the Riemann sphere are the vertices, the mid edge points and mid face points of a regular icosahedron centrally projected on this sphere. Furthermore the following relation holds

(5.52)
$$E^2 - F^3 + 1728V^5 = 0$$

which may be interpreted as a surface in $\mathbf{C}^3$ with an isolated singularity at the origin, pointing to a relation with the classification of isolated singularities on algebraic surfaces.

Let us show on this example the connection between finite subgroups of $SU(2)$ and Dynkin diagrams of simply laced Lie algebras. For this purpose it is best to return to homogeneous polynomials in y_1, y_2 which we denote with the same letter by substituting ${y_1}^{12}V(\frac{y_2}{y_1})$, ${y_1}^{30}F(\frac{y_2}{y_1})$ for $V(z)$, $E(z)$, $F(z)$. These new quantities are at most multiplied by a constant to the powers 12, 30 and 20 or its square to the powers 6, 15, and 10 respectively under modular transformations, factors which do not affect the relation (5.52). This is due to the fact that on y_1 and y_2

T and S act as unitary matrices (generating $\overline{\Gamma^5}$) times $\exp 2i\pi/12$ and $\exp 2i\pi/4$ respectively.

For two variables y'_1 and y'_2 which do transform according to $\overline{\Gamma^5}$ one can however find genuine invariant analogs of V, E, F which we call X, Z and Y, homogeneous of degrees 12, 30 and 20 in y'_1 and y'_2 and which, absorbing irrelevant factors, satisfy a relation similar to (5.52)

$$X^5 + Y^3 + Z^2 = 0. \tag{5.53}$$

The binary icosahedral group $\overline{\Gamma^5}$ has 9 classes and 9 irreducible representations R_i. It has been recognized by McKay that these representations can be made to correspond one to one to a set of fundamental roots of the affine extension $E_8^{(1)}$ of the simply laced Lie algebra E_8. Similar properties hold of course for other members of the A-D-E family (McKay–1981, Ford and McKay–1981). To exhibit the correspondence, plot as nodes of a graph the representations R_i of $\overline{A_5}$. Since this is a subgroup of SU_2, it admits an irreducible representation of dimension 2 (in fact it admits two such inequivalent representations, we pick one of those corresponding to a choice of root of 5). Call R_2 this representation. Then by tensoring with R_2

$$R_2 \otimes R_i = \bigoplus_j f_{ij} R_j. \tag{5.54}$$

The matrix $C = 2 - f$ is the Cartan matrix of $E_8^{(1)}$, and f_{ij} is symmetic, equal to zero or one. We connect the nodes i and j if $f_{ij} = 1$, and obtain the Dynkin diagram of $E_8^{(1)}$ (if one deletes the node R_1 corresponding to the identity representation, one obtains the diagram for E_8). If $\chi_{R_i}(c_\alpha)$ is the character of R_i evaluated on the class c_α, the nonnegative eigenvalues of C are $2 - \chi_2(c_\alpha)$. The eigenvectors are the columns of the character table pertaining to a fixed class c_α. The following table IV gives the characters of this group with u and v as in (5.49). The top row gives the number of elements in each class, the first column the dimension of the representation (as the character on the identity class). The table is completed by the Dynkin diagram.

	1	1	30	20	20	12	12	12	12
R_1	1	1	1	1	1	1	1	1	1
R_2	2	-2	0	-1	1	u	v	$-v$	$-u$
R_2'	2	-2	0	-1	1	v	u	$-u$	$-v$
R_3	3	3	-1	0	0	$-u$	$-v$	$-v$	$-u$
R_3'	3	3	-1	0	0	$-v$	$-u$	$-u$	$-v$
R_4	4	-4	0	1	-1	-1	-1	1	1
R_4'	4	4	0	1	1	-1	-1	-1	-1
R_5	5	5	1	-1	-1	0	0	0	0
R_6	6	-6	0	0	0	1	1	-1	-1

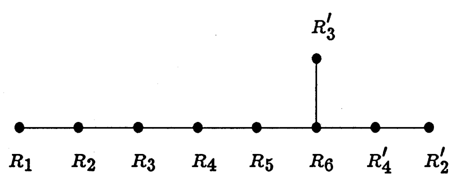


Table IV. Character table for the binary icosahedral group. Associated Dynkin diagram.

From the same data we can easily recapture the fact that the ring of invariant polynomials in y_1', y_2' is generated by three homogeneous polynomials X, Y, Z of degree $2a = 12$, $2b = 20$, $2c = 30$ respectively (the degrees have to be even to take into account the symmetry $(y_1', y_2') \to (-y_1', -y_2')$. Note that with $n = 30$ the Coxeter number (of E_8) we have

$$a + b + c = n + 1$$

a relation also valid for other A-D-E members. Following (Kostant–1984) let $Q(t)$ denote a formal infinite sum over irreducible representations D_j of SU_2 (it could be replaced by a convergent sum for $|q| < 1$ and $\chi_j(\theta) = \sin(2j+1)/\sin\theta$ substituted for D_j)

$$Q(q) = \bigoplus_{2j=0}^{\infty} q^{2j} D_j \tag{5.55}$$

$$D_{\frac{1}{2}} \otimes Q(q) = q^{-1}\left(Q(q) - I\right) + qQ(q)$$

where the second relation expresses that $D_{1/2} \otimes D_j = D_{j+1/2} \oplus D_{j-1/2}$. Restrict now each representation to a finite subgroup $G \subset SU_2$ (here the binary icosahedral group) such that $D_j = \oplus n_{j,k} R_k$ with nonnegative integers $n_{j,k}$. Then

$$Q(q) = \bigoplus_k P_k(q) R_k \qquad P_k(q) = \sum_{2j=0}^{\infty} n_{j,k} q^{2j} \tag{5.56}$$

with $P_k(q)$ a generating function for the number of times $(n_{j,k})$ the representation R_k appears in the decomposition of D_j. In particular $P_1(q)$ will count the invariants. The relation (5.55) can be understood as referring to (tensor) multiplication by R_2 the defining representation of G. Thus

$$\sum_k P_k(q) f_{k\ell} = (q + q^{-1}) P_\ell(q) - q^{-1} \delta_{\ell,1}. \tag{5.57}$$

Using the fact that the matrix f is symmetric and introducing the column vector $P(q) = (P_1(q), \dots)$, $P(0) = (1, 0, 0, \dots)$ we see that

$$P(q) = \left(\frac{1}{1 + q^2 - qf} \right) P(0). \tag{5.58}$$

The matrix f is diagonal in the character basis (here the characters are real). Expanding $P(0)$ as

$$P_k(0) = \delta_{k,1} = \sum_\alpha x_\alpha \chi_k(C_\alpha)$$

and comparing with the orthogonality of characters

$$\sum_\alpha |C_\alpha| \chi_k(C_\alpha) \chi_\ell^*(C_\alpha) = |G| \delta_{k\ell}$$

we find $x_\alpha = \frac{|C_\alpha|}{|G|}$,

$$P_k(q) = \sum_\alpha \frac{|C_\alpha|}{|G|} \frac{\chi_k(C_\alpha)}{1 - q\chi_2(C_\alpha) + q^2} \tag{5.59}$$

and in particular

$$P_1(q) = \sum_\alpha \frac{|C_\alpha|}{|G|} \frac{1}{1 - q\chi_2(C_\alpha) + q^2}.$$

Applying this formula to the binary icosahedral group, we find

$$
\begin{aligned}
P_1(q) &= \frac{1+q^{30}}{(1-q^{12})(1-q^{20})} = \frac{1-q^{60}}{(1-q^{12})(1-q^{20})(1-q^{30})} \\
&\equiv \frac{1-q^{2n}}{(1-q^{2a})(1-q^{2b})(1-q^{2c})}
\end{aligned}
\tag{5.60}
$$

in agreement with the existence of generating polynomials X, Y, Z of degrees 12, 20, 30 satisfying a "homogeneous" equation of degree 60. One can also compute the other P_k's which we list as

$$
P_k(q) = \frac{p_k(q)}{(1-q^{12})(1-q^{20})}
\tag{5.61}
$$

(5.62)

$$
\begin{aligned}
p_1(q) &= 1+q^{30} \\
p_2(q) &= q+q^{11}+q^{19}+q^{29} \\
p_2'(q) &= q^7+q^{13}+q^{17}+q^{23} \\
p_3(q) &= q^2+q^{10}+q^{12}+q^{18}+q^{20}+q^{28} \\
p_3'(q) &= q^6+q^{10}+q^{14}+q^{16}+q^{20}+q^{24} \\
p_4(q) &= q^3+q^9+q^{11}+q^{13}+q^{17}+q^{19}+q^{21}+q^{27} \\
p_4'(q) &= q^6+q^8+q^{12}+q^{14}+q^{16}+q^{18}+q^{22}+q^{24} \\
p_5(q) &= q^4+q^8+q^{10}+q^{12}+q^{14}+q^{16}+q^{18}+q^{20}+q^{22}+q^{26} \\
p_6(q) &= q^5+q^7+q^9+q^{11}+2q^{15}+q^{17}+q^{19}+q^{21}+q^{23}+q^{25}.
\end{aligned}
$$

Note that

$$
\sum_k P_k(q)\chi_k^*(C_\alpha) = \frac{1}{1-q\chi_2(C_\alpha)+q^2}
\tag{5.63}
$$

and in particular

$$
\sum_k P_k(q)\,\dim(R_k) = \frac{1}{(1-q)^2}.
\tag{5.64}
$$

The exponents appearing in the numerators $p_2(q)$ and $p_2'(q)$ corresponding to the two 2-dimensional representations of the group are the Coxeter exponents of E_8 in just the two sets found as indices in the corresponding E_8 affine partition function!

One can go even further by looking at the ring of invariant polynomials in X, Y, Z graded as $[X^{n_1}Y^{n_2}Z^{n_3}] = an_1 + bn_2 + cn_3$ (half the even degree in y'_1, y'_2) modulo those polynomials which are multiples of $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$, $\frac{\partial f}{\partial Z}$, where f is the left hand side of (5.53), $f \equiv X^5 + Y^3 + Z^2$, $[f] = n$ here 30 (Saito–1986). Clearly we find a finite ring. In the present case one works mod (X^4, Y^2, Z) that is we have polynomials

(5.65)

	1	X	Y	X^2	YX	X^3	YX^2	YX^3
grade	0	6	10	12	16	18	22	28
$1+$grade	1	7	11	13	17	19	23	29

such that the set of $(1+\text{grade})$ is the set of Coxeter exponents (here of E_8). A generating function (for the exponents) is the polynomial

$$\begin{aligned}\psi(q) &= q\frac{(1-q^{n-a})(1-q^{n-b})(1-q^{n-c})}{(1-q^a)(1-q^b)(1-q^c)} \\ &= q^{-n}\frac{(q^n-q^a)(q^n-q^b)(q^n-q^c)}{(q^a-1)(q^b-1)(q^c-1)}\end{aligned} \tag{5.66}$$

where we have taken into account that $a+b+c = n+1$. The value $\psi(1)$, the number of exponents, is the rank of the (non affine) Lie algebra (here 8). We reproduce below the data for the exceptional cases.

(5.67)

		a	b	c	n	$f=0$	exponents
tetrahedral	E_6	3	4	6	12	$X^4+Y^3+Z^2$	$1,4,5,7,8,11$
octahedral	E_7	4	6	9	18	$X^3Y+Y^3+Z^2$	$1,5,7,9,$ $11,13,17$
icosahedral	E_8	6	10	15	30	$X^5+Y^3+Z^2$	$1,7,11,13,$ $17,19,23,29$

The connection between singular manifolds and effective potential in a Landau mean field approach is the theme of some recent investigations.

§6. Extensions

It is natural to expect that the W-Z-W models are among the basic building blocks of rational conformal theories. This is strongly suggested by the coset construction of Goddard, Kent and Olive. Therefore one would like to extend the previous classification to higher rank Lie groups. The task looks formidable, but one may hope to see the emergence of some general features. In this last section we describe an (easy) extension to $SU(N)$ models at level one and indicate some ways to deal with rank two Lie groups using the arithmetics of quadratic integers. W-Z-W models based on $SU(N)$ level one involve N characters labelled by an integer λ running from 0 to $N-1$ and identified with the number of boxes in a Young tableau reduced to a single column of length at most $N-1$. The empty Young tableau is assigned to the character based on the identity representation of $SU(N)$, the vacuum state. The modular transformation properties imply that the character $\chi_\lambda(\tau)$ can be considered as an even periodic function of $\lambda \bmod N$,

$$\chi_\lambda(\tau) = \chi_{\lambda+N}(\tau) = \chi_{-\lambda}(\tau) \tag{6.1}$$

such that

$$\begin{aligned} T: &\quad \chi_\lambda(\tau+1) = \exp 2i\pi\left[\frac{\lambda(N-\lambda)}{2N} - \frac{N-1}{24}\right]\chi_\lambda(\tau) \\ S: &\quad \chi_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}}\sum_{\lambda' \bmod N} \exp -\frac{2i\pi\lambda\lambda'}{N}\chi_{\lambda'}(\tau). \end{aligned} \tag{6.2}$$

The corresponding algebra of fusion rules is isomorphic with the algebra of N-th roots of unity. In order to get positive integral invariant partition functions, properly normalized, we can exploit our results of the preceding section, where the formulas were quite similar except for the fact that N was even and the character odd in λ for SU_2. Here we set $n = N/2$ if $2|N$ or $n = N$ if $2 \nmid N$. Apart from this minor change the definition of Ω_δ, $\delta|n$ is the same as in (5.29) except that when $n = N \equiv 1 \bmod 2$ the multiplier ω satisfies $\omega^2 \equiv 1 \bmod N/\alpha^2$ instead of $\bmod 2N/\alpha^2$. A second caveat (a reflection of "charge conjugation" symmetry) is that while the restricted characters $\chi_\lambda(\tau)$ and $\chi_{N-\lambda}(\tau)$ are equal, this ceases to be true if we keep the dependence on (Cartan torus) angles. So we distinguish a partition function of the type $\sum_{\lambda,\lambda'}\overline{\chi}_\lambda Z_{\lambda\lambda'}\chi_{\lambda'}$ from $\sum_{\lambda,\lambda'}\overline{\chi}_\lambda Z_{\lambda,N-\lambda'}\chi_{\lambda'}$, even though as functions of τ only, they appear identical.

Proposition 6.1. *For every divisor δ of N, if N is odd, or of $N/2$ if N is even the partition function*

$$\begin{aligned} Z_{N,\delta}(\tau,\overline{\tau}) &= \sum_{\lambda,\lambda' \bmod N} \overline{\chi}_\lambda(\tau)(\Omega_\delta)_{\lambda,\lambda'}\chi_{\lambda'}(\tau) \\ &= \sum_{\lambda,\lambda' \bmod N/\alpha}\ \sum_{\xi \bmod \alpha} \overline{\chi}_{\alpha\lambda}(\tau)\chi_{\alpha\omega\lambda+\xi N/\alpha}(\tau) \end{aligned} \tag{6.3}$$

is a positive integral invariant partition function for $SU(N)$ at level 1, properly normalized.

A natural conjecture was that these exhaust the consistent $SU(N)$ models at level 1. P. De Giovani has informed me that he succeeded in actually proving the conjecture to be true.

Let us show some non trivial examples apart from the two obvious solutions $\delta = n$, $\sum_{\lambda \bmod N} |\chi_\lambda(\tau)|^2$ and $\delta = 1$ the "conjugate" solution (numerically equal) $\sum_{\lambda \bmod N} \overline{\chi}_\lambda(\tau)\chi_{N-\lambda}(\tau)$. Consider first $SU(9)$. Then we have an extra invariant

$$
\begin{aligned}
& Z_{9,3} = |\psi(\tau)|^2 \\
& \psi(\tau) = \chi_0(\tau) + \chi_3(\tau) + \chi_6(\tau) = \chi_0(\tau) + 2\chi_3(\tau)
\end{aligned}
\tag{6.4}
$$

behaving as

$$
\psi(\tau+1) = \exp\frac{-2i\pi}{3}\psi(\tau) \qquad \psi(-\tau^{-1}) = \psi(\tau). \tag{6.5}
$$

From

$$
\begin{aligned}
\chi_0(\tau) &= q^{-1/3}[1 + 80q + 1376q^2 + \cdots] \\
\chi_3(\tau) &= q^{-1/3}[84q + 1374q^2 + \cdots]
\end{aligned}
\tag{6.6}
$$

it is a pleasure to check that $\psi(\tau)$ coincides with the cube root of the modular invariant $j(\tau)$, namely the unique character of $E_8^{(1)}$ at level one

$$
\psi(\tau) = j(\tau)^{1/3} = q^{-1/3}[1 + 248q + 4124q^2 + \cdots]. \tag{6.7}
$$

More generally for p an odd prime, we have

$$
Z_{p^2,p}(\tau,\overline{\tau}) = \left|\sum_{a \bmod p} \chi_{ap}(\tau)\right|^2 . \tag{6.8}
$$

For $p = 5$, *i.e.* for $SU(25)$ at level 1 the quantity $\chi_0 + 2(\chi_5 + \chi_{10})$ is equal to the modular invariant up to a constant,

$$
\chi_0(\tau) + 2(\chi_5 + \chi_{10}) = j(\tau) - 120. \tag{6.9}
$$

One could of course multiply the examples.

In conclusion I will report a work in progress with M. Bauer for $SU(3)$. Most of what is said applies with little modification to G_2 or $SO(5)$ (and in fact the discussion of general invariants disregarding integrality conditions, extends readily to higher rank Lie groups). The idea is to make use of the fact that the $SU(3)$ weight lattice can be identified (up to scale) with a ring of quadratic integers with unique prime factor decomposition (class field number unity) and one case therefore hope to use this multiplicative structure as we did for $SU(2)$.

For $SU(3)$ we identify the weight lattice with the ring E of integers in $Q(\sqrt{-3})$. Let

$$\omega = \exp\frac{2i\pi}{6} \qquad j = \omega^2 = \exp\frac{2i\pi}{3}. \tag{6.10}$$

We write any integer in E as

$$\lambda = m_1 + m_2\omega \qquad \lambda\bar{\lambda} = m_1^2 + m_2^2 + m_1 m_2 \tag{6.11}$$

with $m_i \in \mathbf{Z}$. The norm is familiar from the discussion in Section 3. Let us consider a model at level k. The Coxeter number of $SU(3)$ being 3, it is convenient to trade k for

$$n = k + 3 \qquad n \geq 3. \tag{6.12}$$

A fundamental domain $\mathcal{B}_n$ for characters at level k is such that the corresponding suffixes λ (quadratic integers) fulfill the inequalities

$$\mathcal{B}_n : \qquad \lambda = m_1 + m_2\omega\,; \quad m_1 > 0,\ m_2 > 0,\ m_1 + m_2 < n. \tag{6.13}$$

The central charge is

$$c = \frac{8k}{k+4} = 8\left(1 - \frac{3}{n}\right). \tag{6.14}$$

The conformal weight h_λ is related to the norm of λ through

$$h_\lambda = \frac{\lambda\bar{\lambda} - 3}{3n} \qquad h_\lambda - \frac{c}{24} = \frac{\lambda\bar{\lambda} - n}{3n}. \tag{6.15}$$

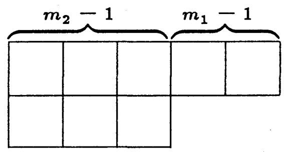
The root lattice is then the ideal generated by the prime

$$\rho = 1 + \omega \tag{6.16}$$

and the ratio of the weight to the root lattice in the field F_3 with three elements, with additive structure $\mathbf{Z}/3\mathbf{Z}$. It was called in older times by physicists the triality t (quark has triality 1, antiquark triality -1). This is

$$E/\rho E = F_3 \qquad t \equiv m_1 - m_2 \bmod 3 \quad t^2 \equiv \lambda\bar{\lambda} \bmod 3. \tag{6.17}$$

The relation with Young tableaux with at most two lines and m_1+m_2-2 boxes is depicted as



For $m_1,\ m_2 > 0$ the representation of $SU(3)$ indexed by λ has dimension

(6.18) $$\dim\lambda = \frac{m_1 m_2 (m_1+m_2)}{2} = \frac{\lambda^3 - \overline{\lambda}^3}{\rho^3 - \overline{\rho}^3}.$$

It is extended over all of E as an odd function under the Weyl group generated by the reflection $\lambda \to \overline{\lambda}$ and the rotation of $2\pi/3$, $\lambda \to j\lambda$. The two Casimir invariants, up to normalization, are

(6.19) $$\begin{aligned} \mathcal{C}_2(\lambda) &= \lambda\overline{\lambda} = m_1^2 + m_2^2 + m_1 m_2 \\ \mathcal{C}_3(\lambda) &= \lambda^3 + \overline{\lambda}^3 = (m_1 - m_2)(m_1 + 2m_2)(2m_1 + m_2). \end{aligned}$$

Note that charge conjugation $\lambda \leftrightarrow \omega\overline{\lambda}$ or $m_1 \leftrightarrow m_2$ leaves $\mathcal{C}_2$ invariant but changes the sign of $\mathcal{C}_3$. The three integers, obtained by the action of the Weyl group on $m_1 - m_2$,

$$d_1 = m_1 - m_2, \quad d_2 = m_1 + 2m_2, \quad d_3 = -(2m_1 + m_2)$$

satisfy a cubic equation

(6.20) $$d^3 - 3d\mathcal{C}_2 + \mathcal{C}_3 = 0$$

and of course one must have ${\mathcal{C}_3}^2 < 4{\mathcal{C}_2}^3$ for real solutions.

The characters of the corresponding affine algebra $A_2^{(1)}$, noted again χ_λ ($\lambda \in \mathcal{B}_n$) can be extended to E as follows. Consider the quadratic integer

(6.21) $$\mathcal{N} = \rho n.$$

It generates an ideal, $\mathcal{N}E$, n times the root lattice, and we set

(6.22) $$E^{\mathcal{N}} = E/\mathcal{N}E$$

a finite ring with $\mathcal{N}\overline{\mathcal{N}} = 3n^2$ elements. It is easily verified that if W stands for the Weyl group the "interior" of $E^{\mathcal{N}}/W$ is $\mathcal{B}_n$, where by interior we mean the orbits with $|W| = 3!$ points. We extend χ_λ on the complete weight lattice, through

$$(6.23) \qquad w \in W \qquad \chi_{w\lambda}(\tau) = \epsilon(w)\chi_\lambda(\tau) \quad \chi_{\lambda+\mathcal{N}}(\tau) = \chi_\lambda(\tau)$$

and $\mathcal{B}_n$ appears as a fundamental domain for a Weyl-Coxeter inhomogeneous group (isomorphic to the one of $A_2^{(1)}$). With ξ a primitive $3n$-th root of unity, $\xi = \exp 2i\pi/3n$, the modular transformation properties of the characters read

$$(6.24) \qquad \begin{aligned} T: &\quad \chi_\lambda(\tau+1) = -\omega\xi^{\lambda\bar{\lambda}}\chi_\lambda(\tau) \\ S: &\quad \chi_\lambda(-\tau^{-1}) = \frac{i}{\sqrt{3n^2}} \sum_{\lambda' \bmod \mathcal{N}} \xi^{\lambda\bar{\lambda}'+\bar{\lambda}\lambda'}\chi_{\lambda'}(\tau) \end{aligned}$$

where the expression of S uses the antisymmetry of χ_λ with respect to W. The restricted characters are invariant under charge conjugation, a property which translates into $\chi_{-\lambda} = -\chi_\lambda$. From the above, we recognize that the diagonal sum

$$(6.25a) \qquad Z_A^{(n)}(\tau,\bar{\tau}) = \sum_{\lambda\in\mathcal{B}_n} |\chi_\lambda(\tau)|^2$$

is a natural principal invariant, to which we can immediately add the (numerically equal) conjugate

$$(6.25b) \qquad Z_{A'}^{(n)}(\tau,\bar{\tau}) = \sum_{\lambda\in\mathcal{B}_n} \overline{\chi}_\lambda(\tau)\chi_{\omega\bar{\lambda}}(\tau).$$

A complementary D series can be described as follows (Bernard–1987, Altschüler, Lacki, Zaugg–1987). The equilateral triangle $\mathcal{B}_n$ is invariant by a rotation of $2\pi/3$ around its center,

$$\lambda \longrightarrow \sigma(\lambda) = j\lambda + n$$

where σ^3 is the identity. With $t \equiv m_1 - m_2 \bmod 3$, we find

Lemma 6.1.

$$(6.26a) \ \text{(i)} \quad t(\lambda) \equiv \pm n \mod 3 \iff \left|\sigma^{\pm1}(\lambda)\right|^2 \equiv |\lambda|^2 \mod 3n$$

$$(6.26b) \ \text{(ii)} \quad t(\sigma^{\pm1}(\lambda)) \equiv \pm n + t(\lambda) \mod 3.$$

The structure of a modular invariant D is then given by

Proposition 6.2. *The D-invariant reads*

$$Z_D^{(n)}(\tau,\overline{\tau}) = \sum_{\lambda,\lambda'\in\mathcal{B}_n} \overline{\chi}_\lambda(\tau) Z^D_{\lambda\lambda'}\chi_{\lambda'}(\tau)$$

(i) $n \equiv 0 \mod 3$

$$Z^D_{\lambda\lambda'} = \begin{cases} \delta_{\lambda,\lambda'} + \delta_{\sigma(\lambda),\lambda'} + \delta_{\sigma^2(\lambda),\lambda'} & \rho|\lambda,\ \rho|\lambda' \\ 0 & \text{otherwise} \end{cases} \tag{6.27}$$

(ii) $n \not\equiv 0 \mod 3$

$$Z^D_{\lambda,\lambda'} = \delta_{\sigma^{nt(\lambda)}(\lambda),\lambda'}.$$

In (i) the conditions $\rho|\lambda$, $\rho|\lambda'$ mean that λ and λ' belong to the root lattice, while in (ii) no such condition applies. To Z_D we may add the conjugate $Z_{D'}$ which may or may not coincide with Z_D. For instance when $n = 6$ using the explicit notation $\chi_\lambda = \chi_{m_1,m_2}$

$$Z_D^{(6)}(\tau,\overline{\tau}) \equiv Z_{D'}^{(6)}(\tau,\overline{\tau}) = |\chi_{1,1} + \chi_{1,4} + \chi_{4,1}|^2 + 3|\chi_{2,2}|^2$$

while for $n = 7$

$$\begin{aligned} Z_D^{(7)}(\tau,\overline{\tau}) =& |\chi_{1,1}|^2 + |\chi_{2,2}|^2 + |\chi_{3,3}|^2 + |\chi_{4,1}|^2 + |\chi_{1,4}|^2 \\ &+ \overline{\chi_{2,1}}\chi_{4,2} + \overline{\chi_{4,2}}\chi_{2,1} + \overline{\chi_{1,2}}\chi_{2,4} + \overline{\chi_{2,4}}\chi_{1,2} + \overline{\chi_{3,1}}\chi_{1,3} \\ &+ \overline{\chi_{1,3}}\chi_{3,1} + \overline{\chi_{5,1}}\chi_{1,5} + \overline{\chi_{1,5}}\chi_{5,1} + \overline{\chi_{3,2}}\chi_{2,3} + \overline{\chi_{2,3}}\chi_{3,2}. \end{aligned}$$

If we use the fact that as functions of τ only, two conjugate characters are equal $\chi_{m_1,m_2} = \chi_{m_2,m_1}$ we find that the difference $Z_A^{(7)} - Z_D^{(7)} = 2|\chi_{2,1} - \chi_{4,2}|^2$ and indeed

$$n = 7 \qquad \chi_{2,1} - \chi_{4,2} = 3.$$

One knows a few more invariants (Christe and Ravanini–1988). They emerge from embeddings into W-Z-W models for higher rank Lie groups at level 1 except $Z_{\mathcal{E}''}^{(12)}$ (Moore and Seiberg–1988) which derives from the $Z_D^{(12)}$ invariant using the identity

$$n = 12 \qquad \chi_{2,2} + \chi_{2,8} + \chi_{8,2} = \chi_{4,4} + 8.$$

The embeddings are in $SU(6)$ level 1 ($n = 8, c = 4$), E_6 level 1 ($n = 12, c = 6$) and E_7 level 1 ($n = 24, c = 24$) all the values of n being divisors of 24. For lack of a better notation they are referred as $Z_{\mathcal{E}}^{(n)}$ invariants (we don't list a possible distinct conjugate).

(6.28)

$$
\begin{aligned}
n = 8 \qquad Z_{\mathcal{E}}^{(8)} &= |\chi_{1,1} + \chi_{3,3}|^2 + |\chi_{3,1} + \chi_{3,4}|^2 \\
&+ |\chi_{1,3} + \chi_{4,3}|^2 + |\chi_{4,1} + \chi_{1,4}|^2 \\
&+ |\chi_{2,3} + \chi_{6,1}|^2 + |\chi_{3,2} + \chi_{1,6}|^2 \\
n = 12 \qquad Z_{\mathcal{E}}^{(12)} &= |\chi_{1,1} + \chi_{10,1} + \chi_{1,10} + \chi_{5,5} + \chi_{5,2} + \chi_{2,5}|^2 \\
&+ 2|\chi_{3,3} + \chi_{3,6} + \chi_{6,3}|^2 \\
Z_{\mathcal{E}''}^{(12)} &= |\chi_{1,1} + \chi_{1,10} + \chi_{10,1}|^2 \\
&+ |\chi_{3,3} + \chi_{3,6} + \chi_{6,3}|^2 \\
&+ 2|\chi_{4,4}|^2 + |\chi_{2,5} + \chi_{5,2} + \chi_{5,5}|^2 \\
&+ |\chi_{1,4} + \chi_{7,1} + \chi_{4,7}|^2 \\
&+ |\chi_{4,1} + \chi_{1,7} + \chi_{7,4}|^2 \\
&+ \overline{\chi_{4,4}}\,(\chi_{2,2} + \chi_{2,8} + \chi_{8,2}) \\
&+ (\overline{\chi_{2,2}} + \overline{\chi_{2,8}} + \overline{\chi_{8,2}})\,\chi_{4,4} \\
n = 24 \qquad Z_{\mathcal{E}}^{(24)} &= |\chi_{1,1} + \chi_{1,22} + \chi_{22,1} + \chi_{5,5} \\
&+ \chi_{14,5} + \chi_{5,14} + \chi_{11,11} + \chi_{11,2} \\
&+ \chi_{2,11} + \chi_{7,7} + \chi_{10,7} + \chi_{7,10}|^2 \\
&+ |\chi_{7,1} + \chi_{1,7} + \chi_{16,1} + \chi_{1,16} \\
&+ \chi_{7,16} + \chi_{16,7} + \chi_{5,8} + \chi_{8,5} \\
&+ \chi_{5,11} + \chi_{11,5} + \chi_{8,11} + \chi_{11,8}|^2
\end{aligned}
$$

Perhaps to no surprise if, for $n = 12$, if we define $z(\tau)$ as the ratio

$$z(\tau) = \sqrt{2}\frac{\chi_{3,3} + \chi_{3,6} + \chi_{6,3}}{\chi_{1,1} + \chi_{10,1} + \chi_{1,10} + \chi_{5,5} + \chi_{5,2} + \chi_{2,5}}$$

we find that it transforms according to the tetrahedral group generated by

$$
\begin{aligned}
&T \qquad z(\tau + 1) = j^2 z(\tau) \\
&S \qquad z(-\tau^{-1}) = \frac{\sqrt{2} - z(\tau)}{\sqrt{2}z(\tau) + 1}
\end{aligned}
$$

while in the case $n = 24$ the similar ratio

$$z(\tau) = \frac{\chi_{7,1} + \chi_{1,7} + \cdots}{\chi_{1,1} + \chi_{22,1} + \chi_{1,22} + \cdots}$$

transforms according to the octahedral group, generated by

$$T: \quad z(\tau+1) = -iz(\tau), \qquad S: \quad z(-\tau^{-1}) = \frac{1-z(\tau)}{1+z(\tau)}.$$

Could there exist an icosahedral invariant not yet listed. One might dream that other finite groups (perhaps subgroups of $SU(3)$) might play a role.

If one forgets for the moment the integrality (and positivity property) one can easily describe the commutant of the linear transformations

$$(6.29) \qquad \begin{aligned} \mathcal{T}_{\lambda,\lambda'} &= \xi^{\lambda\bar{\lambda}}\delta_{\lambda,\lambda'} \qquad \xi^{3n} = 1 \\ \mathcal{S}_{\lambda,\lambda'} &= \frac{1}{\sqrt{3n^2}}\xi^{\lambda\bar{\lambda}'+\bar{\lambda}\lambda'} \end{aligned}$$

where the indices are mod $\mathcal{N}$. Following the technique of finite quantum mechanics of Section 5, we introduce a Hilbert space $\mathcal{H}$ of dimension $3n^2$ spanned by an orthonormal basis $\{|\lambda\rangle\}$ with $\lambda \bmod \mathcal{N}$. The operators $\mathcal{T}$ and $\mathcal{S}$ act in $\mathcal{H}$. Now $E^{\mathcal{N}}$ as an additive group is isomorphic to its dual, this justifies introducing $3n^2$ diagonal operators

$$(6.30) \qquad \begin{aligned} Q^{\mu}|\lambda\rangle &= \xi^{\bar{\mu}\lambda+\mu\bar{\lambda}}|\lambda\rangle \\ Q^{\mu}Q^{\nu} &= Q^{\mu+\nu}, \qquad \mu,\ \nu \bmod \mathcal{N}. \end{aligned}$$

Further define $3n^2$ operators P^{μ}, $\mu \bmod \mathcal{N}$ such that

$$(6.31) \qquad P^{\mu} = \mathcal{S}^{-1}Q^{\mu}\mathcal{S} \qquad \mathcal{S}^{-1}Q^{\mu}\mathcal{S} = Q^{-\mu}$$

with

$$(6.32) \qquad P^{\mu}|\lambda\rangle = |\lambda+\mu\rangle \qquad Q^{\mu}P^{\nu} = \xi^{\mu\bar{\nu}+\bar{\mu}\nu}P^{\nu}Q^{\mu}.$$

The operators P, Q, ξ generate the Heisenberg on $E^{\mathcal{N}}$ and any operators on $\mathcal{H}$ can be expanded as

$$(6.33) \qquad M = \frac{1}{3n^2}\sum_{\mu,\,\nu\in E^{\mathcal{N}}} P^{\mu}Q^{\nu}\mathrm{Tr}(Q^{-\nu}P^{-\mu}M).$$

Define for each "vector" $\underline{\lambda} = (\mu,\ \nu)$ $\mu,\ \nu \in E$ a symbol

$$(6.34) \qquad \begin{aligned} \{\underline{\lambda}\} &= \xi^{B(\lambda)}P^{\mu}Q^{\nu} \equiv \xi^{B(\lambda)}[\underline{\lambda}] \\ \xi^{B(\lambda)} &= \xi^{\frac{\overline{\rho\mu\nu}+\rho\bar{\mu}\nu}{3}} \end{aligned}$$

consistent from the fact that $\overline{\rho}\mu\overline{\nu} + \rho\overline{\mu}\nu$ is a multiple of 3. Thus $\{\underline{\lambda}\}$ is not periodic mod $\mathcal{N}$ as $[\underline{\lambda}] \equiv P^\mu Q^\nu$ but only mod $3n$. If $\underline{\Lambda} = (\alpha, \beta)$, $\{\underline{\lambda} + \mathcal{N}\underline{\Lambda}\} = j^{\alpha\overline{\nu}+\beta\overline{\mu}+\overline{\alpha}\nu+\overline{\beta}\mu}\{\underline{\lambda}\}$. On the other hand under the adjoint action of the modular group generated by *ad* $\mathcal{T}$ and *ad* $\mathcal{S}$ we have the simple relations

$$\begin{aligned} \mathcal{T}^{-1}\{\underline{\lambda}\}\mathcal{T} &= \left\{\underline{\lambda}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right\} \\ \mathcal{S}^{-1}\{\underline{\lambda}\}\mathcal{S} &= \left\{\underline{\lambda}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} \end{aligned} \tag{6.35}$$

so that $\mathcal{S}^{-2}\{\underline{\lambda}\}\mathcal{S}^2 = \{-\underline{\lambda}\}$ and for any element of $\overline{\Gamma}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \mathcal{A} \qquad \mathcal{A}^{-1}\{\underline{\lambda}\}\mathcal{A} = \{\underline{\lambda}A\}. \tag{6.36}$$

The action is the identity if and only if $A \equiv I \bmod 3n$ so that the image group is $G = \overline{\Gamma^{3n}}$. If we act on $[\underline{\lambda}] = P^\mu Q^\nu$ defined mod $\mathcal{N}$ we expect therefore

$$A \in \overline{\Gamma^{3n}} \rightarrow \mathcal{A}^{-1}[\lambda]\mathcal{A} = \xi^{B(\underline{\lambda}A)-B(\underline{\lambda})}[\underline{\lambda}A] \tag{6.37}$$

with $\xi^{B(\underline{\lambda}A)-B(\underline{\lambda})} = \xi^{[ab\mu\overline{\mu}+cd\nu\overline{\nu}+bc(\mu\overline{\nu}+\overline{\mu}\nu)]}$, which we also write for short $\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$. It is readily verified that for any representative of $\underline{\lambda}$ mod $3n$ this is invariant if we change μ and/or ν by a multiple of $\mathcal{N}$, so that it is well defined for $\underline{\lambda}$ mod $\mathcal{N}$. Let $\mathcal{H}_{\underline{\lambda}}$ denote the isotropy group of $\underline{\lambda}$ mod $\mathcal{N}$ in $\overline{\Gamma^{3n}}$, we have

Lemma 6.2. *The adjoint action of the modular group* $\overline{\Gamma^{3n}}$, *formula* (6.37) *is well defined if we make any substitution* ($H \in \mathcal{H}_{\underline{\lambda}}$, $H' \in \mathcal{H}_{\underline{\lambda}A}$)

(i) $\underline{\lambda}H$ *for* $\underline{\lambda}$ (ii) $A \longrightarrow HA$ (iii) $A \longrightarrow AH'$.

We have $[\underline{\lambda}H] = [\underline{\lambda}]$ and $[\underline{\lambda}AH] = [\underline{\lambda}A]$. Pick an arbitrary representative of $\underline{\lambda}$ mod $3n$, then $\underline{\lambda}H$ differs at most from $\underline{\lambda}$ by a vector proportional to $\mathcal{N}$ hence $\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$ is invariant under $\underline{\lambda} \longrightarrow \underline{\lambda}H$. Now change A into AH' then $\xi^{Q(\underline{\lambda}AH',\lambda)} = \xi^{Q(\underline{\lambda}AH',\underline{\lambda}A)}\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$, on the other hand if $A \longrightarrow HA$ we have $\xi^{Q(\underline{\lambda}HA,\underline{\lambda})} = \xi^{Q(\lambda HA,\lambda H)}\xi^{Q(\underline{\lambda}H,\underline{\lambda})} = \xi^{Q(\lambda A,\lambda)}\xi^{Q(\lambda H,\lambda)}$ from (i). Hence to prove (i) and (ii) it is sufficient to show that $\xi^{Q(\underline{\lambda}H,\underline{\lambda})} = 1$ for any $\underline{\lambda}$ mod $\mathcal{N}$ and H in $\mathcal{H}_\lambda$ For convenience $\underline{\lambda} = (\mu, \nu)$ is written as ($\mu = k + k'\rho$, $\nu = \ell + \ell'\rho$) and the condition

$\underline{\lambda}_1 \equiv \underline{\lambda}_2 \bmod \mathcal{N}$ becomes $k_1 \equiv k_2 \bmod 3n$, $\ell_1 \equiv \ell_2 \bmod 3n$, $k_1' \equiv k_2' \bmod n$, $\ell_1' \equiv \ell_2' \bmod n$. Let $H \in \mathcal{H}_\lambda$, $H = \begin{pmatrix} a & b \\ c & b \end{pmatrix}$ $ad - bc = 1 \bmod 3n$, then

$$\begin{pmatrix} k & \ell \\ k' & \ell' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} k & \ell \\ k' + pn & \ell' + qn \end{pmatrix} \bmod 3n$$

for some pair p, q, mod 3. Written in full, we find

$$\xi^{Q(\underline{\lambda}H,\underline{\lambda})} = j^{(kq-\ell p)} = 1$$

where the conclusion follows from equating the determinants of the matrix equality: $(k\ell' - k'\ell) \equiv k(\ell + qn) - (k' + pn)\ell \bmod 3n$, hence $kq - \ell p \equiv 0 \bmod 3$, and the lemma is proved. Let $\underline{\lambda} \bmod \mathcal{N}$ be a fixed representative of an orbit $\mathcal{O} \in E^{\mathcal{N}} \times E^{\mathcal{N}}/\overline{\Gamma^{3n}}$ of the action of the modular group on $E^{\mathcal{N}} \times E^{\mathcal{N}}$, then we have the following

Proposition 6.3. *For any orbit $\mathcal{O}$ we have a unique invariant, up to a phase depending on the representative point $\underline{\lambda} \in \mathcal{O}$,*

$$\underset{C \in \overline{\Gamma^{3n}}}{\lambda \in \underline{\mathcal{O}}} \qquad K_{\underline{\lambda}} = \xi^{Q(\underline{\lambda}C,\lambda)} K_{\underline{\lambda}C} = \frac{1}{|\mathcal{H}_{\underline{\lambda}}|} \sum_{\overline{\Gamma^{3n}}} \mathcal{A}^{-1}[\underline{\lambda}]\mathcal{A}$$

$$(6.38) \qquad = \frac{1}{|\mathcal{H}_\lambda|} \sum_{\overline{\Gamma^{3n}}} [\underline{\lambda}A]\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$$

$$= \sum_{\overline{\Gamma^{3n}}/\mathcal{H}_\lambda} [\underline{\lambda}A]\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$$

and this is a basis of linearly independent nonvanishing elements of the commutant.

Any element of the commutant is a linear combination of such invariants. Two different orbits involve distinct sets of $[\underline{\lambda}]$, and $K_{\underline{\lambda}}$ does not vanish since it can be written as a sum over distinct terms $[\underline{\lambda}\ A]\xi^{Q(\underline{\lambda}A,\underline{\lambda})}$ picking a representative A in each left coset $\overline{\Gamma^{3n}}/\mathcal{H}_\lambda$.

If $\underline{\lambda} = \{\mu,\ \nu\}$ the determinant $\Delta = (\mu\overline{\nu} - \overline{\mu}\nu)\frac{2\omega-1}{3} \bmod 3n$ is obviously constant along each orbit, so is δ the greatest common divisor of μ, ν, and $\mathcal{N}$. Unfortunately this is not quite sufficient to characterize the orbits. M. Bauer has obtained an explicit description which will be reported elsewhere. To continue would require to find an appropriate basis of the commutant with integral valued matrix elements and then introduce the antisymmetry properties of the characters. Work is actively pursued in this direction.

References

[1] S. Altschüler, J. Lacki and Ph. Zaugg, The affine Weyl group and modular invariant partition functions, Phys. Lett., **B288** (1987), 628.

[2] G. E. Andrews, Generalized Frobenius partitions, Memoirs of the American Mathematical Society, **49** No. 301 (1984).

[3] E. Aurell and C. Itzykson, Rational billiards and algebraic curves, To be published in J. of Geom. and Phys., special issue dedicated to I. M. Gelfand.

[4] S. Bernard, String characters from Kac-Moody automorphisms, Nucl. Phys., **B288** (1987), 628-648.

[5] M.V.Berry, Quantizing a classically ergodic system: Sinai's billiard and KKR method, Annals of Physics, **131** (1981), 163-216.

[6] A. Cappelli, C. Itzykson and J.-B. Zuber, The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories, Comm. Math. Phys., **13** (1987), 1-26.

[7] J. L. Cardy, Operator content of two-dimensional conformally invariant theories, Nucl. Phys., **B270** [**FS16**] (1986), 186-204.

[8] H. S. M. Coxeter, "Regular polytopes", Dover, New York, 1973.

[9] B. L. Feigin and D. B. Fuchs, Verma modules over the Virasoro algebra, Funct. Anal. Appl., **17** (1983), 241-248.

[10] D. Ford and J. McKay, Representations and Coxeter graphs, The geometric Vein, The Coxeter Festschrifft, Davis ed., Springer.

[11] D. Friedan, Z. Qiu and S. Shenker, Conformal invariance, unitary and two dimensional critical exponents, "Vertex operators in mathematics and physics", J. Lepowsky, S. Mandelstam and I. Singer eds., Springer, New York, 1985; Details of the non-unitary proof for highest weight representations of the Virasoro algebra Comm. Math. Phys., **107** (1986), 535-547.

[12] D. Gepner and Z. Qiu, Modular invariant partition functions for parafermionic field theories, Nucl. Phys., **B285** [**FS19**] (1987), 423-453.

[13] D. Gepner and E. Witten, String theory on group manifolds, Nucl. Phys., **B278** (1986), 493-549.

[14] C. Itzykson and J.-M. Luck, Arithmetical degeneracies in simple quantum systems, J. Phys., **A9** (1986), 211-239.

[15] V. Kac, "Infinite dimensional Lie algebras", Cambridge University Press, 1985.

[16] A. Kato, Classification of modular invariant partition functions in two dimensions, Mod. Phys. Lett., **A2** (1987), 585-600.

[17] F. Klein, "The icosahedron and the solution of equations of fifth degree", Dover, New York, 1956.

[18] J. McKay, Cartan matrices, finite groups of quaternions and Kleinian singularities, Proc. Amer. Math. Soc. (1981), 153-154.

[19] B. Kostant, On finite subgroups of $SU(2)$, simple Lie algebras and the McKay correspondence, Proc. Nat. Acad, Sci. USA, **81** (1984), 5275-5277.

[20] G. Moore and N. Seiberg, Naturality in conformal field theory, IASSNS HEP/31, Princeton.

[21] W. Nahm, Quantum field theory in one and two dimensions, Duke Math. Journ., **54** (1987), 579-613.

[22] P. J. Richens and M. V. Berry, Pseudo integrable systems in classical and quantum mechanics, Physica, **2D** (1981), 495-512.

[23] A. Rocha-Caridi, Vacuum vector representations of the Virasoro algebra, "Vertex operators", op. cit., 1985.

[24] K. Saito, Regular systems of weights and associated singularities, Advanced studies in pure mathematics, **8** (1986), 479-526.

[25] P. Slodowy, Platonic solids, Kleinian singularities and Lie groups , Springer Lecture Notes in Mathematics, **1008** (1983), 102-138.

[26] A. Weil, Sur les sommes de trois et quatre carrés, L'enseignement mathématique, **XX** (1974), 215-222.

Service de Physique Théorique
C. E. N. Saclay
F-91191 Gif-sur-Yvette Cedex, France