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KdV-Type Equations and W-Algebras

A. A. Belavin

There exists a remarkable connection between the conformal field theory and the theory of KdV-type equations. In this talk I would like to remind you about and to attract your attention to this connection.

The symmetry generators in conformal field theory (CFT) form an associative infinite-dimensional algebra which always contains the Virasoro algebra as a subalgebra. Generators of the Virasoro algebra L_n are Fourier components of the Energy-momentum tensor $T(z) = \sum L_n/z^{n+2}$ with well-known commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{1}$$

However the Virasoro algebra is only a part of the conformal algebra (that is the algebra of symmetries of CFT) in the general case. We know many examples of more general algebras. I mean for example the Neveu-Schwarz algebra, WZW-algebra, parafermionic algebra of F-Z and so on.

Classification of all possible kinds of conformal algebras is the first step to classification of all possible types of CFT. This problem is a very important one and its total investigation is not obtained yet. A very interesting class of such algebras was considered firstly by A. B. Zamolodchikov. It is the so called W-algebras.

The first example of a W-algebra is the Virasoro algebra itself. The next one is formed from the generators L_n and the set of new generators $W_n^{(3)}$ which are components of a spin 3 field $W^{(3)}(z) = \sum W_n^{(3)}/z^{n+3}$.

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Commutation relations in this case have the following form:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

$$[L_n, W^{(3)}_m] = (2n-m)W^{(3)}_{n+m},$$

$$\begin{aligned}[W^{(3)}_n, W^{(3)}_m] &= \frac{16}{22+5c}(n-m)\Lambda_{n+m} \\ &+(n-m)\left(\frac{(n+m+2)(n+m+3)}{15} - \frac{(n+2)(m+3)}{6}\right)L_{n+m} \\ &+ \frac{c}{360}(n^2-4)(n^2-1)n\delta_{n+m,0}\end{aligned} \tag{2}$$

and $$\Lambda_n \overset{\text{def}}{=} \sum_k : L_k L_{n-k} : + \frac{1}{5}x_n L_n,$$

$$x_{2l} = (l+1)(1-l); \quad x_{2l+1} = (2+l)(1-l).$$

Let us note that this algebra is not a Lie-type algebra because of the quadratic terms in the right hand side of $[W, W]$ commutators. However these commutation relations satisfy very rigid limitations following from the associativity of the algebra: for example

$$[W_n, [W_m, W_k]] + \text{ permutations } = 0. \tag{3}$$

In the general case W-algebra is generated by several fields $W^{(j)}(z) \equiv \sum z^{-j-n}W^{(j)}_n$ with higher spins $3 \le j \le N$. Commutation relations have the following form

(4)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

$$[L_n, W^{(j)}_m] = \big((j-1)n - m\big)W^{(j)}_{n+m},$$

$$[W^{(j_1)}_n, W^{(j_2)}_m] = \sum_{\substack{k_1,\cdots,k_s \\ k_1+\cdots+k_s=n+m}} \sum_{\substack{i_1,\cdots,i_s \\ i_1+\cdots+i_s \le j_1+j_2-1}} \alpha^{(j_1,j_2)\{k\}}_{(n,m)\{i\}} \times : W^{(i_1)}_{k_1} \cdot W^{(i_2)}_{k_2} \cdots W^{(i_s)}_{k_s} : .$$

Here in r.h.s. $W^{(2)}_k$ denotes L_k and $W^{(0)}_k$ denotes 1. Again the rigid limitations on constants $\alpha^{\{i\}\ \{k\}}_{(n,m)\{i\}}$ and the set $\{j\}$ arise from the necessity to be in accordance with associativity for commutation relations. The question arises: how many W-algebras exist ? The other problems include constructions of representations of these algebras and the related models of CFT.

The following important fact may be useful to answer these questions. It turns out that W-algebra can be considered as quantum version of a certain structure in the theory of KdV-type equations. The KdV equation has the form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}. \tag{5}$$

This equation is Hamiltonian. This means that it can be written in the Poisson bracket form

$$\frac{\partial u}{\partial t} = \{H, u\}.$$

The Poisson bracket has the form

$$\begin{aligned} \{u(x), u(y)\} &= 2u\delta'(x-y) + u'\delta(x-y) + \delta'''(x-y), \\ &= \Big(u' + 2u\frac{d}{dx} + \frac{d^3}{dx^3}\Big)\delta(x-y), \end{aligned} \tag{6}$$

and

$$H = \int u^2 dx.$$

If one takes a Fourier decomposition of $u(x)$ (provided that $u(x)$ is periodic) then this Poisson bracket is converted to the familiar Virasoro form

$$\{L_n, L_m\} = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{7}$$

where Poisson brackets substitute commutators; so this Poisson bracket can be thought as classical limit of the Virasoro algebra.

The KdV-equation is known to be connected with the second order differential operator $\mathcal{L} = \frac{d^2}{dx^2} + u(x,t)$ and it admits the Lax-form representation, that is it can be written as

$$\frac{\partial \mathcal{L}}{\partial t} = [\mathcal{L}, A], \tag{8}$$

where A is a certain 3rd order differential operator.

The KdV-equation admits a generalization to a system of equations for several functions $u_1(t,x), \cdots, u_N(t,x)$. This generalization is obtained by taking an n–th order differential operator instead of $\mathcal{L}$. In these cases the equations are also Hamiltonian and Poisson brackets are of the form

$$\{u_i(x), u_j(y)\} = \widehat{\mathcal{D}}_{ij}\delta(x-y), \tag{9}$$

where $\widehat{\mathcal{D}}_{ij}$ is a certain differential operator whose coefficients are polynomials in $u_i(x)$ and their derivatives. For $n=3$ one has two functions and their Fourier components L_n and W_n have the following Poisson bracket.

(10)
$$\begin{aligned}\{L_n, L_m\} &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0},\\ \{L_n, W^{(3)}_m\} &= (2n-m)W^{(3)}_{n+m},\\ \{W^{(3)}_n, W^{(3)}_m\} &= \frac{16}{5c}(n-m)\sum L_k L_{n-k}\\ &+(n-m)\left(\frac{(n+m+2)(n+m+3)}{15} - \frac{(n+2)(m+3)}{6}\right)L_{n+m}\\ &+\frac{c}{360}(n^2-4)(n^2-1)n\delta_{n+m,0}.\end{aligned}$$

These relations provide a semiclassical limit of the commutation relations for the spin 3 W-algebra as it was noticed by Feigin and Hovanova. That is (2)$\Rightarrow$(10) if we substitute

(11)
$$\begin{aligned}[\,,\,] &\to \hbar\{\,\},\\ L_n &\to \hbar^{-1}L_n,\\ W_n &\to \hbar^{-1}W_n,\\ c &\to \hbar^{-1}c,\end{aligned}$$

and

(12)
$$\hbar \to 0.$$

This example as well as the former one leads us to the natural conjecture that there exists a similar relation in the general case. If we would have classification and explicit construction of the Hamiltonian structure for general KdV-type equations, we could try to classify and construct W-algebras by means of a kind of quantization of these classical objects. In fact the explicit construction of general KdV-type equations exists. It was achieved by Drinfeld and Sokolov in the year 1980 by means of the Hamiltonian reduction of a natural Hamiltonian structure connected with affine Lie algebras.

Let me present the simplest example of the Drinfeld-Sokolov construction. Consider the $\widehat{\mathfrak{sl}}(2)$-affine algebra. In this case three functions $V^+(x), V^-(x), h(x)$ together with an extra variable k are considered as coordinates on the phase space. The Poisson brackets are given by the following formulas (Berezin-Kirillov-Kostant Poisson bracket):

$$
\begin{aligned}
\{V^+(x), V^-(y)\} &= h(x)\delta(x-y) + k\delta'(x-y),\\
\{h(x), V^\pm(y)\} &= \pm V^\pm(x)\delta(x-y),\\
\{h(x), h(y)\} &= \frac{k}{2}\delta'(x-y).
\end{aligned}
\tag{13}
$$

The set of variables V^+, V^-, h and k can be associated with the linear matrix differential operator

$$
\widehat{\mathcal{L}} = k\frac{d}{dx} + \begin{pmatrix} h & V^+ \\ V^- & -h \end{pmatrix}.
\tag{14}
$$

Let us now consider the group of off diagonal upper triangular matrices G which acts on the phase space by the Gauge transformations

$$
\widehat{\mathcal{L}} \to G^{-1}\widehat{\mathcal{L}}G
\tag{15}
$$

and

$$
G = \begin{pmatrix} 1 & \alpha(x) \\ 0 & 1 \end{pmatrix}.
$$

This action is Hamiltonian. It means that any infinitesimal ($\alpha(x) \ll 1$) variation of coordinates has the Poisson bracket form, that is

$$
\begin{aligned}
\delta V^+ &= \{V^+, H_\alpha\},\\
\delta V^- &= \{V^-, H_\alpha\},\\
\delta h &= \{h, H_\alpha\},
\end{aligned}
\tag{16}
$$

with an appropriate generating function H_α. Namely,

$$
H_\alpha(x) = \int V^-(x)\alpha(x)dx.
\tag{17}
$$

Let us now perform the Hamiltonian reduction with respect to this gauge group. As usual the Hamiltonian reduction consists of two steps. The first step is imposing constraint, namely we must fix the Hamiltonian generators of the gauge group. In our case we put V^- to be equal to one. The specific combination of the phase coordinates $u(x) = V^+(x) + h(x)^2 + h'(x)$ can be shown to commute with the generator of the gauge group V^- under the imposed condition:

$$
\{V^-(x), u(y)\} = 0.
\tag{18}
$$

These combinations are complete set of independent invariants of the gauge group.

The second step of the Hamiltonian reduction amounts to introducing the new phase space and the new Poisson bracket. This phase space consists of orbits of our gauge group in the surface (subspace) defined by the constraints of the first step of reduction. The invariant $u(x) = V^+ + h^2 + h'$ can be then taken as coordinates of the new phase space. It can be verified by the explicit calculation that the Poisson bracket between $u(x)$ and $u(y)$ is expressed through $u(x)$ and its derivatives only and has the form

$$\{u(x), u(y)\} = \left(u' + 2u\frac{d}{dx} + \frac{d^3}{dx^3}\right)\delta(x-y). \tag{19}$$

So the Hamiltonian reduction of the $\widehat{\mathfrak{sl}}(2)$-affine algebra leads one to the Hamiltonian structure of the KdV-equation. Analogously, starting from a Kac-Moody algebra one obtains a Hamiltonian structure of the GKdV equation. Let us turn now to the quantum version of the Drinfeld-Sokolov construction which will allow us to construct W-algebras starting from an affine algebra.

First let us consider again the case of the $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$ algebra. In this case there are three sets of generators I_n^+, I_n^-, I_n^0 and a central charge k. The commutators are well-known. Let us denote as $\mathcal{U}_\mathfrak{g}$ the enveloping algebra of $\widehat{\mathfrak{sl}}(2)$. $\mathcal{U}_\mathfrak{g}$ contains an ideal N, which is by definition

$$n \in N \Longleftrightarrow n = \sum_k x_k(I_k^- - \delta_{k,0}),$$

where $x_k \in \mathcal{U}_\mathfrak{g}$.

Let us find those elements $x \in \mathcal{U}_\mathfrak{g}$ which satisfy the equations

$$[x, I_m^-] \in N \qquad \text{for any } m. \tag{20}$$

The totality of such elements will be denoted by V. This is the quantum version of the first step of the Hamiltonian reduction of the Drinfeld-Sokolov construction.

To make the second step we define W as the factor-space of V/N by the ideal N.

$$W = V/N. \tag{21}$$

This means that W consists of equivalence classes of elements of V:

$$x_1 \sim x_2 \Longleftrightarrow x_1 - x_2 \in N, \tag{22}$$

where $x_1, x_2 \in V$. The following statements hold:

a) W is an associative algebra.

b) Moreover W is the Virasoro algebra with

$$c = \frac{3k}{k+2} - 6k. \tag{23}$$

c) The Virasoro generators can be represented by the expressions

$$L_n = \frac{1}{k+2} \sum I_k^a I_{n-k}^a + nI_n^0 - \frac{k}{4}\delta_{n,0}. \tag{24}$$

Let us turn to the general case. Let $\widehat{\mathfrak{g}}$ be a simple Lie algebra and $X^\alpha, Y^\alpha, H^\alpha$ be its Chevalley generators. Let $\widehat{\mathfrak{g}}$ be the corresponding affine algebra with the generators $X_n^\alpha, Y_n^\alpha, H_n^\alpha$ and the central charge k. Denote the enveloping algebra of $\widehat{\mathfrak{g}}$ by $\mathcal{U}_{\widehat{\mathfrak{g}}}$. $\widehat{\mathfrak{g}}$ contains a subalgebra $\widehat{\mathfrak{n}}$ generated by Y_n^α's.

$\widehat{\mathfrak{n}}$ admits a nontrivial one-dimensional representation $a \to \chi(a)$ defined by the formula $\chi(Y_n^\alpha) = \delta_{n,0}$. Now we can define the ideal $N \subset \mathcal{U}_\mathfrak{g}$ as a set of elements of the form

$$\sum_{a \in \widehat{\mathfrak{n}}} x_a \big(a - \chi(a)\big), \tag{25}$$

where $x_a \in \mathcal{U}_\mathfrak{g}$. Now we are in a position to construct a W-algebra. As the first step we define the space V as the space consisting of the elements whose commutators with $a \in \widehat{\mathfrak{n}}$ belong to N. The second step is the factorization of V by N. Denote $W = V/N$. Then again:

a) W is an associative algebra.

b) W contains the Virasoro algebra as a subalgebra whose generators can be chosen in the form

$$L_n = L_n^{\mathrm{SUG}} + n \sum (\alpha_i, \rho) H_n^i. \tag{26}$$

Here α_i's are the simple roots of $\widehat{\mathfrak{g}}$ and $\rho = \frac{1}{2}\sum \beta$, where β are the positive roots.

Our quantum version of the Hamiltonian reduction has the following BRST analogue proposed by B. L. Feigin. I explain it for the case $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$ again. Let us extend $\widehat{\mathfrak{sl}}(2)$ by the fermion ghosts ψ_n and $\overline{\psi}_n$ whose commutation relations are

$$\begin{aligned} [\psi_n, \psi_m]_+ &= [\overline{\psi}_n, \overline{\psi}_m]_+ = 0, \\ [\psi_n, \overline{\psi}_m]_+ &= \delta_{n+m,0}, \\ [I_n^a, \psi_m] &= 0. \end{aligned} \tag{27}$$

The new algebra will be called $\mathcal{A}$.

Consider the vector space M generated by $I_n^+, I_n^-, I_n^0, \psi_n, \overline{\psi}_n$ from the vacuum vector v which satisfies the following equations:

$$\begin{aligned} I_n^a v &= 0 \qquad n > 0, \\ \psi_n v &= 0 \qquad n > 0, \\ \overline{\psi}_n v &= 0 \qquad n \le 0, \\ I_0^+ v &= 0, \\ I_0^0 v &= lv. \end{aligned} \tag{28}$$

We shall suppose that M is factorised over its submodules and thus irreducible.

Consider the operator

$$Q \overset{\text{def}}{=} \sum_{n \in \mathbf{Z}} (I_n - \delta_{n,0}) \psi_{-n}. \tag{29}$$

Q acts on the module M and

$$Q^2 = 0. \tag{30}$$

By means of Q we can introduce the corresponding cohomologies. Let $\mathcal{U}_{\mathcal{A}}$ be the enveloping algebra $\mathcal{A}$. These elements $x \in \mathcal{U}_{\mathcal{A}}$ for which

$$[x, Q] = 0 \tag{31}$$

obviously act on the cohomologies of Q. It is easy to verify by a direct calculation that

$$[L_n, Q] = 0$$

for

$$L_n \overset{\text{def}}{=} L_n^{\text{SUG}} + nI_n^0 - \frac{k}{4}\delta_{n,0} + L_n^{\text{gh}},$$

where

$$L_n^{\text{gh}} = \sum \left(\frac{n}{2} + k\right) : \psi_k \overline{\psi}_{n-k} : .$$

The L_n generate the Virasoro algebra with the central charge

$$c = \frac{3k}{k+2} - 6k - 2 \tag{32}$$

and

$$\Delta = \frac{l(l+1)}{k+2} - \frac{k}{4}.$$

The module M contains a null-vector if

$$l = \frac{m}{2}(k+2) - \frac{n}{2} \tag{33}$$

for integers m and n. In this case the corresponding weights of the Virasoro algebra Δ and c satisfy the known relation of Kac:

$$\Delta = \frac{c-1}{24} + (\alpha_+ \frac{m}{2} + \alpha_- \frac{n}{2})^2, \tag{34}$$

where

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

and

$$c = \frac{3k}{k+2} - 6k - 2.$$

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L. D. Landau Institute for Theoretical Physics
Moscow, U. S. S. R.