Advanced Studies in Pure Mathematics 19, 1989
Integrable Systems in Quantum Field Theory and Statistical Mechanics
pp. 193–285

Knot Theory based on Solvable Models at Criticality

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Abstract.

A general theory is presented to construct representations of the braid group and link polynomials (topological invariants for knots and links) from exactly solvable models in statistical mechanics at criticality. Sufficient conditions for the existence of the Markov trace are explicitly shown. Application of the theory to IRF and vertex models yields various link polynomials including an infinite sequence of new invariants. The new link polynomials are extended into two-variable link invariants. For the models with crossing symmetry, braid-monoid algebras associated with the link polynomials are derived. It is found that the Yang-Baxter relation gives both an algebraic approach and a graphical approach in knot theory.

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Received February 24, 1989.

§1. Introduction

There are many reasons why theoretical physicists are interested in studying exactly solvable models. First, an exactly solvable model sets a reference system. It gives a test for computer analyses and for analytical methods, in particular, perturbation theory. Second, a non-trivial solvable model reveals an essence of the physical phenomena under consideration. Third, solvability often brings about not only a new physics but also a new mathematics.

Recently, it has been found that the exactly solvable models in statistical mechanics contain an extremely important information on classification of string configurations, knots and links [as a review, see 1,2]. String means a very long and very thin object. In physics, we can cite many examples; vortex filament, magnetic flux, dislocation, polymer, particle trajectory, etc.. In mathematics, classification of knots and links is one of the most fundamental problems. It has a long history and inspite of extensive studies there remained unsolved.

We begin with the most familiar example of exactly solvable models in physics. Suppose that a classical Hamiltonian system with N degrees of freedom,

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \cdots, N, \tag{1.1a}$$

has N conserved quantities $\{I_i\}$. Suppose also that the conserved quantities are *involutive*,

$$\{I_i, I_j\} = 0. \tag{1.1b}$$

Here $\{,\}$ denotes the Poisson bracket. Initial value problem for the system (1.1) can be solved. Such a system is called *completely integrable system.*

Exact solvability of a model in statistical mechanics means that we can evaluate physical quantities such as the free energy and the one-point function (magnetization, density etc.) without any approximation. It is well known that two-dimensional Ising model is exactly solvable [3,4].

When we extend the theory of solitons (theory of integrable systems) into quantum systems, we obtain a unifying picture of exactly solvable models in physics as will be explained below.

Soliton is originally a term for a nonlinear wave with particle property [5,6]. It obeys a nonlinear evolution equation. By the *inverse scattering method* [7,8,9] which is the extension of the Fourier transformation, classical soliton system is shown to be a completely integrable system [10,11]. Since the field variables have an inifinite degrees of freedom, the classical soliton system has an infinite number of involutive conserved quantities.

The inverse scattering method has been applied to quantum systems. This generalization is called *quantum inverse scattering method* [12,13,14,15]. We consider an operator version of an auxiliary linear problem defined on one-dimensional lattice [16,17],

$$\psi_{m+1} = L_m(\mu)\psi_m, \qquad \frac{d\psi_m}{dt} = M_m\psi_m, \tag{1.2}$$

where $L_m(\mu)$ and M_m are $M \times M$ matrix operators, and μ is the spectral parameter. Consistency condition for (1.2) with $\mu_t = 0$ yields Lax equation

$$\frac{dL_m}{dt} = M_{m+1}L_m - L_m M_m. \tag{1.3}$$

A model is completely integrable if we can find a pair of operators $L_m(\mu)$ and M_m such that the Lax equation (1.3) is equivalent to equation of motion of the model. In fact, it is readily shown from (1.3) that a *transfer matrix* $T_N(\mu)$ defined by

$$T_N(\mu) = \mathrm{Tr}[\tau_N(\mu)], \qquad \tau_N(\mu) = L_N(\mu)L_{N-1}(\mu)\cdots L_1(\mu), \tag{1.4}$$

does not depend on time under periodic boundary condition or in an infinite chain. This verifies the existence of an infinite number of conserved operators in an infinite chain.

For a quantum integrable system, direct products of two L_n operators with different spectral parameters satisfy a similarity relation

$$R(\mu,\nu)\cdot[L_n(\mu)\otimes L_n(\nu)] = [L_n(\nu)\otimes L_n(\mu)]\cdot R(\mu,\nu). \tag{1.5}$$

Here symbol $\otimes$ denotes direct product of matrices and $R(\mu,\nu)$ is an $M^2\times M^2$ c-number matrix. The relation (1.5) is the *Yang-Baxter relation* [18,19] for a quantum system defined on a lattice. If $L_n(\mu)$'s with different n commute, we further have

$$R(\mu,\nu)\cdot[\tau_N(\mu)\otimes \tau_N(\nu)] = [\tau_N(\nu)\otimes \tau_N(\mu)]\cdot R(\mu,\nu). \tag{1.6}$$

From (1.6), we find that the transfer matrix

$$T_N(\mu) = \mathrm{Tr}[\tau_N(\mu)] = \sum_{i=1}^{M}[\tau_N(\mu)]_{ii} \tag{1.7}$$

commutes each other:

$$[T_N(\mu), T_N(\nu)] = 0. \tag{1.8}$$

Here [,] is the commutator. The relation (1.8) indicates that $T_N(\mu)$ is a generator of conserved operators. Since μ is arbitrary, μ (or μ^{-1})-expansion of $T_N(\mu)$ gives a set of conserved operators I_j which are involutive; $[I_i, I_j] = 0$. In addition, off-diagonal elements of (1.6) offer an algebraic formulation of the Bethe ansatz method [12,13,14,15].

For quantum field theory, the subscript N of operator $\tau_N(\mu)$ is understood as the system size. Then, the relation (1.8) indicates the existence of an infinite number of involutive conserved operators, which implies the solvability by the Bethe ansatz method.

We may also consider $L_n(\mu)$ and $R(\mu,\nu)$ in (1.5) as the Boltzmann weights of a vertex model in statistical mechanics (this will be shown in the end of section 2.2). Using the Yang-Baxter relation we can calculate physical quantities such as the free energy and the one-point function [20]. The free energy is obtained by the inversion method [21]. The one-point function is obtained by the corner transfer matrix method [20]. We can regard the Yang-Baxter relation as a functional equation to construct a solvable model. Recent discovery of an infinite number of solvable models was accomplished in this way [22].

Thus, we have observed that the exactly solvable models in (1+1)-dimensional quantum field theory and in 2-dimensional classical statistical mechanics have a common property, *commuting transfer matrices.* The Yang-Baxter relation gives the commutability of the transfer matrices, and hence the solvability of the model.

In this paper, new developments in the theory of exactly solvable models, in particular, application to knot theory will be exhibited. In chapter 2, the Yang-Baxter relation is introduced for the S-matrices, the vertex models and the IRF models. This chapter is a starting point. In chapter 3, examples of exactly solvable models in statistical mechanics are displayed. In chapter 4, an introduction to knot theory is given. Some basic ideas of knot theory are explained. In chapter 5, a general theory is presented to construct link polynomials, topological invariants for knots and links, from the exactly solvable models. Sufficient conditions for the existence of link polynomial are explicitly given. Chapter 6 deals with its application to various models. It is shown that the N-state vertex model gives an infinite series of link polynomials. The Jones polynomial corresponds to the case of N=2. Further, the method is applied to the IRF models which are associated to affine Lie algebras $A_{m-1}^{(1)}$, $B_m^{(1)}$, $C_m^{(1)}$ and $D_m^{(1)}$. The B,C,D models corresponding to the $B_m^{(1)}$, $C_m^{(1)}$ and $D_m^{(1)}$ algebras are related to the Kauffman polynomial. In chapter 7, a series of link polynomials constructed from the N-state vertex model is extended into those with two independent variables. The $N = 2$ case is a two variable link polynomial which was found by Freyd, Yetter, Hoste, Lickorish, Millett, Ocneanu, Przytycki and Traczyk. In chapter 8, braid-monoid algebra and graphical approach to the knot theory are presented using the exactly solvable models. The last section is devoted to concluding remarks.

§2. Yang-Baxter relation

In the introduction, we have observed that a common feature of the exactly solvable models is the commutability of the transfer matrix. The Yang-Baxter relation, a sufficient condition for the commuting transfer matrices and then a sufficient condition for the solvability of the model, appears in different forms and often with different names depending on physical situations. We shall describe it with explanation of models.

2.1. Scattering matrix

It is instructive to start the subject from an original usage of the Yang-Baxter relation. In 1967, Yang introduced it as a consistency condition of the Bethe ansatz wavefunctions [18]. The following is a somewhat modern formulation [23,24, 25,26].

We consider n types of particles (particles with n internal degrees of freedom) which are described by generators $R_1^\dagger(u_1)$, $R_2^\dagger(u_2)$, $\cdots$, $R_n^\dagger(u_n)$. The *rapidity* u is defined by

$$E = m\cosh u, \qquad P = m\sinh u, \tag{2.1}$$

where E and P are energy and momentum in unit $c = 1$. The elements of the algebra are all possible combinations of products of the form

$$R_i^\dagger(u_1)R_j^\dagger(u_2)\cdots R_\ell^\dagger(u_N). \tag{2.2}$$

We identify the product (2.2) as the N-particle scattering state. The products arranged in increasing (decreasing) order of the rapidity u correspond to the in-(out-) states. Then, we have the commutation relations for the generators $R_i^\dagger(u)$:

$$R_j^\dagger(u_1)R_i^\dagger(u_2) = \sum_{k,l} S_{jl}^{ik}(u_{21})R_k^\dagger(u_2)R_l^\dagger(u_1), \quad u_1 < u_2. \tag{2.3}$$

Here the coefficient $S_{jl}^{ik}(u_{21})$ is the two-particle scattering matrix (S-matrix, for short) describing the collision process from in-state (i, j) to out-state (k, l) and $u_{21} \equiv u_2 - u_1$ is the rapidity difference (Fig.1).

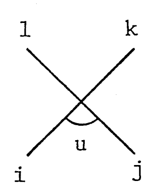


Fig. 1. Scattering matrix (S-matrix) $S_{j\ell}^{ik}(u)$ for the process $(i, j) \to (k, \ell)$.

The commutation relations (2.3) should be reconciled with the requirements of algebraic associativity. That is, the result of pair commutations in the products is independent of the sequence in which the pair commutations are performed. Consider the three particle in-state $R_k^\dagger(u_1)R_j^\dagger(u_2)R_i^\dagger(u_3)$ with $u_1 < u_2 < u_3$. This in-state can be reordered in two different ways into the out-state which is a linear combination of the form $R_p^\dagger(u_3)R_q^\dagger(u_2)R_r^\dagger(u_1)$. The result of two different reorderings (scatterings) should be same.

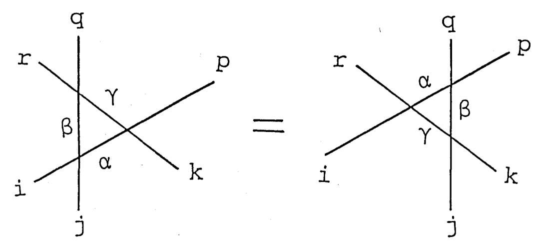


Fig. 2. Yang-Baxter relation for the S-matrices (the factorization equation).

In equations, corresponding to left-hand side of Fig.2, we have

$$
\begin{aligned}
&R_k^\dagger(u_1)R_j^\dagger(u_2)R_i^\dagger(u_3)\\
&\quad=\sum_{\alpha\beta} S_{j\beta}^{i\alpha}(u_{32})R_k^\dagger(u_1)R_\alpha^\dagger(u_3)R_\beta^\dagger(u_2)\\
&\quad=\sum_{\alpha\beta\gamma} S_{j\beta}^{i\alpha}(u_{32})S_{k\gamma}^{\alpha p}(u_{31})R_p^\dagger(u_3)R_\gamma^\dagger(u_1)R_\beta^\dagger(u_2)\\
&\quad=\sum_{\alpha\beta\gamma} S_{j\beta}^{i\alpha}(u_{32})S_{k\gamma}^{\alpha p}(u_{31})S_{\gamma r}^{\beta q}(u_{21})R_p^\dagger(u_3)R_q^\dagger(u_2)R_r^\dagger(u_1),
\end{aligned}
\tag{2.4a}
$$

and, corresponding to right-hand side of Fig.2, we have

$$
\begin{aligned}
&R_k^\dagger(u_1)R_j^\dagger(u_2)R_i^\dagger(u_3)\\
&\quad=\sum_{\beta\gamma} S_{k\gamma}^{j\beta}(u_{21})R_\beta^\dagger(u_2)R_\gamma^\dagger(u_1)R_i^\dagger(u_3)\\
&\quad=\sum_{\alpha\beta\gamma} S_{k\gamma}^{j\beta}(u_{21})S_{\gamma r}^{i\alpha}(u_{31})R_\beta^\dagger(u_2)R_\alpha^\dagger(u_3)R_r^\dagger(u_1)\\
&\quad=\sum_{\alpha\beta\gamma} S_{k\gamma}^{j\beta}(u_{21})S_{\gamma r}^{i\alpha}(u_{31})S_{\beta q}^{\alpha p}(u_{32})R_p^\dagger(u_3)R_q^\dagger(u_2)R_r^\dagger(u_1).
\end{aligned}
\tag{2.4b}
$$

Equating two expressions in (2.4) and putting $u = u_{32}$ and $v = u_{21}$, we obtain

$$\sum_{\alpha\beta\gamma} S_{j\beta}^{i\alpha}(u) S_{k\gamma}^{\alpha p}(u+v) S_{\gamma r}^{\beta q}(v) = \sum_{\alpha\beta\gamma} S_{k\gamma}^{j\beta}(v) S_{\gamma r}^{i\alpha}(u+v) S_{\beta q}^{\alpha p}(u). \tag{2.5}$$

This is the Yang-Baxter relation for the S-matrices.

The Yang-Baxter relation (2.5) is common to completely integrable quantum systems which are characterized by the following properties:

(1) The scattering is restricted by an infinite number of conserved quantities. In particular, the total number of particles, the total momentum and the total energy are conserved. Due to the severe restrictions, only the rearrangement of the momenta of particles occurs during the collisions.

(2) The process of N-particle scattering is reduced to a sequence of pair collisions and the N-paraticle S-matrix is written as a product of $N(N-1)/2$ two-particle S-matrices. Because of this property, such an S-matrix is called *factorized S-matrix* and the relation (2.5) has a name, *factorization equation.*

The properties (1) and (2) can be proved for completely integrable quantum systems such as the quantum nonlinear Schrödinger model by the quantum inverse scattering method [15,25]. It is extremely interesting to notice that the properties (1) and (2) are also common to classical soliton systems such as the Korteweg-de Vries equation [8].

2.2. Vertex model

For 2-dimensional statistical mechanics, we have two types of models, vertex model and IRF (Interaction Round a Face) model [20].

We shall introduce *vertex models.* Let us consider a two-dimensional square lattice.

State variables are located on the edges. We associate the Boltzmann weight (statistical weight) to each vertex configuration. The configuration is defined by the state variables, say, i, j, k, l on the four edges joining together at the vertex (Fig.4).

We denote the energy and the Boltzmann weight of vertex respectively by $\epsilon(i,j,k,l)$ and $w(i,j,k,l)$:

$$w(i,j,k,l) = \exp[-\epsilon(i,j,k,l)/k_B T], \tag{2.6}$$

where k_B is the Boltzmann constant and T is the absolute temperature.

As an example, we describe the 8-vertex model [27]. We set a definite direction, an arrow, to each edge of the lattice. Four edges come together at each lattice point, and so there are 16 distinct types of combinations

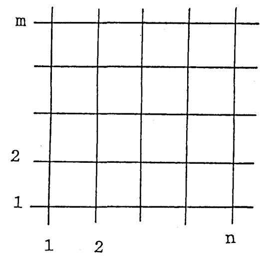


Fig. 3. Square lattice (m rows and n columns).

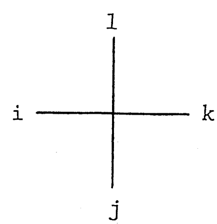


Fig. 4. Vertex $w(i,j,k,\ell)$, the Boltzmann weight of the vertex model.

of arrows. We consider only those configurations of arrows where the number of in-(or out-) arrows at each lattice point is even. The eight allowed configurations are drawn in Fig.5.

We assign energy ϵ_j for type j configuration (j = 1,2, $\cdots$,8). We assume that the energy is invariant under simultaneous inversion of the directions of all the arrows:

$$\epsilon_1 = \epsilon_2, \epsilon_3 = \epsilon_4, \epsilon_5 = \epsilon_6, \epsilon_7 = \epsilon_8. \tag{2.7}$$

This defines the 8-vertex model. With each arrangement of arrows on

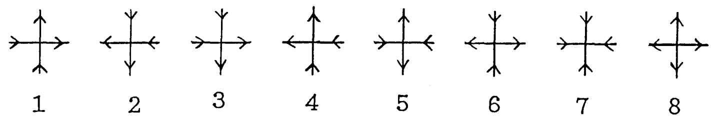


Fig. 5. Configurations of 8-vertex model

the whole lattice, we associate a total energy E:

$$E = \sum_{j=1}^{8} n_j \epsilon_j, \tag{2.8}$$

where n_j is the number of type j vertices in the given configuration. The partition function Z_N and the free energy per site f are given by

$$Z_N = \Sigma \exp(-\beta E), \qquad \beta = 1/k_B T, \tag{2.9}$$

$$f = -k_B T N^{-1} \log Z_N \quad (N \to \infty). \tag{2.10}$$

Here, N is the number of lattice sites and the summation is over all configurations of arrows. As an alternative definition, instead of arrows we may use state variable σ with two values, say ± 1. The state $+1$ corresponds to the arrows going right or upwards, and the state -1 to the arrows going left or downwards.

We now define (row-to-row) transfer matrix. Consider a horizontal row of the lattice and the adjacent vertical edges. Let $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ be the state variables on the lower row of vertical edges, $\beta = \{\beta_1, \cdots, \beta_n\}$ the state variables on the upper row and $\mu = \{\mu_1, \cdots, \mu_n\}$ on the horizontal edges.

The transfer matrix V whose matrix elements are $V_{\alpha\beta}$ is defined by

$$\begin{aligned} V_{\alpha\beta} = \sum_{\mu_1} \cdots \sum_{\mu_n} & w(\mu_1, \alpha_1, \mu_2, \beta_1) w(\mu_2, \alpha_2, \mu_3, \beta_2) \cdots \\ & \cdots w(\mu_n, \alpha_n, \mu_1, \beta_n). \end{aligned} \tag{2.11}$$

With the transfer matrix, we can write the partition function as

$$\begin{aligned} Z_N &= \sum_{\phi_1} \sum_{\phi_2} \cdots \sum_{\phi_m} V_{\phi_1 \phi_2} V_{\phi_2 \phi_3} \cdots V_{\phi_m \phi_1} \\ &= \mathrm{Tr} V^m. \end{aligned} \tag{2.12}$$

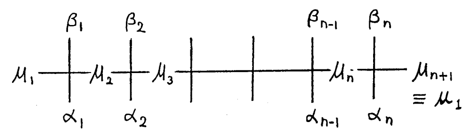


Fig. 6. Row-to-row transfer matrix V for the vertex model.

where ϕ_r denotes the state variables on row r.

Let V' be another transfer matrix where the Boltzmann weight w is replaced by w'. From (2.11), we have

$$(2.13) \qquad \begin{aligned}(VV')_{\alpha\beta} &= \sum_{\gamma} V_{\alpha\gamma}V'_{\gamma\beta} \\ &= \sum_{\mu}\sum_{\nu}\prod_{i=1}^{n} X(\mu_i,\nu_i|\mu_{i+1},\nu_{i+1}|\alpha_i,\beta_i),\end{aligned}$$

where

$$(2.14) \qquad X(\mu,\nu|\mu',\nu'|\alpha,\beta) = \sum_{\gamma} w(\mu,\alpha,\mu',\gamma)w'(\nu,\gamma,\nu',\beta).$$

We regard $X(\alpha,\beta)$ as the matrix with element $X(\mu,\nu|\mu',\nu'|\alpha,\beta)$ in row (μ,ν) and in column (μ',ν'). Then, (2.13) can be written compactly as

$$(2.15) \qquad (VV')_{\alpha\beta} = \mathrm{Tr}X(\alpha_1,\beta_1)X(\alpha_2,\beta_2)\cdots X(\alpha_N,\beta_N).$$

Similarly, we define X' with w and w' interchanged in (2.14). And, we have

$$(2.16) \qquad (V'V)_{\alpha\beta} = \mathrm{Tr}X'(\alpha_1,\beta_1)X'(\alpha_2,\beta_2)\cdots X'(\alpha_N,\beta_N).$$

From (2.15) and (2.16), we see that V and V' commute if there exists a matrix M such that

$$(2.17) \qquad X(\alpha,\beta) = MX'(\alpha,\beta)M^{-1}.$$

The matrix M has rows labelled by (μ,ν) and columns by (μ',ν'). We write the elements as $w''(\nu,\mu,\mu',\nu')$. Multiplying M from the right and using (2.14), we obtain

$$\sum_{\gamma\mu''\nu''} w(\mu,\alpha,\mu'',\gamma)w'(\nu,\gamma,\nu'',\beta)w''(\nu'',\mu'',\mu',\nu') = \sum_{\gamma\mu''\nu''} w''(\nu,\mu,\mu'',\nu'')w'(\mu'',\alpha,\mu',\gamma)w(\nu'',\gamma,\nu',\beta). \tag{2.18}$$

This is the Yang-Baxter relation for the vertex models. We can regard $w''(\nu,\mu,\mu'',\nu'')$ as a Boltzman weight for a vertex model with state variables ν , μ , μ'', ν''. Then, (2.18) has a graphical interpretation depicted in Fig.7.

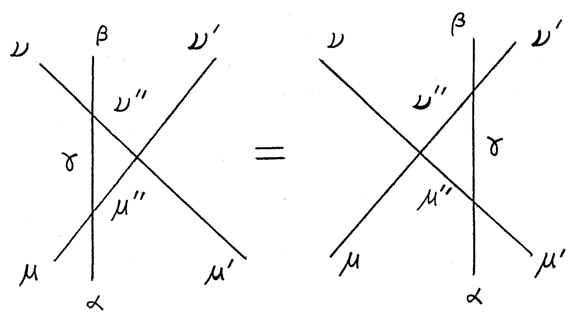


Fig. 7. Yang-Baxter relation for the vertex models.

The Boltzmann weight $w(i,j,k,l)$ can be considered as the S-matrix element S^{ik}_{jl}. In fact, Zamolodchikov found that the factorization equation for the Z_4 model (where the "charges" are conserved with modulo 4) is the same as the Yang-Baxter relation for the 8-vertex model [28]. When we identify

$$\begin{aligned} w(\mu,\alpha,\mu'',\gamma) &= S^{\mu\mu''}_{\alpha\gamma}(u), \\ w'(\nu,\gamma,\nu'',\beta) &= S^{\nu\nu''}_{\gamma\beta}(u+v), \\ w''(\nu'',\mu'',\mu',\nu') &= S^{\nu''\mu'}_{\mu''\nu'}(v), \end{aligned} \tag{2.19}$$

the relation (2.18) becomes the factorization equation (2.5) for the S-matrices from in-state (ν,μ,α) to out-state (μ',ν',β).

Let us recall the Yang-Baxter relation for a quantum system defined on a lattice:

$$R(\mu,\nu)\cdot[L(\mu)\otimes L(\nu)] = [L(\nu)\otimes L(\mu)]\cdot R(\mu,\nu), \tag{2.20}$$

where we have omitted the inessential label n for L_n. The matrix operator $L(\lambda)$ is regarded as a matrix with two sets of indices: two "auxiliary" i and j, and two "quantum" α and β , indices. We identify the matrix L and the Boltzmann weight w as

$$[L_{ij}(\mu)]_{\alpha\beta} = L_{i\alpha,j\beta}(\mu) = w(\alpha,i,\beta,j), \tag{2.21a}$$

$$[L_{ij}(\nu)]_{\alpha\beta} = L_{i\alpha,j\beta}(\nu) = w'(\alpha,i,\beta,j), \tag{2.21b}$$

And, similarly, we choose

$$R_{ij,kl}(\mu,\nu) = w''(i,j,l,k). \tag{2.21c}$$

With (2.21), we have from (2.20) that

$$\begin{aligned}&\sum_{pq\gamma} w''(i,j,q,p)w(\alpha,p,\gamma,m)w'(\gamma,q,\beta,n)\\ =&\sum_{pq\gamma} w'(\alpha,i,\gamma,p)w(\gamma,j,\beta,q)w''(p,q,n,m).\end{aligned} \tag{2.22}$$

Rewriting the indices, we see that (2.22) is the same as (2.18).

2.3. IRF model

We shall introduce *IRF* (Interaction Round a Face) *models* [20]. State variables σ_i are located on the lattice points (sites) of a square lattice. The Boltzmann weight is assigned to each unit face (or plaquette) depending on the state variable configuration round the face. By $\epsilon(a,b,c,d)$, we denote the energy of a face with state variable configuration (a,b,c,d). The corresponding Boltzmann weight is

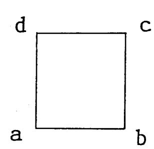


Fig. 8. Boltzmann weight $w(a,b,c,d)$ of the IRF model.

$$w(a,b,c,d) = \exp[-\epsilon(a,b,c,d)/k_BT]. \tag{2.23}$$

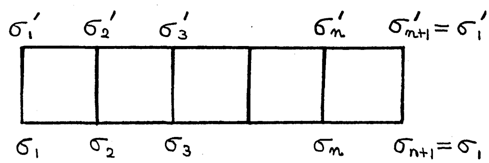


Fig. 9. Row-to-row transfer matrix V for the IRF model.

The IRF model is very general: most of the exactly solvable models are expressed in the form of IRF models. For instance, the Ising model is described [29] as

$$\begin{aligned}\epsilon(a,b,c,d) &= -\frac{1}{2}J_1[(2a-1)(2b-1)+(2c-1)(2d-1)] \\ &\quad -\frac{1}{2}J_2[(2b-1)(2c-1)+(2d-1)(2a-1)], \\ &\quad a,b,c,d=0,1\end{aligned} \tag{2.24}$$

and the 8-vertex model is described as

$$\begin{aligned}\epsilon(a,b,c,d) &= -J(2a-1)(2c-1)-J'(2b-1)(2d-1) \\ &\quad -J_4(2a-1)(2b-1)(2c-1)(2d-1), \\ &\quad a,b,c,d=0,1.\end{aligned} \tag{2.25}$$

Let N be the total number of sites. For the whole lattice, a Hamiltonian is a sum of a face energy over all faces of the lattice:

$$H = \sum_{\text{all faces}} \epsilon(\sigma_i,\sigma_j,\sigma_k,\sigma_l). \tag{2.26}$$

Then, in terms of the Boltzmann weights, the partition function Z_N and the free energy per site f are given by

$$Z_N = \sum_{\sigma_1}\cdots\sum_{\sigma_N}\prod_{(i,j,k,l)} w(\sigma_i,\sigma_j,\sigma_k,\sigma_l), \tag{2.27}$$

$$f = -k_BTN^{-1}\log Z_N \quad (N\to\infty). \tag{2.28}$$

Here, the product is over all faces.

The row-to-row transfer matrix V for the IRF model has matrix elements (Fig.9)

$$V_{\sigma\sigma'} = \prod_{j=1}^{n} w(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j), \tag{2.29}$$

where $\sigma = \{\sigma_1, \cdots, \sigma_n\}$, $\sigma' = \{\sigma'_1, \cdots, \sigma'_n\}$, $\sigma_{n+1} = \sigma_1$, and $\sigma'_{n+1} = \sigma'_1$. Similarly, we define V' with w replaced by w'

$$V'_{\sigma\sigma'} = \prod_{j=1}^{n} w'(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j). \tag{2.30}$$

The elements of the matrix VV' are

$$\begin{aligned} (VV')_{\sigma\sigma'} &= \sum_{\sigma''} V_{\sigma\sigma''} V'_{\sigma''\sigma'} \\ &= \sum_{\sigma''} \prod_{j=1}^{n} X(\sigma_j, \sigma''_j, \sigma'_j | \sigma_{j+1}, \sigma''_{j+1}, \sigma'_{j+1}), \end{aligned} \tag{2.31}$$

where $\sigma'' = \{\sigma''_1, \cdots, \sigma''_n\}$ and

$$X(a, b, c|a', b', c') = w(a, a', b', b) w'(b, b', c', c). \tag{2.32}$$

We regard $X(a, c|a', c')$ as the matrix with element $X(a, b, c|a', b', c')$ in row b and column b'. Then, (2.31) can be written as

$$\begin{aligned} (VV')_{\sigma\sigma'} = \mathrm{Tr} X(\sigma_1, \sigma'_1 | \sigma_2, \sigma'_2) X(\sigma_2, \sigma'_2 | \sigma_3, \sigma'_3) \\ \cdots X(\sigma_n, \sigma'_n | \sigma_1, \sigma'_1). \end{aligned} \tag{2.33}$$

Similarly, we define X' with w and w' interchanged in (2.32). And, we have

$$\begin{aligned} (V'V)_{\sigma\sigma'} = \mathrm{Tr} X'(\sigma_1, \sigma'_1 | \sigma_2, \sigma'_2) X'(\sigma_2, \sigma'_2 | \sigma_3, \sigma'_3) \\ \cdots X'(\sigma_n, \sigma'_n | \sigma_1, \sigma'_1). \end{aligned} \tag{2.34}$$

From (2.33) and (2.34), we see that V and V' commute if there exist matrices $M(a, a')$ such that

$$X(a, a'|b, b') = M(a, a') X'(a, a'|b, b') M(b, b')^{-1}. \tag{2.35}$$

Multiplying $M(b, b')$ from the right and writing the element (c, d) of $M(a, a')$ as $w''(c, a, d, a')$, we get

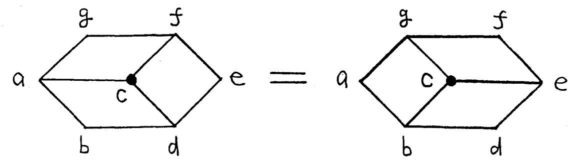


Fig. 10. Yang-Baxter relation for the IRF models (the star-triangle equation).

$$\begin{aligned}&\sum_c w(b,d,c,a)w'(a,c,f,g)w''(c,d,e,f)\\ =&\sum_c w''(a,b,c,g)w'(b,d,e,c)w(c,e,f,g).\end{aligned} \tag{2.36}$$

This is the Yang-Baxter relation for the IRF model (Fig.10). It is also called *star-triangle equation* (this naming originally comes from Onsager's work on the Ising model).

In order to solve the functional equation (2.36), we set

$$w = w(u), \quad w' = w(u+v), \quad w'' = w(v). \tag{2.37}$$

With the relabelling of state variables, the star-triangle equation (2.36) reads as

$$\begin{aligned}&\sum_c w(b,d,c,a;u)w(a,c,f,g;u+v)w(c,d,e,a;v)\\ =&\sum_c w(a,b,c,g;v)w(b,d,e,c;u+v)w(c,e,f,g;u).\end{aligned} \tag{2.38}$$

In the theory of exactly solvable models in statistical mechanics, a trick common to the vertex and IRF models is to work with the Boltzmann weight instead of the energy.

§3. Exactly solvable models

3.1. 8-vertex and 6-vertex models

For an illustration we give the Boltzmann weights of the 8-vertex model [27] and 6-vertex model [30,31]. In later discussions, we shall use elliptic theta functions defined by

$$\theta_1(u;p) = \theta_1(u) = 2p^{1/4}\sin u \prod_{n=1}^{\infty}(1 - 2p^{2n}\cos 2u + p^{4n})(1 - p^{2n}), \tag{3.1}$$

$$\theta_4(u;p) = \theta_4(u) = \prod_{n=1}^{\infty}(1 - 2p^{2n-1}\cos 2u + p^{4n-2})(1 - p^{2n}), \tag{3.2}$$

where the parameter p is called nome of the elliptic functions [32]. In terms of the theta functions the Jacobi's elliptic function $sn(u)$ is expressed as

$$sn(u,k) = sn(u) = k^{-1/2}\frac{\theta_1(u)}{\theta_4(u)}, \tag{3.3}$$

where the modulus k is related to p by

$$k = 4p^{1/2}\prod_{n=1}^{\infty}(\frac{1 + p^{2n}}{1 + p^{2n-1}})^4. \tag{3.4}$$

The allowed configurations for the 8-vertex model are depicted in Fig.5. Let us denote the Boltzmann weights for the configurations as

$$\begin{aligned} a &= \exp(-\epsilon_1/k_BT) = \exp(-\epsilon_2/k_BT), \\ b &= \exp(-\epsilon_3/k_BT) = \exp(-\epsilon_4/k_BT) \\ c &= \exp(-\epsilon_5/k_BT) = \exp(-\epsilon_6/k_BT) \\ d &= \exp(-\epsilon_7/k_BT) = \exp(-\epsilon_8/k_BT). \end{aligned} \tag{3.5}$$

The parametrization of the Boltzmann weights is [20]:

$$\begin{aligned} a &= \frac{sn(\lambda - u)}{sn(\lambda)}, \\ b &= 1, \\ c &= \frac{sn(u)}{sn(\lambda)}, \\ d &= ksn(u)sn(\lambda - u), \end{aligned} \tag{3.6}$$

where u is the spectral parameter and λ is the crossing point.

For the 6-vertex model, the configurations of the state variables i,j,k and l (Fig.4) around a vertex are restricted by the "ice rule"; $i+j=k+l$. The configurations 7 and 8 in Fig.5 are not allowed and therefore the Boltzmann weight d should be 0. Thus, the Boltzmann weights of the 6-vertex model are obtained by taking a "critical limit" $k=0\,(p=0)$ in (3.6).

3.2. Graphical representation of solvable IRF models

We shall present a method to construct solvable IRF models [33,34,1]. The method is also useful in classification of solvable models. We denote by $\{l_i\}$ state variables located on lattice sites and introduce an operator

$$(X_i(u))_{l,p} = \delta(l_1,p_1)\cdots\delta(l_{i-1},p_{i-1})w(l_i,l_{i+1},p_i,l_{i-1};u) \times \delta(l_{i+1},p_{i+1})\cdots\delta(l_n,p_n), \tag{3.7}$$

where $l=\{l_1,l_2,\cdots,l_n\}$, $p=\{p_1,p_2,\cdots,p_n\}$, n is the diagonal size of the lattice and $\delta(\cdot,\cdot)$ is the Kronecker's delta. The operator $X_i(u)$ is a constituent of the diagonal-to-diagonal transfer matrix [20]. From the Yang-Baxter relation (3.2), we can show that the operators $\{X_i(u)\}$ satisfy

$$X_i(u)X_j(v) = X_j(v)X_i(u), \qquad |i-j|\geq 2, \tag{3.8a}$$

$$X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u). \tag{3.8b}$$

A class of solvable models satisfying (3.8) is constructed as follows. Let us introduce the *Temperley-Lieb algebra*. The defining relations for the generators $\{U_i\}$ are [35]

$$U_iU_j = U_jU_i, \qquad |i-j|\geq 2 \tag{3.9a}$$

$$U_iU_{i\pm1}U_i = U_i, \tag{3.9b}$$

$$U_i^2 = q^{1/2}U_i. \tag{3.9c}$$

Using the *Temperley-Lieb operators* $\{U_i\}$, we express the operator $X_i(u)$ as

$$X_i(u) = I + \frac{\sin(u)}{\sin(\lambda-u)}U_i, \tag{3.10}$$

where q and λ are related by

$$q = 4\cos^2\lambda. \tag{3.11}$$

We assume that U_i takes the form

(3.12)

$$(U_i)_{l,p} = \delta(l_1,p_1)\cdots\delta(l_{i-1},p_{i-1})\delta(l_{i-1},l_{i+1})\left(\frac{\psi(l_i)\psi(p_i)}{\psi(l_{i-1})\psi(l_{i+1})}\right)^{1/2} \times \delta(l_{i+1},p_{i+1})\cdots\delta(l_n,p_n).$$

Then, (3.9a) and (3.9b) are always satisfied and (3.9c) gives a set of linear equations (eigenvalue problem) for $\psi(a)$'s;

$$\sum_{b\sim a}\psi(b) = q^{1/2}\psi(a), \tag{3.13}$$

where the symbol $b \sim a$ means that b is admissible to a under the constraint imposed on the model. We refer to a class of solvable models with the Temperley-Lieb algebra structure as *TL class.*

It is convenient to express the constraint on the IRF model by a graph. Each small circle of the graph represents a possible value of state variable and any pair of the circles is connected by a line if it is an allowed pair. For example, the restricted 8VSOS model defined by

$$l_i = 1, 2, \cdots, r-1 \quad (r \geq 4), \tag{3.14a}$$

$$|l_i - l_j| = 1 \quad \text{for adjacent sites } i \text{ and } j, \tag{3.14b}$$

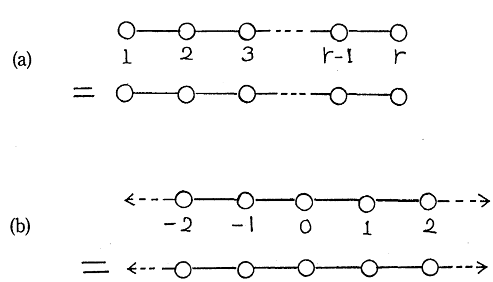


Fig. 11. Graphical representation of IRF models. (a) the restricted 8VSOS model (A type), (b) the unrestricted 8VSOS model.

corresponds to Fig.11(a) and the unrestricted 8VSOS model defined by

(3.15a) $$l_i = 0, \pm 1, \pm 2, \cdots, \pm\infty,$$

(3.15b) $$|l_i - l_j| = 1 \quad \text{for adjacent sites } i \text{ and } j,$$

corresponds to Fig.11(b). Number attached to the circle stands for the value of state variable, and often will not be written explicitly.

It is interesting to identify [36] the graphical representation with the Dynkin diagram which appears in the Cartan's classification of semi-simple Lie algebras [37,38]. In this sense, we may say that the restricted 8VSOS model is A type. However, we need not restrict the graphs to the Dynkin diagrams. For any graph in any dimensional space, we can construct the Temperley-Lieb operators $\{U_i\}$ by solving the eigenvalue problem (3.13) and using the eigenfunctions $\psi(l)$ in (3.12).

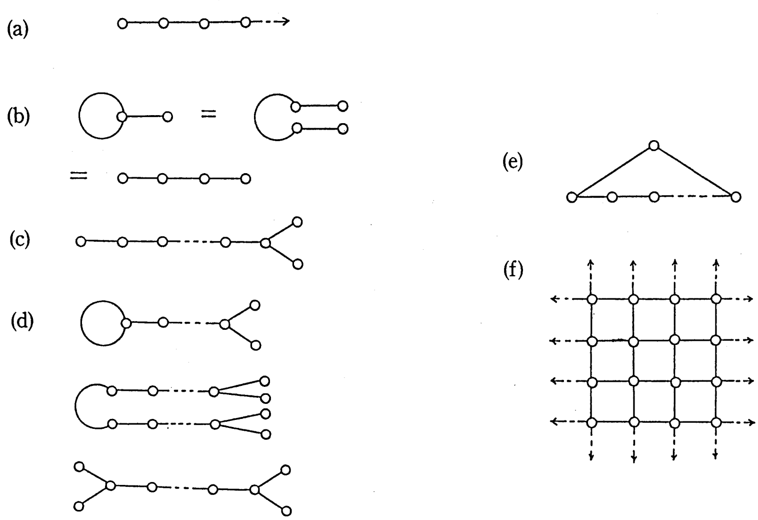


Fig. 12. Graph-state models. (a) half-infinite 8VSOS, (b) hard hexagon model, (c) D type model, (d) special S_2-generalization ($D^{(1)}$ type), (e) periodic 8VSOS model ($A^{(1)}$ type), (f) a two-dimensional square lattice.

The Boltzmann weights of the resulting solvable model which we call *graph-state IRF model* are given by

$$w(a,b,c,d;u) = \delta_{ac} + \delta_{bd}\frac{\sin(u)}{\sin(\lambda-u)}\left(\frac{\psi(a)\psi(c)}{\psi(b)\psi(d)}\right)^{1/2}. \tag{3.16}$$

Examples of the graph-state IRF models are shown in Fig.12.

Depending on the function of parametrization, all the known solutions to the Yang-Baxter relation (3.2) are classified into three cases: (1) elliptic, (2) trigonometric or hyperbolic, (3) rational. Any model in case (1) (resp. case (2)) at criticality corresponds to a model in case (2) (resp. case (3)). Some models in TL class can be extended into non-critical ones where the Boltzmann weights are parametrized in terms of the elliptic theta functions. All the models shown in Figs.11 and 12 (except Fig.12(f)) are such examples. For the unrestricted 8VSOS model, we have

$$\psi(a) = \theta_1(a\lambda+\omega_0), \tag{3.17}$$

and

$$w(l,l+1,l,l-1;u) = w(l,l-1,l,l+1;u) = \frac{\theta_1(\lambda-u)}{\theta_1(\lambda)}, \tag{3.18a}$$

$$w(l+1,l,l-1,l;u) = w(l-1,l,l+1,l;u) = \pm\frac{(\psi(l-1)\psi(l+1))^{1/2}}{\psi(l)}\frac{\theta_1(u)}{\theta_1(\lambda)}, \tag{3.18b}$$

$$w(l,l+1,l,l+1;u) = \frac{\theta_1((l+1)\lambda+\omega_0-u)}{\theta_1((l+1)\lambda+\omega_0)}, \tag{3.18c}$$

$$w(l+1,l,l-1,l;u) = \frac{\theta_1(l\lambda+\omega_0+u)}{\theta_1(l\lambda+\omega_0)}, \tag{3.18d}$$

where λ and ω_0 are arbitrary constants. The Boltzmann weights of the $r-1$-state restricted 8VSOS model are obtained by setting

$$\omega_0 = 0, \qquad \lambda = \pi/r, \tag{3.19}$$

in (3.17) and (3.18). The r-state periodic 8VSOS model (Fig.12(e)), whose state variable l_i takes $l_i = 0, 1, \cdots, r-1$ (modulo r), is obtained by setting

$$\lambda = 2\pi/r, \tag{3.20}$$

and replacing $\theta_1(u)$ in (3.17) and (3.18c,d) by $\theta_4(u)$. The replacement $\theta_1(u) \to \theta_4(u)$, which corresponds to the shift of the parameter ω_0 in complex plane, assures the real-valuedness of the Boltzmann weights.

It is amusing to observe that the parametrization of the unrestricted 8VSOS model, even at off-criticality, plays the role of "plane wave" solution to the star-triangle equation. In this view, the crossing point λ corresponds to the wave number. Then, the discretization (quantization) of λ in the case of the restricted/periodic 8VSOS model is naturally explained as discrete wave number of one-dimensionl wave in a box of size r. Also, arbitrariness of the parameter ω_0 in the unrestricted/periodic 8VSOS model is interpreted as the translational invariance of the graph.

3.3. A, B, C, D models

We shall discuss a generalization of the 6-vertex model and introduce A,B,C,D vertex and IRF models.

As shown in chapter 2, the Boltzmann weights of a solvable vertex model can be interpreted as matrix elements of the factorized S-matrix. The state variables of the vertex model are translated into the quantum numbers (internal degrees of freedom) of scattering particles. For example, we can interpret the Boltzmann weights of the 6-vertex model as scattering amplitudes of relativistic particles. The state variables $\pm 1/2$ of the 6-vertex model correspond to the "charges". That is, there are two kinds of particles; particle with charge $1/2$ and the antiparticle with charge $-1/2$. We generalize the 6-vertex model by introducing more than two kinds of particles, and by assuming the state variables to be vectors.

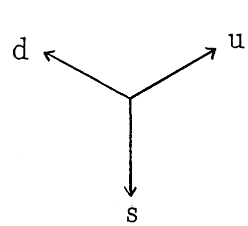


Fig. 13. Weight diagram of $SU(3)$.

Let us consider scattering of particles with $SU(3)$ symmetry. There are three kinds of particles, say u,d,s quarks. In the scattering process we assume the conservation of "charge": For instance, when u and d quarks go into interaction, u and d quarks come out after the scattering. Because of this property matrix elements which involve three particles such as S_{ds}^{us} are zero. The vertex model with the $SU(m)$ symmetry has been studied [39,40]. The Boltzmann weights are given as

$$S_{jl}^{ik}(u) = \rho(u)\{\delta_{il}\delta_{jk}+ + f(u)(e^{\omega\epsilon(k,l)}\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})(1 - \delta_{k\ell})\}. \tag{3.21}$$

Here indices i, j, k, l take integer-values from 1 to m, and functions $\rho(u)$ and $f(u)$ are defined by

$$\rho(u) = \frac{\sinh(\omega - u)}{\sinh(\omega)}, \tag{3.22a}$$

$$f(u) = \frac{\sinh(u)}{\sinh(\omega - u)}, \tag{3.22b}$$

and the symbol $\epsilon(k, l)$ is a sign factor

$$\epsilon(k,l) = \begin{cases} -1, & \text{if } k < l \ ; \\ 1, & \text{if } k > l \ . \end{cases} \tag{3.23}$$

More generally, we may associate vertex models [41] to affine Lie algebras [38]. Similarly, IRF models related to affine Lie algebras are obtained [42,43]. We shall define the unrestricted IRF models related to the fundamental representations of affine Lie algebras. The model corresponding to algebra $A_{m-1}^{(1)}$ ($B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$) is called $A_{m-1}^{(1)}$ ($B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$) model or A_{m-1} (B_m, C_m, D_m) model or simply A (B,C,D) model. The state variables take vector values in the weight space ("weight" is the terminology in Lie algebra and nothing to do with the Boltzmann weight). The Boltzmann weight $w(\vec{a}, \vec{b}, \vec{c}, \vec{d}; u)$ is assigned to the configuration $\{\vec{a}, \vec{b}\, \vec{c}, \vec{d}\}$ round the face as Fig.8. There are constraints on nearest neighbouring pair of state variables. Let Σ be a given set of the weight vectors. For a configuration $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ with fixed $\vec{d}$, we assume that $\vec{a}, \vec{b}$ and $\vec{c}$ take weight vectors which satisfy the following condition:

$$\vec{c} - \vec{d}, \vec{a} - \vec{d}, \vec{b} - \vec{c}, \vec{b} - \vec{a} \in \Sigma \tag{3.24}$$

The state variable $\vec{a}$ is said to be allowable or admissible to $\vec{d}$ when $\vec{a} - \vec{d} \in \Sigma$. We represent this relation as $\vec{a} \sim \vec{d}$. When the configuration $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ round a face does not satisfy the condition (3.24), we set the Boltzmann weight to be zero; $w(\vec{a}, \vec{b}, \vec{c}, \vec{d}; u) = 0$.

Let us write down the Boltzmann weights for the $A_{m-1}^{(1)}$, $B_m^{(1)}$, $C_m^{(1)}$ and $D_m^{(1)}$ models. For the purpose, we introduce the orthonormal vectors $\{\vec{e}_i\}$:

(3.25) $$(\vec{e}_i, \vec{e}_j) = \delta_{ij},$$

where (,) is an inner product on the weight space. The set of weight vectors, Σ, is given as follows;

(3.26a)
for $A_{m-1}^{(1)}$

$$\Sigma = \{\vec{e}_1 - \frac{1}{m}(\vec{e}_1 + \cdots + \vec{e}_m), \cdots, \vec{e}_m - \frac{1}{m}(\vec{e}_1 + \cdots + \vec{e}_m\},$$

(3.26b)
for $B_m^{(1)}$

$$\Sigma = \{0, \pm\vec{e}_1, \cdots, \pm\vec{e}_m\},$$

(3.26c,d)
for $C_m^{(1)}$ and $D_m^{(1)}$

$$\Sigma = \{\pm\vec{e}_1, \cdots, \pm\vec{e}_m\}.$$

We prepare notations:

(3.27) $$d_\mu = \omega(\vec{d} + \vec{\rho}, \vec{\mu}), \qquad \vec{\mu} \in \Sigma, \vec{\mu} \neq 0,$$
$$d_0 = -\frac{\omega}{2},$$

(3.28) $$d_{\mu\nu} = d_\mu - d_\nu, \qquad (d_{\mu\,-\nu} = d_\mu - d_{-\nu})$$

where $\vec{d}$ is a weight vector, $\vec{\rho}$ a fixed vector (the sum of all fundamental weights) and ω a parameter in unrestricted models. The following is remarked:

(3.29a)
$$(\vec{d} + \vec{\mu})_{\nu\kappa} = d_{\nu\kappa} + \omega\delta_{\mu\nu} - \omega\delta_{\mu\kappa}, \qquad \text{for } A_{m-1}^{(1)}, B_m^{(1)}, C_m^{(1)} \text{ and } D_m^{(1)}$$

(3.29b)
$$(\vec{d} + \vec{\mu})_\nu = d_\nu + \omega\delta_{\mu\nu}, \qquad \text{for } B_m^{(1)}, C_m^{(1)} \text{ and } D_m^{(1)}.$$

The Boltzmann weights of the $A_{m-1}^{(1)}$ model are given by

(3.30)

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+2\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d};u)=\frac{\theta_1(\omega-u)}{\theta_1(\omega)},$$

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu}+\overrightarrow{\nu},\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d};u)=\frac{\theta_1(d_{\mu\nu}+u)}{\theta_1(d_{\mu\nu})},$$

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu}+\overrightarrow{\nu},\overrightarrow{d}+\overrightarrow{\nu},\overrightarrow{d};u)$$
$$=\frac{\theta_1(u)}{\theta_1(\omega)}\left(\frac{\theta_1(d_{\mu\nu}+\omega)\theta_1(d_{\mu\nu}-\omega)}{\theta_1(d_{\mu\nu})^2}\right)^{\frac{1}{2}},$$

where $\overrightarrow{\mu},\ \overrightarrow{\nu}\in\Sigma$ and $\overrightarrow{\mu}\neq\overrightarrow{\nu}$. The Boltzmann weights of the $B_m^{(1)}$, $C_m^{(1)}$ and $D_m^{(1)}$ models are given by

(3.31)

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+2\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d};u)$$
$$=\frac{\theta_1(\lambda-u)\theta_1(\omega-u)}{\theta_1(\lambda)\theta_1(\omega)},\qquad \text{for } \overrightarrow{\mu}\neq 0\ ,$$

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu}+\overrightarrow{\nu},\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d};u)$$
$$=\frac{\theta_1(\lambda-u)\theta_1(d_{\mu\nu}+u)}{\theta_1(\lambda)\theta_1(d_{\mu\nu})},\qquad \text{for } \overrightarrow{\mu}\neq\pm\overrightarrow{\nu},$$

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d}+\overrightarrow{\mu}+\overrightarrow{\nu},\overrightarrow{d}+\overrightarrow{\nu},\overrightarrow{d};u)$$
$$=\frac{\theta_1(\lambda-u)\theta_1(u)}{\theta_1(\lambda)\theta_1(\omega)}\left(\frac{\theta_1(d_{\mu\nu}+\omega)\theta_1(d_{\mu\nu}-\omega)}{\theta_1(d_{\mu\nu})^2}\right)^{\frac{1}{2}},\text{for } \overrightarrow{\mu}\neq\pm\overrightarrow{\nu},$$

$$w(\overrightarrow{d}+\overrightarrow{\mu},\overrightarrow{d},\overrightarrow{d}+\overrightarrow{\nu},\overrightarrow{d};u)$$
$$=\frac{\theta_1(u)\theta_1(d_{\mu\,-\nu}+\omega-\lambda+u)}{\theta_1(\lambda)\theta_1(d_{\mu\,-\nu}+\omega)}(g_{d\mu}g_{d\nu})^{\frac{1}{2}}$$
$$+\delta_{\mu\nu}\frac{\theta_1(\lambda-u)\theta_1(d_{\mu\,-\nu}+\omega+u)}{\theta_1(\lambda)\theta_1(d_{\mu\,-\nu}+\omega)},\text{for } \overrightarrow{\mu}\neq 0,$$

$$w(\overrightarrow{d},\overrightarrow{d},\overrightarrow{d},\overrightarrow{d};u)$$
$$=\frac{\theta_1(\lambda+u)\theta_1(2\lambda-u)}{\theta_1(\lambda)\theta_1(2\lambda)}-\frac{\theta_1(u)\theta_1(\lambda-u)}{\theta_1(\lambda)\theta_1(2\lambda)}J_{do},$$

where $\overrightarrow{\mu}, \overrightarrow{\nu} \in \Sigma$ and

$$g_{d\mu} = \sigma \frac{s(d_\mu + \omega)}{s(d_\mu)} \prod_{\kappa \neq \pm\mu, 0} \frac{\theta_1(d_{\mu\kappa} + \omega)}{\theta_1(d_{\mu\kappa})}, \quad \text{for } \overrightarrow{\mu} \neq 0, \tag{3.32a}$$

$$g_{d0} = 1, \tag{3.32b}$$

$$J_{d0} = \sum_{\kappa \neq 0} \frac{\theta_1(d_\kappa + \frac{\omega}{2} - 2\lambda)}{\theta_1(d_\kappa + \frac{\omega}{2})} g_{d\kappa}. \tag{3.33}$$

The sign factor σ, the crossing point λ and the function $s(z)$ are the following; $\sigma = 1$, $\lambda = m\omega/2$ and $s(z) = 1$ for the $A_{m-1}^{(1)}$ model, $\sigma = 1$, $\lambda = (2m-1)\omega/2$ and $s(z) = \theta_1(z)$ for the $B_m^{(1)}$model, $\sigma = -1$ and $\lambda = (m+1)\omega$ and $s(z) = \theta_1(2z)$ for the $C_m^{(1)}$ model and $\sigma = 1$, $\lambda = (m-1)\omega$ and $s(z) = 1$ for the $D_m^{(1)}$ model. By using the addition theorem of the elliptic theta function, we can prove that the Boltzmann weights of those models indeed satisfy the Yang-Baxter relation.

§4. Knot theory

We shall summarize some basic knowledge of knot theory which is familiar in mathematics but seems to be new in physics.

4.1. Knots and links

Let us call a one-dimensional object *string*. We consider a configuration of strings in three-dimensional space. A *knot* is a closed string which does not cross with itself.

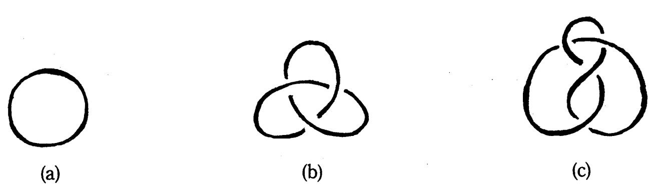
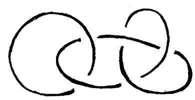
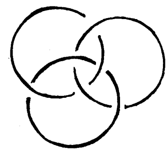


Fig. 14. Knots. (a) trivial knot, (b) trefoil (or clover-leaf) knot, (c) figure eight knot.

In Fig.14, knot (a) describes the simplest knot (a trivial knot). Knot (b) and knot (c) are named respectively as a trefoil (or clover-leaf) knot



(a)



(b)

Fig. 15. Links. (a) a link consisting of trivial and trefoil knots, (b) the Borromean rings.

and a figure eight knot because of their shapes. An assembly of knots is called *link*. Two examples are shown in Fig.15. It is amusing to see that link (b), known as the Borromean rings, exhibits a triadic relation: any two of three strings are unlinked but it is itself linked.

Classification of knots and links is one of the most fundamental questions in topology [44,45,46,47]. The problem is to determine rigorously whether two knots (or links) are different or not. When two knots (or links) are transformed each other by continuous deformations without tearing the string(s), we say that they are *topologically equivalent* or *ambient isotopic*. It is not difficult to observe (Fig.16) that trefoil knot (a) is topologically equivalent to knot (b).

However, it took more than 80 years to find that two knots (a) and (b) in Fig.17 are equivalent. In 1890, C.N. Little presented a classification table where they were different. Around 1974, K.A. Perko found their equivalence [48].

4.2. Braid group and link polynomial

To classify knots and links in a systematic way, it is necessary to find out topological invariant, that is, a quantity which does not change under continuous deformations of strings. When the invariant is expressed in a form of polynomial with some variable, it is called *link polynomial.*

We introduce braid and braid group to describe knots and links. Prepare two horizontal bars and choose n base points on each of them. *Braid* is formed when n points on the upper bar are connected to n points on the lower bar by n strings (Fig.18).

Trivial n-braid is a configuration where no intersection between the strings is present (Fig.19).

An operation of making an intersection, where i-th string (a string from the i-th point on the upper bar) passes *over* $i+1$-th string (a string

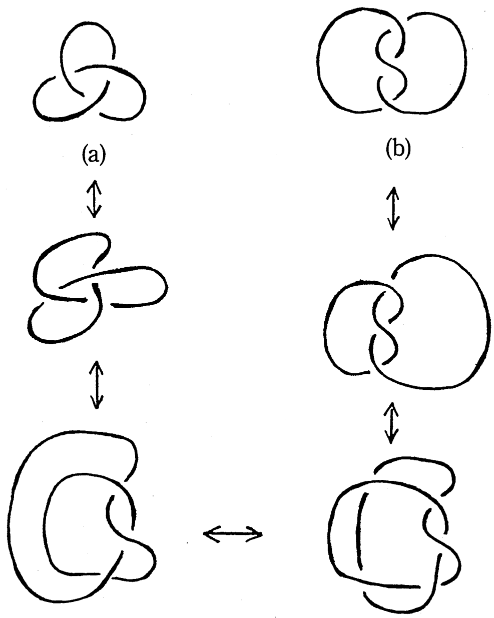


Fig. 16. Trefoil knot (a) is equivalent to knot (b) since they are deformed into each other without tearing the string.

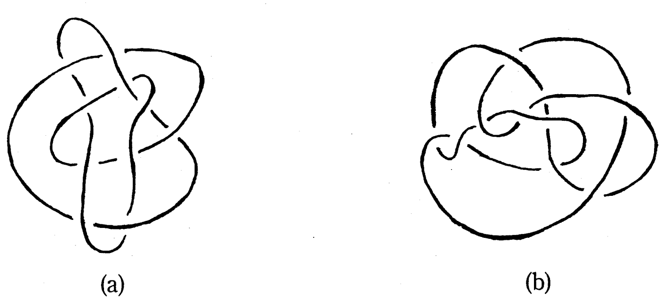


Fig. 17. Two knots (a) and (b) are equivalent.

from the $i+1$-th point on the upper bar) is denoted by b_i, $i = 1,2,\cdots,n-1$.

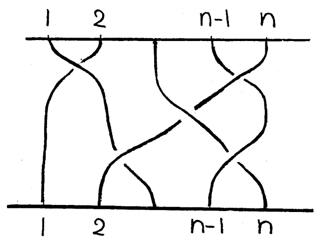


Fig. 18. A general n-braid.

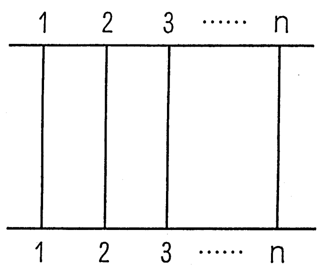


Fig. 19. Trivial n-braid.

Inverse operation of b_i is denoted by b_i^{-1} (Fig.20).

We have two remarks on the braid operation. First, in the operations b_i and b_i^{-1}, other strings except i-th and $i+1$-th strings are fixed. Second,it is not string but base point that is numbered.

A union (product) of two braid operations, say b_1 and b_2^{-1}, is written as $b_1 b_2^{-1}$ (Fig.21).

A general n-braid is constructed from the trivial n-braid by successive applications of operators b_i and their inverses b_i^{-1}. The operators $\{b_i; i = 1, 2, \cdots, n-1\}$ define a group, the *braid group*, which is denoted by B_n. By regarding the trivial n-braid as the identity operation in B_n, we can identify any element in B_n as an n-braid. However, expression of a braid in terms of the braid group elements is not unique. Topological equivalence between seemingly different expressions of a braid is

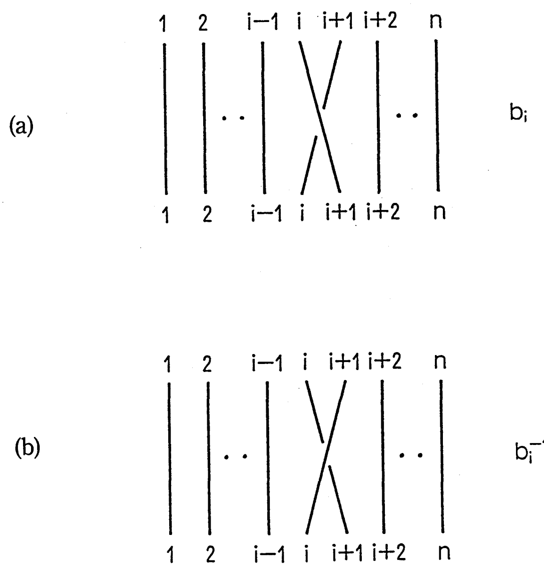


Fig. 20. (a) Braid operation b_i, (b) inverse operation b_i^{-1}. The strings except i and $i+1$ strings are fixed.

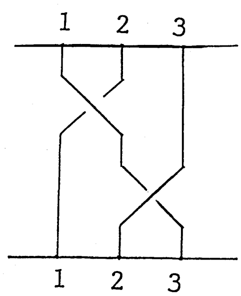


Fig. 21. $b_1 b_2^{-1}$.

guaranteed by the following relations (Fig.22);

(4.1a) $$b_i b_j = b_j b_i, \qquad |i-j| \geq 2,$$

(4.1b) $$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}.$$

This is the defining relation of the braid group by Artin [49].

Under the relation (4.1), each topologically equivalent class of braids

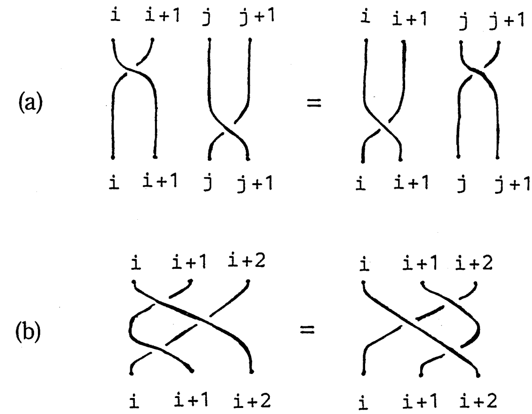


Fig. 22. Braid group. The defining relations (4.1a) and (4.1b) are illustrated.

is identified with an element in B_n. Therefore, any n-braid is expressed as a *word* on B_n (eg. $b_1b_2b_3b_2^{-1}b_1$ on B_4).

Given a braid, we may form a link by tying opposite ends.

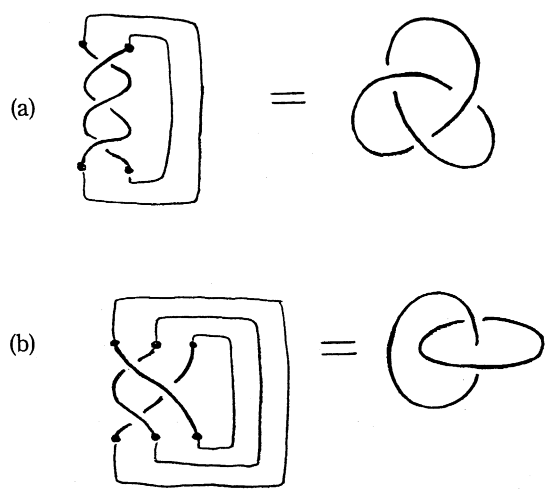


Fig. 23. (a) A closed braid b_1^{-3} is a trefoil knot, (b) a closed braid $b_1b_2b_1$ is a link consisting of two trivial knots.

Rigorously speaking, a *closed braid* represents an oriented link (we shall explain oriented links in §4.3). Conversely, according to Alexander's theorem [50], any oriented link is represented by a closed braid. This fact gives the braid group a fundamental role in the knot theory. However, the representation of a link as a closed braid is highly non-unique. This difficulty was solved by A.A. Markov [51].

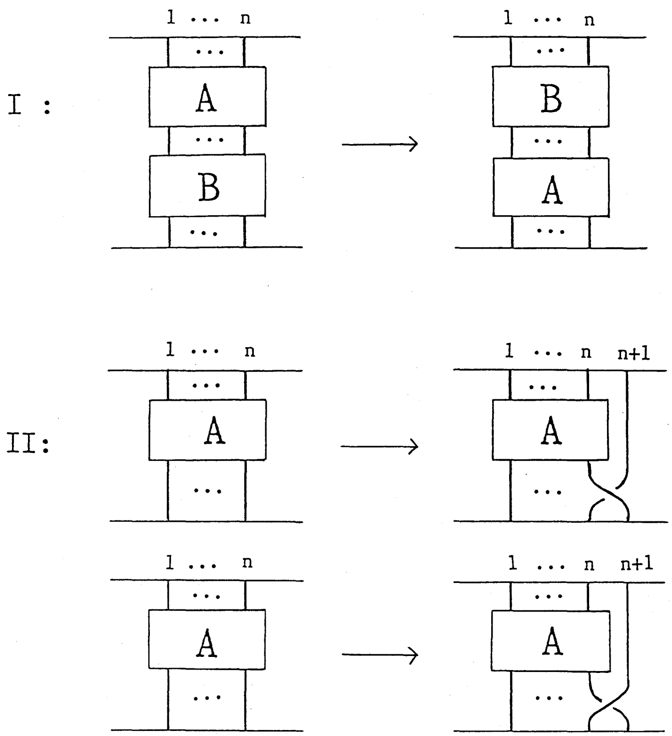


Fig. 24. Markov moves; type I and type II.

The equivalent braids expressing the same link are mutually transformed by successive applications of two types of operations, type I and type II *Markov moves* (Fig.24);

(4.2a) I. $AB \to BA \qquad (A, B \in B_n)$

(4.2b) II. $A \to Ab_n^{\pm 1} \qquad (A \in B_n, b_n^{\pm 1} \in B_{n+1})$.

Hence, we can construct a link polynomial, a topological invariant for knots and links, in the following scheme. We first make a representation of the braid group B_n and then construct a Markov move invariant defined on the representation. We use B_n both for the group and the representation.

Let us denote the representation of b_i by G_i and a link polynomial by $\alpha(\cdot)$. The link polynomial $\alpha(\cdot)$ must satisfy the conditions:

(4.3a)
$$\text{I.} \qquad \alpha(AB) = \alpha(BA) \qquad (A, B \in B_n)$$
(4.3b)
$$\text{II.} \qquad \alpha(AG_n) = \alpha(AG_n^{-1}) = \alpha(A) \qquad (A \in B_n, G_n \in B_{n+1}).$$

This quantity is obtained if we can find a linear functional $\phi(\cdot)$ called the *Markov trace* which have the following properties (the Markov properties):

$$\text{(4.4a)} \qquad \text{I.} \qquad \phi(AB) = \phi(BA), \qquad (A, B \in B_n)$$

$$\text{(4.4b)} \qquad \text{II.} \qquad \begin{aligned} &\phi(AG_n) = \tau\phi(A) \\ &\phi(AG_n^{-1}) = \bar{\tau}\phi(A), \qquad (A \in B_n, G_n^{\pm 1} \in B_{n+1}), \end{aligned}$$

where the parameters τ and $\bar{\tau}$ are given by

$$\text{(4.5)} \qquad \begin{aligned} \tau &= \phi(G_i), \\ \bar{\tau} &= \phi(G_i^{-1}), \qquad \text{for any } i. \end{aligned}$$

In terms of the Markov trace $\phi(\cdot)$, the link polynomial $\alpha(\cdot)$ is expressed as

$$\text{(4.6)} \qquad \alpha(A) = (\tau\bar{\tau})^{-(n-1)/2}(\frac{\bar{\tau}}{\tau})^{e(A)/2}\phi(A), \qquad A \in B_n.$$

Here $e(A)$ is the exponent sum of b_i's appearing in the braid A. For instance, if $A = b_3^2 b_2^{-1} b_1^3$, then $e(A) = 2 - 1 + 3 = 4$.

We prove that the link polynomial $\alpha(\cdot)$ defined by (4.6) satisfies (4.3). The property (4.3a) directly follows from (4.4a). The property (4.3b) is verfied as follows:

$$\text{(4.7)} \qquad \begin{aligned} \alpha(AG_n) &= (\tau\bar{\tau})^{-(n+1-1)/2}(\frac{\bar{\tau}}{\tau})^{e(AG_n)/2}\phi(AG_n) \\ &= (\tau\bar{\tau})^{-(n+1-1)/2}(\frac{\bar{\tau}}{\tau})^{e(A)/2+1/2}\tau\phi(A) \\ &= (\tau\bar{\tau})^{-(n-1)/2}(\frac{\bar{\tau}}{\tau})^{e(A)/2}\phi(A) = \alpha(A). \end{aligned}$$

In the above, (4.4b) and $e(AG_n) = 1 + e(A)$ was used. We pay attention to the meaning of (4.3b). The type II Markov move invariance (4.3b) is a relation between an $n+1-$ *braid* and an $n-$ *braid*. Although the multiplication AG_n is made under the natural inclusion $B_n \subset B_{n+1}$, we still regard A as an $n-$ *braid*, not as an $n+1-$ *braid*.

4.3. Reidemeister moves

Some other concepts in knot theory will be added. First, we define orientation of knots and links. A knot is *oriented* if it has a direction along the string. Mathematically speaking, an oriented knot is an embedding $S^1 \to R^3$ (or S^3) with oriented S^1. Similarly, an oriented link is defined as an embedding $S^1 \bigcup S^1 \bigcup \cdots \bigcup S^1 \to R^3$ (or S^3) with oriented $S^1 \bigcup S^1 \bigcup \cdots \bigcup S^1$.

Second, we shall explain link diagrams and the *Reidemeister moves* [52,46,47]. A link diagram is a projection of a link onto a plane. It does not have multiple points but double points. Over-crossing line is discriminated from under-crossing line at the intersection point. There are three types of the Reidemeister moves as shown in Fig.25.

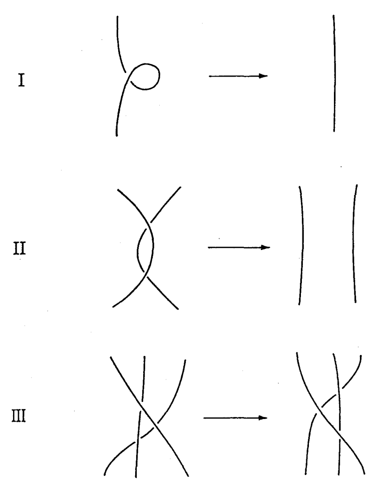


Fig. 25. Reidemeister moves I, II and III.

Each move induces local changes in the link diagram. It is known that link diagrams expressing ambient isotopic links are transformed into each other by a finite sequence of the Reidemeister moves. For oriented links, the Reidemeister moves with orientations should be used.

Third, we introduce the *writhe* of oriented link diagrams. Crossings in the link diagram are classified into two types. We associate a sign $\epsilon(C)$ to each type of crossing C.

Fig. 26. Sign $\epsilon(C)$ of crossing C.

Let $\hat{L}$ be a link diagram for a link L. The writhe $w(\hat{L})$ of the link diagram $\hat{L}$ is defined as a sum of signs for all crossings of the link diagram:

$$w(\hat{L}) = \sum_{C} \epsilon(C). \tag{4.8}$$

The writhe is invariant under the Riedemeister moves II and III, but changed by the Reidemeister move I. Link diagrams are said to be *regularly isotopic* if and only if they are transformed each other by a finite sequence of the Reidemeister moves II and III. Thus, the writhe is a regular isotopy invariant. It is important to realize that the exponent sum of a closed braid is equivalent to the writhe of the link diagram:

$$e(A) = -w(\hat{L}). \tag{4.9}$$

Here, $\hat{L}$ is the link diagram which is equivalent to the closed braid of the braid A.

4.4. Link polynomials, von Neumann algebra and critical statistical systems

As a prelude to the next section where the knot theory is connected with the theory of exactly solvable models in statistical mechanics, we review some earlier developments which lead us to a new formulation of link polynomials.

In 1928, Alexander first invented link polynomial which we now call Alexander polynomial [53]. In 1985, an ingeneous work by Jones [54] created a sensation among mathematicians. A new link polynomial (Jones polynomial) was found after an interval of nearly 60 years. The Jones polynomial detects mirror images (for instance, Fig.14(b) and Fig.16(a)) which the Alexander polynomial fails to do. Another reason for the surprise was his usage of von Neumann algebra which seemed a completely different branch of mathematics from topology.

In the operator algebra theory an important object is *a factor* which means the von Neumann algebra having only trivial center. The factor has been classified into three types, I, II and III, according to the properties of the traces [55]. Among the type II factors, type II_1 factor is characterized by having a unique normalized trace.

In his study of II_1 factors, Jones introduced an algebra $A_{p,n}$ where generators $\{1, e_1, e_2, \cdots, e_n\}$ satisfy the following relations:

$$e_i^* = e_i, \tag{4.10a}$$

$$e_i^2 = e_i, \tag{4.10b}$$

$$e_i e_j = e_j e_i, \qquad |i-j| \geq 2, \tag{4.10c}$$

$$e_i e_{i\pm 1} e_i = p^{-1} e_i. \tag{4.10d}$$

The algebra (4.10) was obtained through the successive extensions of the II_1 factor by its subfactor. The integer n is the number of extensions and the parameter p is the index for subfactors of the II_1 factors. Due to the requirement that there should exist a faithful positive trace with the Markov properties in the inductive limit $n \to \infty$, the possible values of the index for subfactors of hyperfine II_1 factors (II_1 factors generated by ascending sequences of finite dimensional factors) are limited to the following discrete and continuous spectra:

$$p^{-1} = 4\cos^2(\pi/k), \qquad k = 3, 4, \cdots, \tag{4.11}$$

$$\text{or} \qquad p^{-1} \geq 4.$$

The algebra $A_{p,n}$ is utilized to have a representation (Hecke representation) of the braid group. Let us introduce a parameter t by

$$p^{-1} = 2 + t^{-1} + t, \tag{4.12}$$

and a generator G_i by

$$G_i = (t+1)e_i - 1 \qquad (e_i \in A_{p,n-1}). \tag{4.13}$$

Then the *Hecke algebra* $H(t, n)$ is generated by G_i. That is, the generators satisfy both the defining relation (4.1) of the braid group and the following quadratic relation:

$$G_i^2 = (t-1)G_i + t. \tag{4.14}$$

Now we notice that if we set $e_j = p^{-1/2}\hat{U}_j$ and $p = q$, the generators $\{\hat{U}_j\}$ satisfy the Temperley-Lieb algebra (3.9) [56,57,58]. The Temperley-Lieb algebra was introduced in the construction of the transfer matrices of the two-dimensional critical statistical systems such as the ferroelectric models (6-vertex models) and the self-dual Potts models which are exactly solvable [59]. In the case of the restricted 8VSOS model (cf.(3.14)) at the critical point, the generators $\{U_i\}$ defined by (3.7) satisfy the Temperley-Lieb algebra with

$$q = 4cos^2(\pi/r). \tag{4.15}$$

Thus, various important subjects in physics (i.e. exactly solvable models, critical phenomena in two dimensions) and mathematics (i.e. von Neuman algebra, knot theory) are intimately related.

§5. Exactly solvable models and knot theory

In this chapter we present a general theory developed by the authors to construct link polynomials from exactly solvable models in statistical mechanics. The scheme is the following: First, we make a representation of the braid group from the Boltzmann weights of a solvable model. Second, by using the crossing multipliers of the model we introduce the Markov trace on the braid group representation. Then, we obtain a link polynomial from (4.6).

5.1. Basic relations

The Boltzmann weights of the exactly solvable models, $S^{ik}_{j\ell}(u)$ for the vertex model and $w(a,b,c,d;u)$ for the IRF model, satisfy several relations in addition to the Yang-Baxter relation. We introduce the following basic relations [1,2].
1) standard initial condition:

$$S^{ik}_{j\ell}(0) = \delta_{i\ell}\delta_{jk}, \tag{5.1a}$$

$$w(a,b,c,d;0) = \delta_{ac}. \tag{5.1b}$$

There can be index-independent constant factors in the r.h.s.'s of (5.1), since the Yang-Baxter relation is invariant under overall normalization of the Boltzmann weights.
2) inversion relation (unitarity condition):

$$\sum_{m,p} S^{mk}_{p\ell}(u) S^{ip}_{jm}(-u) = \rho(u)\rho(-u)\delta_{i\ell}\delta_{jk}, \tag{5.2a}$$

(5.2b) $$\sum_e w(e,c,d,a;u)w(b,c,e,a;-u) = \rho(u)\rho(-u)\delta_{bd},$$

where $\rho(u)$ is a model-dependent function.
3) second inversion relation (second unitarity condition):

(5.3a)
$$\sum_{m,p} S_{p\ell}^{im}(\lambda-u)S_{mj}^{kp}(\lambda+u)\cdot r(p)r(m)/r(k)r(j) = \rho(u)\rho(-u)\delta_{ij}\delta_{k\ell},$$
(5.3b)
$$\sum_e w(c,e,a,b;\lambda-u)w(a,e,c,d;\lambda+u)\frac{\psi(e)\psi(b)}{\psi(a)\psi(c)} = \rho(u)\rho(-u)\delta_{bd}.$$

We call λ crossing point and $r(i)$ and $\psi(a)$ crossing multipliers [33,60,61].
4) crossing symmetry;

(5.4a) $$S_{j\ell}^{ik}(u) = S_{k\bar{i}}^{j\ell}(\lambda-u)\cdot\{r(i)r(\ell)/r(j)r(k)\}^{1/2},$$
(5.4b)
$$w(a,b,c,d;u) = w(b,c,d,a;\lambda-u)[\frac{\psi(a)\psi(c)}{\psi(b)\psi(d)}]^{1/2}.$$

Here, we have introduced the notation $\bar{j}=-j$ etc. for "charge conjugation".

The crossing multiplier satisfies

(5.4c) $$r(\bar{j}) = \frac{1}{r(j)}.$$

The definition (5.4a) of the crossing symmetry is slightly different from the one in §2.1 and (6.7).
5) reflection symmetry

(5.5a) $$S_{j\ell}^{ik}(u) = S_{ki}^{\ell j}(u),$$
(5.5b) $$w(a,b,c,d;u) = w(c,b,a,d;u).$$

6) charge (or spin) conservation condition

(5.6) $$S_{j\ell}^{ik}(u) = 0, \qquad \text{unless } i+j=k+\ell.$$

For IRF models, this condition corresponds to the single-valuedness of the state variables around a face.

The inversion relation can be derived from the Yang-Baxter relation and the standard initial condition [61]. We can prove the second

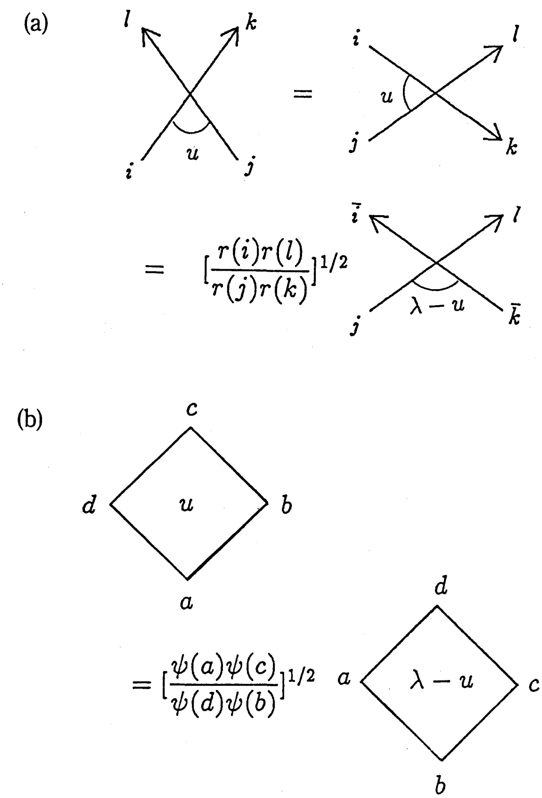


Fig. 27. (a) The crossing symmetry for vertex models.
(b) The crossing symmetry for IRF models.

inversion relation from the crossing symmetry and the inversion relation. Without going into details, we mention that the free energy of two dimensional solvable lattice system can be calculated by using these relations and analyticity [21].

We have defined the crossing multipliers, $r(i)$ and $\psi(a)$, and the crossing point λ by the second inversion relation and the crossing symmetry. These quantities will be important in the general theory. We note that there are models without the crossing symmetry. For example, A_{m-1} vertex model ($m \geq 3$) does not contain antiparticles and then the crossing symmetry does not exist [61]. We also note that the reflection symmetry can be broken by a symmetry breaking transformation [62] without affecting other properties.

The above relations have physical meaning. In terms of the S-matrices, the standard initial condition means that no scattering occurs when the relative velocity of two particles, equivalently the rapidity difference, vanishes. The inversion relation is nothing but the unitarity condition for the S-matrices. The S-matrices with the crossing symmetry is interpreted as scattering amplitudes of relativistic particles. When we regard the variable i as the "charge" of a particle, $\bar{i}$ is considered as the charge of the antiparticle. The crossing symmetry implies that the s-channel scattering is transformed into the "crossing-channel" scattering. The crossing symmetry also implies that the S-matrices are invariant under the CPT transformation:

$$S^{ik}_{j\ell}(u) = S^{\bar{k}\bar{i}}_{\bar{\ell}\bar{j}}(u). \tag{5.7}$$

There is a relation between a vertex model and IRF model, which makes the theory transparent. By the Wu-Kadanoff-Wegner transformation, a configuration $\{a, b, c, d\}$ of an IRF model is transformed into a configuration $\{i, j, k, \ell\}$ of a vertex model [29,62,33].

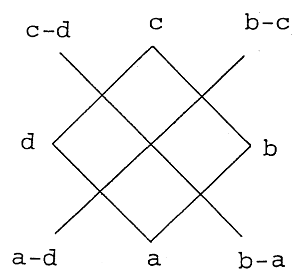


Fig. 28. Wu-Kadanoff-Wegner transformation.

The vertex model thus obtained has the charge conservation property.

5.2. Yang-Baxter algebra and braid group representation

The Yang-Baxter relation (see, Fig.2 or Fig.7) indicates invariance under displacement of one of three lines over an intersection of the other two lines. Interestingly, this scattering diagram looks similar to the graphical illustration Fig.22 of the braid group. If we can discriminate under-crossing from over-crossing at the intersection of the scattering diagram, we will be able to have the braid group representation. This observation was an origin of the idea to construct braid group representations and link polynomials from the exactly solvable models.

To push the above idea forward and make it concrete, we define *Yang-Baxter operator*

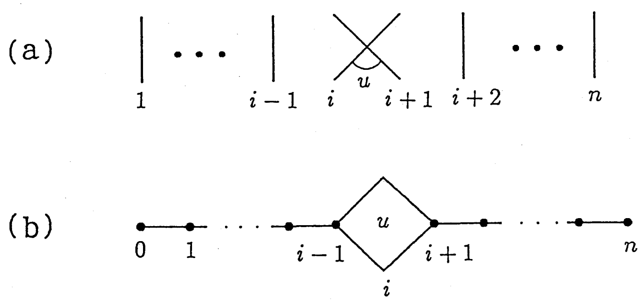


Fig. 29. Yang-Baxter operator $X_i(u)$, (a) vertex model, (b) IRF model.

for vertex models [58,60] by

$$(5.8)\quad X_i(u) = \sum_{k,\ell,m,p} S_{\ell p}^{km}(u) I^{(1)} \otimes \cdots \otimes e_{pk}^{(i)} \otimes e_{m\ell}^{(i+1)} \otimes I^{(i+2)} \cdots \otimes I^{(n)},$$

where $I^{(i)}$ means an identity matrix on i-th position, e_{pk} a matrix whose elements are $(e_{nk})_{ab} = \delta_{pa}\delta_{kb}$. For IRF models [33,63] we define Yang-Baxter operator by

(5.9)

$$[X_i(u)]_{\ell_0 \cdots \ell_n}^{p_0 \cdots p_n} = \begin{cases} \prod_{j=0}^{i-1} \delta(p_j, \ell_j) \cdot w(\ell_i, \ell_{i+1}, p_i, \ell_{i-1}; u) \times \\ \times \prod_{j=i+1}^{n} \delta(p_j, \ell_j), \\ \qquad \text{when } p_{j+1} \sim p_j, \ell_{j+1} \sim \ell_j \text{ for all } j; \\ 0, \qquad \text{otherwise.} \end{cases}$$

Here, $a \sim b$ means that a is admissible to b and $\delta(p, \ell)$ the Kronecker delta. Hereafter we assume that the suffices $\{\ell_0, \ell_1, \cdots, \ell_n\}$ satisfy the relation: $\ell_{j+1} \sim \ell_j$ for all j.

The Yang-Baxter operators $\{X_i(u)\}$ satisfy the following relation (*Yang-Baxter algebra*)[20,60,61]:

$$(5.10a)\qquad X_i(u)X_j(v) = X_j(v)X_i(u), |i - j| \geq 2,$$

$$(5.10b)\qquad X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u).$$

The relation (5.10a) is obvious from the definition ((5.8) and (5.9)). The relation (5.10b) is the Yang-Baxter relation ((2.5) and (2.38)) in operator form.

We notice that if we set $u = u + v = v$ (5.10) reduces to (4.1). This means that we can obtain the braid group representation $\{G_i\}$ from the Yang-Baxter operator. One of the possibilities that $u = u + v = v$ is $u = v = \infty$. In this way, it was found [58,60] that by the limit $u \to \infty$ (with a suitable normalization if necessary), we obtain a representation of the braid group from the Yang-Baxter operator $X_i(u)$:

(5.11a) $$G_i = \lim_{u\to\infty} X_i(u)/\rho(u), \quad i = 1, 2, \cdots, n,$$

(5.11b) $$G_i^{-1} = \lim_{u\to\infty} X_i(-u)/\rho(-u), \quad i = 1, 2, \cdots, n.$$

The limit $u \to \infty$ implies a certain direction in the complex u-plane. It is sometimes convenient to use "weight matrix", $\sigma^{(\pm)}_{\ell k,ij}$ for vertex models [58,60] and $\sigma(a, b, c, d; \pm)$ for IRF models [33]:

(5.12) $$\lim_{u\to\infty} S^{ik}_{j\ell}(\pm u)/\rho(\pm u) = \sigma^{(\pm)}_{\ell k,ij},$$

(5.13) $$\lim_{u\to\infty} w(a, b, c, d; \pm u)/\rho(\pm u) = \sigma(a, b, c, d; \pm).$$

Then, the braid operator G_i is given for vertex models by

(5.14a) $$G_i = \sum_{k,\ell,m,p} \sigma^{(+)}_{pm,k\ell} I^{(1)} \otimes \cdots \otimes e^{(i)}_{pk} \otimes e^{(i+1)}_{m\ell} \otimes I^{(i+2)} \cdots \otimes I^{(n)},$$

and for IRF models by

(5.14b) $$[G_i]^{p_0\cdots p_n}_{\ell_0\cdots\ell_n} = \delta(p_0, \ell_0)\cdots\delta(p_{i-1}, \ell_{i-1})\sigma(\ell_i, \ell_{i+1}, p_i, \ell_{i-1}; +)\times \times \delta(p_{i+1}, \ell_{i+1})\cdots\delta(p_n, \ell_n).$$

The existence of the limit $u \to \infty$ requires that the model should be critical; the Boltzmann weights are parametrized by hyperbolic or trigonometric functions. In the scattering theory, $u \to \infty$ corresponds to the high energy limit. That is, we may regard the matrix elements of the braid operator as the S-matrix elements of ultra relativistic particles [61]. There is another possibility for the condition $u = u + v = v$. That is $u = v = 0$. Due to the standard initial condition if we set $u = 0$ in $X_i(u)$, we have

(5.15) $$X_i(0) = I_n,$$

where I_n is the identity operator in the representation of B_n. From the inversion relation (5.2), we have

(5.16) $$G_i \cdot G_i^{-1} = G_i^{-1} \cdot G_i = I_n.$$

The formula (5.11) is applicable to any solvable model at criticality. Corresponding to an exactly solvable model, we obtain a representation of the braid group by using the formula (5.11).

5.3. Construction of Markov trace

We construct the Markov trace $\phi(\cdot)$ on the representation of the braid group. With the Markov trace $\phi(\cdot)$, (4.6) yields a link polynomial. We remark that the link polynomials thus obtained are invariants for oriented links.

We first consider the vertex model (or the S-matrix). The matrix representation of the braid group B_n given in (5.14a) is defined on the n-tensor product of vector spaces, and the identity operation in the braid group is expressed as the tensor product of identity matrices. Having these in mind, we assume a trace $\phi(\cdot)$ has a form [58,60]:

(5.17a) $$\phi(A) = Tr(H^{(n)}A)/Tr(H^{(n)}), \qquad A \in B_n,$$

where

(5.17b) $$H^{(n)} = h^{(1)} \otimes h^{(2)} \otimes \cdots \otimes h^{(n)},$$

and $h^{(i)}$ is a diagonal matrix whose elements are

(5.17c) $$h_{pq} = r^2(p)\delta_{pq}.$$

Here $r(p)$ is the crossing multiplier of the model. We have normalized $\phi(\cdot)$ as $\phi(I) = 1$. The trace $\phi(\cdot)$ becomes the Markov trace when the following conditions hold [60]:

(5.18) $$\sigma^{(\pm)}_{pm,k\ell} \neq 0, \qquad \text{only when } p+m = k+\ell,$$

(5.19) $$r^2(p)r^2(m) = r^2(\ell)r^2(k), \qquad \text{when } p+m = k+\ell.$$

(5.20a) $$\sum_\ell \sigma^{(+)}_{k\ell,k\ell} r^2(\ell) = \chi(\lambda) \quad \text{(independent of } k\text{)},$$

(5.20b) $$\sum_\ell \sigma^{(-)}_{k\ell,k\ell} r^2(\ell) = \bar{\chi}(\lambda) \quad \text{(independent of } k\text{)},$$

The condition (5.18) is the charge conservation (or spin conservation) condition [60,62].

Let us show that the conditions (5.18)–(5.20) are sufficient for $\phi(\cdot)$ to be the Markov trace. For the Markov property I:

$$\phi(AB) = \phi(BA), \qquad \text{for } A, B \in B_n, \tag{5.21}$$

it is sufficient to have

$$[H^{(n)}, A] = 0, \qquad \text{for } A \in B_n. \tag{5.22}$$

Since any element of the braid group is expressed in the product of the generators, the relation (5.22) reduces to

$$[H^{(n)}, G_i] = 0, \qquad \text{for } i = 1, 2, \cdots, n-1. \tag{5.23}$$

Because of the form of $H^{(n)}$ in (5.17b), it is enough to prove (5.23) for $n = 2$:

$$h^{(1)} \otimes h^{(2)} \cdot G = G \cdot h^{(1)} \otimes h^{(2)}, \tag{5.24}$$

where G is the matrix representation of B_2,

$$G = \sum_{k,\ell,m,p} \sigma^{(+)}_{pm,k\ell} e^{(1)}_{pk} \otimes e^{(2)}_{m\ell}. \tag{5.25}$$

Substituting (5.17c) and (5.25) into (5.24), we have

$$r^2(k) r^2(\ell) \sigma^{(+)}_{pm,k\ell} = r^2(p) r^2(m) \sigma^{(+)}_{pm,k\ell}, \tag{5.26}$$

which holds due to (5.18) and (5.19). Thus, the conditions (5.18) and (5.19) are sufficient for the Markov property I.

In general, the state variables of vertex models take vector values $\vec{p}$. For most of vertex models the crossing multipliers take exponential forms:

$$r(\vec{p}) = \exp q(\vec{p}), \tag{5.27}$$

where $q(\vec{p})$ is a linear functional (or projection on some direction) of $\vec{p}$. The state variables k, ℓ, m, p in (5.19) are considered as the components or the projections on some direction of the vectors $\vec{k}, \vec{\ell}, \vec{m}, \vec{p}$. Thus the condition (5.26) holds if the following is satisfied:

$$\vec{k} + \vec{\ell} = \vec{p} + \vec{m}. \tag{5.28}$$

Let us consider the Markov property II:

(5.29a) $$\phi(AG_n) = \tau \cdot \phi(A), \quad \text{for } A \in B_n, AG_n \in B_{n+1},$$

(5.29b) $$\phi(AG_n^{-1}) = \bar{\tau} \cdot \phi(A), \quad \text{for } A \in B_n, AG_n^{-1} \in B_{n+1},$$

where

(5.29c) $$\tau = \phi(G_i), \quad \bar{\tau} = \phi(G_i^{-1}), \quad i = 1, 2, \cdots, n.$$

We shall calculate the l.h.s. of (5.29a). We introduce a quantity (recall that λ is the crossing point):

(5.30) $$\xi(\lambda) = \sum_p r^2(p).$$

Then using (5.17b,c) we have

(5.31) $$\mathrm{Tr}(H^{(n)}) = \sum_{k_1 \cdots k_n} r^2(k_1) r^2(k_2) \cdots r^2(k_n) = \{\xi(\lambda)\}^n.$$

From (5.17) and (5.31) the l.h.s. of (5.29a) is

(5.32) $$\phi(AG_n) = \frac{\mathrm{Tr}(H^{(n+1)} AG_n)}{\mathrm{Tr}(H^{(n+1)})} = \{\xi(\lambda)\}^{-(n+1)} \mathrm{Tr}(H^{(n+1)} AG_n).$$

We write the matrix elements of the representation of the braid group as

(5.33) $$A = \sum_{k_1 \cdots k_n} \sum_{p_1 \cdots p_n} [A]_{p_1 \cdots p_n}^{k_1 \cdots k_n} e_{k_1 p_1}^{(1)} \otimes \cdots \otimes e_{k_n p_n}^{(n)},$$

then we have

(5.34) $$\begin{aligned} \mathrm{Tr}(H^{(n+1)} AG_n) &= \sum_{k_1 \cdots k_{n+1}} \sum_{p_1 \cdots p_{n+1}} r^2(k_1) \cdots r^2(k_{n+1}) \times \\ &\qquad \times [A]_{p_1 \cdots p_{n+1}}^{k_1 \cdots k_{n+1}} [G_n]_{p_1 \cdots p_{n+1}}^{k_1 \cdots k_{n+1}} \\ &= \sum_{k_1 \cdots k_{n+1}} r^2(k_1) \cdots r^2(k_{n+1}) [A]_{k_1 \cdots k_n}^{k_1 \cdots k_n} [G_n]_{k_n k_{n+1}}^{k_n k_{n+1}} \\ &= \sum_{k_1 \cdots k_n} r^2(k_1) \cdots r^2(k_n) [A]_{k_1 \cdots k_n}^{k_1 \cdots k_n} [\sum_\ell \sigma_{k_n \ell, k_n \ell}^{(+)} r^2(\ell)]. \end{aligned}$$

In the above the charge conservation condition has been used. Using (5.20a) in (5.34) we have

$$\begin{aligned}\mathrm{Tr}(H^{(n+1)}AG_n) &= \chi(\lambda)\cdot\sum_{k_1\cdots k_n} r^2(k_1)\cdots r^2(k_n)[A]^{k_1\cdots k_n}_{k_1\cdots k_n} \\ &= \chi(\lambda)Tr(H^{(n)}A).\end{aligned} \tag{5.35}$$

Thus the Markov property II holds with $\tau = \chi(\lambda)/\xi(\lambda)$ because

$$\begin{aligned}\phi(AG_n) &= [\xi(\lambda)]^{-(n+1)}\mathrm{Tr}(H^{(n+1)}AG_n) \\ &= \frac{\chi(\lambda)}{\xi(\lambda)}\cdot[\xi(\lambda)]^{-n}\mathrm{Tr}(H^{(n)}A) \\ &= \frac{\chi(\lambda)}{\xi(\lambda)}\cdot\phi(A).\end{aligned} \tag{5.36}$$

In the same way, we can show from (5.20b) and (5.18) that (5.29b) holds with $\bar{\tau} = \bar{\chi}(\lambda)/\xi(\lambda)$. Thus, the conditions (5.18) and (5.20) are sufficient for the Markov property II. To summarize, (5.18)–(5.20) are sufficient conditions for the trace $\phi(\cdot)$ to be the Markov trace.

We add a comment. If, for a vertex model, the representation of the braid group satisfies the reflection symmetry, the trace defined by (5.17) satisfies the Markov property I. But the charge conservation condition seems to be necessary for the Markov property II if the crossing multiplier is not trivial.

It is amusing to explain graphically the matrix elements of the braid group representation and the Markov trace. A braid A is considered as orbits of particles going upward with time and interacting at intersections. To each part of the diagram of the braid A we assign one of the state values of the vertex model. The assigned value may change only at intersections. We specify weights of the braid representation at the intersections of the diagram. We denote the values of lower edges of A by $p_1\dots p_n$ and those of upper edges by $q_1,\dots,q_n$. The edges of the diagram are referred to as external lines and other parts of the orbits as internal lines. Then, the matrix element of the representation of of the braid A is interpreted as the scattering amplitude $\langle q_1,\dots,q_n|p_1,\cdots,p_n\rangle$.

We consider a closed braid as closed orbits of the particles. The Markov trace is denoted by black circles on the upper part of the strings of the closed braid.

We next consider the IRF model. We assume that a trace $\phi(\cdot)$ has the form [33,63]:

$$\phi(A) = \hat{\mathrm{T}}\mathrm{r}(H^{(n)}A)/\hat{\mathrm{T}}\mathrm{r}(H^{(n)}), \quad A\in B_n, \tag{5.37a}$$

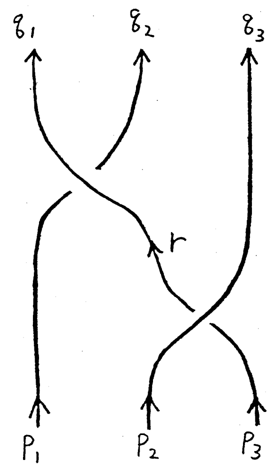


Fig. 30. A configuration of the orbits of moving particles corresponding to the braid A. We take summations over possible values of internal lines.

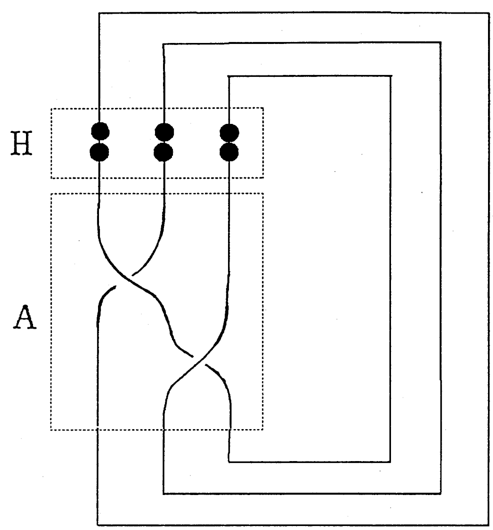


Fig. 31. Markov trace (for vertex and IRF models). Crossing multipliers are depicted by black circles.

$$
\begin{aligned}
[H^{(n)}]^{p_0\cdots p_n}_{\ell_0\cdots\ell_n} &= \delta(p_0,\ell_0)\frac{\psi(\ell_1)}{\psi(\ell_0)}\cdot\delta(p_1,\ell_1)\frac{\psi(\ell_2)}{\psi(\ell_1)}\cdots \\
&\qquad\cdots\delta(p_{n-1},\ell_{n-1})\frac{\psi(\ell_n)}{\psi(\ell_{n-1})}\cdot\delta(p_n,\ell_n) \\
&= \delta(p_0,\ell_0)\delta(p_1,\ell_1)\cdots\delta(p_n,\ell_n)\frac{\psi(\ell_n)}{\psi(\ell_0)}.
\end{aligned} \tag{5.37b}
$$

Here $\psi(\ell)$ is the crossing multiplier and $\hat{\mathrm{Tr}}(\cdot)$ is the constrained trace defined by

$$
\hat{\mathrm{Tr}}(A) = \widetilde{\sum_{\substack{\ell_1\cdots\ell_n \\ \ell_0:\text{fixed}}}} A^{\ell_0\cdots\ell_n}_{\ell_0\cdots\ell_n}, \quad A\in B_n. \tag{5.37c}
$$

We use the symbol $\tilde{\sum}$ for the summation under the constraint imposed on the model. We fix the state variable ℓ_0 . In the unrestricted IRF models, it is easy to see that the trace (5.37c) does not depend on the fixed value ℓ_0 [33]. From (5.37), we have [33]

$$
\hat{\mathrm{Tr}}(H^{(n)}A) = \widetilde{\sum_{\substack{\ell_1\cdots\ell_n \\ \ell_0:\text{fixed}}}} A^{\ell_0\cdots\ell_n}_{\ell_0\cdots\ell_n}\cdot\frac{\psi(\ell_n)}{\psi(\ell_0)}. \tag{5.38}
$$

We shall show that the following conditions are sufficient for $\phi(\cdot)$ defined by (5.37) to be the Markov trace [33]:

$$
\sum_{b\sim a}\sigma(a,b,a,c;+)\psi(b)/\psi(a) = \chi(\lambda) \quad \text{(independent of } c,a\text{)}, \tag{5.39a}
$$

$$
\sum_{b\sim a}\sigma(a,b,a,c;-)\psi(b)/\psi(a) = \bar{\chi}(\lambda) \quad \text{(independent of } c,a\text{)}, \tag{5.39b}
$$

$$
\sum_{b\sim a}\psi(b)/\psi(a) = \xi(\lambda) \quad \text{(independent of } a\text{)}, \tag{5.40}
$$

where the summations are over all values of b admissible to a and $\psi(a)$ is the crossing multiplier of the model. It is remarked that the condition (5.40) is not trivial.

For the IRF models, the trace $\phi(\cdot)$ automatically satisfies the Markov property I. Let us observe it explicitly. In the same way as the vertex models, we can show that the following relation is sufficient for $\phi(\cdot)$ to satisfy the Markov property I:

$$(5.41) \qquad [H^{(2)}, G] = 0,$$

where G is the representation of the braid generator of B_2, and $H^{(2)}$ is defined on the representation of B_2. Since the products $H^{(2)}G$ and $GH^{(2)}$ are

$$(5.42) \qquad \begin{aligned} [H^{(2)}G]^{p_0p_1p_2}_{\ell_0\ell_1\ell_2} &= \delta(p_0,\ell_0)\sigma(\ell_1,\ell_2,p_1,\ell_0)\frac{\psi(\ell_2)}{\psi(\ell_0)}\delta(p_2,\ell_2), \\ [GH^{(2)}]^{p_0p_1p_2}_{\ell_0\ell_1\ell_2} &= \delta(p_0,\ell_0)\sigma(\ell_1,\ell_2,p_1,\ell_0)\frac{\psi(\ell_2)}{\psi(\ell_0)}\delta(p_2,\ell_2), \end{aligned}$$

we find that (5.41) holds without any additional condition. Thus, $\phi(\cdot)$ defined by (5.37) satisfies the Markov property I.

In IRF models, the constraint of the model implies the charge conservation condition. This is an another reason why the Markov property I is automatically satisfied for the IRF models.

Let us show that the conditions (5.39) and (5.40) are sufficient for the trace $\phi(\cdot)$ to satisfy the Markov property II (5.29). Using (5.40) we have

$$(5.43) \qquad \begin{aligned} \hat{\mathrm{Tr}}(H^{(n)}) &= \widetilde{\sum_{\substack{\ell_1\cdots\ell_n \\ \ell_0:\mathrm{fixed}}}} \frac{\psi(\ell_1)}{\psi(\ell_0)}\cdot\frac{\psi(\ell_2)}{\psi(\ell_1)}\cdots\frac{\psi(\ell_n)}{\psi(\ell_{n-1})} \\ &= [\xi(\lambda)]^n. \end{aligned}$$

Then, the l.h.s. of (5.29a) is

$$(5.44) \qquad \begin{aligned} \phi(AG_n) &= \frac{\hat{\mathrm{Tr}}(H^{(n+1)}AG_n)}{\hat{\mathrm{Tr}}(H^{(n+1)})} \\ &= [\xi(\lambda)]^{-(n+1)}\hat{\mathrm{Tr}}(H^{(n+1)}AG_n). \end{aligned}$$

We calculate $\hat{\mathrm{Tr}}(H^{(n+1)}AG_n)$. Using the expression (5.14b) and (5.39a) we have

(5.45)

$$\begin{aligned}\hat{\mathrm{Tr}}(H^{(n+1)}AG_n) &= \widetilde{\sum_{\substack{\ell_1\cdots\ell_{n+1}\\ \ell_0:\mathrm{fixed}}}} [AG_n]^{\ell_0\cdots\ell_{n+1}}_{\ell_0\cdots\ell_{n+1}} \frac{\psi(\ell_{n+1})}{\psi(\ell_0)} \\ &= \widetilde{\sum_{\substack{\ell_1\cdots\ell_{n+1}\\ \ell_0:\mathrm{fixed}}}} [A]^{\ell_0\cdots\ell_n}_{\ell_0\cdots\ell_n} \sigma(\ell_n,\ell_{n+1},\ell_n,\ell_{n-1};+) \frac{\psi(\ell_{n+1})}{\psi(\ell_0)} \\ &= \widetilde{\sum_{\substack{\ell_1\cdots\ell_n\\ \ell_0:\mathrm{fixed}}}} [A]^{\ell_0\cdots\ell_n}_{\ell_0\cdots\ell_n} \frac{\psi(\ell_n)}{\psi(\ell_0)} \times \\ &\qquad \times \sum_{\ell_{n+1}\sim\ell_n} [\sigma(\ell_n,\ell_{n+1},\ell_n,\ell_{n-1};+)\frac{\psi(\ell_{n+1})}{\psi(\ell_n)}] \\ &= \chi(\lambda)\cdot \widetilde{\sum_{\substack{\ell_1\cdots\ell_n\\ \ell_0:\mathrm{fixed}}}} [A]^{\ell_0\cdots\ell_n}_{\ell_0\cdots\ell_n} \frac{\psi(\ell_n)}{\psi(\ell_0)} \\ &= \chi(\lambda)\hat{\mathrm{Tr}}(H^{(n)}A).\end{aligned}$$

Thus we obtain the Markov property II with $\tau = \chi(\lambda)/\xi(\lambda)$ [33]

$$\begin{aligned}\phi(AG_n) &= [\xi(\lambda)]^{-(n+1)}\hat{\mathrm{Tr}}(H^{(n+1)}AG_n) \\ &= \frac{\chi(\lambda)}{\xi(\lambda)}\cdot[\xi(\lambda)]^{-n}\hat{\mathrm{Tr}}(H^{(n)}A) \\ &= \frac{\chi(\lambda)}{\xi(\lambda)}\cdot\phi(A).\end{aligned} \tag{5.46}$$

In the same way, we obtain (5.29b) with $\bar{\tau} = \bar{\chi}(\lambda)/\xi(\lambda)$. Thus, the conditions (5.39) and (5.40) are sufficient for the trace $\phi(\cdot)$ defined by (5.37) to satisfy the Markov property II. To summarize, when the conditions (5.39) and (5.40) are satisfied the trace $\phi(\cdot)$ is the Markov trace.

The representation of the braid group and the Markov trace (5.37) can also be graphically explained. Let us consider a diagram of a braid A.

In the diagram there are domains separated by strings. The state variables are specified on the domains. The identity operator I_n represents a configuration such that there are $n+1$ parallel domains. We refer to domains encircled by strings as internal domains and other domains as external domains. To the intersections of the strings, weight matrices of the braid group representation are assigned. A matrix element of the

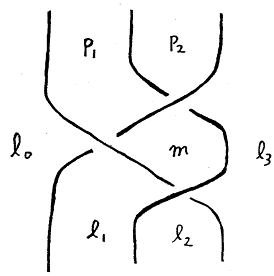


Fig. 32. A configuration of domains on a plain.

braid group representation can be considered as a sum of configurations over all possible internal domains. In this graphical representation, the Markov trace (5.37) is illustrated by putting the crossing multipliers on the upper part of the strings of the closed braid (Fig.31).

Finally, we give a remark on the orientations of strings. Due to the reflection symmetry (5.6), the Markov trace for an oriented link is the same as that for a link whose orientations of strings are reversed. Orientations of links are detected by the writhe, equivalently by the exponent sum (cf.(4.9)) in (4.6).

5.4. Extended Markov property

The conditions (5.20) for the vertex models ((5.39) and (5.40) for the IRF models) are combined into a single relation which holds even with finite u [33,2]:

(5.47a)
$$\sum_{\ell} S_{\ell k}^{k\ell}(u) r^2(\ell) = H(u;\lambda)\rho(u), \qquad \text{(independent of } k\text{)},$$

(5.47b)
$$\sum_{b\sim a} w(a,b,a,c;u)\frac{\psi(b)}{\psi(a)} = H(u;\lambda)\rho(u), \qquad \text{(independent of } a,c\text{)}.$$

We call (5.47) the *extended Markov property* and $H(u;\lambda)$ the *characteristic function.* The τ factors in the Markov property II are given by

$$\tau = \lim_{u\to\infty} \frac{H(u;\lambda)}{H(0;\lambda)}, \tag{5.48a}$$

$$\bar{\tau} = \lim_{u\to\infty} \frac{H(-u;\lambda)}{H(0;\lambda)}. \tag{5.48b}$$

What is significant in this theory is that the extended Markov property (and the charge conservation condition for vertex models) is sufficient for the existence of the Markov trace.

Furthermore, when the model has the crossing symmetry, the projection relation [63]

$$X_i(\lambda)X_i(u) = \beta(u)X_i(\lambda), \tag{5.49}$$

is equivalent to the extended Markov property. The function $\beta(u)$ is related [63] to the characteristic function $H(u;\lambda)$ as $\beta(u) = H(\lambda - u;\lambda)\rho(\lambda - u)$.

Before we proceed to the application of the theory, we consider a relation between unrestricted IRF model and vertex model. The Boltzmann weights of unrestricted IRF model contain an arbitrary parameter $\vec{\omega}_0$. For example, the Boltzmann weights of the unrestricted 8VSOS model (3.18) have an arbitrary parameter ω_0. In general, $\vec{\omega}_0$ is a vector and we set

$$\vec{\omega}_0 = \omega_0 \vec{n}_0, \tag{5.50}$$

where $\vec{n}_0$ is a unit vector. It is known that, at the criticality an IRF model becomes equivalent to the corresponding vertex model in the limit $\omega_0 \to \pm i\infty$ [33]. The Boltzmann weights and the crossing multipliers are transformed as [2,33]

$$w(a,b,c,d;u) \to S^{ik}_{j\ell}(u), \qquad (\omega_0 \to \pm i\infty) \tag{5.51}$$

$$\frac{\psi(b)}{\psi(a)} \to r^2(j), \qquad (\omega_0 \to \pm i\infty) \tag{5.52}$$

where

$$\begin{aligned} \vec{i} &= \vec{a} - \vec{d}, \\ \vec{j} &= \vec{b} - \vec{a}, \\ \vec{k} &= \vec{b} - \vec{c}, \\ \vec{\ell} &= \vec{c} - \vec{d}. \end{aligned} \tag{5.53}$$

Under this transformation, the Markov trace (5.37) for the IRF model reduces to the one in (5.17) for the vertex model in the limit $\omega_0 \to \pm i\infty$ [33]. Therefore, when a vertex model is obtained from an IRF model by the Wu-Kadanoff-Wegner transformation and the limit $\omega_0 \to \pm i\infty$, both characteristic functions $H(u;\lambda)$ have the same form with a replacement such as $\lambda \to i\lambda$.

§6. New link polynomials

In the previous two chapters we have given a general theory to construct link polynomials from the exactly solvable models. We shall apply it to various solvable models and obtain new link polynomials.

6.1. N-state vertex model

The N-state vertex model [62] is a generalization of the 6-vertex model where state variables take 2 values. For $N = 3$, it is the 19-vertex model [64]. For general N, it has $N(2N^2+1)/3$ non-zero vertices.

We write the Boltzmann weights of the N-state vertex model by $\{S_{j\ell}^{ik}(u)\}$. We define "spin" (or "charge") s by

$$N = 2s + 1. \tag{6.1}$$

The state variables i, j, k and ℓ of the Boltzmann weights $S_{j\ell}^{ik}(u)$ take the following values:

$$i, j, k, \ell = -s, -s+1, \cdots, s-1, s. \tag{6.2}$$

In this way we may regard the Boltzmann weights of the N-state vertex model as the S-matrices of two spin-s particles. The model satisfies the charge (or spin) conservation condition :

$$S_{j\ell}^{ik}(u) = 0 \qquad \text{unless } i + j = k + \ell. \tag{6.3}$$

In addition, the model has the properties:
1) standard initial condition

$$S_{j\ell}^{ik}(u=0) = \delta_{i\ell}\delta_{jk}. \tag{6.4}$$

2) unitarity condition

$$\sum_{p,q} S_{pk}^{q\ell}(-u) S_{jq}^{ip}(u) = \rho(u)\rho(-u)\delta_{ik}\delta_{j\ell}. \tag{6.5}$$

3) CPT invariances

$$\begin{aligned} S_{j\ell}^{ik}(u) &= S_{-j-\ell}^{-i-k}(u), && \text{C-invariance,} \\ &= S_{ik}^{j\ell}(u), && \text{P-invariance,} \\ &= S_{\ell j}^{ki}(u), && \text{T-invariance.} \end{aligned} \tag{6.6}$$

4) crossing symmetry

$$S_{j\ell}^{ik}(u) = S_{j\ell}^{-k-i}(\lambda - u), \tag{6.7}$$

with the crossing point λ. We note that the crossing multiplier for the N-state vertex model is equal to 1.

Explicit parametrizations of the Boltzmann weights for N=2,3,4 cases are given in below (Only a part of the Boltzmann weights are shown since all other non-zero weights are obtained by the crossing symmetry and the CPT invariances). For general N, we have the recursive formula [62] for the Boltzmann weights and the algebraic construction [14,61] for the Yang-Baxter operator.

1) $N = 2$ $(s = 1/2)$ case (the 6-vertex model)

$$\begin{aligned} S^{1/2\,1/2}_{1/2\,1/2}(u) &= \frac{\sinh(\lambda - u)}{\sinh\lambda}, \\ S^{1/2\,-1/2}_{-1/2\,1/2}(u) &= 1. \end{aligned} \tag{6.8}$$

2) $N = 3$ $(s = 1)$ case (the 19-vertex model)

$$\begin{aligned} S^{1\,1}_{1\,1}(u) &= \frac{\sinh(\lambda - u)\sinh(2\lambda - u)}{\sinh\lambda\sinh 2\lambda}, \\ S^{1\,-1}_{-1\,1}(u) &= 1, \\ S^{1\,1}_{0\,0}(u) &= \frac{\sinh u\sinh(\lambda - u)}{\sinh\lambda\sinh 2\lambda}, \\ S^{1\,0}_{0\,1}(u) &= \frac{\sinh(\lambda - u)}{\sinh\lambda}, \\ S^{0\,0}_{0\,0}(u) &= \frac{\sinh\lambda\sinh 2\lambda - \sinh u\sinh(\lambda - u)}{\sinh\lambda\sinh 2\lambda}. \end{aligned} \tag{6.9}$$

3) $N = 4$ (s=3/2) case (the 44-vertex model)

(6.10)

$$S^{3/2\,3/2}_{3/2\,3/2}(u) = \frac{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}{\sinh\lambda\sinh 2\lambda\sinh 3\lambda},$$

$$S^{3/2\,-3/2}_{-3/2\,3/2}(u) = 1,$$

$$S^{3/2\,3/2}_{1/2\,1/2}(u) = \frac{\sinh u\sinh(\lambda-u)\sinh(2\lambda-u)}{\sinh\lambda\sinh 2\lambda\sinh 3\lambda},$$

$$S^{3/2\,1/2}_{1/2\,3/2}(u) = \frac{\sinh(\lambda-u)\sinh(2\lambda-u)}{\sinh\lambda\sinh 2\lambda},$$

$$S^{3/2\,-1/2}_{-1/2\,3/2}(u) = \frac{\sinh(\lambda-u)}{\sinh\lambda},$$

$$S^{3/2\,3/2}_{-1/2\,-1/2}(u) = \frac{\sinh u}{\sinh\lambda}\frac{\sinh(\lambda+u)\sinh(\lambda-u)}{\sinh 2\lambda\sinh 3\lambda},$$

$$S^{3/2\,1/2}_{-1/2\,1/2}(u) = \frac{2\sqrt{\sinh\lambda\sinh 3\lambda}\cdot\cosh\lambda\sinh u\sinh(\lambda-u)}{\sinh\lambda\sinh 2\lambda\sinh 3\lambda},$$

$$S^{1/2\,1/2}_{1/2\,1/2}(u) = \frac{[\sinh 2\lambda\sinh 3\lambda - \sinh u\sinh(\lambda-u)]\sinh(\lambda-u)}{\sinh\lambda\sinh 2\lambda\sinh 3\lambda},$$

$$S^{1/2\,-1/2}_{-1/2\,1/2}(u) = \frac{\sinh\lambda\sinh 3\lambda - 2\cosh\lambda\sinh u\sinh(\lambda-u)}{\sinh\lambda\sinh 3\lambda},$$

The function $\rho(u)$ for general N is given by

$$\rho(u) = \prod_{n=1}^{N-1}\frac{\sinh(n\lambda-u)}{\sinh(n\lambda)}. \tag{6.11}$$

In order to construct an "interesting" braid group representation from the N-state vertex model, we take the following procedure: We asymmetrize the model and then apply the formula (5.11). Before we proceed, we should explain a meaning of "interesting". Even without asymmetrization, we obtain some braid group representation which leads to a new link polynomial. Primary interest, however, was to construct more powerful (hence interesting) polynomials than the Jones polynomial. We introduce symmetry breaking transformation [62]:

$$S^{ik}_{j\ell}(u) \to \tilde{S}^{ik}_{j\ell}(u) = \exp[\frac{1}{2}(j+k-i-\ell)u]\cdot S^{ik}_{j\ell}(u). \tag{6.12}$$

If the model satisfies charge conservation condition, then the transformation is compatible with the Yang-Baxter relation; the transformed weights $\tilde{S}^{ik}_{j\ell}(u)$ also satisfy the Yang-Baxter relation. Through the transformation we have asymmetrized N-state vertex model in that the crossing multipliers are non-trivial. For an illustration we present the asymmetrized weights for N=2 (the 6-vertex model):

$$
\begin{aligned}
&\tilde{S}^{1/2\,1/2}_{1/2\,1/2}(u) = \tilde{S}^{-1/2\,-1/2}_{-1/2\,-1/2}(u) = \frac{\sinh(\lambda-u)}{\sinh\lambda},\\
&\tilde{S}^{1/2\,1/2}_{-1/2\,-1/2}(u) = \tilde{S}^{-1/2\,-1/2}_{1/2\,1/2}(u) = \frac{\sinh u}{\sinh\lambda},\\
&\tilde{S}^{1/2\,-1/2}_{-1/2\,1/2}(u) = e^{-u},\\
&\tilde{S}^{-1/2\,1/2}_{1/2\,-1/2}(u) = e^{u}.
\end{aligned}
\tag{6.13}
$$

Let us construct representation of the braid group. Applying the formula (5.11) to the asymmetrized weights $\tilde{S}^{ik}_{j\ell}(u)$ with $\rho(u) = \sinh(\lambda - u)/\sinh\lambda$, we obtain the weight matrices $\sigma^{(+)}_{\ell k,ij}$. Resulting representation of the braid operator G_i contains one-parameter t:

$$t = \exp(2\lambda). \tag{6.14}$$

The N-state vertex model has two parameters, the spectral parameter u and the crossing point λ. The braid group representation is obtained by the limit $u \to \infty$, and therefore one variable λ survives in the representation. For the $N = 2$ case , the weight matrices are

$$\sigma_{1/2\,1/2,1/2\,1/2} = \sigma_{-1/2\,-1/2,-1/2\,-1/2} = 1 \tag{6.15}$$

$$
\begin{pmatrix}
\sigma_{1/2\quad -1/2,-1/2\quad 1/2} & \sigma_{-1/2\quad 1/2,1/2\quad -1/2}\\
\sigma_{1/2\quad -1/2,1/2\quad -1/2} & \sigma_{-1/2\quad 1/2,-1/2\quad 1/2}
\end{pmatrix}
=
\begin{pmatrix}
0 & -t^{1/2}\\
-t^{1/2} & 1-t
\end{pmatrix}
$$

The weight matrices for the general N are given as follows. By $\sigma^{(c)}$ we denote the submatrix acting in the sector of total charge c:

$$
\begin{aligned}
\sigma^{(c)} &= T^{(N-|c|)}(t^{|c|}),\\
T^{(N)}_{nm} &= (-1)^{n+m}[Q_{m-1,N-n}Q_{n-1,N-m}]^{1/2},\\
Q_{pm}(z) &= \frac{(t;p)(zt;p)}{(t;p-m)(t;m)(zt;m)}z^m t^{m^2},
\end{aligned}
\tag{6.16}
$$

where $(z;n) = (1-z)(1-zt)\cdots(1-zt^{n-1})$.

Since the matrix representation is of finite dimensions the braid operator satisfies a relation (the minimal polynomial for the matrix) which we call *reduction relation* [58,60]. From the asymmetrized N-state vertex model we obtain the braid operator which satisfies N-th order reduction relation:

$$(G_i - c_1)(G_i - c_2)\cdots(G_i - c_N) = 0, \tag{6.17a}$$

where

$$c_j = (-1)^{j+N} t^{\frac{1}{2}N(N-1)-\frac{1}{2}j(j-1)}, \qquad j = 1, 2, \cdots, N. \tag{6.17b}$$

In the N=2 case the braid operator is a representation of the generator of the Hecke algebra(see, (4.14)).

We shall construct the Markov trace on the representation and then obtain link polynomials. We apply the formula (5.17). Since the crossing multipliers for the asymmetrized N-state vertex model are given by

$$r(k) = \exp(-\lambda k) = t^{-k/2}, \tag{6.18}$$

the Markov trace $\phi(\cdot)$ is (cf.(5.17))

$$\phi(A) = \mathrm{Tr}(H^{(n)}A)/\mathrm{Tr}(H^{(n)}), \qquad \text{for } A \in B_n, \tag{6.19a}$$

where

$$H^{(n)} = h^{(1)} \otimes h^{(2)} \otimes \cdots \otimes h^{(n)}, \tag{6.19b}$$

and

$$h_{pq} = t^{-p}\delta_{pq}, \quad p, q = -s, -s+1, \cdots, s. \tag{6.19c}$$

The Markov trace (6.19) is a generalization of the Powers state [65]. We can show that the extended Markov property (5.47a) is satisfied with the characteristic function $H(u;\lambda)$ as

$$H(u;\lambda) = \frac{\sinh(N\lambda - u)}{\sinh(\lambda - u)}. \tag{6.20}$$

From (6.20) and (5.48) the τ and $\bar{\tau}$ factors are

$$\begin{aligned} \tau(t) &= 1/(1 + t + \cdots + t^{N-1}), \\ \bar{\tau}(t) &= t^{N-1}/(1 + t + \cdots + t^{N-1}). \end{aligned} \tag{6.21}$$

Thus, there exists an infinite sequence of link polynomials corresponding to the$N$-state vertex model ($N = 2, 3, 4, 5, \cdots$). The link polynomial $\alpha(A)$ for an element $A \in B_n$ is given by

$$\alpha(A) = [t^{-\frac{N-1}{2}}(1 + t + t^2 + \cdots + t^{N-1})]^{n-1}[t^{\frac{N-1}{2}}]^{e(A)}\phi(A), \tag{6.22}$$

where $e(A)$ is the exponent sum of b_i's appearing in the braid A.

Using the reduction relation (6.17) and the τ-factors (6.21), we obtain the *skein relations* (the *Alexander-Conway relations* [53,66]) for the link polynomials [58,60]:

(6.23a)
$$\alpha(L_+) = (1-t)t^{1/2}\alpha(L_0) + t^2\alpha(L_-), \qquad (N=2)$$

(6.23b)
$$\alpha(L_{2+}) = t(1-t^2+t^3)\alpha(L_+) + (t^4-t^5+t^7)\alpha(L_0) - t^8\alpha(L_-), \quad (N=3)$$

(6.23c)
$$\alpha(L_{3+}) = t^{3/2}(1-t^3+t^5-t^6)\alpha(L_{2+}) + t^6(1-t^2+t^3+t^5-t^6+t^8)\alpha(L_+) + t^{25/2}(-1+t-t^3+t^6)\alpha(L_0) - t^{20}\alpha(L_-), \qquad (N=4).$$

In (6.23a), by L_+, L_0 and L_- we have denoted links which have the configuration of b_i, b_i^0 and b_i^{-1}, at an intersection.

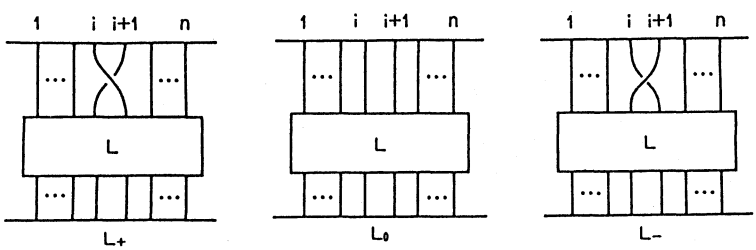


Fig. 33. Links L_+, L_0 and L_-.

Similarly, L_{2+}, L_+, L_0 and L_- in (6.23b) and L_{3+}, L_{2+}, L_+, L_0 and L_- in (6.23c) should be understood.

The $N=2$ case corresponds to the Jones polynomial. In the $N \geq 3$ cases we have new link polynomials. The Jones polynomial (and its two-variable extension) is not complete in the sense that there exist infinitely many different links which have the same polynomial [67,68]. We shall give in §6.4 an example of two different links which can not be classified by the Jones polynomial but can be by the N=3 link polynomial [69]. The link polynomial constructed from the N-state vertex model has also been studied from the viewpoint of q-deformation of universal enveloping algebra $U_q(sl(2))$ [70].

6.2. IRF models

a) Unrestricted 8VSOS model

The unrestricted 8VSOS model is defined by (3.15). The Boltzmann weights, given by (3.18), satisfy the basic relations (5.1)–(5.6) with the crossing multiplier (3.17), the crossing point λ and the function $\rho(u)$:

$$\rho(u) = \frac{\theta_1(\lambda - u, p)}{\theta_1(\lambda, p)}. \tag{6.24}$$

To obtain representation of the braid group we apply the general formula (5.11). The existence of the limit $u \to \infty$ requires the nome p to be 0: The model is critical. The Boltzmann weights at the criticality are

$$w(a,b,c,d;u) = \delta_{ac} + \delta_{bd} \cdot \frac{\sin u}{\sin(\lambda - u)} \cdot \frac{\sqrt{\psi(a)\psi(c)}}{\psi(b)}, \tag{6.25}$$

where the crossing multiplier is

$$\psi(a) = \sin(a\lambda + \omega_0). \tag{6.26}$$

By taking the limit $u \to i\infty$ in (6.25), we get the weight matrix

$$\sigma(a,b,c,d;t) = \delta_{ac} - \delta_{bd} e^{-i\lambda} \cdot \frac{\sqrt{\psi(a)\psi(c)}}{\psi(b)}. \tag{6.27}$$

Substituting this into (5.14), we obtain representaion of the braid operator. The braid operator G_i satisfies a quadratic reduction relation

$$(G_i - I_n)(G_i + tI_n) = 0, \tag{6.28}$$

with

$$t = e^{-2i\lambda}. \tag{6.29}$$

Using (6.25) and (6.26), we can show that the extended Markov property (5.47b) holds with the characteristic function

$$H(u;\lambda) = \frac{\sin(2\lambda - u)}{\sin(\lambda - u)}. \tag{6.30}$$

Thus, we obtain the Markov trace by (5.37) and the link polynomial by (4.6). The τ-factors in the Markov property II is given by

$$\tau = \frac{H(i\infty;\lambda)}{H(0;\lambda)} = \frac{1}{1+t}, \qquad \bar{\tau} = \frac{H(-i\infty;\lambda)}{H(0;\lambda)} = \frac{t}{1+t}. \tag{6.31}$$

The reduction relation (6.28) indicates that the link polynomial is equivalent to the Jones polynomial.

The parameter ω_0 in the crossing multiplier $\psi(\ell) = \sin(\ell\lambda+\omega_0)$ does not appear in (6.30), which leads to ω_0-independence of $\hat{\mathrm{Tr}}(\cdot)$. This also leads to ℓ_0-independence of $\hat{\mathrm{Tr}}(\cdot)$ since a change $\omega_0 \to \omega_0 - k\lambda$ $(k:$ integer) is equivalent to the parallel shift of all heights $\ell_i \to \ell_i + k$ $(i = 0, ...n)$.

We shall show that in the limit $\omega_0 \to i\infty$ the Markov trace for the critical unrestricted 8VSOS model reproduces that for the 6-vertex model. The Wu-Kadanoff-Wegner transformation gives an equivalence between the unrestricted 8VSOS model (with $\omega_0 \to i\infty$) and the 6-vertex model. By taking the limit $\omega_0 \to i\infty$, we have the crossing multiplier as

$$\frac{\psi(b)}{\psi(a)} \to e^{-i\lambda(b-a)}, \tag{6.32}$$

and the Boltzmann weights as

$$w(a,b,c,d;u) \to \delta_{ac} + \delta_{bd} \cdot \frac{\sin u}{\sin(\lambda - u)} \cdot e^{-i\lambda(a+c-2b)/2}. \tag{6.33}$$

Setting

$$S_{j\ell}^{ik}(u) = S_{b-a\,c-d}^{a-d\,b-c}(u) = \lim_{\omega_0 \to i\infty} w(a,b,c,d;u), \tag{6.34}$$

we obtain the Boltzmann weights of the 6-vertex model (6.13). The corresponding braid group representations and then the Markov trace are equivalent. In particular, the extended Markov property for the unrestricted 8VSOS model becomes

$$\sum_{b\sim a} S_{b-a\,a-c}^{a-c\,b-a}(u) \cdot e^{-i\lambda(b-a)} = H(u;\lambda), \tag{6.35}$$

which is the extended Markov property for the 6-vertex model.

b) General graph-state IRF models

The constraint imposed on the IRF models can be expressed as a graph (see §3.2). For this class of models (graph-state models), the Boltzmann weights (resp. the weight matrices) have the same form as (6.25) (resp.(6.27)). The crossing multipliers $\{\psi(a)\}$ and the quantity $q^{1/2}$ are the components of the eigenvector and the eigenvalue for the relation (3.13), respectively. The crossing point λ is defined by

(6.36) $$2\cos\lambda = q^{1/2}.$$

For graph-state models, the extended Markov property (5.47b) is easily proved and the characteristic function is given by

(6.37) $$H(u;\lambda) = q^{1/2} + \frac{\sin u}{\sin(\lambda - u)}.$$

The quantity $q^{1/2}$ and the crossing point λ are model-dependent. The τ-factors in the Markov property II are calculated as

(6.38) $$\tau = \frac{H(i\infty;\lambda)}{H(0;\lambda)} = 1 - q^{-1/2}e^{-i\lambda}, \qquad \bar{\tau} = \frac{H(-i\infty;\lambda)}{H(0;\lambda)} = 1 - q^{-1/2}e^{i\lambda},$$

The link polynomial satisfies the second order skein relation

(6.39) $$\alpha(L_+) = (1 - e^{-2i\lambda})e^{-i\lambda}\alpha(L_0) + e^{-4i\lambda}\alpha(L_-).$$

6.3. A, B, C, D IRF models

We apply the general theory to the IRF models related to the affine Lie algebras (for notation, see §3.3). The Boltzmann weights of the $B_m^{(1)}, C_m^{(1)}$ and $D_m^{(1)}$ models satisfy the basic relations (5.1)–(5.5). The $A_{m-1}^{(1)}$ model satisfies the basic relations (5.1)–(5.3) and (5.5) but does not the crossing symmetry (5.4). The function $\rho(u)$ is defined as

(6.40a)
$$\rho(u) = \frac{\theta_1(\omega - u)}{\theta_1(\omega)}, \qquad \text{for } A_{m-1}^{(1)},$$

(6.40b)
$$\rho(u) = \frac{\theta_1(\lambda - u)\theta_1(\omega - u)}{\theta_1(\lambda)\theta_1(\omega)}, \qquad \text{for } B_m^{(1)}, C_m^{(1)} \text{ and } D_m^{(1)}.$$

The crossing multiplier $\psi(\vec{d})$ is defined as

(6.41a) $$\psi(\vec{d}) = \prod_{1\le i<j\le m} \theta_1(d_{ij}), \quad \text{for } A^{(1)}_{m-1},$$

(6.41b) $$= \epsilon(\vec{d}) \prod_{\kappa=1}^{m} \theta_1(d_\kappa) \prod_{1\le i<j\le m} \theta_1(d_{ij})\theta_1(d_{i\,-j}),$$

$$\text{for } B^{(1)}_m, C^{(1)}_m \text{ and } D^{(1)}_m,$$

with $\epsilon(\vec{d})$ being a sign factor satisfying $\epsilon(\vec{d}+\vec{\mu})/\epsilon(\vec{d}) = \sigma$. For the $B^{(1)}_m$ model $\sigma = 1$ and $\lambda = (2m-1)\omega/2$, for the $C^{(1)}_m$ model $\sigma = -1$ and $\lambda = (m+1)\omega$, and for the $D^{(1)}_m$ model $\sigma = 1$ and $\lambda = (m-1)\omega$. The quantity $g_{d\mu}$ defined in (3.32) is given by

(6.42) $$g_{d\mu} = \frac{\psi(\vec{d}+\vec{\mu})}{\psi(\vec{d})}.$$

Representation of the braid group is obtained by the formula (5.11). The weight matrix elements $\sigma(\vec{a},\vec{b},\vec{c},\vec{d};+)$ of the braid operator G_i are in the following forms.

1) $A^{(1)}_{m-1}$ IRF model[63]

(6.43)
$$\sigma(\vec{a}+\vec{\mu},\vec{d}+2\vec{\mu},\vec{d}+\vec{\mu},\vec{d};+) = 1,$$
$$\sigma(\vec{d}+\vec{\mu},\vec{d}+\vec{\mu}+\vec{\nu},\vec{d}+\vec{\nu},\vec{d};+) = -\gamma\left(\frac{\sin(d_{\mu\nu}+\omega)\sin(d_{\mu\nu}-\omega)}{\sin^2 d_{\mu\nu}}\right)^{\frac{1}{2}},$$
$$\sigma(\vec{d}+\vec{\mu},\vec{d}+\vec{\mu}+\vec{\nu},\vec{d}+\vec{\mu},\vec{d};+) = 1-\gamma\frac{\sin(d_{\mu\nu}+\omega)}{\sin d_{\mu\nu}},$$

where $\vec{\mu},\vec{\nu}\in\Sigma, \vec{\mu}\neq\vec{\nu}$ and

(6.44) $$\gamma = e^{-i\omega}.$$

The weights of the inverse operator G_i^{-1} are obtained by letting $\gamma\to\gamma^{-1}$ in (6.43). We find that the braid operator satisfies a quadratic reduction relation:

(6.45) $$(G_i - I)(G_i+\gamma^2 I) = 0.$$

Setting $t = \gamma^2$, we see that the braid operator G_i is a representation of the generators of the Hecke algebra. The irreducible representations of the Hecke algebra were obtained in different contexts [71].

2) $B_m^{(1)}, C_m^{(1)}, D_m^{(1)}$ IRF models [63]

(6.46)

$$\sigma(\vec{d}+\vec{\mu}, \vec{d}+2\vec{\mu}, \vec{d}+\vec{\mu}, \vec{d}; +) = 1, \quad \text{for } \vec{\mu} \neq 0,$$

$$\sigma(\vec{d}+\vec{\mu}, \vec{d}+\vec{\mu}+\vec{\nu}, \vec{d}+\vec{\mu}, \vec{d}; +) = 1 - \gamma\frac{\sin(d_{\mu\nu}+\omega)}{\sin d_{\mu\nu}}, \quad \text{for } \vec{\mu} \neq \pm\vec{\nu},$$

$$\sigma(\vec{d}+\vec{\mu}, \vec{d}+\vec{\mu}+\vec{\nu}, \vec{d}+\vec{\nu}, \vec{d}; +) = -\gamma\left(\frac{\sin(d_{\mu\nu}+\omega)\sin(d_{\mu\nu}-\omega)}{\sin^2 d_{\mu\nu}}\right)^{\frac{1}{2}},$$

$$\text{for } \vec{\mu} \neq \pm\vec{\nu},$$

$$\sigma(\vec{d}+\vec{\mu}, \vec{d}, \vec{d}+\vec{\nu}, \vec{d}; +) = -(1-\gamma\ell_\mu(d_\nu))((g^c_{d\mu}g^c_{d\nu})^{\frac{1}{2}} - \delta_{\mu\nu}), \quad \text{for } \vec{\mu} \neq 0,$$

$$\sigma(\vec{d}, \vec{d}, \vec{d}, \vec{d}; +) = -\gamma\frac{\sin\omega}{\sin 2\lambda}(1 - J^c_{do}),$$

where $\vec{\mu}, \vec{\nu} \in \Sigma$ and $\ell_\mu(d_\nu) = \sin(d_\mu + d_\nu + 2w)/\sin(d_\mu + d_\nu + w)$. The superscript c means that the value is at criticality: $p = 0$. The weights of the inverse operator G_i^{-1} are again obtained by letting $\gamma \to \gamma^{-1}$ in (6.46). We find that the braid operator G_i satisfies a cubic reduction relation:

$$(G_i - \beta I)(G_i - I)(G_i + \gamma^2 I) = 0, \tag{6.47}$$

$$\beta = \sigma e^{-i(2\lambda+\omega(1+\sigma))}, \quad \gamma = \exp(-i\omega). \tag{6.48}$$

The braid group representations constructed from $B_m^{(1)}$, $C_m^{(1)}$ and $D_m^{(1)}$ IRF models satisfy the defining relations of Birman-Wenzl-Murakami (BWM) algebra [72,73].

Constructions of the link polynomials are straightforward. Using the crossing multipliers we have the Markov trace (5.37) on the representation of the braid group. In [63], the extended Markov property is proved and the characteristic functions for the A, B, C, D-IRF models are calculated.

1) $A_{m-1}^{(1)}$ IRF model [63]

$$H(u;\lambda) = \frac{\sin(m\omega - u)}{\sin(\omega - u)}. \tag{6.49}$$

From (6.49) the τ and $\bar{\tau}$ factors are

$$\tau = e^{i(m-1)\omega} \cdot \frac{\sin\omega}{\sin(m\omega)}, \qquad \bar{\tau} = e^{-i(m-1)\omega} \cdot \frac{\sin\omega}{\sin(m\omega)}. \tag{6.50}$$

2) $B_m^{(1)}, C_m^{(1)}, D_m^{(1)}$ IRF models [63]

$$H(u;\lambda) = \frac{\sigma\sin(2\lambda - u)\sin(\sigma\omega + \lambda - u)}{\sin(\lambda - u)\sin(\omega - u)}. \tag{6.51}$$

From (6.51) the τ and $\bar{\tau}$ factors are

$$\tau = e^{i(2\lambda+\omega(\sigma-1))}\frac{\sin\lambda\sin\omega}{\sin 2\lambda\sin(\sigma\omega+\lambda)}, \tag{6.52a}$$

$$\bar{\tau} = e^{-i(2\lambda+\omega(\sigma-1))}\frac{\sin\lambda\sin\omega}{\sin 2\lambda\sin(\sigma\omega+\lambda)}. \tag{6.52b}$$

Using the reduction relations and the Markov traces, we obtain the skein relations.
1) $A_{m-1}^{(1)}$ IRF model

$$\alpha(L_+) = (1-t)t^{(m-1)/2}\alpha(L_0) + t^m\alpha(L_-), \tag{6.53}$$

where

$$t = \exp(-2i\omega). \tag{6.54}$$

The Alexander polynomial is obtained by the limit $m \to 0$, while $m = 2$ corresponds to the Jones polynomial.
2) $B_m^{(1)}, C_m^{(1)}, D_m^{(1)}$ IRF models

$$\begin{aligned}\alpha(L_{2+}) &= (1-t+\beta)e^{-i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_+)\\ &\quad + (t+\beta t-\beta)e^{-2i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_0)\\ &\quad - t\beta e^{-3i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_-),\end{aligned} \tag{6.55}$$

where

$$t = \exp(-2i\omega), \qquad \beta = \sigma\exp(-i(2\lambda+\omega(1+\sigma))). \tag{6.56}$$

By L_{2+}, L_+, L_0 and L_-, we have denoted links which have, at a particular intersection, the configuration represented by b_i^2, b_i^1, b_i^0 and b_i^{-1}, respectively.

These polynomials are one-variable link invariants for each fixed m. We note that m appears in the τ-factors and the ratio $\bar{\tau}/\tau$ is a function of m. We regard m as a continuous parameter which is independent of t. The link polynomial constructed from $A_{m-1}^{(1)}$ IRF model corresponds to the two-variable extension of the Jones polynomial [74,75]. The link polynomials from $B_m^{(1)}, C_m^{(1)}, D_m^{(1)}$ IRF models correspond to the Kauffman polynomial [76] which has a reduction relation $G_i^3 = (b + 1/a)G_i^2 - (1 + b/a)G_i + 1/a$ with

$$\begin{aligned} a &= \frac{\sqrt{-1}\cdot\gamma}{\beta}, \\ b &= \sqrt{-1}(\gamma - \gamma^{-1}). \end{aligned} \tag{6.57}$$

We thus have explicit realizations of the two-variable extension of the Jones polynomial and the Kauffman polynomial.

The Yang-Baxter operators for A type composite IRF models [77,43] can be constructed by an algebraic way [61,63]. We combine $(N-1)$ strings into a composite string. The braid operator for the composite model satisfies the N-th order reduction relation (6.16) with t defined by (6.54). The characteristic function of the model is given by

$$H(u;\lambda) = \prod_{n=1}^{N-1} \frac{\sin((m+n-1)\omega - u)}{\sin(n\omega - u)}. \tag{6.58}$$

Two-variable extension of the link polynomial will be discussed in chapter 7.

We may work with vertex models. The Boltzmann weights of the critical unrestricted $A_{m-1}^{(1)}, B_m^{(1)}, C_m^{(1)}$ and $D_m^{(1)}$ IRF models reduce to A_{m-1}, B_m, C_m and D_m type vertex models when we set $\vec{\rho} \to \vec{\rho} + \vec{\omega}_0$ and consider the limit $\omega_0 \to i\infty$. Since the characteristic function $H(u;\lambda)$ is independent of $\vec{\omega}_0$, the extended Markov property holds for the vertex model with the same characteristic function. Thus we obtain the braid group representations and the Markov traces for the vertex models. The link polynomials are equivalent both for the vertex and IRF models. They are also obtained from studying the vertex models [78] and q-deformation of universal enveloping algebras [79].

6.4. Birman's example

In this section we study an example of two links whose difference are not detected by the Jones polynomial but by the N=3 link polynomial [69].

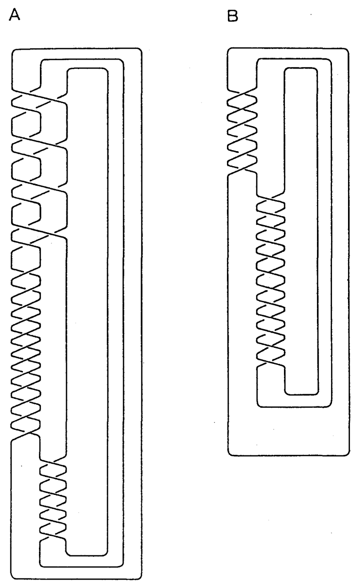


Fig. 34. $A = (b_1 b_2 b_1)^4 b_1^{-12} b_2^6$ and $B = b_1^{-6} b_2^{12}$.

We shall consider two braids:

$$
(6.59)\qquad \begin{aligned} A &= (b_1 b_2 b_1)^4 b_1^{-12} b_2^6, \\ B &= b_1^{-6} b_2^{12}. \end{aligned}
$$

Birman has proved [67] that the two closed-braids in (6.59) can not be distinguished by the Jones polynomial. In fact, if we use the N=2 case (the Jones polynomial) where

$$
(6.60)\qquad G_i^2 = (1-t)G_i + t, \qquad t = e^{2\lambda},
$$

we have

(6.61)
$$\begin{aligned}\alpha(A) &= \alpha(B)\\ &= (t^{18} - t^{17} + 2t^{16} - 3t^{15} + 4t^{14} - 5t^{13} + 6t^{12} - 6t^{11}\\ &+ 6t^{10} - 6t^{9} + 6t^{8} - 5t^{7} + 6t^{6} - 4t^{5} + 4t^{4} - 3t^{3}\\ &+ 2t^{2} - t + 1)/t^{3}\end{aligned}$$

On the other hand, $\alpha(A)$ and $\alpha(B)$ are different in the N=3 case. After some calculation, we have

(6.62)
$$\begin{aligned}\alpha(B) = (&t^{54} - t^{53} + 2t^{51} - 2t^{50} - t^{49} + 4t^{48} - 3t^{47} - 2t^{46} + 6t^{45}\\ &- 4t^{44} - 2t^{43} + 8t^{42} - 4t^{41} - 4t^{40} + 10t^{39} - 6t^{38} - 5t^{37}\\ &+ 12t^{36} - 7t^{35} - 5t^{34} + 12t^{33} - 7t^{32} - 5t^{31} + 13t^{30} - 7t^{29}\\ &- 5t^{28} + 12t^{27} - 7t^{26} - 5t^{25} + 12t^{24} - 6t^{23} - 6t^{22}\\ &+ 12t^{21} - 5t^{20} - 6t^{19} + 12t^{18} - 4t^{17} - 6t^{16} + 10t^{15} - 3t^{14}\\ &- 6t^{13} + 8t^{12} - 2t^{11} - 5t^{10} + 6t^{9} - t^{8} - 3t^{7}\\ &+ 4t^{6} - 2t^{4} + 2t^{3} - t + 1)/t^{12}.\end{aligned}$$

We may evaluate $\alpha(A)$ as follows. Using the generalized Alexander-Conway relation (6.23b), we have

$$\begin{aligned}\alpha(A) &= \alpha(\Delta^4 b_1^{-12} b_2^9) = \alpha(b_1 b_2^3 b_1 b_2^{-9} b_1 b_2^9)\\ &= a\alpha(b_1 b_2^2 b_1 b_2^{-9} b_1 b_2^{-9}) + b\alpha(b_1 b_2 b_1 b_2^{-9} b_1 b_2^9)\\ &+ c\alpha(b_1^2 b_2^{-9} b_1 b_2^9).\end{aligned} \tag{6.63a}$$

where

$$\begin{aligned}a &= t - t^3 + t^4,\\ b &= t^4 - t^5 + t^7,\\ c &= -t^8,\end{aligned} \tag{6.63b}$$

and Δ is a half twist of three strings: $\Delta = b_1 b_2 b_1$. Each term in (6.63) may be calculated as

(6.64)
$$\begin{aligned}&\alpha(b_1 b_2^2 b_1 b_2^{-9} b_1 b_2^9)\\ &= t^3(t^{15} - t^{14} - t^{13} + 2t^{12} - 2t^{11} - t^{10} + 3t^{9} - 2t^{8}\end{aligned}$$

$$- t^7 + 3t^6 - t^5 + 2t^3 + 1),$$

(6.65)

$$\begin{aligned}
&\alpha(b_1 b_2 b_1 b_2^{-9} b_1 b_2^9) \\
&= (t^{54} - t^{53} + 2t^{51} - 2t^{50} - t^{49} + 4t^{48} - 3t^{47} - 2t^{46} + 7t^{45} \\
&- 4t^{44} - 4t^{43} + 9t^{42} - 3t^{41} - 6t^{40} + 9t^{39} - 3t^{38} - 5t^{37} \\
&+ 9t^{36} - 6t^{35} - 4t^{34} + 14t^{33} - 10t^{32} - 7t^{31} + 18t^{30} \\
&- 9t^{29} - 10t^{28} + 19t^{27} - 7t^{26} - 12t^{25} + 19t^{24} - 6t^{23} \\
&- 10t^{22} + 15t^{21} - 4t^{20} - 7t^{19} + 11t^{18} - 5t^{17} - 4t^{16} \\
&+ 11t^{15} - 7t^{14} - 5t^{13} + 11t^{12} - 4t^{11} - 6t^{10} + 8t^9 - t^8 \\
&- 4t^7 + 4t^6 - 2t^4 + 2t^3 - t + 1)/t^{16},
\end{aligned}$$

(6.66)

$$\begin{aligned}
&\alpha(b_1^2 b_2^{-9} b_1 b_2^9) \\
&= (t^{57} - 2t^{56} + 4t^{54} - 5t^{53} - t^{52} + 9t^{51} - 8t^{50} - 5t^{49} \\
&+ 16t^{48} - 9t^{47} - 10t^{46} + 20t^{45} - 8t^{44} - 12t^{43} + 20t^{42} \\
&- 11t^{41} - 9t^{40} + 25t^{39} - 20t^{38} - 10t^{37} + 38t^{36} - 26t^{35} \\
&- 20t^{34} + 48t^{33} - 23t^{32} - 28t^{31} + 48t^{30} - 18t^{29} - 30t^{28} \\
&+ 45t^{27} - 14t^{26} - 27t^{25} + 40t^{24} - 14t^{23} - 20t^{22} + 36t^{21} \\
&- 18t^{20} - 16t^{19} + 36t^{18} - 18t^{17} - 20t^{16} + 34t^{15} - 10t^{14} \\
&- 21t^{13} + 24t^{12} - 3t^{11} - 14t^{10} + 14t^9 - 2t^8 - 8t^7 + 8t^6 \\
&- t^5 - 5t^4 + 4t^3 - 2t + 1)/t^{20}.
\end{aligned}$$

Therefore, we obtain

(6.67)

$$\begin{aligned}
&\alpha(A) \\
&= (t^{54} - t^{53} + 2t^{51} - 2t^{50} - t^{49} + 4t^{48} - 2t^{47} - 2t^{46} + 4t^{45} \\
&- 3t^{44} + 5t^{42} - 7t^{41} + t^{40} + 11t^{39} - 11t^{38} - 4t^{37} \\
&+ 16t^{36} - 8t^{35} - 9t^{34} + 14t^{33} - 4t^{32} - 8t^{31} + 11t^{30} \\
&- 3t^{29} - 7t^{28} + 12t^{27} - 5t^{26} - 7t^{25} + 13t^{24} - 7t^{23} - 6t^{22} \\
&+ 14t^{21} - 7t^{20} - 9t^{19} + 16t^{18} - 3t^{17} - 10t^{16} + 10t^{15} \\
&+ t^{14} - 5t^{13} + 4t^{12} - t^{11} - t^{10} + 4t^9 - 4t^8 - t^7 + 5t^6 \\
&- 2t^5 - 2t^4 + 3t^3 - t + 1)/t^{10}
\end{aligned}$$

We see that $\alpha(B)$ in (6.61) and $\alpha(A)$ in (6.67) are different.

This shows that the $N=3$ theory is more powerful than the $N=2$ theory. We expect that the larger N theory is more powerful and that a set of link polynomials for $N = 2, 3, 4, \cdots$ provides us a systematic method to classify knots and links.

§7. Two-Variable Extension

Soon after the discovery by Jones, many researchers [74,75] independently extended the Jones polynomial into a two-variable link polynomial. The two-variable Jones polynomial (sometimes, called HOMFLY polynomial after names of 6 researchers [74]) includes the Alexander polynomial and the Jones polynomial as special cases. We shall present a two-variable extension of the new link polynomials constructed from the N-state vertex models [80,81]. The extension also contains the composite models of A type models.

7.1. Composite string representation

Let us introduce the Hecke algebra $H(t,n)$. The following relations for operators $g_1, g_2, \cdots, g_{n-1}$ define the Hecke algebra

$$g_i g_j = g_j g_i, \quad |i-j| \geq 2, \tag{7.1a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \tag{7.1b}$$

$$g_i^2 = (1-t) g_i + t. \tag{7.2}$$

The braid operator constructed from the Yang-Baxter operator of the 6-vertex model (6.13) (the 8VSOS model, A type vertex and IRF models) satisfies the defining relations of the Hecke algebra (7.1) and (7.2).

We start from the generators $\{g_i\}$ of the Hecke algebra to construct composite braid operators $\{G_i\}$ [80,81]. We shall make use of only the defining relations of the Hecke algebra (7.1) and (7.2) for the operators $\{g_i\}$. This remark is important since representations of the Hecke algebra may be restricted by some additional conditions.

We form a *composite string* by combining $(N-1)$ strings and attaching a projector $P^{(N)}$ at each end.

This means the followings. We first make a multiplet of $(N-1)$ particles, and then extract a fully symmetric component by the projectors (the theory is generalized into any symmetry in §7.3). For the case of spin 1/2 particles, the fully symmetric component has the spin $(N-1)/2$. In this sense, the N-state vertex model describes the composition of spin 1/2 particles into spin $s = (N-1)/2$ particle. It was explicitly shown

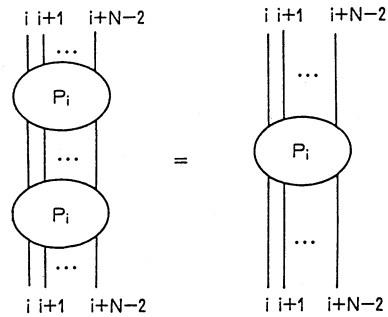


Fig. 35. A composite string. Two diagrams are equivalent since $P_i^2 = P_i$.

[80] that the symmetric projector $P_i^{(N)}$ for $N = 2, 3, 4, \cdots$, is derived through a recursion formula;

$$P_i^{(N)} = P_i^{(N-1)} h_{i+N-3}^{(N)} P_i^{(N-1)}, \quad P_i^{(2)} \equiv 1, \tag{7.3}$$

where

$$h_j^{(N)} = \frac{\tau_{N-2}}{\tau_{N-1}}\left(\frac{t^{N-2}}{\tau_{N-2}} + g_i\right), \quad \tau_m = 1 + t + t^2 + \cdots + t^{m-1}. \tag{7.4}$$

Recursion relation (7.3) gives, for instance,

$$P_i^{(4)} = \frac{1}{(1+t)(1+t+t^2)}\{t^3 + t^2(g_i + g_{i+1}) + t(g_i g_{i+1} + g_{i+1} g_i) + g_i g_{i+1} g_i\}. \tag{7.5}$$

It is remarked that in the limit $t \to 1$ the Hecke algebra reduces to the group algebra of the symmetric group. That is, the operators in the Hecke algebra become those in the symmetric group algebra when $t = 1$. Hereafter, we write P_i instead of $P_i^{(N)}$ when no confusion arises. The projector P_i defined by (7.3) satisfies the relations

(7.6a)
(i) $P_i^2 = P_i$,

(7.6b)
(ii) $P_i \Delta_i^2 = P_i$,

(7.6c)
(iii) $P_i(g_{i+N-2}g_{i+N-3}\cdots g_i) = (g_{i+N-2}g_{i+N-3}\cdots g_i)P_{i+1}$,
$P_i(g_{i+N-2}^{-1}g_{i+N-3}^{-1}\cdots g_i^{-1}) = (g_{i+N-2}^{-1}g_{i+N-3}^{-1}\cdots g_i^{-1})P_{i+1}$,

(7.6d)
(iv) $P_i g_j = P_i$, for $j = i, i+1, \cdots, i+N-3$,

where the operator Δ_i is a half-twist:

(7.7) $$\Delta_i = (g_i g_{i+1}\cdots g_{i+N-3})(g_i g_{i+1}\cdots g_{i+N-4})\cdots(g_i).$$

Let us denote "spin s" representation of B_n by $B_n^{[s]}$. We shall refer to $B_n^{[s]}$ as *composite string representation* or *spin s composite representation.* Using the composite string, we introduce generators $\{G_i; i = 1, 2, \cdots, n-1\}$ in $B_n^{[s]}$ as follows. For notational simplicity, we sometime use

(7.8) $$k \equiv N - 1 = 2s.$$

We prepare n sets of k strings and combine k strings into a composite string with projectors at both ends. The generator G_i is depicted in Fig.36.

To describe this, we introduce an operator $\bar{G}_i^{(N)}$ by

(7.9) $$\bar{G}_i^{(N)} = g_i^{(1)} g_i^{(2)} \cdots g_i^{(N-1)},$$

where

(7.10) $$g_i^{(\ell)} = g_{ik+1-\ell}\, g_{ik+2-\ell}\cdots g_{(i+1)k-\ell}, \qquad \ell = 1, 2, \cdots, N-1.$$

Then, the generator G_i of $B_n^{[s]}$ is expressed as

(7.11) $$G_i = P_{(i-1)k+1}^{(N)} P_{ik+1}^{(N)} \bar{G}_i^{(N)} P_{(i-1)k+1}^{(N)} P_{ik+1}^{(N)}.$$

Using (7.1) and (7.6), we can show that the generators $G_1, G_2, \cdots, G_{n-1}$ satisfy the defining relations of the braid group:

(7.12a) $$G_i G_j = G_j G_i, \qquad |i-j| \geq 2,$$

(7.12b) $$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1}.$$

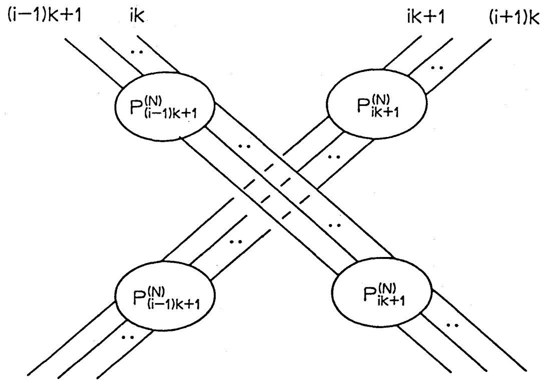


Fig. 36. Generator G_i of $B_n^{[s]}$. Note that $k = N - 1$.

In addition, the operators $\{G_i\}$ satisfy the N-th order reduction relations [80,81]

(7.12c) $$(G_i - c_1)(G_i - c_2)\cdots(G_i - c_N) = 0,$$

where for $r = 1, 2, \cdots, N$,

(7.12d) $$c_r = (-1)^{N+r} t^{\frac{1}{2}N(N-1) - \frac{1}{2}r(r-1)}.$$

All what we have used so far is the Hecke algebra. Using a representation of the Hecke algebra constructed from vertex or IRF models, we have obtained explicit realization of the "spin s" composite representation characterized by the relation (7.12). It is remarked that the matrix representation of the braid group satisfies additional relations. For example, the braid operator G_i constructed from (6.15) satisfies

$$1 - G_i - G_{i+1} + G_i G_{i+1} + G_{i+1} G_i - G_i G_{i+1} G_i = 0.$$

As we shall see in (7.34), the l.h.s. is the antisymmetrizer for 3-strings. For the braid operator constructed from A_{m-1} (vertex or IRF) model the antisymmetrizer for $m + 1$ strings vanishes, while the antisymmetrizers in the Hecke algebra do not.

We can construct the Yang-Baxter operator for the N-state vertex model, *composite Yang-Baxter operator*, from the Yang-Baxter operator of the 6-vertex model [14,61]. In the same way, we can also construct composite Yang-Baxter operator from A_{m-1} (vertex and IRF) models [61]. The discussion is parallel to the composite string representation given in this section.

7.2. Two-variable link invariants

Let us discuss two-variable extension of link polynomials constructed from the N-state vertex model and A_{m-1} type composite model [80,81]. For the Jones polynomial, which is the $N = 2s + 1 = 2$ polynomial, the two variable extension was made in a combinatorial way and in an algebraic way [74,75]. As the latter approach, Ocneanu introduced a trace function $\psi(\cdot)$ defined on $B_n^{[1/2]}$ [74,56]. The trace $\psi(\cdot)$ satisfies the normalization condition

$$\psi(I) = 1, \qquad I : \text{identity in } B_n^{[1/2]}, \tag{7.13}$$

and the Markov properties

$$\text{I.} \quad \psi(AB) = \psi(BA), \quad (A, B \in B_n^{[1/2]}). \tag{7.14a}$$

$$\text{II.} \quad \psi(Ag_n) = z\psi(A), \quad (A \in B_n^{[1/2]}, g_n \in B_{n+1}^{[1/2]}), \tag{7.14b,c}$$
$$\psi(Ag_n^{-1}) = \bar{z}\psi(A), \quad (A \in B_n^{[1/2]}, g_n \in B_{n+1}^{[1/2]}),$$

where

$$z = \psi(g_j), \tag{7.15a}$$
$$\bar{z} = \psi(g_j^{-1}), \quad \text{for all } j, \tag{7.15b}$$

Recall that the Jones polynomial has one parameter t. This z is another parameter independent of t. In this way, a pair (t, z) enters into the two-variable Jones polynomial. Other choices of two variables are possible. We introduce a variable ω by

$$\omega = \bar{z}/z = \psi(g_j^{-1})/\psi(g_j), \tag{7.16}$$

and change a set of variables (t, z) into (t, ω). From (7.2), (7.15) and (7.16), we have

$$z = \frac{1-t}{1-\omega t}, \qquad \bar{z} = \frac{\omega(1-t)}{1-\omega t}. \tag{7.17}$$

Generalizing the Ocneanu's approach, we obtain a two-variable extension of the link polynomials for $N = 2s + 1 \geq 3$ [80,81]. Let us define a trace (generalized Ocneanu's trace) $\psi^{[s]}(\cdot)$ by

$$\psi^{[s]}(A) = \frac{\psi(A)}{[\psi(P_j)]^n}, \quad A \in B_n^{[s]}. \tag{7.18}$$

Note that $A \in B_n^{[s]}$ consists of the generators in $B_{(N-1)n}^{[1/2]}$. We evaluate the following quantities;

$$(7.19)\qquad \begin{aligned} Z &= \psi^{[s]}(G_j) = \frac{\psi(G_j)}{[\psi(P_j)]^2} = \frac{z^{N-1}}{\psi(P_i)}, \\ \bar{Z} &= \psi^{[s]}(G_j^{-1}) = \frac{\psi(G_j^{-1})}{[\psi(P_j)]^2} = \frac{\bar{z}^{N-1}}{\psi(P_i)}. \end{aligned}$$

The value of $\psi(P_j^{(N)})$ can be calculated recursively by using (7.3). Substituting the result into (7.19), we get

$$(7.20)\qquad \begin{aligned} Z &= \frac{(1-t)(1-t^2)\cdots(1-t^{N-1})}{(1-\omega t)(1-\omega t^2)\cdots(1-\omega t^{N-1})}, \\ \bar{Z} &= \frac{\omega^{N-1}(1-t)(1-t^2)\cdots(1-t^{N-1})}{(1-\omega t)(1-\omega t^2)\cdots(1-\omega t^{N-1})}. \end{aligned}$$

We can show [80,81] that the generalized Ocneanu's trace $\psi^{[s]}(\cdot)$ satisfies the Markov properties:

$$(7.21)\qquad \text{I.}\quad \psi^{[s]}(AB) = \psi^{[s]}(BA), \quad (A, B \in B_n^{[s]}),$$

$$(7.22)\qquad \text{II.}\quad \psi^{[s]}(AG_n) = Z\psi^{[s]}(A), \quad (A \in B_n^{[s]}, G_n \in B_{n+1}^{[s]}),$$

$$\psi^{[s]}(AG_n^{-1}) = \bar{Z}\psi^{[s]}(A), \quad (A \in B_n^{[s]}, G_n \in B_{n+1}^{[s]}).$$

By using the generalized Ocneanu's trace, two-variable link polynomial $\alpha_\omega^{[s]}(\cdot)$ is expressed as

$$(7.23)\qquad \alpha_\omega^{[s]}(A) = (\bar{Z}Z)^{-(n-1)/2}\left(\frac{\bar{Z}}{Z}\right)^{e(A)/2}\psi^{[s]}(A), \quad (A \in B_n^{[s]}),$$

where $e(A)$ is the exponent sum of the generators in $B_n^{[s]}$. In fact, we can prove that $\alpha_\omega^{[s]}(\cdot)$ is invariant under the Markov moves:

(7.24a)

$$\text{I.}\quad \alpha_\omega^{[s]}(AB) = \alpha_\omega^{[s]}(BA), \qquad (A, B \in B_n^{[s]}),$$

(7.24b)

$$\text{II.}\quad \alpha_\omega^{[s]}(AG_n) = \alpha_\omega^{[s]}(AG_n^{-1}) = \alpha_\omega^{[s]}(A), \quad (A \in B_n^{[s]}, G_n \in B_{n+1}^{[s]}).$$

Corresponding to the N-th order reduction relation (7.12c), the two-variable link invariants satisfies the generalized skein relation such as

$$\alpha(L_{2+}) = \omega^{\frac{1}{2}}(1-t)\alpha(L_+) + \omega t\alpha(L_0), \quad \text{for } N=2, \tag{7.25}$$

$$\begin{aligned}\alpha(L_{3+}) &= \omega(1-t^2+t^3)\alpha(L_{2+}) + \omega^2(t^2-t^3+t^5)\alpha(L_+)- \\ &- \omega^3 t^5 \alpha(L_0), \quad \text{for } N=3,\end{aligned} \tag{7.26}$$

$$\begin{aligned}\alpha(L_{4+}) &= \omega^{\frac{3}{2}}(1-t^3+t^5-t^6)\alpha(L_{3+}) \\ &+ \omega^3(t^3-t^5+t^6+t^8-t^9+t^{11}) \times \alpha(L_{2+}) \\ &+ \omega^{\frac{9}{2}}(-t^8+t^9-t^{11}+t^{14})\alpha(L_+) \\ &- \omega^6 t^{14}\alpha(L_0), \quad \text{for } N=4.\end{aligned} \tag{7.27}$$

We have comments: (1) The braid operator constructed from the composite Yang-Baxter operator for A_{m-1} type model corresponds to the composite string representation. (2) For fixed m the link polynomial from the composite A_{m-1} model is one variable restriction of $\alpha_\omega^{[s]}(\cdot)$ with $\omega = t^{m-1}$. If we consider m as a continuous parameter independent of t, then the link polynomial from the composite A_{m-1} model is equivalent to $\alpha_\omega^{[s]}(\cdot)$. (3) When we set $\omega = t$ the two-variable link polynomial $\alpha_\omega^{[s]}(\cdot)$ reduces to the new link polynomial for the N-state vertex model. (4) While $\alpha_\omega^{[1/2]}(\cdot)$ is the two-variable Jones polynomial, $\alpha_\omega^{[s]}(\cdot)$ for $s = 1, 3/2, 2, \cdots$, are new [80, 81, 82]. It is interesting to notice that the $N = 3$ two-variable polynomial is different from the Kauffman polynomial [76]. (5) Since the $N = 2$ case contains the Alexander polynomial as the limit $\omega \to 1/t$, the two-variable link polynomial $\alpha_\omega^{[s]}(\cdot)$ may also be considered as the generalizations of the Alexander polynomial.

7.3. Further development of the composite string representation

We discuss a generalization of the composite string representation. By combining $(N-1)$ strings into a composite string and projecting it to symmetric sub-space, we have made the composite string representation $B_n^{[s]}$ from the braid group $B_{n(N-1)}^{[1/2]}$. We have defined the Markov trace (generalized Ocneanu's trace) on $B_n^{[s]}$ and obtained the two-variable link invariants. In the construction of the Markov trace, a remarkable fact is that the projector is compatible with the Markov property II. We may ask other possibilities of forming the composite strings. The answer is:

The choice of the projector is not unique. Let us denote the symmetry of the projector by λ. Various types of projectors, $(P_i^\lambda)^2 = P_i^\lambda$, with different symmetries are possible as far as the following relation holds [80];

$$P_i^\lambda \Delta_i^2 = \alpha_\lambda P_i^\lambda, \quad \alpha_\lambda \text{ :constant,} \tag{7.28}$$

which indicates that the projector is an eigenvector of the full twist Δ_i^2 with eigenvalue α_λ. It is remarked that the relation (7.28) assures the Markov property II of the trace defined by (7.18). Moreover, we find that the relation (7.28) is the sufficient and necessary condition for the projector to be compatible with the Markov property II.

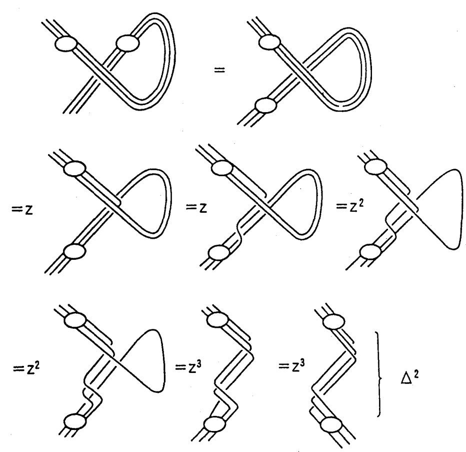


Fig. 37. Graphical proof for the Markov property II. Circles denote the projectors P_i^λ When the deformation is substituted into the trace $\psi^{[\lambda]}(\cdot)$, we obtain the Markov property II. To simplify the diagram, the case of three strings is shown.

The proof is depicted in Figure 37 [80].

Let us formulate the generalization. The generator G_i of $B_n^{[\lambda]}$ is expressed as (cf.(7.11))

$$G_i = P_{(i-1)k+1}^\lambda P_{ik+1}^\lambda \bar{G}_i^{(N)} P_{(i-1)k+1}^\lambda P_{ik+1}^\lambda . \tag{7.29}$$

We define a trace (generalized Ocneanu's trace) $\psi^{[\lambda]}(\cdot)$ by

$$\psi^{[\lambda]}(A) = \frac{\psi(A)}{[\psi(P_j^{\lambda})]^n}, \quad A \in B_n^{[\lambda]}. \tag{7.30}$$

The Z and $\bar{Z}$ factors are calculated as

$$\begin{aligned} Z_\lambda &= \psi^{[\lambda]}(G_j) = \frac{\psi(Gj)}{[\psi(P_j^{\lambda})]^2} = \frac{\alpha_\lambda z^{N-1}}{\psi(P_i^{\lambda})}, \\ \bar{Z}_\lambda &= \psi^{[\lambda]}(G_j^{-1}) = \frac{\psi(G_j^{-1})}{[\psi(P_j^{\lambda})]^2} = \frac{\alpha_\lambda \bar{z}^{N-1}}{\psi(P_i^{\lambda})}. \end{aligned} \tag{7.31}$$

In terms of the generalized Ocneanu's trace (7.30), two-variable link polynomial $\alpha_\omega^{[\lambda]}(\cdot)$ is expressed as

$$\alpha_\omega^{[\lambda]}(A) = (\bar{Z}_\lambda Z_\lambda)^{-(n-1)/2}(\frac{\bar{Z}_\lambda}{Z_\lambda})^{e(A)/2}\psi^{[\lambda]}(A), \quad (A \in B_n^{[\lambda]}), \tag{7.32}$$

where $e(A)$ is the exponent sum of the generators in $B_n^{[\lambda]}$.

Let us discuss construction of the projectors. For a generic value of the parameter t, the Hecke algebra is isomorphic to the symmetric group algebra [40,56]. As mentioned before, in the limit $t \to 1$, the defining relations of the Hecke algebra (7.1) and (7.2) reduce to those for the symmetric group. Therefore the projectors in (7.28) at $t = 1$ become the Young operators in the theory of the symmetric group. The symmetries of the Young operators are registered by the Young diagrams (or the Dynkin coefficients). Thus we construct the projector P_i^{λ} in (7.28) corresponding to the Young diagram of the symmetric group. We have chosen the symmetrizer in the sense that it satisfies $P_i^{(+)} g_j = +P_i^{(+)}$, $j = i, i+1, \cdots, i+N-3$, at $t = 1$. We may also choose the anti-symmetrizer satisfying $P_i^{(-)} g_j = -P_i^{(-)}$, $j = i, i+1, \cdots, i+N-3$ at $t = 1$. However, $P_i^{(+)}$ and $P_i^{(-)}$ give the equivalent results since they are transformed into each other by

$$t \to t^{-1} \quad \text{and} \quad g_i \to -t^{-1} g_i. \tag{7.33}$$

More generally, we may have the projectors with mixed symmetries [80]. For example, in the $N = 4$ case, we can construct the following projectors:

(7.34)

$$P_i^s = \frac{1}{(1+t)(1+t+t^2)}\{t^3 + t^2(g_i + g_{i+1}) + t(g_i g_{i+1} + g_{i+1} g_i) + g_i g_{i+1} g_i\},$$

$$P_i^{ms} = \{2t^{3/2} + t^{1/2}(1 - t - t^{1/2})(g_i + g_{i+1}) + (-1 + t - t^{1/2})(g_i g_{i+1} + g_{i+1} g_i) + 2g_i g_{i+1} g_i\}/2t^{1/2}(1+t+t^2),$$

$$P_i^{ma} = \{2t^{3/2} + t^{1/2}(1 - t + t^{1/2})(g_i + g_{i+1}) + (1 - t - t^{1/2})(g_i g_{i+1} + g_{i+1} g_i) - 2g_i g_{i+1} g_i\}/2t^{1/2}(1+t+t^2),$$

$$P_i^a = \frac{1}{(1+t)(1+t+t^2)}\{1 - (g_i + g_{i+1}) + (g_i g_{i+1} + g_{i+1} g_i) - g_i g_{i+1} g_i\}.$$

Here, P_i^s is the symmetrizer for the Young diagram with one row, and P_i^a the antisymmetrizer with one column. P_i^{ms} and P_i^{ma} the projectors for the mixed symmetry. By the transformation (7.33), P_i^s (P_i^{ms}) is related to P_i^a (P_i^{ma}). The projectors satisfy a relation

$$P^s + P^a + P^{ms} + P^{ma} = 1. \tag{7.35}$$

It is instructive to observe that the projectors in (7.34) reduce to the Young operators of the symmetric group:

$$\begin{aligned} P_i^s &= \frac{1}{6}(1 + s_i + s_{i+1} + s_i s_{i+1} + s_{i+1} s_i + s_i s_{i+1} s_i), \\ P_i^{ms} &= \frac{1}{6}(2 - s_i - s_{i+1} - s_i s_{i+1} - s_{i+1} s_i + 2 s_i s_{i+1} s_i) \\ P_i^{ma} &= \frac{1}{6}(2 + s_i + s_{i+1} - s_i s_{i+1} - s_{i+1} s_i - 2 s_i s_{i+1} s_i) \\ P_i^a &= \frac{1}{6}(1 - s_i - s_{i+1} + s_i s_{i+1} + s_{i+1} s_i - s_i s_{i+1} s_i), \end{aligned} \tag{7.36}$$

where $\{s_i\}$ are generators of the symmetric group. Since the dimension of the irreducible representation corresponding to the mixed symmetry is not equal to 1, the choice of the projector for the mixed symmetry is not unique. For instance, we find that the following is also the projector for the mixed symmetry

$$\begin{aligned} P_i^m = \{&t^{3/2} + t^{1/2}(1-t)g_i - t g_{i+1} \\ &+ (-1 + t - t^{1/2}) g_i g_{i+1} + g_i g_{i+1} g_i\}/t^{1/2}(1+t+t^2). \end{aligned} \tag{7.37}$$

We have chosen the projectors P_i^{ms} and P_i^{ma} so as to satisfy the relation (7.35). The projectors in (7.34) satisfy the relations(7.28) as

$$\begin{aligned} P_i^s \Delta_i^2 &= P_i^s, \\ P_i^{ms} \Delta_i^2 &= t^3 P_i^{ms}, \\ P_i^{ma} \Delta_i^2 &= t^3 P_i^{ma}, \\ P_i^a \Delta_i^2 &= t^6 P_i^a. \end{aligned} \tag{7.38}$$

We have also other possibilities of forming the composite strings. Let us define a "decorated" generator G_i by

$$G_i = \alpha_{(i-1)k+1} \beta_{ik+1} \bar{G}_i^{(N)} \gamma_{ik+1} \delta_{(i-1)k+1}, \tag{7.39}$$

where $\bar{G}_i^{(N)}$ has been defined in (7.9). In order to satisfy the Markov move II, the "decorations" α_i, β_i, γ_i and δ_i should satisfy either of the following conditions

$$\alpha_i \beta_i \gamma_i \delta_i \Delta_i^2 = P_i, \tag{7.40a}$$

$$\alpha_i \beta_i \gamma_i \delta_i \Delta_i^2 = I_i. \tag{7.40b}$$

The former case corresponds to the theory in §7.1 and §7.2. In particular we have chosen $\alpha_i = \beta_i = \gamma_i = \delta_i = P_i$. The latter case (called parallel links or cablings) was studied [82] with a choice $\alpha_i = \Delta_i^{-2}$, $\beta_i = \gamma_i = \delta_i = I_i$ (more correctly for links, $\alpha_i \gamma_i = \beta_i \delta_i = \Delta_i^{-1}$). We note that the projectors [80] are important in theoretical and practical purposes in the knot theory.

§8. Related Topics

8.1. Braid-monoid algebra

Solvable models with the crossing symmetry offer the Temperley-Lieb algebra [61]. In the following we use the factorized S-matrices (vertex models). The discussion also goes parallel for IRF models [61,63]. From the crossing symmetry and the standard initial condition, we have

$$S_{j\ell}^{ik}(\lambda) = [\frac{r(i)r(\ell)}{r(j)r(k)}]^{\frac{1}{2}} S_{\bar{k}\bar{i}}^{j\ell}(0) = r(i)r(\ell)\delta_{\ell\bar{k}}\delta_{i\bar{j}}, \tag{8.1}$$

where we have used $r(\bar{j}) = 1/r(j)$. We define an operator U_i by

$$U_i = X_i(\lambda). \tag{8.2}$$

Using (8.1) in (5.8a), we have

$$U_i = \sum_{k,\ell,m,p} r(p)r(k)\delta_{p\bar{m}}\delta_{k\bar{\ell}} I^{(1)} \otimes \cdots \otimes e_{pk}^{(i)} \otimes e_{m\ell}^{(i+1)} \otimes \cdots I^{(n)}. \tag{8.3}$$

It is easy to show that the operator $\{U_i\}$ satisfies the Temperley-Lieb algebra (3.9) with $q^{1/2} = \sum_a r^2(a)$. Thus, the Yang-Baxter operators which have the crossing symmetry and the standard initial condition satisfy the Temperley-Lieb algebra at the crossing point λ .

Let us explain the above property by diagrams [61]. We choose the time direction upward. We assume that a particle going backward in time is the antiparticle advancing forward in time. The crossing symmetry corresponds to the 90°-rotation of the diagram and relates the "scattering" channel with the "crossing" channel.

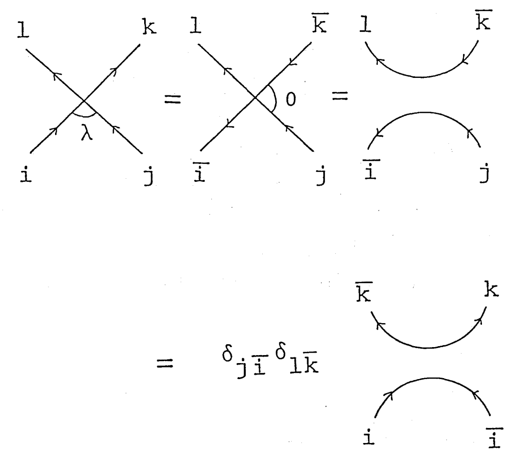


Fig. 38. Scattering with $u = \lambda$ can be considered as the scattering with $u = 0$ in the crossing channel and corresponds to the annihilation-creation diagram (monoid diagram).

The scattering with $u = \lambda$ is considered as $u = 0$ in the crossing channel. Therefore, this can be interepreted as an annihilation-creation process of a particle-antiparticle pair. We call the creation-annihilation diagram *monoid diagram*. The monoid diagram corresponds to the Temperley-Lieb operator.

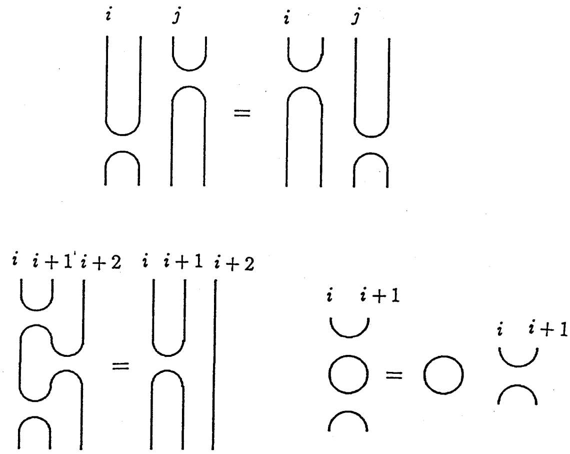


Fig. 39. Graphical representation of the Temperley-Lieb algebra.

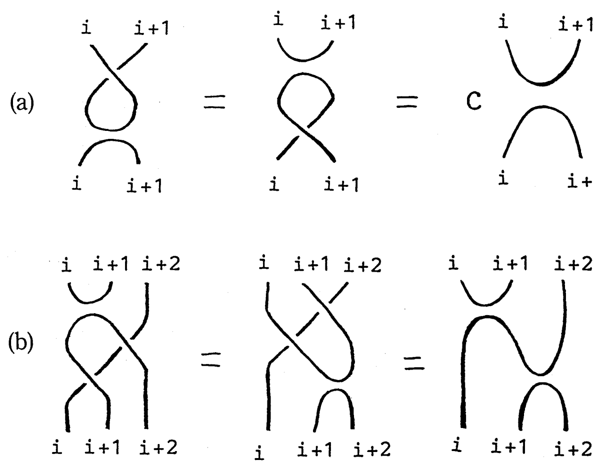


Fig. 40. Graphical representation of braid-monoid algebra. (a) relation (8.4d) (b) relation for (8.4g).

In fact, the defining relations of the Temperley-Lieb algebra can be interpreted in terms of the monoid diagrams (Fig.39).

We have derived braids and monoids from the Yang-Baxter operator. In terms of the factorized S-matrices, the S-matrices at $u = \infty$ (high energy limit) correspond to braid operators, while the S-matrices at $u = \lambda$ correspond to monoid operators [61]. From the models with the crossing symmetry and the Markov property II, we find that the braids and monoids operators satisfy an algebra which we call *braid-monoid algebra* [61,63]:

(8.4a) $$f(G_i) = 0,$$

(8.4b) $$E_i = g(G_i),$$

(8.4c) $$E_i^2 = q^{1/2} E_i,$$

(8.4d) $$G_i E_i = E_i G_i = c E_i,$$

(8.4e) $$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1},$$

(8.4f) $$E_i E_{i\pm 1} E_i = E_i,$$

(8.4g) $$E_i G_{i\pm 1} G_i = G_{i\pm 1} G_i E_{i\pm 1} = E_i E_{i\pm 1},$$

(8.4h) $$G_{i\pm 1} E_i G_{i\pm 1} = G_i^{-1} E_{i\pm 1} G_i^{-1},$$

(8.4i) $$G_{i\pm 1} E_i E_{i\pm 1} = G_i^{-1} E_{i\pm 1},$$

(8.4j) $$E_{i\pm 1} E_i G_{i\pm 1} = E_{i\pm 1} G_i^{-1},$$

(8.4k) $$E_i G_{i\pm 1} E_i = c^{-1} E_i,$$

(8.4l) $$G_i G_j = G_j G_i, \qquad \text{for } |i-j| \geq 2$$

(8.4m) $$E_i E_j = E_j E_i, \qquad \text{for } |i-j| \geq 2,$$

where $f(G_i), g(G_i), q$ and c are model-dependent.

In the derivation of the relations (8.4) we have chosen the normalization of the Yang-Baxter operator as follows. We use the Boltzmann weights which satisfy the standard initial condition (5.1) and the crossing symmetry (5.4). In terms of the Yang-Baxter operator (5.8), we define the monoid operator E_i by

(8.5) $$E_i = X_i(\lambda).$$

To construct the braid operator we normalize the Yang-Baxter operator as

(8.6) $$\tilde{X}_i(u) = X_i(u)/[\rho(\lambda - u)\rho(u)]^{1/2},$$

with the function $\rho(u)$ in (5.2). And we define the braid operator G_i by

(8.7) $$G_i^{\pm 1} = \lim_{u\to\infty} \tilde{X}_i(\pm u).$$

The quantities $q^{1/2}$ and c are related to the characteristic function of the extended Markov property as

$$q^{1/2} = (\tau\bar{\tau})^{-1/2} = H(0;\lambda), \tag{8.8}$$

$$c = \lim_{u\to\infty} \frac{H(\lambda-u;\lambda)\rho(\lambda-u)}{[\rho(u)\rho(\lambda-u)]^{1/2}}. \tag{8.9}$$

For the N-state vertex model [61], the model-dependent parts of (8.4) are

$$(G_i - c_1 I)(G_i - c_2 I)\cdots(G_i - c_N I) = 0, \tag{8.10a}$$

$$E_i = \frac{q^{1/2}}{(c_1-c_2)(c_1-c_3)\cdots(c_1-c_N)}(G_i - c_2 I)\times \times (G_i - c_3 I)\cdots(G_i - c_N I) \tag{8.10b}$$

where

$$c_r = (-i)^{N-1}(-1)^{r+1} t^{\frac{1}{4}(N^2-1)-\frac{1}{2}r(r-1)} \tag{8.10c}$$

$$c = (-i)^{N-1} t^{\frac{1}{4}(N^2-1)} \tag{8.10d}$$

$$q^{\frac{1}{2}} = \frac{\sinh N\lambda}{\sinh\lambda} = t^{\frac{(N-1)}{2}} + \cdots + t^{-\frac{N-1}{2}}, \tag{8.10e}$$

and for B, C, D IRF models [63], they are

$$(G_i - \gamma^{-1}I)(G_i + \gamma I)(G_i - \gamma^{-1}\beta I) = 0, \tag{8.11a}$$

$$E_i = \frac{\gamma}{\beta(\gamma - \gamma^{-1})}(G_i - \gamma^{-1}I)(G_i + \gamma I) \tag{8.11b}$$

where

$$c = \beta\gamma^{-1} = \sigma e^{-i(2\lambda + \sigma\omega)}, \tag{8.11c}$$

$$q^{1/2} = \frac{\sin 2\lambda \sin(\sigma\omega + \lambda)}{\sin\lambda \sin\omega}. \tag{8.11d}$$

For the $N = 3$ case and $B_m^{(1)}, C_m^{(1)}$ and $D_m^{(1)}$ IRF models, the braid-monoid algebra becomes BWM algebra with

$$\begin{aligned} \ell &= \sqrt{-1}t^{-2}, \\ m &= \sqrt{-1}(t - t^{-1}), \qquad \text{for the } N = 3 \text{ case,} \end{aligned} \tag{8.12}$$

$$
(8.13) \quad \begin{aligned} \ell &= \sqrt{-1}\gamma\beta^{-1}, \\ m &= \sqrt{-1}(\gamma - \gamma^{-1}), \quad \text{for } B_m^{(1)}, C_m^{(1)} \text{ and } D_m^{(1)} \text{ IRF models,} \end{aligned}
$$

where ℓ and m are constants in the algebra (actually, a and b in (6.57) are ℓ and m). BWM algebra was derived from the Kauffman polynomial [72,73].

Let us discuss the braid-monoid algebras from the viewpoint of the Reidemeister moves (see §4.3). The Reidemeister move I corresponds to the Markov property II. The Reidemeister moves II and III are generators of regular isotopy. The basic relations of the model correspond to the Reidemeister moves II and III. The relations (8.4e) $\sim$ (8.4j) implies the invariance of the elements of the braid-monoid algebra under the regular isotopy moves. The relations (8.4d) and (8.4k) correspond to the Reidemeister move I with a factor c .

8.2. Graphical formulation of the link polynomials

The link polynomials constructed from the exactly solvable models with the crossing symmetry can be formulated directly on the link diagrams. For a given link diagram we shall calculate the link polynomial by using the weights matrices and the crossing multipliers for the model. The key observations are that the monoid operator (8.1) can be divided into two parts, $r(i)\delta_{i\bar{j}}$ and $r(\ell)\delta_{\ell\bar{k}}$, which respectively correspond to the annihilation process and the creation process, and as discussed in the previous section the matrix elements of the braid monoid algebra are invariant under the Reidemeister moves. The monoid diagram and the weights for the creation and annihilation diagrams were first introduced for the bracket polynomial [46,83].

Let us explain the algorithm of the calculation. We consider a diagram $\hat{L}$ of an oriented link L. Each string has a direction. We denote the writhe of the link diagram $\hat{L}$ by $w(\hat{L})$. The link diagram is decomposed into the following elements; annihilation, creation, braid with $w = -1$, braid with $w = 1$ and line diagrams (Fig.41). To them, we assign weights $r(i)\delta_{i\bar{j}}, r(\ell)\delta_{\ell\bar{k}}, \sigma^{(+)}_{\ell k,ij}, \sigma^{(-)}_{\ell k,ij}, \delta_{jk}$, where the normalization of the weights $\sigma^{(\pm)}_{\ell k,ij}$ are defined by (8.6).

Considering the charge conservation condition, we take a summation over all possible state variables for the link diagram $\hat{L}$. We denote this sum by $\mathrm{Tr}(\hat{L})$, which is a regular isotopy invariant. We multiply the factor $c^{-w(\hat{L})}$, which is also a regular isotopy invariant, to $\mathrm{Tr}(\hat{L})$. Then

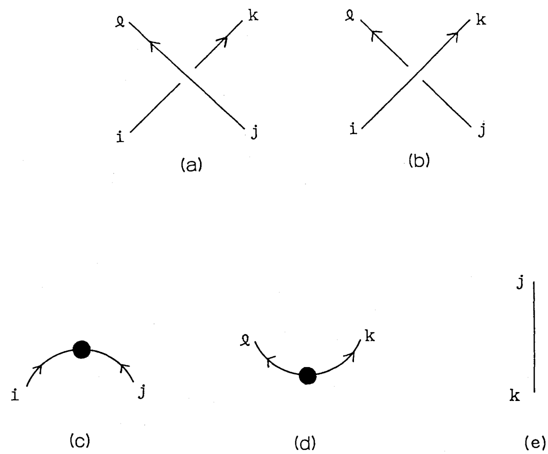


Fig. 41. (a) Braid diagram with $w = -1$, $\sigma^{(+)}_{\ell k,ij}$, (b) Braid diagram with $w = 1$, $\sigma^{(-)}_{\ell k,ij}$, (c) Annihilation diagram $r(i)\delta_{i\bar{j}}$, (d) Creation diagram $r(\ell)\delta_{\ell\bar{k}}$, (e) Line diagram δ_{jk} . Dots denote the crossing multipliers.

we find that a link polynomial is expressed as

$$\alpha(L) = c^{-w(\hat{L})}\mathrm{Tr}(\hat{L})/\mathrm{Tr}(\hat{K}_0), \tag{8.14}$$

where $\mathrm{Tr}(\hat{K}_0)$ is the sum for the trivial knot diagram $\hat{K}_0$ (a loop). Let us evaluate the sum for the trivial knot. Since the trivial knot consists of annihilation and creation diagrams, we get

$$\mathrm{Tr}(\hat{K}_0) = \sum_j r^2(j) = q^{1/2}. \tag{8.15}$$

This quantity $q^{1/2}$ appeared in the defining relations of the Temperley-Lieb algebra.

For the IRF models, the monoid diagram is decomposed into two parts and we assign the weights $[\psi(a)/\psi(b)]^{1/2}$ and $[\psi(c)/\psi(b)]^{1/2}$ to the annihilation and creation diagrams, respectively (Fig.42). To the braid diagrams with $w = -1$ and $w = 1$, we give the weights $\sigma(a,b,c,d;+)$ and $\sigma(a,b,c,d;-)$, whose normalizations are defined by (8.6). (In the limit $\omega_0 \to \pm i\infty$, the weights for unrestricted IRF models reduce to those for the vertex models by the Wu-Kadanoff-Wegner transformation.)

In the same way as the vertex models we get

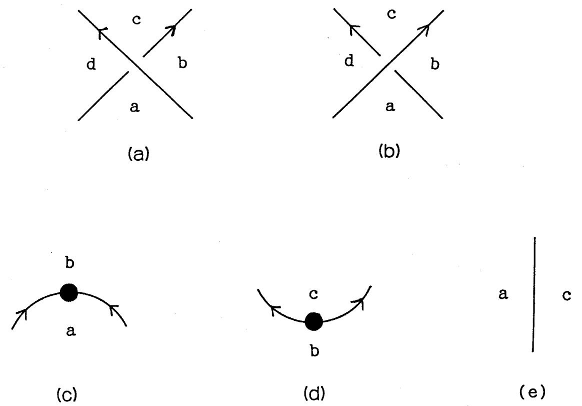


Fig. 42. (a) Braid diagram with $w=-1$ $\sigma(a,b,c,d;+)$, (b) Braid diagram with $w=1$ $\sigma(a,b,c,d;-)$, (c) Annihilation diagram $(\psi(a)/\psi(b))^{1/2}$, (d) Creation diagram $(\psi(c)/\psi(b))^{1/2}$, (e) Line diagram $\delta_{c\sim a}$ ($\delta_{c\sim a}$ is 1 when $c\sim a$ and 0 otherwise). Dots denote the crossing multipliers.

$$\text{Tr}(\hat{K}_0)=\sum_{b\sim a}\frac{\psi(b)}{\psi(a)}=q^{1/2}. \tag{8.16}$$

Let us give some examples. First we use the $N=2$ vertex model where the weight matrices are

$$\begin{gathered}\sigma^{(+)}_{1/2\quad 1/2,1/2\quad 1/2}=\sigma^{(+)}_{-1/2\quad -1/2,-1/2\quad -1/2}=it^{-1/4}.\\ \begin{pmatrix}\sigma^{(+)}_{1/2\quad -1/2,1/2\quad -1/2} & \sigma^{(+)}_{1/2\quad -1/2,-1/2\quad 1/2}\\ \sigma^{(+)}_{-1/2\quad 1/2,1/2\quad -1/2} & \sigma^{(+)}_{-1/2\quad 1/2,-1/2\quad 1/2}\end{pmatrix}\\ =\begin{pmatrix}0 & -it^{1/4}\\ -it^{1/4} & it^{-1/4}(1-t)\end{pmatrix},\end{gathered} \tag{8.17}$$

and the crossing multipliers $\{r(j)\}$ and the constant c are

$$r(1/2)=t^{-1/4},\quad r(-1/2)=t^{1/4},\quad c=-it^{3/4}. \tag{8.18}$$

The graphical calculation with (8.17) and (8.18) is nothing but the bracket polynomial for the Jones polynomial [46].

Next we make use of the $N = 3$ vertex model. For an illustration we calculate the link polynomial for a link $L = 2_1^2$ [45] depicted in Fig.43.

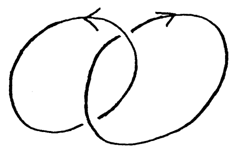


Fig. 43. A link L which is named as 2_1^2.

The weight matrices are

$$
\begin{aligned}
&\sigma^{(+)}_{11,11} = \sigma^{(+)}_{-1-1,-1-1} = -t^{-1} \\
&\begin{pmatrix} \sigma^{(+)}_{10,10} & \sigma^{(+)}_{10,01} \\ \sigma^{(+)}_{01,10} & \sigma^{(+)}_{01,01} \end{pmatrix} = \begin{pmatrix} \sigma^{(+)}_{0-1,0-1} & \sigma^{(+)}_{0-1,-10} \\ \sigma^{(+)}_{-10,0-1} & \sigma^{(+)}_{-10,-10} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & t - t^{-1} \end{pmatrix}, \\
&\begin{pmatrix} \sigma^{(+)}_{1-1,1-1} & \sigma^{(+)}_{1-1,00} & \sigma^{(+)}_{1-1,-11} \\ \sigma^{(+)}_{00,1-1} & \sigma^{(+)}_{00,00} & \sigma^{(+)}_{00,-11} \\ \sigma^{(+)}_{-11,1-1} & \sigma^{(+)}_{-11,00} & \sigma^{(+)}_{-11,-11} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -t \\ 0 & -1 & t^{1/2}(t^{-1} - t) \\ -t & t^{1/2}(t^{-1} - t) & (1 - t^2)(1 - t^-1) \end{pmatrix}.
\end{aligned}
\tag{8.19}
$$

The crossing multipliers and the constant c are

$$
\begin{aligned}
&r(1) = t^{-1/2}, r(0) = 1, r(-1) = t^{1/2}, \\
&c = -t^2.
\end{aligned}
\tag{8.20}
$$

Under the charge conservation condition, there are 19 configurations for the link diagram. For these configurations we have

$$
\mathrm{Tr}(\hat{L}) = t^{-4} + t^{-3} + t^{-2} + t^{-1} + 1 + t + t^2 + t^3 + t^4. \tag{8.21}
$$

Since the writhe for the link $\hat{L}$ is -2 and $\mathrm{Tr}(\hat{K}_0) = t + 1 + t^{-1}$, we have from (8.14)

$$
\alpha(L) = t(1 + t^3 + t^6). \tag{8.22}
$$

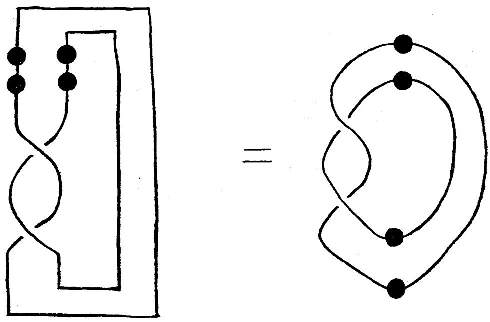


Fig. 44. Markov trace showing the equivalence of graphical and algebraic formulations.

The result agrees with what was obtained from the $N = 3$ link polynomial [69].

We have remarks. First, the graphical formulation applied to closed braids yields the Markov trace. Thus, the two approaches, graphical formulation and construction of the Markov trace, are equivalent. This fact gives an intuitive explanation for the braid-plat equivalence [84]. Second, $\mathrm{Tr}(\hat{L})$ does not depend on the orientation of link. The orientation is taken into account only through the writhe. Third, the link diagrams for knots and links are the "Feynmann diagrams' for the high energy process: At the lowest point there occurs a pair creation and at the highest point a pair annihilation.

§9. Concluding Remarks

In this article, we have presented a general theory to construct link polynomials from exactly solvable models in statistical mechanics. The main results are the following:

(1) The Yang-Baxter algebra is a generalization of the braid group. A representation of the braid group is obtained from the Yang-Baxter operators $\{X_i(u)\}$ by sending the spectral parameter u to infinity. The existence of the well-defined limit requires the model to be at criticality, having trigonometric/hyperbolic parametrization.

(2) The Markov trace on the braid group representation is constructed by using the crossing multipliers of the model. The extended Markov property, which holds for finite spectral parameter, assures the existence of the Markov trace.

Two variable link invariants are constructed by using the Hecke algebra. The Hecke algebra is an extension of the symmetric group algebra, and the projectors in the Hecke algebra are considered as the generalized Young operators. From this algebraic structure, we have made two-variable extension of the link polynomials.

The critical condition for the model is vital in the theory. The representation of the braid group and also the extended Markov property are obtained only at criticality. It will be extremely interesting to apply the knot theory based on solvable models at criticality to other subjects in mathematics and physics.

Acknowledgements

We would like to express our sincere thanks to Professor K. Murasugi, Professor C. Itzykson, Professor D. Olive, Professor R.J. Baxter, Professor A.B. Zamolodchikov, Professor I.B. Frenkel, Professor H.J. de Vega and Professor J. Cardy for continuous encouragement and interest in our work.

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