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# Yang-Mills Connections and Einstein-Hermitian Metrics

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## Introduction

The Yang-Mills connections, which were originally introduced by theoretical physicist, have been studied from several points of view by many mathematicians. Twistor method of Penrose allowed Atiyah, Drinfeld, Hitchin and Manin [ADHM] to describe self-dual Yang-Mills connections on $S^4$ in terms of holomorphic vector bundles on $\mathbb{P}^3(\mathbb{C})$ (see also [Wa], [AHS]). Through the study of Yang-Mills equations Uhlenbeck [Uh1], [Uh2] obtained *a priori* estimates, the compactness theorem and the removable singularities theorem for Yang-Mills connections on 4-manifolds. Using her results, Sedlacek [Se] observed the bubble-off phenomena which occur when the curvatures of a sequence of connections become concentrated. The existence of self-dual connections on 4-manifolds which are not covered by twistor methods were given by Taubes [Ta1], [Ta2]. These results led Donaldson to discover the link between the moduli space of self-dual connections and the topology of base manifolds. Then he obtained surprising applications to 4-dimensional topology [Do1], [Do3].

In algebraic geometry it is important to construct good moduli spaces for holomorphic vector bundles over algebraic varieties. For algebraic curves, this was done by Mumford [Mu] using his notion of stability for vector bundles over curves. This stability was further generalized for higher dimensional algebraic varieties by Takemoto, Bogomolov, Gieseker, Maruyama and Barth. Algebraic constructions of moduli spaces of holomorphic vector bundles in general cases were carried out by Maruyama [Ma]. On the other hand Atiyah and Bott [AB] pointed out the relation between stable bundles and Yang-Mills connections on Riemann surface, which in particular allows them to calculate the cohomology of the moduli spaces by the Morse theory. Inspired by Bogomolov's work on the inequality for Chern numbers, S.Kobayashi introduced a differential geometric notion of Einstein-Hermitian connections for arbitrary Kähler manifolds and showed that the existence of such connections on indecomposable holomorphic vector bundles implies the stability of the vector bundles [Kb1]. (It was also shown by Lübke [Lü].) It was conjectured that the converse is true. For algebraic curves it had already been known to be true by Narasimhan and Seshadri [NS]. When the base manifolds are projective algebraic surfaces, it was obtained by Donaldson [Do2]. General cases are solved by Donaldson [Do4], Uhlenbeck and Yau [UY], which enables us to study moduli spaces of vector bundles by differential geometric methods. The first named author [It3] showed that when the base manifold is a Kähler surface, the moduli space of Einstein-Hermitian connections has a natural Kähler structure. It was generalized to higher dimensions by Koiso [Ko], Kim [Ki], Lübke and Okonek [LO] and Cho [Ch].

Twistor theory was studied intensively by many mathematicians. For instance it was generalized by Salamon to quaternionic Kähler cases (see Bérard-Bergery and Ochiai [BO] for a further generalization). Then he pointed out the correspondence of certain connections (we call them $B_2$-connections according to Nitta [Ni1]) on quaternionic Kähler manifolds with holomorphic vector bundles on their twistor spaces. When the twister space is Kähler, Nitta [Ni1], [Ni2] proved that the pull-back of $B_2$-connections to the twister space became Einstein-Hermitian connections, and he studied the relationship between the moduli space of $B_2$-connections on a quaternionic Kähler manifold and that of Einstein-Hermitian connections on the twistor space.

The purpose of this paper is to give a survey of basic results on the Yang-Mills connections and Einstein-Hermitian metrics. In Chapter 1, we fix notation and give some preliminary results. In Chapter 2, we study local structures of various moduli spaces. In particular, they are smooth manifolds with natural Riemannian structure and if the base space are Kähler manifolds, they inherit the natural complex structure. Chapter 3 is devoted to review of the twistor theory on self-dual manifolds and quaternionic Kähler manifolds. In Chapter 4, we state Uhlenbeck's *a priori* estimates for Yang-Mills connections [Uh3] which in particular enable us to prove a compactness theorem of moduli spaces for Yang-Mills connections over base manifolds of arbitrary dimensions (see also [Na1]). As another application we have a simplified proof of the removable singularity theorem [Uh1] for Yang-Mills connection. In Chapter 5, two existence theorems for Yang-Mills connections will be given. One is Taubes' implicit function theorem [Ta1], [Ta2] and the other is the existence result by Uhlenbeck and Yau [UY] on Einstein-Hermitian metrics over Kähler manifolds. Finally in Chapter 6 we shall study the universal connections over moduli spaces and explain the relation between the determinant line bundles and the Donaldson's functional.

We do not cover all the topics on Yang-Mills connections and Einstein-Hermitian metrics. For instance, we cannot refer to the application of Yang-Mills connections to 4-dimensional differential topology.

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# Chapter 1. Notation and Preliminaries

## §1.1. Yang-Mills functional and Yang-Mills connections

In this section we prepare the notation and the preliminaries for Yang-Mills connections. For detail the reader should consult [FU], [La] (or Mogi-Itoh's book [MI] for the reader who reads Japanese).

Let $P$ be a principal $G$-bundle over an $n$-dimensional compact connected oriented smooth Riemannian manifold $(X, g)$, where the structure group $G$ is a compact Lie group. We denote by $E$ the associated vector bundle $P \times_\rho \mathbb{R}^r$ of rank $r$ for some faithful representation $\rho: G \to O(r)$. We introduce two fiber bundles $\mathrm{Aut}(P)$ and $\mathrm{Ad}(P)$ associated with $P$. The first one $\mathrm{Aut}(P)$ is the bundle of automorphisms of $P$, i.e.

$$\mathrm{Aut}(P) = P \times_\sigma G,$$

where $\sigma(s)(g) = sgs^{-1}$. A section of $\mathrm{Aut}(P)$ is identified with an automorphism of $P$ covering the identity map. The space of sections of $\mathrm{Aut}(P)$ is called the gauge group and denoted by $\mathcal{G}(P)$. The other bundle $\mathrm{Ad}(P)$ is the vector bundle associated with the adjoint representation $\mathrm{Ad}: G \to GL(\mathfrak{g})$, i.e.

$$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of $G$. We denote by $\Omega^k(\mathrm{Ad}(P))$ the vector space of all smooth $\mathrm{Ad}(P)$-valued $k$-forms (i.e. sections of $\Lambda^k(M) \otimes \mathrm{Ad}(P)$). We put a fiber metric $(\ ,\ )$ on $\mathrm{Ad}(P)$ by some Ad-invariant metric on $\mathfrak{g}$. This metric induces that on $\Lambda^k(M) \otimes \mathrm{Ad}(P)$ which we also denote by $(\ ,\ )$.

The bracket operation on the Lie algebra $\mathfrak{g}$ induces bilinear maps $[\cdot \wedge \cdot]: \Omega^k(\mathrm{Ad}(P)) \times \Omega^l(\mathrm{Ad}(P)) \to \Omega^{k+l}(\mathrm{Ad}(P))$ by

$$\begin{aligned}&[\varphi \wedge \psi](X_1, \cdots, X_{k+l}) \\ =&\frac{1}{k!\, l!} \sum_{\sigma \in \Sigma_{k+l}} \mathrm{sgn}(\sigma)[\varphi(X_{\sigma(1)}, \cdots, X_{\sigma(k)}), \psi(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)})].\end{aligned}$$

We shall identify the connection $A$ on $P$ with the covariant differential operator $\nabla_A$ which acts on the space of all sections of $E$ and $\mathrm{Ad}(P)$. Combining $\nabla_A$ with the Levi-Civita connection of $(X, g)$, we have an induced connection (also denoted by $\nabla_A$) on $\otimes^k T^*X \otimes \mathrm{Ad}(P)$. Skew-symmetrizing $\nabla_A$, we define the covariant exterior differential operators $d_A: \Omega^k(\mathrm{Ad}(P)) \to \Omega^{k+1}(\mathrm{Ad}(P))$ by

$$(d_A\varphi)(X_1, \cdots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (\nabla_A \varphi)(X_i; X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}).$$

We denote by $d_A^*\colon \Omega^{k+1}(\mathrm{Ad}(P)) \to \Omega^k(\mathrm{Ad}(P))$ the formal adjoint operator of $d_A$. It can be written as

$$d_A^*(\varphi)(X_1,\cdots,X_{k-1}) = -\sum_{i=1}^{n}(\nabla_A\varphi)(E_i; E_i, X_1,\cdots,X_{k-1}),$$

where $(E_1,\cdots,E_n)$ is a local oriented orthonormal frame field of $TX$. Note that it does not hold $d_A \circ d_A = 0$, in general. In fact, we have

$$d_A \circ d_A\varphi = [R_A \wedge \varphi],$$

where $R_A$ is the curvature form of $A$ which is an element in $\Omega^2(\mathrm{Ad}(P))$.

Let $\mathcal{A}(P)$ denote the space of all $G$-connections on $P$. Fixing a connection $A_0$, we can identify $\mathcal{A}(P)$ with $\Omega^1(\mathrm{Ad}(P))$, since the difference of two connections is an $\mathrm{Ad}(P)$-valued 1-form. Thus we can define Sobolev norms on $\mathcal{A}(P)$. The completion of $\mathcal{A}(P)$ with respect to such a norm becomes a Banach space (see [Do1], [FU], [La] for detail).

We define a Yang-Mills functional $\mathcal{YM}\colon \mathcal{A}(P) \to \mathbb{R}^+$ by

$$\mathcal{YM}(A) = \frac{1}{2}\int_X |R_A|^2 dV_g \qquad \text{for } A \in \mathcal{A}(P),$$

where $|R_A|^2 = (R_A, R_A)$.

**Definition 1.1.** A connection $A$ is called a *Yang-Mills connection* when $A$ is a critical point of the Yang-Mills functional $\mathcal{YM}$, namely for any smooth family $A_t$ $(-\varepsilon < t < \varepsilon)$ with $A_0 = A$, it holds

$$\frac{d}{dt}\mathcal{YM}(A_t)\Big|_{t=0} = 0.$$

Examples of Yang-Mills connections are given in the next two sections.

**Proposition 1.2.** *A connection $A$ is a Yang-Mills connection if and only if*

$$d_A^* R_A = 0.$$

*Proof.* Let $A_t$ $(-\varepsilon < t < \varepsilon)$ be a smooth family of connections with $A_0 = A$. Then we have

$$\frac{d}{dt}R_{A_t}\Big|_{t=0} = d_A\alpha,$$

where

$$\alpha = \left.\frac{d}{dt}A_t\right|_{t=0}.$$

Hence

$$\left.\frac{d}{dt}\mathcal{YM}(A_t)\right|_{t=0} = \int_X (d_A\alpha, R_A)dV_g = \int_X (\alpha, d_A^* R_A)dV_g.$$

Since an arbitrary $\mathrm{Ad}(P)$-valued 1-form can be taken for $\alpha$, this implies the assertion. Q.E.D.

Since we have the Bianchi identity

$$d_A R_A = 0$$

for any connection $A$, the connection $A$ is a Yang-Mills connection if and only if it satisfies

$$\Delta_A R_A = 0,$$

where $\Delta_A = d_A d_A^* + d_A^* d_A$.

Even if $X$ is not a compact manifold or the Yang-Mills functional is not necessarily finite, we say a connection $A$ is a Yang-Mills connection if it satisfies $d_A^* R_A = 0$.

The gauge group $\mathcal{G}(P)$ acts on the space $\mathcal{A}(P)$ from the right by the pull back of the connection. As a covariant differential operator acting on the section of $E$, the action is written as

$$\nabla_{g(A)}\xi = g^{-1}\nabla_A(g\xi) \qquad \text{for } \xi \in \Gamma(E).$$

The group $\mathcal{G}(P)$, more precisely the completion of $\mathcal{G}(P)$ by a suitable Sobolev norm, has a structure of an infinite dimensional Lie group with its Lie algebra $\Omega^0(\mathrm{Ad}(P))$ and the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is differentiable (see [Do1], [FU], [La]). Since the curvature of $g(A)$ is given by

$$R_{g(A)} = \mathrm{Ad}(g^{-1})R_A,$$

we have $|R_A| = |R_{g(A)}|$. This shows that the Yang-Mills functional is invariant under the action of the gauge group $\mathcal{G}(P)$. In particular, the space $\mathcal{A}_{YM}(P)$ of all Yang-Mills connections on $P$ preserved by the action of $\mathcal{G}(P)$. (It can be proved directly that $d_A^* R_A = 0$ if and only if $d_{g(A)}^* R_{g(A)} = 0$.) The quotient space of $\mathcal{A}_{YM}(P)$ by the action of the gauge group $\mathcal{G}(P)$ is called by *the moduli space of Yang-Mills connections on* $P$ and denoted by $\mathcal{M}_{YM}(P)$.

## §1.2. Self-dual connections

When the dimension of the base manifold is equal to 4, there is certain reduction, called self-duality equation, of the Yang-Mills equation. Its solutions are called self-dual connections and intensively studied by Donaldson [Do1], [Do3] to give applications to 4-dimensional differential topology. As a differential equation with respect to the connection form, the Yang-Mills equation is the second order. On the other hand the self-duality equation is the first order and relatively simple. This is similar to the relation between harmonic maps and conformal maps between Riemann surfaces.

Let $*: \Lambda^2(X) \to \Lambda^2(X)$ be the Hodge star operator of a 4-dimensional compact oriented Riemannian manifold $(X, g)$. In dimension 4, we have

$$** = \mathrm{Id},$$

so there is an eigenspace decomposition

$$\Lambda^2(X) = \Lambda^+ \oplus \Lambda^-,$$

where $\Lambda^\pm$ is the $\pm 1$-eigenspace of $*$. If we take a local oriented orthonormal frame field $(E_1, E_2, E_3, E_4)$ of $T^*X$, then

$$\Lambda^\pm = \mathrm{span}\{E_1 \wedge E_2 \pm E_3 \wedge E_4, E_1 \wedge E_3 \pm E_4 \wedge E_2, E_1 \wedge E_4 \pm E_2 \wedge E_3\}.$$

The Hodge star operator $*$ extends naturally to an operator on $\Lambda^2(X) \otimes \mathrm{Ad}(P)$ by $* \otimes \mathrm{Id}$. For simplicity we denote it also by $*$. Just as above we have an orthogonal decomposition

$$\Lambda^2(X) \otimes \mathrm{Ad}(P) = (\Lambda^+ \otimes \mathrm{Ad}(P)) \oplus (\Lambda^- \otimes \mathrm{Ad}(P)),$$

and this induces a decomposition of the space of $\mathrm{Ad}(P)$-valued 2-forms

$$\Omega^2(\mathrm{Ad}(P)) = \Omega^+(\mathrm{Ad}(P)) \oplus \Omega^-(\mathrm{Ad}(P)).$$

In particular, the curvature form $R_A$ decomposes into two components:

$$R_A = R_A^+ + R_A^-. \tag{1.3}$$

**Definition 1.4.** A connection $A$ is called *self-dual* if $R_A^- = 0$. Similarly $A$ is called *anti-self-dual* if $R_A^+ = 0$.

We denote by $\mathcal{M}_\pm(P)$ the moduli space of (anti-)self-dual connections on $P$ which is the quotient space of the space of (anti-)self-dual connections by the action of the gauge group. In Chapter 2 we shall show that an open subset of $\mathcal{M}_{\pm}(P)$ has a structure of a finite dimensional smooth manifold.

Since the decomposition (1.3) is orthogonal, it holds

$$|R_A|^2 = |R_A^+|^2 + |R_A^-|^2.$$

So we have

$$\mathcal{YM}(A) = \frac{1}{2}\int_X (|R_A^+|^2 + |R_A^-|^2) dV_g.$$

On the other hand by the Chern-Weil theory, it holds

$$2\pi^2 p_1(E) = \frac{1}{2}\int_X (|R_A^+|^2 - |R_A^-|^2) dV_g$$

where $p_1(E)$ is the first Pontrjagin number of the bundle $E$ which is a topological invariant, and independent of a connection $A$. Hence we have

**Proposition 1.5.** *On a compact oriented* 4*-dimensional Riemannian manifold* $(X, g)$, *(anti-)self-dual connections minimize the Yang-Mills functional* $\mathcal{YM}$. *In particular, they are Yang-Mills connections. If there exists at least one (anti-)self-dual connection on* $P$, *any other minimizer is also (anti-)self-dual (respectively).*

To give examples of self-dual connections we consider the $SO(3)$-bundle $\Lambda^+$ and the connection $\nabla$ induced from the Levi-Civita connection of the Riemannian manifold $(X, g)$. It is well-known that the curvature operator $R: \Lambda^2(X) \to \Lambda^2(X)$ can be written as a block matrix relative to the direct sum decomposition $\Lambda^2(X) = \Lambda^+ \oplus \Lambda^-$

$$R = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $B \in \mathrm{Hom}(\Lambda^+, \Lambda^-)$ and $A \in \mathrm{End}\,\Lambda^+$, $C \in \mathrm{End}\,\Lambda^-$ are self-adjoint. We have

$$\begin{aligned}
&\operatorname{tr} A = \operatorname{tr} C = \frac{1}{4}S, \qquad B = \mathrm{Ric} - \frac{1}{4}Sg \\
&A - \frac{1}{12}S\ \mathrm{Id} = W_+ \quad \text{(self-dual Weyl tensor)}, \\
&C - \frac{1}{12}S\ \mathrm{Id} = W_- \quad \text{(anti-self-dual Weyl tensor)},
\end{aligned}$$

where Ric is the Ricci curvature and $S$ is the scalar curvature. The curvature of $\Lambda^+$ is given by the first row of the block decomposition of the curvature tensor. Thus we have the following:

**Proposition 1.6** [AHS]. *A 4-dimensional Riemannian manifold $(X, g)$ is Einstein if and only if the connection $\nabla$ on $\Lambda^{\pm}$ induced from the Levi-Civita connection is (anti-)self-dual.*

At the end of this section we remark the special feature of the Yang-Mills functional in dimension 4. It is the conformal invariance, namely for $G$-principal bundles $P \to X$ and $Q \to Y$, if $\overline{f}: P \to Q$ is a bundle map such that the induced map $f: (X, g) \to (Y, h)$ is a conformal diffeomorphism, then

$$\mathcal{YM}(D) = \mathcal{YM}(\overline{f}^*(D)),$$

where $\overline{f}^*(D)$ is an induced connection by the map $\overline{f}$. As a corollary of the above we can say that $D$ is a Yang-Mills connection if and only if $\overline{f}^*(D)$ is a Yang-Mills connection. Moreover since the Hodge star operator $*$ is conformally invariant, $D$ is (anti-)self-dual if and only if $\overline{f}^*(D)$ is (anti-)self-dual.

## §1.3. Einstein-Hermitian connections

In this section we treat Einstein-Hermitian connections which are special solutions of Yang-Mills equation. For more details one can see S. Kobayashi's book [Kb2].

We assume that the manifold $(X, g)$ is Kähler with the Kähler form $\omega$ and the structure group $G$ of the bundle $P$ is the unitary group $U(r)$. We denote by $E$ the associated vector bundle and by $H$ the fiber metric on $E$. Let $m = n/2$ be the complex dimension of $X$. We have a decomposition of the exterior algebra of the cotangent bundle of $X$:

$$\Lambda^k_{\mathbb{C}}(X) = \sum_{p+q=k} \Lambda^{p,q}(X).$$

For the vector bundles $E$ and $\mathrm{Ad}(P)$ we have the similar decompositions

$$\Lambda^k_{\mathbb{C}}(E) = \sum_{p+q=k} \Lambda^{p,q}(E),$$

$$\Lambda^k_{\mathbb{C}}(\mathrm{Ad}(P)) = \sum_{p+q=k} \Lambda^{p,q}(\mathrm{Ad}(P)).$$

The exterior differential operator $d_A$ decomposes according to the above decomposition

$$d_A = \partial_A + \overline{\partial}_A : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E) \oplus \Omega^{p,q+1}(E).$$

Similarly we have a decomposition of the curvature tensor $R_A$

$$R_A = R_A^{2,0} + R_A^{1,1} + R_A^{0,2} \in \Omega^{2,0}(\mathrm{Ad}(P)) \oplus \Omega^{1,1}(\mathrm{Ad}(P)) \oplus \Omega^{0,2}(\mathrm{Ad}(P)).$$

Since $A$ preserves the fiber metric $H$, we have

$$R_A^{0,2} = -\overline{{}^t R_A^{2,0}}.$$

**Definition 1.7.** A connection $A$ is said to be an *integrable connection* if it satisfies

$$\overline{\partial}_A^2 = 0.$$

It is easy to see that $A$ is integrable if and only if $R_A^{2,0} = R_A^{0,2} = 0$. By the theorem of Newlander-Nirenberg (see [AHS]) a connection $A$ is integrable if and only if there is a holomorphic structure on $E$ such that $\overline{\partial}_A = \overline{\partial}$. The holomorphic structure means precisely that there is a holomorphic trivialization

$$E|U \cong U \times \mathbb{C}^r.$$

Then the $\overline{\partial}$ part of the exterior differential operator is well-defined since the trivialization changes holomorphically. We shall identify the holomorphic structure with the $\overline{\partial}$ operator. If $E$ has a holomorphic structure $\overline{\partial}$, there exists a unique integrable connection $A$ such that $\overline{\partial}_A = \overline{\partial}$. In fact, we can determine $\partial_A$ from the formula

$$\overline{\partial} H(\xi, \zeta) = H(\overline{\partial}\xi, \zeta) + H(\xi, \partial_A \zeta) \qquad \text{for } \xi, \zeta \in \Omega^0(E).$$

For a connection $A$ we define a mean curvature $K_A \in \Omega^0(\mathrm{Ad}(P)) \otimes \mathbb{C}$ by

$$K_A = \mathrm{tr}_\omega R_A = \sum_{i=1}^{m} R_A(E_i, \overline{E}_i),$$

where $(E_1, \cdots, E_m)$ is a local unitary frame field for the holomorphic tangent bundle $T'X$.

**Definition 1.8.** We say an integrable connection $A$ is an *Einstein-Hermitian connection* when

$$K_A = \lambda(E)\,\mathrm{Id}$$

for some constant $\lambda(E)$.

Using the Chern-Weil formula, we see that the constant $\lambda(E)$ is given by

$$\lambda(E) = \frac{2\pi}{r(m-1)!\operatorname{vol} X} \int_X c_1(E) \wedge \omega^{m-1}.$$

For a general connection $A$ (not necessarily integrable), we have an identity

$$m(m-1)\operatorname{tr}(R_A \wedge R_A) \wedge \omega^{m-2} = (|R_A^{1,1}|^2 - |K_A|^2 - 2|R_A^{2,0}|^2)\omega^m.$$

Then by the Chern-Weil formula,

$$\frac{1}{2}\int_X (|R_A|^2 - |K_A - \lambda(E)\operatorname{Id}|^2 - 4|R_A^{2,0}|^2)\,\omega^m$$

is a topological invariant independent of a connection $A$. So we have

**Proposition 1.9.** *On a compact Kähler manifold $(X, g)$, Einstein-Hermitian connections minimize the Yang-Mills functional $\mathcal{YM}$. In particular, they are Yang-Mills connections. If there exists at least one Einstein-Hermitian connection on $E$, any other minimizer is also an Einstein-Hermitian connection.*

We remark that for the holomorphic tangent bundle $T'X$ the Levi-Civita connection $\nabla$ is an Einstein-Hermitian connection if and only if $g$ is an Einstein-Kähler metric.

We denote by $\mathcal{M}_{EH}(E)$ the moduli space of Einstein-Hermitian connections on $E$, the quotient space of the space of all Einstein-Hermitian connections divided by the action of the gauge group. In Chapter 2 it will be shown that an open subset of $\mathcal{M}_{EH}(E)$ has a structure of a finite dimensional smooth manifold.

On a Kähler surface (namely $m = \dim_{\mathbb{C}} X = 2$), there is a relation between Einstein-Hermitian connections and anti-self-dual connections. For a 2-form $\alpha$ the self-dual part $\alpha^+$ is given by

$$\alpha^+ = \alpha^{2,0} + \alpha^{0,2} - \frac{\sqrt{-1}}{2}\sum_{i=1}^{2} \alpha(E_i, \overline{E}_i)\omega.$$

Thus we have

**Proposition 1.10** [It2]. *On a Kähler surface $(X, g)$ a connection $A$ on a bundle $P$ is an Einstein-Hermitian connection if and only if $R_A^0 = R_A - \frac{1}{r}\operatorname{tr} R_A \operatorname{Id}$ is anti-self-dual and $\operatorname{tr} R_A$ is a harmonic form of type $(1,1)$.*

## §1.4. Moment maps

Atiyah and Bott [AB] pointed out that some properties of Yang-Mills connections over Riemann surface can be explained in the language of the moment map which is a familiar tool in symplectic geometry. Since their observation is also essential in our study of Yang-Mills connections in the following chapters, we briefly recall the notion of the moment map for an action on a symplectic manifold. For more details, one can see Guillemin and Sternberg's book [GS].

Let $\omega$ be a symplectic form, namely a smooth non-degenerate closed 2-form, on a manifold $Y$ which may be infinite dimensional. A transformation $f$ of $Y$ is called *symplectic* if $f^*\omega = \omega$. Suppose that a Lie group $G$ (which may be infinite dimensional) acts on $Y$ from the right symplectically. Denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{g}^*$ its dual. We say a map $\mu: Y \to \mathfrak{g}^*$ is a *moment map* for the action of $G$ on $Y$ when it satisfies

(a) $\mu(y.g) = \mathrm{Ad}_g^*(\mu(y))$,
(b) $\langle a, d\mu_y(v)\rangle = \omega(a_y, v)$ for $a \in \mathfrak{g}$, $v \in T_yY$, $y \in Y$,

where $\mathrm{Ad}^*$ is the coadjoint operator on $\mathfrak{g}^*$, $a_y \in T_yY$ is the tangent vector defined by $a \in \mathfrak{g}$ through the action of $G$ and $\langle\ ,\ \rangle$ is the dual pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Take $c \in \mathfrak{g}^*$ which satisfies $\mathrm{Ad}_g^*(c) = c$ for all $g \in G$ and consider the quotient space $\mu^{-1}(c)/G$. We assume further more that $\mu$ satisfies the following conditions which imply that $\mu^{-1}(c)/G$ has a manifold structure:

(c) $d\mu_y: T_yY \to \mathfrak{g}^*$ is surjective for all $y \in \mu^{-1}(c)$,
(d) the action of $G$ on $\mu^{-1}(c)$ is free and there is a slice $S_y \subset \mu^{-1}(c)$ at each point $y \in \mu^{-1}(c)$.

Then the *reduction theorem* of Marsden and Weinstein states

**Proposition 1.11** [MW]. *There is a unique symplectic form $\omega_c$ defined on $\mu^{-1}(c)/G$ such that*

$$\pi^*\omega_c = i^*\omega \qquad \text{on } \mu^{-1}(c),$$

*where $\pi: \mu^{-1}(c) \to \mu^{-1}(c)/G$ is the natural projection and $i: \mu^{-1}(c) \to Y$ is the inclusion map.*

The typical examples of symplectic manifolds are Kähler manifolds. Suppose $Y$ is a Kähler manifold, namely a complex manifold with a metric $g$ whose Kähler form $\omega$ is closed. When $G$ acts on $Y$ so as to preserve $g$ and $\omega$ (and hence the complex structure) we have

**Proposition 1.12** [HKLR]. *The complex structure and the metric on $Y$ descend to the quotient space $\mu^{-1}(c)/G$ and the associated Kähler form coincides with $\omega_c$ constructed in Proposition* 1.11.

Next suppose that $(Y, g)$ is a hyperkähler manifold, namely it admits three almost complex structures $I$, $J$, $K$ which are parallel and satisfy the quaternion relation $IJ = -JI = K$. Associated with these complex structures we have three Kähler forms $\omega_I$, $\omega_J$, $\omega_K$ defined by

$$\omega_I(v, w) = g(Iv, w), \quad \omega_J(v, w) = g(Jv, w),$$
$$\omega_K(v, w) = g(Kv, w).$$

Suppose that the Lie group $G$ acts on $Y$ so as to preserve the metric $g$ and $I$, $J$, $K$. Moreover we assume that there exists the hyperkähler moment map $\mu = (\mu_I, \mu_J, \mu_K): Y \to \mathfrak{g}^* \otimes \mathbb{R}^3$ which is the set of three moment maps $\mu_I$, $\mu_J$, $\mu_K$ associated with three Kähler forms $\omega_I$, $\omega_J$, $\omega_K$ respectively. Then we have

**Proposition 1.13** [HKLR]. *The hyperkähler structure $(I, J, K)$ descends to the quotient space $\mu^{-1}(c)/G$.*

Now we explain the relation between the moment map and the concept of stability defined in algebraic geometry [KN], [Ki].

Let $Y$ again be a general Kähler manifold with a Kähler form $\omega$. Assume that there exists a holomorphic line bundle $\pi: L \to Y$ having a metric and connection $\nabla$ with curvature $-2\pi\sqrt{-1}\omega$. Suppose that a reductive algebraic group $G^{\mathbb{C}}$ acts holomorphically on $Y$ and there exists a lift of the action of $G^{\mathbb{C}}$ to $L$. We assume moreover that a maximal compact subgroup $G \subset G^{\mathbb{C}}$ acts so as to preserve the Kähler form $\omega$, the metric and connection on $L$. For $a \in \mathfrak{g}$, $\xi \in L$ we take a curve $c: (-\varepsilon, \varepsilon) \ni t \mapsto \pi(\xi).\exp(ta)$ in $Y$. Then $s: (-\varepsilon, \varepsilon) \ni t \mapsto \xi.\exp(ta)$ defines a section of the induced bundle $c^*L$. Differentiating with respect to the induced connection, we define $\widehat{\mu}(a, \xi) \in \mathbb{R}$ by

$$\left.\frac{\nabla}{dt}\right|_{t=0} s = 2\pi\sqrt{-1}\widehat{\mu}(a, \xi)\xi.$$

In other words $\widehat{\mu}(a, \xi)$ is defined by the following decomposition of the tangent vector $\overline{a}_\xi \in T_\xi L$ defined by $a \in \mathfrak{g}$ through the action of $G$ into the horizontal part and vertical part:

$$\overline{a}_\xi = \overline{a}_\xi^h + 2\pi\sqrt{-1}\widehat{\mu}(a, \xi)\xi.$$

Then $\widehat{\mu}(a, \cdot)$ depends only on $\pi(\xi)$ and induces a function on $Y$. We have

**Proposition 1.14** [GS]. *The function $\mu: Y \to \mathfrak{g}^*$ defined by*

$$\langle a, \mu(y) \rangle = \widehat{\mu}(a, \xi) \qquad \text{where } \pi(\xi) = y$$

*is a moment map.*

*Proof.* Let $g \in G$, $a \in \mathfrak{g}$ and $\xi \in L$ with $\pi(\xi) = y$. Let $R_g$ denote the diffeomorphism $\xi \in L \mapsto \xi.g \in L$. Since $G$ preserves the connection on $L$,

$$\overline{\mathrm{Ad}_{g^{-1}}(a)}_{\xi.g} = R_{g*}\overline{a}_\xi = R_{g*}\overline{a}^h_\xi + 2\pi\sqrt{-1}\widehat{\mu}(a, \xi)\xi.g$$

gives the decomposition of $\overline{\mathrm{Ad}_{g^{-1}}(a)}_{\xi.g}$ into the horizontal part and vertical part. Hence we have

$$\langle \mathrm{Ad}_{g^{-1}}(a), \mu(y.g) \rangle = \langle a, \mu(y) \rangle.$$

This shows that $\mu$ is equivariant.

Next we check the equation $\langle a, d\mu_y(v) \rangle = \omega(a_y, v)$ for $v \in T_yY$. We take a map $f: (-\varepsilon, \varepsilon) \to Y$ with $\dot{f}(0) = v$ and a parallel section $\psi$ of $f^*L$ with $\psi(0) = \xi$. Define a map $F$ by

$$\begin{aligned} &F: (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to Y \\ &F(s, t) = f(s). \exp(ta). \end{aligned}$$

Then $\Psi(s, t) = \psi(s). \exp(ta)$ gives a section of $F^*L$. Hence we have

$$2\pi\sqrt{-1}\widehat{\mu}(a, \psi(s))\psi(s) = \left.\frac{\nabla}{dt}\right|_{t=0} \Psi(s, t).$$

Since $\psi$ is a parallel section, we differentiate the above to get

$$\begin{aligned} 2\pi\sqrt{-1}\langle a, d\mu_y(v) \rangle &= \left.\frac{\nabla}{ds}\frac{\nabla}{dt}\right|_{s,t=0} \Psi(s, t) \\ &= R(v, a_y) = -2\pi\sqrt{-1}\omega(v, a_y). \end{aligned}$$

Thus we have obtained $\langle a, d\mu_y(v) \rangle = \omega(a_y, v)$. Q.E.D.

For the action of the complexification $G^{\mathbb{C}}$, the vertical part of the tangent vector $z_\xi \in T_\xi L$ defined by $z = a + \sqrt{-1}b \in \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$ is given by

$$2\pi\sqrt{-1}(\widehat{\mu}(a, \xi) + \sqrt{-1}\widehat{\mu}(b, \xi))\xi.$$

Let us introduce a function $m: Y \times G^{\mathbb{C}} \to \mathbb{R}$ measuring the distortion of the norm in $L$ caused by $G^{\mathbb{C}}$:

$$|\xi.g|^2 = e^{m(y,g)}|\xi|^2 \qquad \text{for } g \in G^{\mathbb{C}}, y = \pi(\xi) \in Y.$$

Fix a point $y \in Y$ and consider $m(y, \cdot)$ as a function on $G^{\mathbb{C}}$. Then for $\sqrt{-1}b \in \sqrt{-1}\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$

$$\left.\frac{d}{dt}\right|_{t=0} m(y, \exp(t\sqrt{-1}b)) = -2\pi\widehat{\mu}(b, \xi)$$

for $\xi \in L$ with $\pi(\xi) = y$. Since $m(y, \cdot)$ is $G$-invariant and $G^{\mathbb{C}} = G\exp(\sqrt{-1}\mathfrak{g})$, the above calculation shows that $dm(y, \cdot) = 0$ if and only if $\mu(y) = 0$. Namely the critical points of $m$ are precisely the zeroes of the moment map $\mu$.

The most important case of this situation is when $Y \subset \mathbb{CP}^n$, $L = \mathcal{O}(1)$ and $G$ acts on $Y$ via some representation $G \to SU(n+1)$ with the complexification $G^{\mathbb{C}} \to SL(n+1; \mathbb{C})$. Let $\widehat{Y} \subset \mathbb{C}^{n+1}$ be the affine cone over $Y$. The manifold $Y$ has a canonical Kähler metric $g$ which is the restriction of the Fubini-Study metric on $\mathbb{CP}^n$. We denote the associated Kähler form by $\omega$. In this situation the moment map $\mu$ is given by

$$\mu(y) = \frac{1}{2\pi\sqrt{-1}}\left(\frac{\widehat{y}^*\widehat{y}}{|\widehat{y}|^2} - \mathrm{Id}\right),$$

where $\mathfrak{su}(n+1)^*$ is identified with $\mathfrak{su}(n+1)$ by the inner product defined by $(A, B) = \mathrm{tr}(AB^*)$ and $\widehat{y} \in \widehat{Y} \subset \mathbb{C}^{n+1}$ over $y$ is considered as $(1, n+1)$ matrix. By definition $y \in Y$ is *stable* if and only if for any $\widehat{y} \in \widehat{Y}$ over $y$ the orbit $G\widehat{y}$ is closed in $\mathbb{C}^{n+1}$ and the stabilizer of $y$ is finite ([MF]).

There is a characterization of stability using the moment map:

**Proposition 1.15** [KN],[Kir]. *The following conditions on $y \in Y$ such that the stabilizer of $y$ is finite, are equivalent:*

(1) *$y$ is stable,*
(2) *there is a point $y'$ in the orbit $G^{\mathbb{C}}y$ which satisfy $\mu(y') = 0$,*
(3) *$m(y, \cdot): G^{\mathbb{C}} \to \mathbb{R}$ has a critical point.*

*For an invariant set $W \subset Y$, all of whose points are stable, the symplectic quotient $\mu^{-1}(0) \cap W/G$ coincides with the quotient $W/G^{\mathbb{C}}$.*

Now we explain the relationship between Einstein-Hermitian connections and the moment map. Let $P$ be a principal $U(r)$-bundle over an $m$-dimensional compact Kähler manifold $(X, g)$ with Kähler form $\omega$.

We denote by $E$ the associated hermitian vector bundle. We consider the space $\mathcal{A}^{(1,1)}(P)$ of all integrable connections on $P$ which is a submanifold of $\mathcal{A}(P) \cong \Omega^1(\mathrm{Ad}(P))$. (Strictly speaking we must take a certain open subset of $\mathcal{A}^{(1,1)}(P)$ to avoid possible singularities in $\mathcal{A}^{(1,1)}(P)$. See Chapter 2 for detail.) The $L^2$-inner product

$$(\alpha, \beta) = -\int_X \mathrm{tr}(\alpha \wedge *\beta)$$

on $\Omega^1(\mathrm{Ad}(P))$ induces a Riemannian metric on $\mathcal{A}^{(1,1)}(P)$. The almost complex structure $J$ on $X$ induces that on $\Omega^1(\mathrm{Ad}(P))$ and we also denote it by $J$. In fact, if we extend it to the complexification $\Omega^1(\mathrm{Ad}(P)) \otimes \mathbb{C}$, it is given by

$$J\alpha = -\sqrt{-1}\alpha^{1,0} + \sqrt{-1}\alpha^{0,1},$$

where $\alpha^{1,0}$ (resp. $\alpha^{0,1}$) is a $(1,0)$-form (resp. $(0,1)$-form). Then $\mathcal{A}^{(1,1)}(P)$ is a complex submanifold of $\Omega^1(\mathrm{Ad}(P))$. Moreover the Riemannian metric $(\ ,\ )$ is the Kähler metric and the associated Kähler form $\Omega(\cdot,\cdot) = (J\cdot,\cdot)$ is given by

$$\Omega(\alpha, \beta) = -\frac{1}{(m-1)!}\int_X \mathrm{tr}(\alpha \wedge \beta) \wedge \omega^{m-1} \qquad \text{for } \alpha,\ \beta \in T\mathcal{A}^{(1,1)}(P).$$

The action of the gauge group $\mathcal{G}(P)$ on $\mathcal{A}^{(1,1)}(P)$ preserves the metric and the complex structure. The Lie algebra of $\mathcal{G}(P)$ is $\Omega^0(\mathrm{Ad}(P))$ and its dual is identified with $\Omega^{2m}(\mathrm{Ad}(P))$ by

$$\langle \alpha, \beta\omega^m \rangle = -\int_X \mathrm{tr}(\alpha\beta)\omega^m \quad \text{for } \alpha \in \Omega^0(\mathrm{Ad}(P)),\ \beta\omega^m \in \Omega^{2m}(\mathrm{Ad}(P)).$$

Then the moment map is given by

$$\mu(A) = -\frac{1}{(m-1)!}R_A \wedge \omega^{m-1}.$$

In fact, the identity $R_{g(A)} = \mathrm{Ad}(g^{-1})R_A$ shows that $\mu$ is equivariant. Next we check the equation $\langle \alpha, d\mu_A(v)\rangle = \omega(\alpha_A, v)$ for $\alpha \in \Omega^0(\mathrm{Ad}(P))$, $v \in T_A\mathcal{A}^{(1,1)} \subset \Omega^1(\mathrm{Ad}(P))$. We have

$$d\mu_A(v) = -\frac{1}{(m-1)!}d_A v \wedge \omega^{m-1}.$$

So for $\alpha \in \Omega^0(\mathrm{Ad}(P))$

$$\begin{aligned}\langle \alpha, d\mu_A(v)\rangle &= \frac{1}{(m-1)!}\int_X \mathrm{tr}(\alpha\; d_A v)\wedge\omega^{m-1}\\ &= -\frac{1}{(m-1)!}\int_X \mathrm{tr}(d_A\alpha\wedge v)\wedge\omega^{m-1}\\ &= \Omega(d_A\alpha, v).\end{aligned}$$

Since the vector field on $\mathcal{A}^{(1,1)}(P)$ associated with $\alpha$ is given by $\alpha_A = d_A\alpha$, the above shows that $\mu$ is the moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}^{(1,1)}(P)$.

The Einstein-Hermitian tensor is given by

$$\sqrt{-1}R_A\wedge\omega^{m-1} = \frac{1}{m}K_A\omega^m.$$

Hence $A \in \mathcal{A}^{(1,1)}(P)$ is an Einstein-Hermitian connection if and only if

$$\mu(A) = \frac{\sqrt{-1}\lambda(E)}{m!}\ \mathrm{Id}\ \omega^m.$$

Hence the moduli space $\mathcal{M}_{EH}(P)$ can be described as a symplectic quotient $\mu^{-1}(\frac{\sqrt{-1}\lambda(E)}{m!}\ \mathrm{Id}\ \omega^m)/\mathcal{G}(P)$. This observation is essential in studying geometries of moduli space (see Chapter 2). Moreover the formal analogy of stability and the function $m$ is possible under certain conditions although the manifold and the acting group are infinite dimensional (see Chapters 5, 6).

## Chapter 2. Local Goometries of Moduli Spaces

### §2.1. Smooth structure on moduli space

The moduli space of (anti-)self-dual Yang-Mills connections over a 4-dimensional Riemannian manifold and the moduli space of Einstein-Hermitian connections over a Kähler manifold are primarily dominated by elliptic complexes arising from infinitesimal deformations of (anti-) self-dual (resp., Einstein-Hermitian) connections.

In this chapter we will extract from these complexes geometrical aspects of the moduli spaces and show by means of the moment map, a familiar tool in symplectic geometry, that they inherit further a natural Kähler (resp., hyperkähler) structure provided the base manifold is Kähler (resp., hyperkähler).

Since there is no essential difference between the situations of (anti-) self-dual connections and of Einstein-Hermitian connections, we focus on the moduli space of Einstein-Hermitian connections over a compact Kähler surface $(X, g)$ for convenience' sake.

Let $P$ be a $U(r)$-principal bundle over $X$ and $E = P \times_\rho \mathbb{C}^r$ the associated complex vector bundle. Let $A$ be an Einstein-Hermitian connection on $P$, that is, an integrable connection with mean curvature $K_A = \lambda(E)\,\mathrm{Id}$ ($\lambda(E)$ is the constant depending only on $E$). Associated with this Einstein-Hermitian connection $A$ there are the following complexes:

$$\begin{array}{ccccc}
0 \to \Omega^0(\mathrm{Ad}(P)) & \xrightarrow{d_A} & \Omega^1(\mathrm{Ad}(P)) & \xrightarrow{d_A^+} & \Omega^+(\mathrm{Ad}(P)) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \Omega^{0,0}(\mathrm{Ad}(P)^{\mathbb{C}}) & \xrightarrow{\overline{\partial}_A} & \Omega^{0,1}(\mathrm{Ad}(P)^{\mathbb{C}}) & \xrightarrow{\overline{\partial}_A} & \Omega^{0,2}(\mathrm{Ad}(P)^{\mathbb{C}}) \to 0
\end{array}$$

first of which represents infinitesimal deformations of Einstein-Hermitian connections because of the Bianchi identity and the gauge action, where $d_A^+ = p_+ \circ d_A$ and $p_+ \colon \Lambda^2(X) \to \Lambda^+$ is the projection. The lower complex, the twisted Dolbeault complex, is valid not only for Einstein-Hermitian connections but for integrable connections ([AHS], [It1], [Kb2]).

Then similarly as the de Rham complex the operators $d_A$ and $d_A^+$ (resp., $\overline{\partial}_A$) together with the $L^2$-adjoint operators define the Laplacians $\Delta_A^i$ (resp., $\Delta_A^{0,i}$), ($i = 0, 1, 2$). Each kernel of the Laplacians $H_A^i = \mathrm{Ker}\,\Delta_A^i$ (resp., $H_A^{0,i} = \mathrm{Ker}\,\Delta_A^{0,i}$), the $i$-th harmonic space, is isomorphic to the corresponding cohomology group. Between these harmonic spaces we have isomorphisms over $\mathbb{R}$ ([It1]):

$$H^0 \otimes \mathbb{C} \cong H^{0,0},\ H^1 \cong H^{0,1} \text{ and } H^2 \cong H^0 \oplus H^{0,2},$$

last of which generalizes the formula in Kähler geometry; $b_+(X) = 1 + 2p_g(X)$, where $b_+(X)$ is the number of positive eigenvalues of the intersection form of $X$ and $p_g(X)$ is the geometric genus of $X$.

The unitary group $U(r)$ has the nontrivial center $Z$ which should be taken into account in the subsequent argument. Actually, the adjoint bundle $\mathrm{Ad}(P)$ decomposes as $\mathrm{Ad}(P) = \mathrm{Ad}_c(P) \oplus \mathrm{Ad}_s(P)$ according to the decomposition of the Lie algebra $\mathfrak{u}(r) = \mathfrak{c} \oplus \mathfrak{su}(r)$, where $\mathrm{Ad}_c(P) = P \times_{\mathrm{Ad}} \mathfrak{c}$ is the rank one product bundle and $\mathrm{Ad}_s(P) = P \times_{\mathrm{Ad}} \mathfrak{su}(r)$

in such a way that the covariant derivative $d_A$ reduces to the ordinary derivative $d$ on the subbundle $\mathrm{Ad}_c(P)$. Then the above diagram splits as

$$\begin{array}{ccccc} 0 \to \Omega^0 \oplus B^0 & \xrightarrow{(d,d_A)} & \Omega^1 \oplus B^1 & \xrightarrow{(d^+,d_A^+)} & \Omega^+ \oplus B^+ \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \Omega^{0,0} \oplus C^{0,0} & \xrightarrow{(\bar\partial,\bar\partial_A)} & \Omega^{0,1} \oplus C^{0,1} & \xrightarrow{(\bar\partial,\bar\partial_A)} & \Omega^{0,2} \oplus C^{0,2} \to 0 \end{array}$$

where $B^i = \Omega^i(\mathrm{Ad}_s(P))$, $C^{0,i} = \Omega^{0,i}(\mathrm{Ad}_s(P)^{\mathbb{C}})$. The harmonic spaces for the bundles $\mathrm{Ad}_c(P)$ and $\mathrm{Ad}_s(P)$ decompose into $H^i_A = H^i(X) \oplus H^i_{s,A}$ and $H^{0,i}_A = H^{0,i}(X) \oplus H^{0,i}_{s,A}$, $(i = 0,1,2)$ where $H^i(X)$ and $H^{0,i}(X)$ are the ordinary harmonic spaces of $X$.

Denote by $\widehat{\mathcal{A}}_{EH}(P)$ the set of all generic Einstein-Hermitian connections on $P$ and by $\widehat{\mathcal{M}}_{EH}(P)$ the moduli space $\widehat{\mathcal{A}}_{EH}(P)/\mathcal{G}(P)$ of generic Einstein-Hermitian connections. Here an Einstein-Hermitian connection $A$ is called *generic* when $H^0_{s,A} = \{0\}$ and $H^2_{s,A} = \{0\}$. More generally we say a connection $A$ to be *irreducible* if the isotropy $\Gamma_A = \{g \in \mathcal{G}(P) \mid g(A) = A\}$ consists only of constant gauge transformations taking values in $Z$. So, $H^0_{s,A} = \{0\}$ if and only if $A$ is irreducible.

**Proposition 2.1.** *The moduli space $\widehat{\mathcal{M}}_{EH}(P)$ of generic Einstein-Hermitian connections on $P$ is a $C^\infty$-manifold of dimension $b^1(X) + \dim H^1_s$. It is actually written locally as a product of the de Rham cohomology group $H^1(X)$ and the moduli space $\widehat{\mathcal{M}}_{EH,s}(P)$ of generic Einstein-Hermitian connections on $P$ with fixed trace curvature* $\operatorname{tr} R_A$.

*The dimension of $\widehat{\mathcal{M}}_{EH,s}(P)$ is computed by the Atiyah-Singer index theorem as*

$$\begin{aligned} &\dim_{\mathbb{R}} \widehat{\mathcal{M}}_{EH,s}(P) \\ =&2\{(2rc_2(E) - (r-1)c_1^2(E))[X] - (r^2-1)(1 - q(X) + p_g(X))\} \end{aligned}$$

*where $q(X)$ is the irregularity of $X$.*

This proposition is shown by using a slice lemma argument on gauge fixing condition and applying the Kuranishi method which integrates the non-linear Einstein-Hermitian equations along directions in $H^1_A$. The slice lemma states [FU] that for any connection $A$ there is a neighborhood $S$ of $0$ in the kernel of $d_A^*: \Omega^1(\mathrm{Ad}(P)) \to \Omega^0(\mathrm{Ad}(P))$ being transversal to gauge orbits so that the quotient $S/\Gamma_A$ by $\Gamma_A$ represents a neighborhood of $[A]$ in the orbit space $\mathcal{A}(P)/\mathcal{G}(P)$.

We let $S_{EH}$ be the intersection $S \cap \widehat{\mathcal{A}}_{EH}(P)$. Then any point $[A]$ of $\widehat{\mathcal{M}}_{EH}(P)$ has a neighborhood corresponding to the quotient $S_{EH}/\Gamma_A$.

Now let $A$ be a generic Einstein-Hermitian connection. Then a connection $A+\alpha$ is Einstein-Hermitian if and only if

$$d_A^+\alpha_s + \frac{1}{2}p_+[\alpha_s \wedge \alpha_s] = 0, \tag{2.2}$$

$$d\alpha_c = 0, \tag{2.3}$$

where $\alpha = \alpha_c + \alpha_s$ is the decomposition of $\alpha \in \Omega^1(\mathrm{Ad}(P))$ ([It1]). So, as the center part $\alpha_c$ and the trace free part $\alpha_s$ are chosen separately, the slice $S_{EH}$ is written exactly as a product $S_{EH}^c \times S_{EH}^s$, where

$$S_{EH}^c = \left\{\sqrt{-1}\alpha\,\mathrm{Id} \mid \alpha \in H^1(X), |\alpha| < \varepsilon\right\}, \tag{2.4}$$

$$S_{EH}^s = \left\{\alpha_s \in \Omega^1(\mathrm{Ad}_s(P)) \;\middle|\; \begin{array}{c} |\alpha_s| < \varepsilon', d_A^*\alpha_s = 0 \\ \text{and (2.2) holds} \end{array}\right\}. \tag{2.5}$$

The isotropy $\Gamma_A$ acts trivially on the adjoint bundle so that $S_{EH} = S_{EH}^c \times S_{EH}^s$ precisely represents a neighborhood of $[A]$.

Define a map which is a gauge analog of the Kuranishi map for the Kodaira-Spencer deformation theory of complex manifolds, as

$$\begin{aligned} K = K_A &: \Omega^1(\mathrm{Ad}_s(P)) \to \Omega^1(\mathrm{Ad}_s(P)), \\ K(\alpha_s) &= \alpha_s + \frac{1}{2}d_A^{+*}G_A^2(p_+[\alpha_s \wedge \alpha_s]), \end{aligned} \tag{2.6}$$

where $G_A^2$ is the Green operator associated to the Laplacian $\Delta_A^2$. In this definition we should regard in mind the space $\Omega^1(\mathrm{Ad}_s(P))$ as a Banach space completed in a $W^{l,2}$-norm. Then the Fréchet derivative $DK$ at $\alpha_s = 0$ is the identity.

The following is a key lemma for a local description of the moduli space.

**Lemma 2.7.** *A trace free one form $\alpha_s$ satisfies $d_A^*\alpha_s = 0$ and (2.2) if and only if $K(\alpha_s) \in H_{s,A}^1$ and the harmonic part of $p_+[\alpha_s \wedge \alpha_s]$ vanishes in $\Omega^+(\mathrm{Ad}_s(P))$.*

This lemma is based on the elliptic complex arising from infinitesimal deformations. So, since $H_{s,A}^2 = \{0\}$ for generic $A$, a neighborhood $U$ of 0 in $H_{s,A}^1$ is through $K$ homeomorphic to the slice $S_{EH}^s$ and moreover it is seen that neighborhoods of the form $S^c_{EH} \times U \subset H^1(X) \times H^1_{s,A}$ for all $A$ in $\widehat{\mathcal{A}}_{EH}(P)$ yield a smooth manifold structure on $\widehat{\mathcal{M}}_{EH}(P)$.

We remark that each tangent space to $\widehat{\mathcal{M}}_{EH}(P)$ is identified from (2.7) with the harmonic space $H^1_A$.

The moduli space $\widehat{\mathcal{M}}_{EH}(P)$ which turned out to be smooth admits a natural Riemannian structure. In fact, the Riemannian structure is defined on the moduli space by restricting the $L^2$-inner product

$$(\alpha, \beta) = -\int_X \mathrm{tr}(\alpha \wedge *\beta)$$

on $\Omega^1(\mathrm{Ad}(P))$ to each harmonic space $H^1_A$ (for more details especially on the curvature formula, see [It3], [Ko1] or Chapter 6).

We have in particular for the (anti-)self-dual case

**Theorem 2.8.** *Let $(X, g)$ be a compact connected oriented 4-dimensional Riemannian manifold and $P$ a $G$-principal bundle over $X$ ($G$ is compact and semi-simple). Then the moduli space of generic (anti-) self-dual connections carries a smooth Riemannian manifold structure.*

## §2.2. Complex Kähler structure on moduli spaces

The next geometrical investigation on the moduli space is that $\widehat{\mathcal{M}}_{EH}(P)$ is endowed with a complex manifold structure and also with a Kähler structure. Actually, as we will see in the following lemma, the moduli space of generic Einstein-Hermitian connections has an almost complex structure induced naturally from the complex structure of $X$.

**Lemma 2.9.** *The harmonic space $H^1_A$ is invariant under the action of the almost complex structure $J$ on $\Omega^1(\mathrm{Ad}(P))$.*

See for the proof [It1]. Here $J$ is introduced as the endomorphism $J_X \otimes \mathrm{Id}$ of $\Lambda^1(X) \otimes \mathrm{Ad}(P)$ and $J_X$ is the almost complex structure defining the complex structure of $X$. Remark that the almost complex structure $J_X$ which is an endomorphism of the tangent bundle $TX$ naturally induces an almost complex structure, also denoted by $J_X$, on the cotangent bundle $T^*X$ by

$$J_X\alpha(v) = -\alpha(J_X v) \quad \text{for } \alpha \in T^*X,\ v \in TX.$$

As $J$ preserves the $L^2$-inner product, the naturally defined Riemannian structure becomes Hermitian on the almost complex manifold $(\widehat{\mathcal{M}}_{EH}(P), J)$.

To assert the integrability of $J$ we require the notion of simple integrable connection. An integrable connection is called *simple* when $H^{0,0}_{s,A} = \{0\}$, in other words, $\mathrm{End}(E)$ has no holomorphic sections other than constant multiples of the identity. We consider the set $\mathcal{A}^{(1,1)}_s(P)$ of all simple integrable connections on $P$. While $\mathcal{G}(P)$ fixes $\mathcal{A}^{(1,1)}_s(P)$ invariant, we can enlarge the invariance group $\mathcal{G}(P)$ complexifically. Actually the group of complex gauge transformations $\mathcal{G}(P)^{\mathbb{C}} = \Gamma(X; P \times_{\mathrm{Ad}} GL(r;\mathbb{C}))$ which contains $\mathcal{G}(P)$ acts in a twisted way as $\overline{\partial}_{g(A)} = g^{-1} \circ \overline{\partial}_A \circ g$ (see §5.2). Then the quotient space $\mathcal{A}^{(1,1)}_s(P)/\mathcal{G}(P)^{\mathbb{C}}$, denoted by $\mathcal{M}_{hol}(P)$, parametrizes complex gauge equivalence classes of simple integrable connections. Denote by $\widehat{\mathcal{M}}_{hol}(P)$ the subset $\{[A] \in \mathcal{M}_{hol}(P) \mid H^{0,2}_{s,A} = \{0\}\}$. It is open and dense in $\mathcal{M}_{hol}(P)$. Likewise the Einstein-Hermitian case the complex gauge version of the slice lemma and of the Kuranishi map associated to the twisted Dolbeault complex can be applied to obtain the following ([It1]).

**Theorem 2.10.** *The moduli space $\widehat{\mathcal{M}}_{hol}(P)$ of simple integrable connections with $H^{0,2}_{s,A} = \{0\}$ is a complex manifold of complex dimension $h^{0,1}(X) + \dim_{\mathbb{C}} \widehat{\mathcal{M}}_{hol,s}(P)$, which has a local product structure of the abelian variety of $X$ and a complex manifold $\widehat{\mathcal{M}}_{hol,s}(P)$, where $\widehat{\mathcal{M}}_{hol,s}(P)$ denotes the subset $\{[A] \in \widehat{\mathcal{M}}_{hol}(P)$ with fixed trace curvature $\mathrm{tr}\, R_A\}$.*

*The dimension of $\widehat{\mathcal{M}}_{hol,s}(P)$ is given as*

$$\dim_{\mathbb{C}} \widehat{\mathcal{M}}_{hol,s}(P) = \frac{1}{2} \dim_{\mathbb{R}} \widehat{\mathcal{M}}_{EH,s}(P). \tag{2.11}$$

It should be remarked that $\widehat{\mathcal{M}}_{hol}(P)$ may not be Hausdorff.

Since $\mathcal{G}(P) \subset \mathcal{G}(P)^{\mathbb{C}}$ and $\widehat{\mathcal{A}}_{EH}(P) \subset \mathcal{A}^{(1,1)}_s(P)$, we get a canonical map $\psi$ between the moduli spaces by

$$\begin{array}{ccc} \widehat{\mathcal{M}}_{EH}(P) & \hookrightarrow & \mathcal{A}^{(1,1)}_s(P)/\mathcal{G}(P) \\ & \psi \searrow & \downarrow \\ & & \mathcal{A}^{(1,1)}_s(P)/\mathcal{G}(P)^{\mathbb{C}} = \mathcal{M}_{hol}(P). \end{array} \tag{2.12}$$

Then the map $\psi$ is shown to be injective by using the moment map due to Donaldson (see [It1]). Moreover the differential $\psi_*$ of $\psi$, between tangent spaces, is given by the canonical isomorphism $H^1_A \to H^{0,1}_A$;

$$\alpha = \alpha' + \alpha'' \mapsto \alpha''$$

so that $\psi$ is an open map of full rank and satisfies $\psi_* J\alpha = \sqrt{-1}\psi_*\alpha$ .

**Theorem 2.13.** *The moduli space $\widehat{\mathcal{M}}_{EH}(P)$ of generic Einstein-Hermitian connections inherits a complex manifold structure with respect to the almost complex structure $J$ in such a way that $\psi\colon \widehat{\mathcal{M}}_{EH}(P) \to \widehat{\mathcal{M}}_{hol}(P)$ is a holomorphic embedding.*

See [It1] for a proof on the anti-self-dual case which is available also to the Einstein-Hermitian case.

In order to show that the Hermitian structure ( , ) on the moduli space is Kähler it suffices to verify that the fundamental form $\Omega(\cdot,\cdot) = (J\cdot,\cdot)$ is closed. For this we make use of the idea of moment map (see §1.4).

We notice that $\Omega$ precisely coincides when regarded as a 2-form on a slice neighborhood with the symplectic form $(\alpha,\beta) \mapsto -\int_X \mathrm{tr}(\alpha\wedge\beta)\wedge\omega$ restricted to the tangent space $T_{[A]}\widehat{\mathcal{M}}_{EH}(P) \cong H^1_A$.

Since the action of the gauge group $\mathcal{G}(P)$ on $\mathcal{A}^{(1,1)}$ is not free, we must consider the quotient group $G = \mathcal{G}(P)/U(1)$ where $U(1)$ is considered as the group of the constant scalars of absolute value 1. Let $\mathfrak{g} = \Omega^1(\mathrm{Ad}(P))/\mathfrak{u}(1)$ be its Lie algebra. Then define a map $\mu\colon \widehat{\mathcal{A}}^{(1,1)}(P) \to \mathfrak{g}^*$ by

$$\langle\alpha,\mu(A)\rangle = \int_X \mathrm{tr}(\alpha R_A)\wedge\omega + \frac{\sqrt{-1}\lambda(E)}{2}\,\mathrm{tr}(\alpha)\wedge\omega^2 \quad \text{for } \alpha\in\mathfrak{g}, \tag{2.14}$$

where $\widehat{\mathcal{A}}^{(1,1)}(P)$ is for the moment the set of generic integrable connections, i.e., integrable connections with vanishing $H^0_{s,A}$ and $H^2_{s,A}$. It is seen that this is a moment map for the action of the group $G$, and $\mu(A) = 0$ if and only if $A$ is an Einstein-Hermitian connection.

Since $\mu^{-1}(0)$ is gauge invariant, we have the reduced phase space $\mu^{-1}(0)/G$ giving the moduli space of generic Einstein-Hermitian connections. To apply the reduction theorem (Propositions 1.11, 1.12) to the canonical symplectic form we need to see the surjectivity of the differential $d\mu$ and the existence of a slice at any $A\in\mu^{-1}(0)$. The differential $d\mu_A$ is written as

$$\langle\alpha, d\mu_A(v)\rangle = \int_X \mathrm{tr}(\alpha d_A v)\wedge\omega \quad \text{for } \alpha\in\mathfrak{g},\ v\in T\widehat{\mathcal{A}}^{(1,1)} \subset \Omega^1(\mathrm{Ad}(P)).$$

It is then an easy exercise from the Hodge theory to see the equation $d_A v\wedge\omega = \phi\otimes\omega^2$ has a solution $\alpha\in T\widehat{\mathcal{A}}^{(1,1)}$ for all $\phi\in\Omega^0(\mathrm{Ad}(P))$

which is orthogonal to constant scalars. The action of the quotient group $G = \mathcal{G}(P)/U(1)$ on $\mu^{-1}(0)$ is free.

On the other hand the slice lemma can be entirely applied to $\mu^{-1}(0)$ $(\subset \widehat{\mathcal{A}}^{(1,1)}(P)) = \{$ generic Einstein-Hermitian connections $\}$ in such a way that as before there exists a slice neighborhood $S$ centered at $A$ inside $\mu^{-1}(0)$, which is diffeomorphic to a neighborhood of $H^1_A$ and satisfies

$$T_A(\mu^{-1}(0)) = T_A(G(A)) \oplus T_A(S).$$

So, the symplectic form $(\alpha, \beta) \mapsto -\int_X \operatorname{tr}(\alpha \wedge \beta) \wedge \omega$ descends to be a closed form which is actually the form $\Omega$.

Thus we can summarize this as

**Theorem 2.15.** *Let $(X, g)$ be a compact Kähler surface and $P$ a $C^\infty$ $U(r)$-principal bundle over $X$. Then the complex structure and the Kähler structure on $X$ naturally induce a complex manifold structure and a Kähler structure on the moduli space $\widehat{\mathcal{M}}_{EH}(P)$.*

*Remarks* 2.16. 1) This theorem is also valid for the moduli space $\mathcal{M}_-(P)$ of anti-self-dual connections and for the moduli space of Einstein-Hermitian connections over an $m$-dimensional Kähler manifold ([It3], [Ki], [Kb2] and [Ko]).

2) The moduli space $\widehat{\mathcal{M}}_{EH}(P)$ corresponds completely to the moduli space of stable, generic integrable connections which can be identified with the moduli of stable holomorphic bundle structure on the bundle $E$ (§5.2).

## §2.3. Hyperkähler structure of the moduli space

A Riemannian manifold $(X, g)$ is called *hyperkähler* if it admits almost complex structures $I_X$, $J_X$, $K_X$ which are parallel and satisfy the quaternionic relations $I_X J_X = -J_X I_X = K_X$. So each tangent space is endowed with an $\mathbb{H}$-vector space structure. A hyperkähler manifold has the holonomy group in a symplectic group and hence is a Ricci-flat Kähler manifold. Moreover a hyperkähler manifold is a complex symplectic manifold, i.e., a complex manifold carrying a nondegenerate holomorphic 2-form.

Let $P$ be a $U(r)$-principal bundle over a compact 4-dimensional hyperkähler manifold $(X, g)$. We regard $(X, g)$ as a complex Kähler surface with respect to $I_X$. Let denote $\omega_I$, $\omega_J$, $\omega_K$ three Kähler forms associated with $I$, $J$, $K$. As in §2.2 the hyperkähler structure on $X$ induces a hyperkähler structure $I$, $J$, $K$ on the space $\mathcal{A}$ of all connections on $P$. Then we define the hyperkähler moment map $\mu = (\mu_I, \mu_J, \mu_K): \mathcal{A} \to \mathfrak{g} \otimes \mathbb{R}^3$ for the action of $G = \mathcal{G}(P)/U(1)$ by

$$\langle \alpha, \mu_I(A) \rangle = \int_X \operatorname{tr}(\alpha R_A) \wedge \omega_I + \frac{\sqrt{-1}\lambda(E)}{2} \operatorname{tr}(\alpha) \wedge \omega_I^2,$$

$$\langle \alpha, \mu_J(A) \rangle = \int_X \operatorname{tr}(\alpha R_A) \wedge \omega_J,$$

$$\langle \alpha, \mu_K(A) \rangle = \int_X \operatorname{tr}(\alpha R_A) \wedge \omega_K \qquad \text{for } \alpha \in \mathfrak{g}.$$

It is easy to see the connection $A \in \mathcal{A}$ is an Einstein-Hermitian connection if and only if $\mu(A) = 0$. Hence by Proposition 1.13 we have the following

**Theorem 2.17.** *Over a compact* 4*-dimensional hyperkähler manifold the moduli space $\widehat{\mathcal{M}}_{EH}(P)$ is a hyperkähler manifold with respect to the natural almost complex structures $I$, $J$, $K$.*

See also [It2] for another proof using the Kuranishi mapping.

The moduli space $\widehat{\mathcal{M}}_{EH}(P)$ is a complex symplectic manifold and hence the canonical line bundle is trivial so that the first Chern class $c_1$ is zero. (See [Kb2] for the argument concerning the holomorphic analog of the moment map.)

# Chapter 3. Twistor Theory and Yang-Mills Connection

## §3.1. Self-dual connections on self-dual manifolds

Self-dual connections on a special 4-manifold (called self-dual manifold) can be described in terms of holomorphic vector bundles on a complex manifold (called twistor space). Therefore we can study self-dual connections using algebro-geometric methods. This procedure was carried out for $S^4$ by Atiyah, Drinfeld, Hitchin and Manin [ADHM].

Let $(X, g)$ be an oriented 4-manifold. Using the Hodge star operator, we have a decomposition

$$\Lambda^2 T^* X = \Lambda^+ \oplus \Lambda^-.$$

Let $\pi: Z = S(\Lambda^-) \to X$ be the unit sphere bundle of the vector bundle $\Lambda^-$, which is a fiber bundle over $X$ whose fiber is a 2-sphere. If we identify 2-forms with skew-symmetric endomorphisms of $TX$, each point $z \in Z$ defines an endomorphism $J = J_z$ of $T_xX$ for $x = \pi(z)$ such that

$$g(Jv, w) = -g(v, Jw) \quad \text{for } v,\, w \in T_xX$$
$$J^2 = -\operatorname{Id}.$$

The Levi-Civita connection naturally induces a connection on $\Lambda^-$, and hence we have a natural splitting of the tangent bundle of $Z$ into the vertical and horizontal spaces:

$$TZ = T^vZ \oplus T^hZ.$$

Then we define an almost complex structure $J$ on $T_zZ$ by taking $\pi^*J_z$ on $T^h_zZ \cong \pi^*T_xX$ and the standard one (as $S^2$) on $T^vZ$. The almost complex manifold $Z$ is called the *twistor space* of $(X, g)$. We have

**Theorem 3.1** [AHS]. *The almost complex structure $J$ on $Z$ is integrable if and only if the anti-self-dual Weyl tensor $W_-$ of $(X, g)$ vanishes.*

An oriented 4-manifold $(X, g)$ is called *self-dual* if $W_- = 0$.

*Examples* 3.2. 1) Consider the standard 4-sphere $(S^4, g_{std})$. Since it is conformally flat, it is clearly self-dual. Its twistor space is the 3-dimensional projective space $\mathbb{P}^3(\mathbb{C})$.

2) The projective plane $\mathbb{P}^2(\mathbb{C})$ with the Fubini-Study metric is self-dual. Its twistor is the flag manifold $F_{1,2} = \{(z, w) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid \langle w, z\rangle = 0\}$ where $V$ is a 3-dimensional complex vector space.

3) Consider the flat complex torus $T^2 = \mathbb{C}^2/\mathbb{Z}^4$. It is a typical example of hyperkähler manifolds (see §1.4 for the definition). Its twistor space is a fiber bundle over $\mathbb{P}^1(\mathbb{C})$ with fiber $T^2$.

The twistor space has a *real structure* $\tau\colon Z \to Z$ which is induced from the mapping $z \mapsto -z$ for $z \in \Lambda^-$. It is an anti-holomorphic involution which preserves each fiber. The restriction on each fiber $S^2$ is the antipodal map.

The twistor space has a natural Riemannian metric (also denoted by $g$) induced from that of $X$. We have a natural Kähler form $\omega$ defined by $\omega(v, w) = g(Jv, w)$ for $v$, $w \in TZ$, though it is not necessarily closed. When $X$ is $S^4$ or $\mathbb{P}^2(\mathbb{C})$, $\omega$ is a Kähler-Einstein form of positive scalar curvature.

We now consider a complex vector bundle $E$ over a self-dual 4-manifold $X$, a hermitian fiber metric $h$ on $E$ and a self-dual connection $A$ on $E$. Let $F = \pi^*E$, and $B = \pi^*A$ be the pull back of the bundle and connection (respectively) on $Z$. The curvature form $R_B$ of $B$ is equal to $\pi^* R_A$, and it is easy to see $R_B \in \Omega_0^{1,1}(\mathrm{Ad}(F))$, namely that $R_B$ is a (1,1)-form which is orthogonal to the canonical Hermitian form $\omega$. In particular, $B$ is an integrable connection. Conversely if $R_B \in (\Omega^{1,1} \cap \pi^*\Omega^2)(\mathrm{Ad}(F))$, $R_A$ is self-dual. More precisely we have

**Theorem 3.3** [AHS]. *We have a one-to-one correspondence between the gauge equivalent classes of self-dual connections on $X$ and the isomorphism classes of holomorphic vector bundles $F$ on the twistor space $Z$ which satisfy the following:*

(1) *$F$ is holomorphically trivial on each fiber,*
(2) *There is a holomorphic isomorphism $\sigma: \tau^*\overline{F} \to F^*$ such that $\sigma$ induces a positive definite hermitian structure on the space $H^0(Z_x, \mathcal{O}(F))$ of holomorphic sections of $F$ on each fiber by*

$$(s_1, s_2) = \sigma(\overline{s_2(\tau(z))})s_1(z)$$

*for fixed $z \in Z_x$.*

*Proof.* We shall only show that a self-dual connections on $X$ give rise to a holomorphic vector bundle on $Z$. The proof of the converse is left to the reader.

For a self-dual connection $A$, the bundle $F = \pi^* E$ with a connection $B = \pi^* A$ is a holomorphic vector bundle. It has a hermitian structure induced from the one of $E$.

Along each fiber $\mathbb{P}^1(\mathbb{C})$ the pull back connection $B$ is trivial, so its holomorphic structure is also trivial along each fiber. We have a natural isomorphism

$$H^0(Z_x, \mathcal{O}(F)) \to F_z$$

defined by evaluation of a holomorphic section at a point $z$ in the fiber $Z_x$. Thus we can identify the left-hand side with $E_x$. In particular we have a hermitian metric on $H^0(Z_x, \mathcal{O}(F))$ induced from the fiber metric on $E_x$.

Let $(e_1, \cdots, e_r)$ be a local unitary frame of $E$. Then $(f_1, \cdots, f_r)$ with $f_i = \pi^* e_i$ is a local unitary frame of $F$ and constant along each fiber. We define an isomorphism $\sigma$ by

$$\sigma: \tau^*\overline{F}_z \ni \overline{f}_i(\tau(z)) \mapsto f_i^*(z) \in F_z^*.$$

It is standard to see $\sigma$ is holomorphic. For $s_1 = \sum_i \lambda_i f_i, s_2 = \sum_i \mu_i f_i \in$

$H^0(Z_x, \mathcal{O}(F))$ we have

$$(s_1(z), s_2(z)) = \sum_i \lambda_i \overline{\mu}_i = \sum_i \sigma(\overline{\mu}_i \overline{f}_i(\tau(z)))\lambda_i f_i(z)$$
$$= \sigma(\overline{s_2(\tau(z))}) s_1(z) = (s_1, s_2).$$

Q.E.D.

We remark that if the base manifold $(X, g)$ is $S^4$ or $\mathbb{P}^2(\mathbb{C})$, then the pull back connection $B$ on $F$ is an Einstein-Hermitian connection with respect to the natural Kähler form on the twistor space $Z$ (see Proposition 3.10).

Suppose a structure group $G$ is compact and semi-simple. For a self-dual connection $A$ on a $G$-principal bundle $P$ we have the following complex (see Chapter 2):

$$0 \longrightarrow \Omega^0(\mathrm{Ad}(P)) \xrightarrow{d_A} \Omega^1(\mathrm{Ad}(P)) \xrightarrow{d_A^-} \Omega^-(\mathrm{Ad}(P)) \longrightarrow 0,$$

where $d_A^- = p_- \circ d_A$ and $p_-: \Lambda^2 \to \Lambda^-$ is the projection. Similarly as the de Rham complex, $d_A$ and $d_A^-$ together with the $L^2$ adjoint operators define the Laplacian. Let $H_A^i$ be the $i$-th harmonic space of the above complex. In our situation, since the center of $G$ is zero-dimensional, $H_A^0 = 0$ if $A$ is *irreducible*. Moreover we have $H_A^2 = 0$ if $(X, g)$ has positive scalar curvature ([AHS]).

**Theorem 3.4** [AHS]. *Let $(X, g)$ be a compact self-dual 4-manifold with positive scalar curvature. Let $P$ be a principal $G$-bundle on $X$ where $G$ is a compact semi-simple Lie group. Then the moduli space $\widehat{\mathcal{M}}_-(P)$ of irreducible self-dual connections on $P$ is a $C^\infty$ manifold, and its tangent space at $[A]$ is naturally isomorphic to $H_A^1$.*

## §3.2. $B_2$-connections on quaternionic Kähler manifolds

Salamon [Sa] generalized the twister theory to higher dimensional manifolds called quaternionic Kähler manifolds. He pointed out the correspondence between certain connections (we call them $B_2$-connections after [Ni1]) and holomorphic vector bundles on the twister space. Nitta [Ni1], [Ni2] studied further the relation between the moduli space of $B_2$-connections and that of holomorphic vector bundles on the twister space.

**Definition 3.5.** An $n = 4k (k \geqq 2)$-dimensional Riemannian manifold $(X, g)$ is called *quaternionic Kähler* if there exists a covering $U_i$ of $X$ and three almost complex structures $I$, $J$ and $K$ on $U_i$ for each $i$ such that

(1) $g$ is hermitian for $I$, $J$ and $K$ on $U_i$,
(2) $K = IJ = -JI$,
(3) the subbundle of $\mathrm{End}(TX)$ spanned by $I$, $J$ and $K$ patches together on the intersection $U_i \cap U_j$, and is parallel with respect to the Levi-Civita connection $\nabla$. (We denote by $A_2'$ this vector bundle of rank 3.)

We remark that a quaternionic Kähler manifold $(X, g)$ is Einstein.

Towards the end of this section we assume $(X, g)$ is a quaternionic Kähler manifold. We define three local 2-forms $\omega_I$, $\omega_J$, $\omega_K$ and a 4-form $\Omega$ by

$$\begin{aligned} &\omega_I(v, w) = g(Iv, w), \\ &\omega_J(v, w) = g(Jv, w), \\ &\omega_K(v, w) = g(Kv, w) \quad \text{for } v,\ w \in TX, \\ &\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2. \end{aligned}$$

Then $\Omega$ is globally well-defined, parallel (i.e. $\nabla\Omega = 0$), non-degenerate (i.e. $\Omega^k \neq 0$) and called the *fundamental* 4*-form* of $(X, g)$.

Let $\mathbb{H} = \mathbb{C} + j\mathbb{C}$ be the quaternion field. The symplectic group $Sp(k)$ acts $\mathbb{H}^k$ on the left. We denote by $j^{(k)}$ the multiplication on $\mathbb{H}^k$ by $j$ from the right. We consider the real structure of $\mathbb{H}^k \otimes \mathbb{H}^1$ defined by

$$\overline{h \otimes h'} = j^{(k)} h \otimes j^{(1)} h' \qquad \text{for } h \in \mathbb{H}^k, h' \in \mathbb{H}^1. \tag{3.6}$$

Let us write $(\mathbb{H}^k \otimes \mathbb{H}^1)_{\mathbb{R}}$ the corresponding real form. It is known that $(X, g)$ is quaternionic Kähler if and only if its holonomy group is contained in $Sp(k)Sp(1) = Sp(k) \times Sp(1)/\mathbb{Z}_2$. Hence we have a reduction of the frame bundle of the tangent bundle $TX$ to a principal $Sp(k)Sp(1)$-bundle $P$. Then the tangent bundle $TX$ can be regarded as the associated vector bundle

$$P \times_{Sp(k)Sp(1)} (\mathbb{H}^k \otimes \mathbb{H}^1)_{\mathbb{R}}.$$

Let $Z$ be the unit sphere bundle of $A_2'$, and $\pi\colon Z \to X$ be the projection. We call $Z$ the *twistor space* of the quaternionic Kähler manifold

$(X, g)$. The Levi-Civita connection $\nabla$ induces a connection on $A_2'$ and also a splitting of the tangent bundle

$$TZ = T^v Z \oplus T^h Z$$

into the vertical and horizontal subspaces. Each point $z \in Z$ is identified with an almost complex structure $aI + bJ + cK$ on $T_{\pi(z)}X$ where $a^2 + b^2 + c^2 = 1$. Since $T_z^h Z$ is identified with $T_{\pi(z)}X$ via the map $\pi^*$, we have a natural almost complex structure on $T_z^h Z$. The vertical subspace $T_z^v Z$ also has a natural almost complex structure induced from that of the fiber $S^2$. Combining these, we have an almost complex structure on $TZ$. Then we have the following:

**Theorem 3.7** [Sa]. *The almost complex structure on the twistor space $Z$ of a quaternionic Kähler manifold is integrable.*

The twistor space $Z$ has a natural real structure $\tau: Z \to Z$ as in the self-dual case. It is an anti-holomorphic map preserving each fiber. We can define a natural Riemannian structure on the twistor space $Z$ as in self-dual case. It is a Kähler-Einstein metric if $(X, g)$ has positive scalar curvature.

As before we identify the space $\mathrm{Skew}(T_xX)$ of skew-symmetric endomorphisms of $T_xX$ with $\Lambda^2 T_x^*X$ for $x \in X$. So we consider $(A_2')_x$ as a subspace of $\Lambda^2 T_x^*X$. Let $(B_2)_x$ be the centralizer of $(A_2')_x$ in $\mathrm{Skew}(T_xX)$. Namely

$$A \in (B_2)_x \Longleftrightarrow AI - IA = AJ - JA = AK - KA = 0.$$

We have an orthogonal decomposition of the bundle $\Lambda^2 T^*X$:

$$\Lambda^2 T^*X \cong B_2 \oplus A_2' \oplus A_2'',$$

where $A_2''$ is the orthogonal complement of $B_2 \oplus A_2'$.

Now let $(E, H)$ be a Hermitian vector bundle on $(X, g)$.

**Definition 3.8** [CS],[Ni1]. A connection $A$ on $(E, H)$ is called *$B_2$-connection* if its curvature form $R_A$ has its value in $\mathrm{Ad}(E) \otimes B_2$. (In [CS] it is called *self-dual connection.*)

We denote by $\mathcal{M}_{B_2}(E)$ the moduli space of $B_2$-connections on $E$. The definition of $B_2$-connections is a natural generalization of that of self-dual connections on 4-manifolds. It was shown in [CS] that there exist constants $c_1$, $c_2$ and $c_3$ with $0 < c_3 < -c_1 < c_2$ such that for a general connection $A$

$$\int_X \operatorname{tr}(R_A \wedge R_A) \wedge \Omega^{k-1} = -\int_X \sum c_i |R_i|^2 \Omega^k,$$

where $R_A = R_1 + R_2 + R_3$ is the decomposition of the curvature according to the decomposition $\Lambda^2 T^*X \cong B_2 \oplus A_2' \oplus A_2''$. Since the left-hand side is a topological invariant, we find

**Proposition 3.9** [CS],[Ni1]. *If $A$ is a $B_2$-connection, it is a Yang-Mills connection. In fact, it minimizes the Yang-Mills functional.*

When the manifold $(X, g)$ has positive scalar curvature, its twistor space has a natural Kähler metric. In this case we have

**Proposition 3.10** [Ni1]. *When $(X, g)$ has positive scalar curvature, there is a natural bijective correspondence induced by $\pi^*$ between a $B_2$-connection $A$ on a bundle $(E, H)$ and a Ricci-flat Einstein-Hermitian connection $B = \pi^* A$ on the pull back bundle $(\widetilde{E}, \widetilde{H}) = (\pi^* E, \pi^* H)$ satisfying the following:*

(1) *$B$ is trivial along each fiber,*
(2) *$B$ commutes with the real structure $\tau$.*

*Moreover the natural mapping*

$$\pi^* \colon \mathcal{M}_{B_2}(E) \to \mathcal{M}_{EH}(\widetilde{E})$$

*is injective.*

The proof is similar to the self-dual case. The key point is the observation that the pull backs of 2-forms with value in $B_2$-part are orthogonal to the Kähler form, so lie in $\Omega_0^{1,1}(\mathrm{Ad}(\widetilde{E}))$.

For an integer $r$ with $r \geqq 2$, define $A_r^{\mathbb{C}}$ by $\Lambda^r(\mathbb{H}^k)^* \otimes_{\mathbb{C}} S^r(\mathbb{H}^1)^*$, and $B_r^{\mathbb{C}}$ by its orthogonal complement. Then $A_r^{\mathbb{C}}$ and $B_r^{\mathbb{C}}$ admit real forms $A_r$ and $B_r$ respectively induced by the real structure (3.6). Hence we have a decomposition $\Lambda^r(\mathbb{H}^k \otimes \mathbb{H}^1)^*_{\mathbb{R}} = A_r \oplus B_r$. Since the cotangent bundle $T^*X$ is the associated vector bundle $P \times_{Sp(k)Sp(1)} (\mathbb{H}^k \otimes \mathbb{H}^1)^*_{\mathbb{R}}$, we have a corresponding decomposition

$$\Lambda^r T^* X = A_r \oplus B_r.$$

Now we consider the moduli space $\mathcal{M}_{B_2}(E)$ of $B_2$-connections. First we study its tangent space. Let $\Omega_+^r(\mathrm{Ad}(E))$ be $\Omega^0(A_r \otimes \mathrm{Ad}(E))$. We have the following elliptic complex ([CS], [Ni1])

$$0 \longrightarrow \Omega^0(\mathrm{Ad}(E)) \xrightarrow{d_A} \Omega^1(\mathrm{Ad}(E)) \xrightarrow{d_A^+} \Omega^2_+(\mathrm{Ad}(E)) \xrightarrow{d_A^+}$$
$$\xrightarrow{d_A^+} \Omega^3_+(\mathrm{Ad}(E)) \xrightarrow{d_A^+} \cdots \xrightarrow{d_A^+} \Omega^{2k}_+(\mathrm{Ad}(E)) \longrightarrow 0,$$

where $d_A^+ = p_+ \circ d_A$ and $p_+$ is the projection to $A_r \otimes \mathrm{Ad}(E)$. As in §2 we consider the $i$-th harmonic space $H^i_A$. We say a $B_2$-connection $A$ is *generic* if it satisfies $H^0_A = 0$ and $H^2_A = 0$.

**Theorem 3.11** [Ni1]. *The moduli space $\widehat{\mathcal{M}}_{B_2}(E)$ of generic $B_2$-connections on $E$ is a $C^\infty$-manifold and its tangent space is isomorphic to $H^1_A$.*

Now we assume $(X,g)$ has positive scalar curvature. As remarked in the above the twistor space has the natural Kähler-Einstein form $\omega$. Recall that the tangent space of the moduli space $\widehat{\mathcal{M}}_{EH}(\widetilde{E})$ of generic Einstein-Hermitian connections at $[B] = [\pi^* A]$ is isomorphic to $H^1_B(Z, \mathrm{Ad}(\widetilde{E}))$ (we write base spaces and bundles of cohomology groups precisely to avoid the confusion). We have the natural map

$$\pi^*: H^1_A(X, \mathrm{Ad}(E)) \to H^1_B(Z, \mathrm{Ad}(\widetilde{E})).$$

This map is the differential of the map $\pi^*: \mathcal{M}_{B_2}(E) \to \mathcal{M}_{EH}(\widetilde{E})$, and it is shown in [Ni2] that this map is injective. In §2 we define a Riemannian structure ( , ) and an almost complex structure $J_Z$ on $H^1_B(Z, \mathrm{Ad}(\widetilde{E}))$ by

$$(\alpha, \beta) = \int_Z (\alpha, \beta) dV$$
$$J_Z \alpha = (J \otimes \mathrm{Id})\alpha \qquad \text{for } \alpha, \beta \in H^1_B(Z, \mathrm{Ad}(\widetilde{E})),$$

where $J$ is the almost complex structure of $Z$. If we fix $x \in X$, we have

$$\int_{\pi^{-1}(x)} (\pi^*(\alpha), J_Z \pi^*(\beta)) dV = 0$$

for $\alpha, \beta \in T^*_x X \otimes \mathrm{Ad}(E)_x$. Thus it holds $(\pi^*\alpha, J_Z \pi^*\beta) = 0$ for $\alpha, \beta \in H^1_A(X, \mathrm{Ad}(E))$. Namely the inclusion map $\pi^*: \widehat{\mathcal{M}}_{B_2}(E) \to \widehat{\mathcal{M}}_{EH}(\widetilde{E})$ is totally real.

Since $\widetilde{E} = \pi^* E$ is trivial on each fiber of $\pi: Z \to X$ the real structure $\tau$ induces a bundle map (also denoted by $\tau$) which makes the following commutative diagram:

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\tau"] \arrow[d] & \widetilde{E} \arrow[d] \\ Z \arrow[r, "\tau"] & Z \end{tikzcd}$$

Then $\tau$ induces a mapping (also denoted by $\tau$)

$$\begin{aligned} \mathcal{M}_{EH}(\widetilde{E}) &\to \mathcal{M}_{EH}(\widetilde{E}) \\ [B] &\mapsto [\tau^*(B)]. \end{aligned}$$

Since $\tau: Z \to Z$ is anti-holomorphic, the above map $\tau$ is also anti-holomorphic. Moreover it is easy to see $\tau$ is an isometry with respect to the metric ( , ) induced from the $L^2$-inner product. Since $\pi^*(A)$ commutes with $\tau$ (3.10), $\pi^*(\mathcal{M}_{B_2}(E))$ is fixed by $\tau: \mathcal{M}_{EH}(\widetilde{E}) \to \mathcal{M}_{EH}(\widetilde{E})$. Thus we have

**Theorem 3.12** [Ni2]. *The natural map $\pi^*: \widehat{\mathcal{M}}_{B_2}(E) \to \widehat{\mathcal{M}}_{EH}(\widetilde{E})$ induced by the pull-back gives a totally real embedding and its image is fixed by the real structure $\tau$. Moreover we have $\dim_{\mathbb{R}} \widehat{\mathcal{M}}_{B_2}(E) = \dim_{\mathbb{C}} \widehat{\mathcal{M}}_{EH}(\widetilde{E})$.*

See §4.5 for examples of $B_2$-connections.

## Chapter 4. Analysis for Yang-Mills Equations

### §4.1. A priori estimates for Yang-Mills connections

In this section we shall state Uhlenbeck's *a priori* estimates for Yang-Mills connections [Uh2], [Uh3]. They relate to a compactification of the moduli space. In dimension $n \leqq 4$ the analytical treatment for the Yang-Mills equation was established by Uhlenbeck in [Uh1], [Uh2]. In fact, she derived $C^0$-estimates of the curvature, and hence higher derivatives, under the assumption that the $L^{n/2}$ norm of the curvature is small. But in dimension $n > 4$ this is not a natural condition. The natural condition is that the normalized $L^2$ norm of the curvature is small. Under this condition the estimates for Yang-Mills type equation (including lower order term) were obtained in [Uh3]. They were used in constructing Einstein-Hermitian connections on stable bundles in the joint work of herself with Yau [UY]. For the pure Yang-Mills equation a simpler theory was obtained by the second named author [Na1].

Let $B_\tau(x)$ be a metric ball in an $m$-dimensional Kähler manifold $(X, g)$. Suppose that $\tau$ is smaller than the injectivity radius of $(X, g)$, and the following estimates hold for some positive constants $K_0$ at $y \in B_\tau(x)$

$$
\begin{aligned}
&|g_{i\bar{j}}(y) - \delta_{i\bar{j}}| \leqq K_0 d(x,y)^2, \\
&|\frac{\partial}{\partial z^k} g_{i\bar{j}}(y)| \leqq K_0 d(x,y),
\end{aligned}
\tag{4.1}
$$

where $(g_{i\bar{j}})$ is a metric tensor in holomorphic normal coordinates on $B_\tau(x)$. The constant $K_0$ depends on the curvature of $g$. We assume that $K_0\tau^2 \leqq 1/2$. It is satisfied if we take sufficiently small $\tau$.

We fix a $U(r)$-principal bundle $P$ and the associated complex vector bundle $E$ with the fiber metric $H$. Let $A$ be an integrable connection on a bundle $E$ over $B_\tau(x)$. Suppose a connection $A$ satisfies the equation

$$K_A = K$$

for some $K \in \Omega^0(\mathrm{Ad}(P)) \otimes \mathbb{C}$.

As a partial differential equation, the Yang-Mills equation is not elliptic because of the invariance of the Yang-Mills functional under the action of the gauge group. In order to get an elliptic equation, we must take a good local trivialization. Via a local (unitary) trivialization it can be written as

$$
\begin{aligned}
&E|B_\tau(x) \cong B_\tau(x) \times \mathbb{C}^r \\
&D = d + A,
\end{aligned}
$$

where $d$ is the product connection on $B_\tau(x) \times \mathbb{C}^r$ and $A$ is a $\mathfrak{u}(r)$-valued 1-form. Changes of local trivialization can be done by a composition of a map $s\colon B_\tau(x) \to U(r)$. The connection form $A$ changes as

$$A' = s^{-1}ds + s^{-1}As.$$

**Definition 4.2.** A local trivialization $E|B_\tau(x) \cong B_\tau(x) \times \mathbb{C}^r$ is called a *Coulomb gauge* for a connection $D$, if the connection form $A$ satisfies

$$d^*A = 0,$$

where $d^*$ is the adjoint operator of the product connection $d$.

We write the two equations $d^*A = 0$, $K_A = K$ and the integrability of $A$ in terms of $A$:

$$\begin{cases} d^*A = 0 \\ (dA + [A,A])^{2,0} = 0 \\ \mathrm{tr}_\omega(dA + [A,A]) = K. \end{cases}$$

Then this is a first order elliptic system on $A$. So we can apply the standard theory for elliptic equations to get *a priori estimates* on $A$.

Thus the remaining problem is *when can we take Coulomb gauges for a connection $D$* ? It was answered in [Uh2]. Fixing a local trivialization $P|B_\tau(x) \cong B_\tau(x) \times U(r)$, finding a Coulomb gauge is equivalent to solving the following equation on $s\colon B_\tau(x) \to U(r)$:

$$d^*(s^{-1}ds + s^{-1}As) = 0.$$

We remark that this equation coincides with the harmonic map equation when $A = 0$.

**Theorem 4.3** [Uh2],[Uh3]. *There exists $\varepsilon_m = \varepsilon_m(m, r, K_0\tau^2) > 0$ such that if*

$$\|R_A\|_{L^m(B_\tau(x))} \leqq \varepsilon_m, \tag{4.4}$$

*then there exists a unique local trivialization (up to constant rotation of the fiber ) in which $d^*A = 0$ on $B_\tau(x)$, $\nu \lrcorner A = 0$ on $\partial B_\tau(x)$ (where $\nu$ is the normal vector field on $\partial B_\tau(x)$), and it holds*

$$\int_{B_\tau(x)} |\nabla A|^m dV_g + \left(\int_{B_\tau(x)} |A|^{2m} dV_g\right)^{1/2} \leqq C \int_{B_\tau(x)} |R_A|^m dV_g,$$

*where $C = C(m, r, K_0\tau^2)$.*

We remark that this theorem holds for general connections. We do not require them to satisfy the equation $K_A = K$.

The following theorem is the main result of this section.

**Theorem 4.5** [Uh3]. *Suppose an integrable connection $A$ on a vector bundle $E$ of rank $r$ over $B_\tau(x)$ satisfies $K_A = K$ for some $K \in \Omega^0(\mathrm{Ad}(P)) \otimes \mathbb{C}$. Then there exists $\varepsilon = \varepsilon(m, r, K_0\tau^2) > 0$ such that if*

$$\tau^{2-m}\|R_A\|_{L^2(B_\tau(x))} + \tau^2 \sup_{B_\tau(x)} |K| \leqq \varepsilon,$$

*then for $p > 2$*

$$\tau^{2p-2m}\|R_A\|_{L^p(B_{\tau/2}(x))} \leqq C(\tau^{2-m}\|R_A\|_{L^2(B_\tau(x))} + \tau^2 \sup_{B_\tau(x)} |K|),$$

*where* $C = C(m, r, K_0\tau^2, p)$.

We should remark that each term appeared in the above (e.g., $\tau^{2-m}\|R_A\|_{L^2(B_\tau(x))}$) is invariant under the scale change $(X, g) \to (X, cg)$ for a positive constant $c$.

From Theorem 4.3 and Theorem 4.5 we obtain a compactness result of moduli spaces of integrable connections with a universal bound of Einstein-Hermitian tensor.

**Theorem 4.6.** *Let* $\{A_i\}$ *be a sequence of integrable connections on a bundle* $E \to X$ *such that*

$$\int_X |R_{A_i}|^2 dV \leqq R, \quad \sup_X |K_{A_i}| \leqq R$$

*for some constant* $R$. *Then there exist a subsequence* $\{A_j\} \subset \{A_i\}$, *a sequence of gauge transformations* $\{\gamma_j\}$ *and a compact set* $S \subset X$ *with finite* $(2m-4)$*-dimensional Hausdorff measure* $H_{2m-4}(S) < \infty$ *such that* $\gamma_j^*(A_j)$ *converges to* $A_\infty$ *weakly in* $W^{1,p}_{loc}(X \setminus S)$ *for all* $p > 1$ *where* $A_\infty$ *is an integrable connection on* $E|X \setminus S$.

The proof is standard and given in [Se], [Sc], [Na].

## §4.2. Removable singularities theorem

In this section we shall prove the removable singularities theorem for Yang-Mills connections on 4-manifolds [Uhl]. The proof given here is same as that for the removable singularities theorem for Einstein metrics proved by the second-named author in the joint work with Bando and Kasue [BKN]. The key of the proof is Yau's trick [SSY] (see [Na3] for a survey of Yau's trick appeared in other situation).

**Theorem 4.7** [Uhl]. *Let* $A$ *be a Yang-Mills connection on a bundle* $P$ *over the punctured ball* $B^* = B \setminus \{0\}$ *in* $\mathbb{R}^4$. *If* $A$ *satisfies*

$$\int_{B^*} |R_A|^2 dx < \infty,$$

*then the bundle* $P$ *and the connection* $A$ *are extended smoothly to the whole ball* $B$.

We only show that the curvature $R_A$ is bounded on $B^*$. The remainder of the proof is standard (cf. [Uhl], [FU]).

**Lemma 4.8.** *Let $R_A$ be the curvature of a self-dual connection. Then for $\delta = 1/2$ we have*

$$(1+\delta)|(\nabla R_A, R_A)|^2 \leqq |\nabla R_A|^2 |R_A|^2,$$

*where $(\nabla R_A, R_A)$ is a 1-form defined by $(\nabla R_A, R_A)(v) = (\nabla_v R_A, R_A)$ for a tangent vector $v$.*

*Proof.* We use the index notation for tensors. Let $R = R_A$. We may suppose 1-form $(\nabla R, R) = \sum_{ij}(\nabla_p R_{ij}, R_{ij})$ is non-zero only if $p = 1$. From the Schwartz inequality we have

$$|(\nabla R, R)| \leqq |R|^2 |\nabla_1 R|^2,$$

where $\nabla_1 R$ means a contraction by $\partial/\partial x^1$. Since $R$ is self-dual, we have $\nabla R = *\nabla R$. Hence we have

$$\begin{aligned}
&|\nabla_1 R|^2 = 2(|\nabla_1 R_{23}|^2 + |\nabla_1 R_{24}|^2 + |\nabla_1 R_{34}|^2) \\
=&2(|\nabla_2 R_{13} - \nabla_3 R_{12}|^2 + |\nabla_2 R_{14} - \nabla_4 R_{12}|^2 + |\nabla_3 R_{14} - \nabla_4 R_{13}|^2) \\
\leqq&4(|\nabla_2 R_{13}|^2 + |\nabla_3 R_{12}|^2 + |\nabla_2 R_{14}|^2 \\
&\qquad + |\nabla_4 R_{12}|^2 + |\nabla_3 R_{14}|^2 + |\nabla_4 R_{13}|^2) \\
\leqq&2(|\nabla_2 R|^2 + |\nabla_3 R|^2 + |\nabla_4 R|^2),
\end{aligned}$$

where in the second equality we have used the Bianchi identity. So we get

$$\frac{3}{2}|(\nabla R, R)|^2 \leqq |R|^2 |\nabla R|^2.$$

Q.E.D.

**Corollary 4.9.** *If $A$ is a Yang-Mills connection, then*

$$\Delta |R_A|^{1-\delta} \geqq -C|R_A|^{2-\delta},$$

*for some $C$.*

*Proof.* First suppose $R_A$ is self-dual. By Weitzenböck formula, we have

$$\Delta |R_A|^2 \geqq 2|\nabla R_A|^2 - C|R_A|^3$$

for some constant $C$. On the other hand by Lemma 4.8, we get

$$|\nabla R_A|^2 \geqq (1+\delta)|(\nabla R_A, R_A)|^2 |R_A|^{-2} \geqq (1+\delta)|\nabla |R_A||^2.$$

Combining these we obtain

$$\begin{aligned}
&\Delta|R_A|^{1-\delta}\\
&\geqq -(1-\delta)(1+\delta)|R_A|^{-1-\delta}|\nabla|R_A||^2 + (1-\delta)|\nabla R_A|^2|R_A|^{1-\delta}\\
&\qquad - C|R_A|^{2-\delta}\\
&\geqq -C|R_A|^{2-\delta}.
\end{aligned}$$

This shows the conclusion in case of self-dual connections. In general case, we decompose $R_A = R_A^+ + R_A^-$. The above discussion shows $\Delta|R_A^+|^{1-\delta} \geqq -C|R_A||R_A^+|^{1-\delta}$, and $\Delta|R_A^-|^{1-\delta} \geqq -C|R_A||R_A^-|^{1-\delta}$. Adding these, we find $\Delta|R_A|^{1-\delta} \geqq -C|R_A|^{2-\delta}$. Q.E.D.

Combining the above differential inequality with the following analytical lemma, we immediately get the boundedness of $R_A$ on $B^*$.

**Lemma 4.10** cf. [BKN, Lemmas 5.8, 5.9]. *Suppose positive functions $f \in L^2$ and $u \in L^q (q > 2)$ satisfy $\Delta u \geqq -fu$ on $B^*$. Then $u$ is in $L^p$ for any $p \in (1, \infty)$. If $f \in L^q$ $(q > 2)$, then $u$ is bounded on $B^*$.*

For our proof the precise value for $\delta$ is not important. It is sufficient to prove Lemma 4.8 for some $\delta > 0$. But it becomes important when we study the decay order of $|R_A(x)|$ as $|x| \to \infty$ in the exterior region $\mathbb{R}^4 \setminus B$. For instance for the Einstein metric we have a better estimate $\delta = 2/3$ using more symmetries of the curvature tensor of Einstein metric than that of self-dual connection (see [BKN]). Then the decay order becomes $O(|x|^{-6})$, though we have only $O(|x|^{-4})$ for Yang-Mills connection.

## §4.3. Examples of convergence of Yang-Mills connections

In this section we shall give examples of convergence of Yang-Mills connections. The results in this section were obtained by Doi and Okai [DO] and Nitta [Ni2] independently. Other examples of convergence will be given in the proof of Theorem 5.9.

Let $L$ be the tautological quaternionic line bundle on the quaternionic projective $n$-space $\mathbb{P}^n(\mathbb{H})$. We consider $E$ the orthogonal complement of $L$ in the trivial bundle $\mathbb{P}^n(\mathbb{H}) \times \mathbb{H}^{n+1}$. We denote by $\mathcal{M}_{B_2}(E)$ the moduli space of $B_2$ connections on $E$. Then we have

**Theorem 4.11** [DO],[Ni2]. *The moduli space $\mathcal{M}_{B_2}(E)$ is isomorphic to the space $SL(n+1, \mathbb{H})/Sp(n+1)$.*

When the dimension $n = 1$, the above theorem was shown by Atiyah, Hitchin and Singer [AHS].

The homogeneous space $SL(n+1,\mathbb{H})/Sp(n+1)$ is identified with the set of positive definite quaternionic Hermitian matrices

$$\mathcal{P} = \{h \in SL(n+1,\mathbb{H}) \mid {}^t\overline{h} = h, h > 0\}.$$

We can consider a compactification of this set.

**Theorem 4.12** [DO],[Ni2]. *A compactification of the moduli space $\mathcal{M}_{B_2}(E)$ is given by*

$$\overline{\mathcal{P}} = \{h \in M(n+1,\mathbb{H}) \mid {}^t\overline{h} = h, h \geqq 0, h \neq 0\}/\mathbb{R}^+.$$

Suppose a sequence $\{h_i\} \subset \mathcal{P}$ converges to $h_\infty \in \overline{\mathcal{P}}$ in the usual topology for the homogeneous space. Then the corresponding connections $A_i$ converges to $A_\infty$ outside the set

$$\mathcal{S}_{h_\infty} = \{x \in \mathbb{P}^n(\mathbb{H}) \mid h_\infty \widetilde{x} = 0\},$$

where $\widetilde{x}$ is a representative of $x$ in $\mathbb{H}^{n+1}$. Hence we have

$$\dim_{\mathbb{R}} \mathcal{S}_{h_\infty} \leqq 4n-4,$$

and the equality holds if and only if $\operatorname{rank}_{\mathbb{H}} h_\infty = 1$.

## Chapter 5. Existence Theorems for Yang-Mills Connections

### §5.1. Anti-self-dual connections on 4-manifolds

Existence results of anti-self-dual connections on 4-manifolds which are not covered by twister methods were obtained by Taubes [Ta1], [Ta2]. In this section we shall give the outline of the proof of his results. For details the reader should see the original papers.

We prepare the notation. Let $P$ be a principal $SU(2)$-bundle over a compact oriented connected 4-dimensional Riemannian manifold $(X,g)$ with $c_2(P) = k$. We denote by $b_+$ the dimension of the space $H^2_+(X)$ of closed self-dual 2-forms on $(X,g)$. We take an $L^2$-orthonormal basis $\{\omega_n\}_{n=1}^{b_+}$ for $H^2_+(X)$. Let $\pi: F^+ \to X$ denote the orthonormal frame bundle associated with the vector bundle of self-dual 2-forms $\Lambda^+(X)$. Each point $y \in F^+$ defines a linear isomorphism, also denoted by $y$ between $\Lambda^+(X)$ and $\mathbb{R}^3$. Define the space $F(k)$ by

$$F(k) := \{(y_i,\lambda_i)_{i=1}^k \in \prod^k (F^+ \times \mathbb{R}^+) \mid \pi(y_i) \neq \pi(y_j) \quad \text{if } i \neq j\}/\Sigma_k,$$

where the permutation group $\Sigma_k$ acts by the interchange of factors. We define a map $h: F(k) \to \mathbb{R}^{3b_+}$ by

$$h(p) = \Big(\sum_{i=1}^{k} \lambda_i^2\, y_i\,(\omega_n(x_i))\,\Big)_{n=1}^{b_+} \quad \text{for } p = (y_i, \lambda_i)_{i=1}^{k},$$

where $x_i = \pi(y_i)$. Using the map $h$, we can give a sufficient condition for the existence of anti-self-dual connections on $P$ as follows.

**Theorem 5.1** [Ta2] (see also [Do3]). *If there exists $p \in F(k)$ such that*

(1) $h(p) = 0$,
(2) $dh_p: T_pF(k) \to \mathbb{R}^{3b_+}$ *is surjective,*

*then there exists an anti-self-dual connection on all principal $SU(2)$-bundles $P \to X$ with $c_2(P) = k$.*

As applications of the above theorem, we have a sufficient condition for the existence of anti-self-dual connections on $P$ under certain relations between $c_2(P)$ and $b_+$

**Theorem 5.2** [Ta2]. *If the following relation between $c_2(P)$ and $b_+$ is satisfied, then there exists irreducible anti-self-dual connection on $P$:*

(1) $c_2(P) \geqq \max(\frac{4}{3}b_+, 1)$ *when* $b_+ \neq 2$,
(2) $c_2(P) \geqq 4$ *when* $b_+ = 2$.

In the following we give an outline of the proof of Theorem 5.1. Fix a connection $A_0 \in \mathcal{A}(P)$. Any connection $A \in \mathcal{A}(P)$ can be written as $A = A_0 + \alpha$ for some $\alpha \in \Omega^1(\mathrm{Ad}(P))$. Then $A$ is anti-self-dual if and only if $\alpha$ satisfies

$$d_{A_0}^+\alpha + \frac{1}{2}[\alpha \wedge \alpha]^+ + R_{A_0}^+ = 0 \tag{5.3}$$

where "+" means the projection to the self-dual part $\Omega^+(\mathrm{Ad}(P))$. Since the equation (5.3) for $\alpha$ is not elliptic, we write $\alpha = d_{A_0}^* u$ for $u \in \Omega^+(\mathrm{Ad}(P))$ and replace (5.3) by

$$d_{A_0}^+ d_{A_0}^* u + \frac{1}{2}[d_{A_0}^* u \wedge d_{A_0}^* u]^+ + R_{A_0}^+ = 0. \tag{5.4}$$

The idea to solve the equation (5.4) is to use the implicit function theorem. The elliptic operator $d_{A_0}^+ d_{A_0}^*$ has discrete spectra consisting on nonnegative eigenvalues $\{\lambda_i\}$. If we can take a connection $A_0$ so that the certain norm of $R^+_{A_0}$ is sufficiently small in comparison with the first eigenvalue $\lambda_1$, we can expect to find a connection near $A_0$ which is anti-self-dual using the implicit function theorem for the operator defined by the left-hand side of (5.4). We shall construct $A_0$ to be close to the trivial connection in a certain sense. For the trivial connection, the kernel of $d^+d^*: \Omega^+(X) \to \Omega^+(X)$ corresponds to the space of closed self-dual forms. So when $b_+ = 0$, we can use the implicit function theorem directly [Ta1], [FU], [Na2]). But if $b_+ > 0$, this proof does not work directly. Hence we divide the equation (5.4) into the two equations:

$$(5.5) \qquad d^+_{A_0} d^*_{A_0} u + \Pi^{\perp}_E(A_0)(\frac{1}{2}[d^*_{A_0} u \wedge d^*_{A_0} u]^+ + R^+_{A_0}) = 0,$$

$$(5.6) \qquad \Pi_E(A_0)(\frac{1}{2}[d^*_{A_0} u \wedge d^*_{A_0} u]^+ + R^+_{A_0}) = 0,$$

where $\Pi_E(A_0): \Omega^+(\mathrm{Ad}(P)) \to \Omega^+(\mathrm{Ad}(P))$ is the $L^2$-orthogonal projection to the sum of the eigenspaces of the operator $d^+_{A_0} d^*_{A_0}$ for eigenvalues less than $E > 0$, and $\Pi_E(A_0)^{\perp} = \mathrm{Id} - \Pi_E(A_0)$. If we choose suitably $E$ and $A_0$, the equation (5.5) has a unique solution $u$ in the image of $\Pi_E(A_0)$ which is close to 0. Hence the problem is reduced to solving the equation (5.6). In order to find a solution of (5.6), we introduce a family of connections $\mathcal{N}$ parametrized by a finite dimensional manifold. When the connection $A_0$ varies in $\mathcal{N}$, the left-hand side of (5.6) defines a section of a certain vector bundle over $\mathcal{N}$. Thus the problem is to find a zero set of this section. Roughly speaking an element of $\mathcal{N}$ corresponds to points where the curvature of the connection concentrates and radii which represent how concentrates the curvature.

Recall that on $S^4$ we have a principal $SU(2)$-bundle $P_0$ with $c_2(P_0) = 1$ and a family of anti-self-dual connections $A_\lambda$ for $\lambda > 0$ defined by

$$A_\lambda(x) = \mathrm{Im}\left(\frac{\overline{x} \cdot dx}{\lambda^2 + |x|^2}\right) \qquad \text{for } x \in S^4 \setminus \{\infty\} \cong \mathbb{H},$$

$$A_\lambda(y) = \mathrm{Im}\left(\frac{\lambda^2 \overline{y} \cdot dy}{1 + \lambda^2 |y|^2}\right) \qquad \text{for } y = x^{-1} \in S^4 \setminus \{0\} \cong \mathbb{H},$$

with the transition function $\overline{x}/|x| \in Sp(1)$ on $\mathbb{H} \setminus \{0\}$. The curvature of $A_\lambda$ is given on $\mathbb{H}$ by

$$R_\lambda(x) = \frac{\lambda^2 d\overline{x} \wedge dx}{(\lambda^2 + |x|^2)^2}.$$

Let $\pi: F(X) \to X$ be the orthonormal frame bundle over $X$ with the structure group $SO(4)$. A point $f \in F(X)$ defines an orientation preserving isometry (denoted also by $f$) from $\mathbb{R}^4$ to $T_xX$ for $x = \pi(f)$. Composing with the exponential map at $x$ we have a map $\exp_f = \exp \circ f: \mathbb{R}^4 \to X$ for each $f \in F(X)$. Let $i(X)$ be the injective radius of $X$.

Let $\lambda_0 = \frac{i(x)^2}{16}$. For a positive integer $k$ and $\lambda \in (0, \lambda_0)$, we define an open subset $\mathcal{N}_1(k, \lambda)$ in $\prod^k (F(X) \times (0, \lambda))$ by the following; $z = (f_i, \lambda_i)_{i=1}^k \in \mathcal{N}_1(k, \lambda)$ if and only if

(i) $d(z) := \min_{i \neq j} \operatorname{dist}(\pi(f_i), \pi(f_j)) > 0$, and

(ii) $\lambda_i \leqq \dfrac{1}{64} d^2(z)$ for each $i$.

Fix a cut-off function $\beta: [0, \infty) \to [0, 1]$ such that

(i) $\beta(r) \equiv 1$ on $[0, 1]$,

(ii) $\beta(r) \equiv 0$ on $[2, \infty)$,

(iii) $|d\beta| \leqq 2$,

(iv) $\beta$ is monotone decreasing.

For $x \in \mathbb{R}^4$ and $r > 0$, set $\beta_r(x) := \beta(\dfrac{|x|}{r})$.

For each $z = (f_i, \lambda_i)_{i=1}^k \in \mathcal{N}_1$ we shall construct a principal $SU(2)$ bundle $P_z$ and a connection $A_z$. First for a point $x$ in a small ball $B_{3\sqrt{\lambda_i}}(\pi(f_i))$ we define a map $F$ to $S^4$ by

$$F(x) = \frac{y}{\beta_{2\sqrt{\lambda_i}}(y)} \qquad \text{for } y = (\exp_{f_i})^{-1}(x)$$

where we identify $S^4$ with $\mathbb{R}^4 \cup \{\infty\}$. We pull back the bundle $P_0 \to S^4$ and a connection $A_{\lambda_i}$ by the map $F$ on $B_{3\sqrt{\lambda_i}}(\pi(f_i))$ on each $i$. Since $B_{3\sqrt{\lambda_i}}(\pi(f_i)) \setminus B_{2\sqrt{\lambda_i}}(\pi(f_i))$ is mapped to the north pole of $S^4$, we can extend the bundle and the connection to the whole $X$ as the trivial bundle and the trivial connection on $X \setminus \bigcup_{i=1}^k B_{3\sqrt{\lambda_i}}(\pi(f_i))$. We denote by $P_z$ and $A_z$ the bundle and the connection respectively.

Since $c_2(P_z) = k$ for all $z$, all the bundles $P_z$ are topologically the same. We denote this by $P$. Choosing an identification $g_z : P \to P_z$ we have a map

$$\begin{aligned} \Psi: \mathcal{N}_1(k, \lambda_0) &\to \mathcal{A}(P)/\mathcal{G}(P) \\ z &\mapsto [g_z^*(A_z)]. \end{aligned}$$

An orthonormal frame $f \in F(X)$ naturally induces an orthonormal frame $\rho(f) \in F^+$ for $\Lambda^+(X)$. This gives the bundle maps $\rho_+: F(X) \to F^+$ and $\rho_+: \prod^k (F(X) \times (0, \lambda_0)) \to \prod^k (F_+ \times (0, \lambda_0))$. It is easy to see the map $\Psi$ factors through $\mathcal{N}_2(k,\lambda_0)=\rho(\mathcal{N}_1(k,\lambda_0))$. Obviously a permutation of the factors of $z=(f_i,\lambda_i)_{i=1}^k$ changes nothing. We denote by $\overline{\mathcal{N}}(k,\lambda_0)$ the quotient of $\mathcal{N}_2(k,\lambda_0)$ by the permutation group $\Sigma_k$.

Now we study properties of the connection $A_z$ constructed above. First the connection $A_z$ is almost anti-self-dual, namely $\|R^+_{A_z}\|_{L^p}$ is small. Secondly we can estimate eigenvalues of $d^+_{A_z}d^*_{A_z}$ as follows. Let $E_0(X)$ be the half of the lowest nonzero eigenvalue of $d^+d^*\colon\Omega^+(X)\to\Omega^+(X)$. For $n=1,\cdots,b_+$ and $a=1,2,3$, we take an orthonormal basis $\{\omega_n\otimes\sigma_a\}_{n=1}^{b_+}{}_{a=1}^{3}$ for $H^2_+(X)\otimes\mathfrak{su}(2)$, where $\sigma_a\in\mathfrak{su}(2)$ is considered as the parallel section of the trivial bundle $X\times\mathfrak{su}(2)$. By the construction of the bundle $P_z$, $\sigma_a$ can also be identified with a section of $\mathrm{Ad}(P)$ over $X\setminus\{\pi(f_1),\cdots,\pi(f_k)\}$. Then we set

$$\omega_{z;n,a}:=\prod_{i=1}^{k}(1-\beta_{\sqrt{\lambda_i}}\circ\exp^{-1}_{f_i})\omega_n\otimes\sigma_a,$$

which is an $\mathrm{Ad}(P)$-valued 2-forms. The basis $\{\omega_{z;n,a}\}_{n=1}^{b_+}{}_{a=1}^{3}$ is nearly orthonormal and satisfies

$$\|d^*_{A_z}\omega_{z;n,a}\|_{L^2}\leqq C\max_i\lambda_i$$

for some constant $C$. If we take $\lambda$ sufficiently smaller than $C^{-1}E_0(X)$, there exist exactly $3b_+$ eigenvectors of $d^+_{A_z}d^*_{A_z}$ with eigenvalues smaller than $C\lambda$ for $y=\rho_+(z)\in\overline{\mathcal{N}}(k,\lambda)$, and the linear subspace spanned by $\{\omega_{z;J,a}\}_{J=1}^{b_+}{}_{a=1}^{3}$ coincides with $\mathrm{Im}(\Pi_{C\lambda}(A_z))$.

Estimating the $R^+_{A_z}$ carefully, one can show by the implicit function theorem technique

**Proposition 5.7** [Ta2, Theorems(3.2),(4.7)]. *For $k>0$ and $0<E\leqq E_0(X)$, there exists $0<\lambda=\lambda(E,k)\leqq\lambda_0$ such that for each $y=\rho_+(z)\in\overline{\mathcal{N}}(k,\lambda)$ there exists a unique solution $u=u_z$ of the equation* (5.5) *which satisfies $\Pi_E(A_z)(u)=0$ and*

$$\|\nabla_{A_z}d^*_{A_z}u\|_{L^2}+\|d^*_{A_z}u\|_{L^2}\leqq C\delta_E(A_z),$$
$$\|d^*_{A_z}u\|_{L^2}\leqq C\zeta_E(A_z)\|R^+_{A_z}\|_{L^{4/3}},$$

*where $C=C(k,E,X)$, and*

$$\zeta_E(A_z):=E^{-1/2}(1+E+\|R^+_{A_z}\|^3_{L^3})^{1/2},$$
$$\delta_E(A_z):=\|R^+_{A_z}\|_{L^2}+\zeta_E(A_z)\|R_{A_z}\|_{L^{4/3}}(1+\|R_{A_z}\|_{L^4}).$$

*Moreover the projection* $\Pi_E(A_z)$ *has the constant rank* $3b_+$ *and the image is isomorphic to* $\mathbb{R}^{3b_+}$ *through* $\{\omega_{z;n,a}\}_{n=1}^{b_+}{}_{a=1}^{3}$.

In particular if $b_+ = 0$, Proposition 5.7 implies the existence of an anti-self-dual connection on $P$ with $c_2(P) = k \geqq 1$.

We define the map $\Phi: \overline{\mathcal{N}}(k,\lambda) \to \mathbb{R}^{3b_+}$ by

$$\Phi(y) = \Pi_E(A_z)(\frac{1}{2}[d^*_{A_z} u_z \wedge d^*_{A_z} u_z] + R^+_{A_z}),$$

where image$(\Pi_E(A_z))$ is identified with $\mathbb{R}^{3b_+}$ by Proposition 5.7. Then $\Phi(y)$ equals to zero if and only if the connection $A_z$ is anti-self-dual.

It is difficult to prove that $\Phi$ has a zero on $\overline{\mathcal{N}}(k,\lambda)$ directly. But one can prove that $h' = \Phi - h$ is small in comparison with $h$. Hence if $h$ has a zero and its differential at a zero does not vanish, then we can prove that $\Phi$ must have a zero by using the implicit function theorem in the finite dimension. In fact, we have [Ta2, Proposition 5.4]

$$|\Phi(y) - h(y)| \leqq O(\max_i \lambda_i^{5/2}).$$

This estimate can be proved by studying the behavior of $A_z$ for $\rho_+(z) \in \overline{\mathcal{N}}(k,\lambda)$ when $\lambda$ tends to 0.

## §5.2. Einstein-Hermitian connections on stable bundles

The moduli space $\mathcal{M}^0_{EH}(P)$ of irreducible Einstein-Hermitian connections over a Kähler manifold is identified with the moduli space of stable holomorphic bundles. Half of this assertion was proved by S.Kobayashi [Kb1]; namely that the holomorphic structure associated to an irreducible Einstein-Hermitian connection is stable. To prove the converse one must solve a nonlinear partial differential equation of elliptic type. Using the continuity method the most essential part is to derive estimates under the assumption that the bundle is stable. It was done by Uhlenbeck and Yau [UY]. Donaldson [Do4] also proved that when the base manifold $X$ is projective. See also [LY] for a generalization to non-Kähler manifolds.

Let $(X, g)$ be an $m$-dimensional compact Kähler manifold with Kähler form $\omega$. We fix a principal $U(r)$-bundle $P$, the associated $C^\infty$ vector bundle $E$ and a fiber metric $H_1(\cdot,\cdot)$ on $E$. An integrable connection $A$ on $E$ (see Definition 1.7) defines a locally free sheaf which is denoted by $\mathcal{O}(E, \overline{\partial}_A)$.

**Definition 5.8.** We say a locally free sheaf $\mathcal{E}$ is *stable* (in the sense of Mumford-Takemoto) if for every coherent subsheaf $\mathcal{F}$ with $0 < \operatorname{rk}\mathcal{F} < \operatorname{rk}\mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$ where $\mu$ is defined by

$$\mu(\mathcal{F}) = \frac{1}{\operatorname{rk}\mathcal{F}} \int_X c_1(\det \mathcal{F}) \wedge \omega^{m-1}.$$

Similarly $\mathcal{E}$ is *semistable* if $\mu(\mathcal{F}) \leqq \mu(\mathcal{E})$ for all $\mathcal{F}$. An integrable connection $A$ is said to be (semi)stable if the associated locally free sheaf $\mathcal{O}(E, \overline{\partial}_A)$ is (semi)stable (respectively).

Let $\mathcal{A}_{st}$ be the set of all stable integrable connections on $P$. The action of the gauge group $\mathcal{G} = \mathcal{G}(P)$ on $\mathcal{A}_{st}$ is extended to an action of its complexification $\mathcal{G}^{\mathbb{C}} \subset \Gamma(X; \operatorname{End} E)$ by

$$\overline{\partial}_{g(A)} = g^{-1} \circ \overline{\partial}_A \circ g,$$
$$\partial_{g(A)} = g^* \circ \partial_A \circ g^{*-1}$$

where $g^*$ is the adjoint of $g$ with respect to the metric $H_1(\cdot,\cdot)$. If $g$ preserves $H_1(\cdot,\cdot)$, namely $g \in \mathcal{G}$, then we have $g^* = g^{-1}$ and

$$d_{g(A)} = g^{-1} \circ d_A \circ g.$$

Denote by $\mathcal{M}_{st}$ the quotient space $\mathcal{A}_{st}/\mathcal{G}^{\mathbb{C}}$. It parametrizes isomorphism classes of stable holomorphic structure on $P$. The following theorem is the formal analog of Proposition 1.15 which is proved in the finite dimensional case.

**Theorem 5.9** [Kb1],[Lü],[UY],[Do4]. *There is a natural bijective correspondence between the moduli space $\mathcal{M}_{st}$ of stable holomorphic structures on $P$, and the moduli space $\mathcal{M}^0_{EH}$ of irreducible Einstein-Hermitian connections on $P$.*

As observed in Chapter 2, the above correspondence is in fact holomorphic. The advantage of $\mathcal{M}^0_{EH}$ is its natural Kähler metric. For example one can compute the curvature (see [It3]).

We shall restrict our concern to the existence of an Einstein-Hermitian connection on $\mathcal{G}^{\mathbb{C}}$-orbit of a stable integrable connection. For the proof of the converse the reader should see [Kb2]. We shall review the proof of Uhlenbeck and Yau [UY]. See their paper for detail.

Let $A$ be a stable integrable connection on $P$. We want to find a complex gauge transformation $g$ of $P$ such that

$$K^0_{g(A)} = 0,$$

where $K^0_{g(A)}$ is the traceless part of $K_{g(A)}$, namely $K^0_{g(A)} = K_{g(A)} - \lambda(E)\,\mathrm{Id}$ for some constant $\lambda(E)$ depending on the topological structure of $E$. By multiplying a unitary gauge transformation of $E$, we may assume that $g$ is a positive self-adjoint endomorphism.

Before starting the proof of Theorem 5.9, we explain the identification between the quotient space $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ and the space $\mathcal{H}$ of fiber metrics on $E$. The complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts on $\mathcal{H}$ as follows; for a metric $H \in \mathcal{H}$ and a complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$, a new metric $g.H$ is given by

$$(g.H)(v,w) = H(g^{-1}v, g^{-1}w) \qquad \text{for } v,\, w \in E.$$

For two metrics $H$ and $H'$, we take an endomorphism $h$ by

$$H'(v,w) = H(hv, w).$$

In terms of local trivialization, it is given by $h = H^{-1}H'$. Then $H$ and $H'$ are related by $H' = h^{-1/2}.H$. Thus the action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{H}$ is transitive. Therefore $\mathcal{H}$ is identified with the quotient space $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ where $\mathcal{G}$ is the stabilizer at the fixed metric $H_1 \in \mathcal{H}$.

It is more convenient for our purpose to write the equation in terms of fiber metrics rather than complex gauge transformations. Fixing a metric $H_1$, we define an endomorphism $K(H)$ of $E$ by

$$K(H) = K^0_A - \mathrm{tr}_\omega\, \overline{\partial}_A(h^{-1}\partial_A h) = g \circ K^0_{g(A)} \circ g^{-1},$$

where $h = H_1^{-1}H$, $g = h^{-1/2}$. Then the equation $K^0_{g(A)} = 0$ turns out to be

$$K(H) = 0. \tag{5.10}$$

To solve the equation (5.10), we consider the following equation with parameter:

$$K(H) + t\log H_1^{-1}H = 0. \tag{5.11}$$

Since the endomorphism $h = H_1^{-1}H$ is self-adjoint with respect to the metric $H_1$, we can diagonalize $h$ in the form $\sum_i f_i e_i^* \otimes e_i$. Then $\log h$ is defined by $\sum_i \log f_i\; e_i^* \otimes e_i$.

Let $T = \{t \in (0,1] \mid (5.11) \text{ has a solution at } t\}$. First we observe that we may assume $1 \in T$. In fact, by changing the background metric $H_1$ to $H'(\cdot,\cdot) = H_1(\exp K(H_1)\cdot,\cdot)$, the equation (5.11) is rewritten as

$$K(H) + t\log(K^{-1}H) = K(H) + t\log(\exp(-K(H_1))H_1^{-1}H) = 0$$

and this has a solution $H = H_1$ at $t = 1$.

The plan of our proof is as follows. First we shall show that $T$ is both open and closed, which implies $(0,1] = T$. Next we shall study the behavior of the solution $H_t$ of (5.11) as $t \to 0$. Then we shall prove that, if $A$ is a stable integrable connection, $H_t$ converges to $H_0$, which is a solution of (5.10). In other words, we shall show that, if $H_t$ does not converge, there exists a subsheaf of $\mathcal{O}(E, \overline{\partial}_A)$ which contradicts the stability condition.

By a conformal change of the metric we may assume that $\operatorname{tr}(K_A^0) = 0$, where the trace is taken over $E$ ($K_A$ is an endomorphism of $E$). Then any solution of (5.11) for each $t$ satisfies $\det H_1^{-1} H = 1$.

Now we prove the openness of $T$. We define the inner product on $\operatorname{End} E$ by

$$(f, g) = \operatorname{tr}(f g^*),$$

where $g^* \in \Gamma(\operatorname{End} E)$ is the adjoint of $g$ with respect to the metric $H_1$. This inner product induces an $L^p$-norm on the space of sections of $\operatorname{End} E$. For $p > 2m$ and $j \in \mathbb{N}$, let $W^{j,p}(X; \operatorname{End} E)$ be the Sobolev space of endomorphisms on $E$ whose derivatives up to order $j$ are in $L^p$. We consider the submanifold $\Phi^{j,p} = \{h \in W^{j,p}(X; \operatorname{End} E) \mid h \text{ is positive and self-adjoint with respect to the metric } H_1\}$. Its tangent space is given by $\Psi^{j,p} = \{\psi \in W^{j,p}(X; \operatorname{End} E) \mid \psi \text{ is self-adjoint with respect to } H_1\}$. We define a smooth map $P\colon [0,1] \times \Phi^{2,p} \to \Psi^{0,p}$ by

$$P(t, h) = (K(H_1 h) + t \log h) h^{-1}.$$

Suppose that for some $t > 0$ we have a solution $P(t, h) = 0$. Then the linearization of $P$ at $(t, h)$ defines a linear elliptic operator $\delta P$. We have

$$\begin{aligned}
&((\delta P)\psi_1, \psi_2)_{L^2} \\
&= -(\operatorname{tr}_\omega \overline{\partial}_A (h^{-1} \partial_A (\psi_1 h^{-1}) h) h^{-1}, \psi_2)_{L^2} + t((\delta \log)_h \psi_1 \cdot h^{-1}, \psi_2)_{L^2} \\
&= (h^{-1/2} \partial_A (\psi_1 h^{-1}) h^{1/2}, h^{-1/2} \partial_A (\psi_2 h^{-1}) h^{1/2})_{L^2} \\
&\qquad + t((\delta \log)_h \psi_1 \cdot h^{-1}, \psi_2)_{L^2},
\end{aligned}$$

where $(\delta \log)_h$ is the differential of log at $h$. Then a direct calculation shows that $((\delta P)\cdot, \cdot)_{L^2}$ defines a positive definite symmetric bilinear form. This implies that $\delta P$ is an isomorphism, and hence $T$ is open.

Next we prove the closedness of $T$. This will be proved by *a priori* estimates. Let $H_t$ $(a < t < b)$ be a smooth 1-parameter family of solutions of (5.11). A direct calculation shows ([UY, Lemma(2.4)])

$$-\operatorname{tr}(K_A^0 \log h_t) \geqq \left(-\frac{1}{2}\Box + t\right) |\log h_t|^2$$

where $h_t = H_1^{-1} H_t$, $\square = \sum_{\alpha,\beta} g^{\alpha\overline{\beta}} \dfrac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta}$. By the maximal principle, we have

$$\sup_X |\log h_t| \leqq \frac{1}{t} \sup_X |K_A^0|. \tag{5.12}$$

In particular, if $t \geqq a > 0$, we have a bound of $\log h_t$ independent of $t$ (possible depending on $a$). By the elliptic estimates [GT, Theorem 8.17], we also have

$$\sup_X |\log h_t| \leqq C_1(\|\log h_t\|_{L^2} + \sup_X |K_A^0|) \tag{5.13}$$

for some constant $C_1$ independent of $t$.

By differentiating the equation (5.11) with respect to $t$, we get

$$\operatorname{tr}_\omega \overline{\partial}_A(h_t^{-1}\partial_A(\dot{h}_t\, h_t^{-1})h_t) + t(\delta \log)_{h_t} \dot{h}_t + \log h_t = 0 \tag{5.14}$$

where $\dot{h}_t = \dfrac{d}{dt} h_t$. Put $\partial_t = \operatorname{Ad}(h_t^{-1/2}) \circ \partial_A \circ \operatorname{Ad}(h_t^{1/2})$ and $\overline{\partial}_t = \operatorname{Ad}(h_t^{1/2}) \circ \overline{\partial}_A \circ \operatorname{Ad}(h_t^{-1/2})$. Then $\psi_t := h_t^{-1/2}\, \dot{h}_t\, h_t^{-1/2}$ satisfies

$$\square|\psi_t|^2 = -\operatorname{Re}(\sqrt{-1}\operatorname{tr}_\omega(\partial_A\overline{\partial}_A - \overline{\partial}_A\partial_A)\psi_t, \psi_t) + |\partial_t\psi_t|^2 + |\overline{\partial}_t\psi_t|^2.$$

Hence

$$\begin{aligned}
&\square|\psi_t|^2 - 2\operatorname{tr}(\psi_t \log h_t)\\
=&|\partial_t\psi_t|^2 + |\overline{\partial}_t\psi_t|^2 + t\operatorname{Re}\operatorname{tr}\left\{(\delta\log)_{h_t}\dot{h}_t(h_t^{-1}\dot{h}_t + \dot{h}_t h_t^{-1})\right\}\\
\geqq&|\partial_t\psi_t|^2 + |\overline{\partial}_t\psi_t|^2 + t|\psi_t|^2 \geqq 0.
\end{aligned}$$

By the elliptic estimates [GT, Theorem 8.17], we have

$$\sup_X |\psi_t| \leqq C_2(\|\psi_t\|_{L^2} + \sup_X |\log h_t|) \tag{5.15}$$

for some constant $C_2$ independent of $t$. On the other hand, integrating (5.14), we get

$$\begin{aligned}
&\int_X |\overline{\partial}_t\psi_t|^2 \leqq -2\int_X \operatorname{tr}(\psi_t \log h_t)\\
&\leqq \varepsilon \int_X |\psi_t|^2 + \varepsilon^{-1}\int_X |\log h_t|^2
\end{aligned} \tag{5.16}$$

for $\varepsilon > 0$ which will be specified later. Since $\mathcal{O}(E, \overline{\partial}_A)$ is simple (i.e. $\operatorname{End} E$ has no holomorphic sections other than constant multiples of the identity), there exists a constant $C_3$ such that

$$\int_X |f|^2 \leqq C_3 \int_X |\overline{\partial}_A f|^2$$

holds for any trace-free section $f$ of $\operatorname{End} E$. We can substitute $h_t^{-1}\,\dot{h}_t$ for $f$ in this inequality, since $\det h_t = 1$ implies $\operatorname{tr}(h_t^{-1}\,\dot{h}_t) = 0$. Thus

$$\begin{aligned}\int_X |\psi_t|^2 &= \int_X |h_t^{-1}\,\dot{h}_t|^2 \\ &\leqq C_3 \int_X |\overline{\partial}_A(h_t^{-1}\,\dot{h}_t)|^2 = C_3 \int_X |\overline{\partial}_t \psi_t|^2.\end{aligned}$$

Taking $\varepsilon = 1/2C_3$ in (5.16), this inequality together with (5.15) and (5.16) gives an estimate

$$\sup_X |\psi_t| \leqq C_4, \tag{5.17}$$

provided that

$$\int_X |\log h_t|^2 \leqq M \tag{5.18}$$

for some constant $M$ independent of $t > a$ whenever $a > 0$. The estimate (5.18) gives an upper bound of $\sup_X |\log h_t|$ by (5.13), and then using the equation (5.14), we have ([UY, Proposition 3.2])

$$\|\dot{h}_t\|_{W^{2,p}} \leqq C_5(M)(\|h_t\|_{W^{2,p}} + 1)$$

for some constant $C_5(M)$ depending on $M$. Integrating the above inequality, we get

$$\|h_t\|_{W^{2,p}} \leqq C_6(M)(\|h_1\|_{W^{2,p}} + 1)$$

for some constant $C_6(M)$. In particular, for $t > a > 0$ the inequality (5.12) yields the estimate of $\|h_t\|_{W^{2,p}}$ independent of $t$, and hence $T$ is closed in $(0, 1]$. Moreover, even if $a = 0$, the metric $H_t$ converges to a solution of (5.10) as $t \to 0$, whenever we have (5.18).

Now we study the limiting behavior of the solution $H_t$ as $t \to 0$. We rewrite the equation again in terms of the complex gauge transformation $g_t = h_t^{-1/2}$. We find

$$\begin{aligned}K^0_{g_t(A)} &= t g_t^{-1} \circ \log h_t \circ g_t \\ &= t \log h_t,\end{aligned} \tag{5.19}$$

since $g_t = h_t^{-1/2}$ commutes with $\log h_t$. By (5.13), we have

$$\sup_X |K_{g_t(A)}| \leqq C_7$$

for some constant $C_7$ independent of $t$. Since

$$\int_X |R_{g_t(A)}|^2 - |K^0_{g_t(A)}|^2 \ \omega^m$$

is a topological invariant independent of $t$, we also have

$$\int_X |R_{g_t(A)}|^2 \omega^m \leqq C_8$$

for some $C_8$. Thus the conditions in Theorem 4.9 are satisfied for the sequence of connections $\{g_t(A)\}$. Hence for any sequence $\{t_i\}$ converging to 0 there exist a subsequence $\{t_j\}$ and a sequence of unitary gauge transformations $\{u_j\}$ such that $(g_j u_j)(A)$ converges outside a singular set $S$ of finite $2m-4$ dimensional Hausdorff measure, where we write $g_j$ for $g_{t_j}$. We denote by $A_\infty$ the limit integrable connection on $X \setminus S$. Since the endomorphism $g_j u_j$ gives a holomorphic section of $\mathcal{O}(E, \overline{\partial}_{g_i u_i(A)})^* \otimes \mathcal{O}(E, \overline{\partial}_A)$, Bochner's formula [Kb2, Chapter 3 (1.8)] shows

$$\begin{aligned}\Box |g_j u_j|^2 &\geqq -\operatorname{tr}(K_A g_j u_j \ K_{g_j u_j(A)} (g_j u_j)^*) \\ &\geqq -C_9 |g_j u_j|^2\end{aligned}$$

for some constant $C_9$ independent of $j$. This implies

$$\sup_X |g_j u_j| \leqq C_{10} \|g_j u_j\|_{L^2}. \tag{5.20}$$

We take a positive constant $a_j$ such that

$$\|a_j(g_j u_j)\|_{L^2} = 1.$$

Then $\|a_j(g_j u_j)\|_{W^{1,2}}$ is uniformly bounded, and hence (taking a subsequence if necessary) we see that $a_j(g_j u_j)$ converges to an endomorphism $g$ of $\mathcal{O}(E, \overline{\partial}_{A_\infty})$ to $\mathcal{O}(E, \overline{\partial}_A)$ over $X \setminus S$. Take a compact subset $K$ of $X \setminus S$ with $\operatorname{vol}(X \setminus K) \leqq \frac{1}{2C_{10}}$. Then the estimate (5.20) implies

$$\|a_j(g_j u_j)\|_{L^2(K)} \geqq \frac{1}{2}.$$

Since the convergence is strong in $L^2$ on $K$, this implies the limit $g$ is a nonzero endomorphism.

We first assume $\mathrm{rk}(\mathrm{Im}\, g) = r$, where the remaining case $0 < \mathrm{rk}(\mathrm{Im}\, g) < r$ will be treated later. Then $g$ gives an isomorphism between $\mathcal{O}(E, \overline{\partial}_{A_\infty})$ and $\mathcal{O}(E, \overline{\partial}_A)$. Since $|\det(a_j g_j u_j)| = a_j^r$, this implies that $a_j$ is bounded from below by a positive constant. Thus we have a uniform $C^0$-estimate independent of $j$ both for $g_j$ and $g_j^{-1}$, and hence $\log h_j$. Since $t_i$ is an arbitrary sequence converging to 0, this means that we have the estimate (5.18) uniformly in $t$. So $H_t$ converges to a solution of (5.10), as required.

Now we assume $0 < \mathrm{rk}(\mathrm{Im}\, g) < r$. By the sheaf extension theorem [Si], the sheaf $\mathrm{Im}\, g$ can be extended to a subsheaf of $\mathcal{O}(E, \overline{\partial}_A)$ on the whole $X$. Then for contradiction, we shall show that $\mu(\mathrm{Im}\, g) \geqq \mu(E)$ in view of the stability of $\mathcal{O}(E, \overline{\partial}_A)$. Since $\mathrm{Im}\, g$ is a torsion-free sheaf, it is a locally free outside an analytic subset of $\mathrm{codim}_{\mathbb{C}} \geqq 2$ in $X$. Hence by replacing $S$ if necessary, we may assume that $\mathrm{Ker}\, g$ defines a *subbundle* of $E$ over $X \setminus S$. Then as a $C^\infty$-vector bundle, $\mathrm{Im}\, g$ is identified with $\mathrm{Ker}\, g^\perp \subset E$ over $X \setminus S$ and carries a metric induced by that of $E$. We compute $c_1(\mathrm{Im}\, g)$ via the Chern-Weil formula using this metric.

Since $a_j g_j u_j$ converges to $g$, $(a_j g_j u_j)^* a_j g_j u_j = a_j^2 u_j^{-1} g_j^2 u_j$ converges to $g^* g$. The endomorphism $g^* g$ does not vanish on $\mathrm{Ker}\, g^\perp = \mathrm{Im}(g^*)$, and therefore $t_j \log(a_j^2 u_j^{-1} h_j u_j)$ converges to 0 on $\mathrm{Ker}\, g^\perp$. Thus on $\mathrm{Ker}\, g^\perp$,

$$\begin{aligned} K^0_{g_j u_j(A)} &= t_j u_j^{-1} \log h_j u_j \\ &= t_j \log(a_j^2 u_j^{-1} h_j u_j) - 2t_j \log a_j \,\mathrm{Id} \end{aligned}$$

converges to $c\,\mathrm{Id}$ for some constant $c$. By the assumption, $a_j$ tends to 0, and hence $c$ is nonnegative. We induce respectively connections $\nabla_{\mathrm{Ker}\, g}$ on $\mathrm{Ker}\, g$ and $\nabla_{\mathrm{Ker}\, g^\perp}$ on $\mathrm{Ker}\, g^\perp$ from the connection $\nabla_{A_\infty}$. We have the following expression of the curvature $R_{A_\infty}$:

$$R_{A_\infty} = \begin{pmatrix} R_{\mathrm{Ker}\, g} - \beta^* \wedge \beta & \partial_A \beta^* \\ -\overline{\partial}_A \beta & R_{\mathrm{Ker}\, g^\perp} - \beta \wedge \beta^* \end{pmatrix},$$

where $\beta\colon \mathrm{Ker}\, g \to \mathrm{Ker}\, g^\perp$ is the second fundamental form of $\mathrm{Ker}\, g$ in $E$, namely so that $\beta\xi = \nabla_{A_\infty}\xi - \nabla_{\mathrm{Ker}\, g}\xi$ for $\xi \in \Gamma(\mathrm{Ker}\, g)$. Since $\mathrm{tr}(\beta \wedge \beta^*)$ is a positive current, we have

$$K_{\mathrm{Ker}\, g^\perp} \geqq K_{A_\infty} | \mathrm{Ker}\, g^\perp \geqq \lambda(E)\, \mathrm{Id}_{\mathrm{Ker}\, g^\perp}\,.$$

This implies

$$2m\pi \int_X c_1(\operatorname{Im} g) \wedge \omega^{m-1} = \int_X \operatorname{tr}(K_{\operatorname{Ker} g^\perp})\omega^m$$
$$\geqq 2m\pi \frac{\operatorname{rk} \operatorname{Im} g}{r} \int_X c_1(E) \wedge \omega^{m-1}.$$

But this contradicts the stability of $\mathcal{O}(E, \overline{\partial}_A)$. Note that, though $\operatorname{tr}(R_{\operatorname{Ker} g^\perp})$ is only defined outside $S$, it can be extended on the whole $X$ as a current. Because the positive current $\operatorname{tr}(\beta \wedge \beta^*)$ on $X \setminus S$ can be also defined on the whole $X$ by [Ha]. Moreover this procedure does not affect the computation, since $H^2(X; \mathbb{R}) \cong H^2(X \setminus S; \mathbb{R})$.

# Chapter 6. Vector Bundles over Moduli Spaces

## §6.1. Universal connections

In this section we shall investigate vector bundles having connections, called universal bundles and universal connections, over moduli spaces. In [Do3] the universal bundles were used to define new invariants which play very important roles in studying the differential topology of base manifolds. Here we shall study the universal bundles from a view point of differential geometry.

Let $\mathcal{A}$ be the space of all irreducible connections on a $G$-principal bundle $P$ over a compact connected oriented Riemannian manifold $(X, g)$ ($G$ is compact and semisimple). Since the stabilizer of each connections is the center $Z(G)$ of $G$, the action of the gauge group $\mathcal{G}$ on $\mathcal{A}$ induces a free action of $\mathcal{G}_0 = \mathcal{G}/Z(G)$ on $\mathcal{A}$. The slice lemma [FU] states that for any connection $A \in \mathcal{A}$ there is a neighborhood $S$ of 0 in the kernel of $d_A^*\colon \Omega^1(\operatorname{Ad}(P)) \to \Omega^0(\operatorname{Ad}(P))$ being transversal to gauge orbits. Then the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}_0$ has a natural structure of an infinite dimensional differentiable manifold such that $\mathcal{A} \to \mathcal{B}$ is a principal bundle with the structure group $\mathcal{G}_0$. The adjoint bundle $\operatorname{Ad}(\mathcal{A})$ of $\mathcal{A} \to \mathcal{B}$ is given by $\mathcal{A} \times_{\mathcal{G}_0} \Omega^0(\operatorname{Ad}(P))$. Moreover we have a natural connection on this bundle defined by the splitting of the tangent space at $A \in \mathcal{A}$:

$$T_A\mathcal{A} = \mathcal{V}_A + \mathcal{H}_A,$$

where $\mathcal{V}_A$ is the vertical subspace defined by $\{d_A\alpha \mid \alpha \in \Omega^0(\operatorname{Ad}(P))\}$ and $\mathcal{H}_A$ is the horizontal subspace defined by $\{\beta \in \Omega^1(\operatorname{Ad}(P)) \mid d_A^*\beta = 0\}$. Its connection form $\omega\colon T\mathcal{A} \to \Omega^0(\operatorname{Ad}(P))$ and the curvature form $R_\omega \in \Omega^2(\mathcal{B}; \operatorname{Ad}(\mathcal{A}))$ are given as follows (see [It4]):

**Proposition 6.1.**

$$\omega(\alpha) = G_A(d_A^*\alpha), \qquad \text{for } \alpha \in T_A\mathcal{A} \cong \Omega^1(\mathrm{Ad}(P)),$$
$$R_\omega(\alpha,\beta) = -2G_A(\{\alpha,\beta\}), \qquad \text{for } \alpha,\, \beta \in T_{[A]}\mathcal{B} \cong \mathcal{H}_A,$$

*where $G_A$ is the Green operator on $\Omega^0(\mathrm{Ad}(P))$ and $\{\cdot,\cdot\}$ is the bilinear operation defined by*

$$\{\alpha,\beta\} = \sum_{i,j} g^{ij}[\alpha_i,\beta_j]$$

*for $\alpha = \sum \alpha_i dx^i$, $\beta = \sum \beta_i dx^i$ ($g_{ij}$ is the metric tensor).*

Now we introduce a universal bundle with a natural connection following [AS]. Suppose $Z(G) = 1$, for simplicity. The gauge group $\mathcal{G}_0$ acts freely on the product $P \times \mathcal{A}$ and gives a principal $G$-bundle

$$\mathbb{P} = P \times_{\mathcal{G}_0} \mathcal{A} \to X \times \mathcal{B}$$

by taking a quotient. Its adjoint bundle $\mathrm{Ad}(\mathbb{P})$ is $\mathrm{Ad}(P) \times_{\mathcal{G}_0} \mathcal{A}$. The space $P \times \mathcal{A}$ has a natural Riemannian metric invariant under $G \times \mathcal{G}$. In fact, for $(p, A) \in P \times \mathcal{A}$ we have an inner product on $T_{(p,A)}P \times \mathcal{A}$ such that

(a) The decomposition $T_{(p,A)}P \times \mathcal{A} = T_pP \oplus T_A\mathcal{A}$ is orthogonal,
(b) The metric on $T_pP$ is defined by ones on $G$ and $X$ using the decomposition induced from the connection $A$,
(c) The restriction to $T_A\mathcal{A} \cong \Omega^1(\mathrm{Ad}(P))$ is given by the $L^2$-inner product.

This metric descends to a $G$-invariant metric on $\mathbb{P}$ and the orthogonal complement to orbits of $G$ defines a universal connection $\mathbb{A}$ on $\mathbb{P}$. This connection $\mathbb{A}$ can be reformulated by the above connection $\omega$ on $\mathcal{A} \to \mathcal{B}$ as follows. Let $\pi\colon P \to X$ be the projection and $\mathrm{ev}_x\colon \Omega^0(\mathrm{Ad}(P)) \to (\mathrm{Ad}(P))_x$ the evaluation map at $x$. At $[p, A] \in \mathbb{P} = P \times_{\mathcal{G}} \mathcal{A}$ the connection form $\mathbb{A}$ is given by

$$\mathbb{A}|_{T_{[p,A]}\mathbb{P}} = \widetilde{p} \circ \mathrm{ev}_x \circ \omega|_{T_A\mathcal{A}} + A|_{T_pP},$$

where $x = \pi(p)$ and $\widetilde{p}\colon \Omega^0(\mathrm{Ad}(P))_x \to \mathfrak{g}$ is the differential of the trivialization $P_x \cong G$ such that $p$ is mapped to the unit element $e \in G$. A direct calculation shows that $\mathbb{A}$ is well-defined and coincides with the above definition. In particular, the restriction of the connection $\mathbb{A}$ to $X \times \{[A]\}$ is isomorphic to $A$. The formulas for the curvature form $\mathbb{R} \in \Omega^2(X \times \mathcal{B}; \mathrm{Ad}(\mathbb{P}))$ are as follows:

**Proposition 6.2.** *At each point* $(x, [A])$ *the curvature form* $\mathbb{R}$ *is given by*

$$\begin{aligned}
&\mathbb{R}(v, w) = R_A(v, w), \\
&\mathbb{R}(\alpha, v) = \alpha(v), \\
&\mathbb{R}(\alpha, \beta) = -2\,\mathrm{ev}_x\, G_A(\{\alpha, \beta\}) \\
&\qquad\qquad\qquad \text{for } v,\ w \in T_x X,\ \alpha,\ \beta \in T_{[A]}\mathcal{B} \cong \mathcal{H}_A,
\end{aligned}$$

*where* $\mathrm{Ad}(\mathbb{P})_{(x,[A])}$ *is identified with* $\mathrm{Ad}(P)_x$.

The above formulas enable us to compute the Chern class of the universal bundle $\mathbb{P}$ via the Chern-Weil theory (see [AS] for details).

*Remark* 6.3. If $X$ is a Kähler manifold, the connection $\mathbb{A}$ is an integrable connection when $\mathbb{P}$ is restricted to $X \times \mathcal{M}_{EH}$. This can be proved by showing that the curvature form $\mathbb{R}$ is a $(1,1)$-form via the above formula ([It4]).

Next we compute the curvature of the moduli space $\mathcal{M}_-$ of anti-self-dual connections. Let $X$ be a 4-dimensional compact Riemannian manifold and $A$ be an anti-self-dual connection on a $G$-principal bundle over $X$. Consider the operator $D_A$ defined by

$$D_A = (d_A^*, d_A^+) \colon \Omega^1(\mathrm{Ad}(P)) \to (\Omega^0 \oplus \Omega^+)(\mathrm{Ad}(P)).$$

(For the notation see Chapters 1, 2.) Let $\widehat{\mathcal{M}}_-$ denote the moduli space of generic anti-self-dual connections which is a subset of $\mathcal{B}$. The connection $\omega$ on the $\mathcal{G}$-principal bundle $\mathcal{A} \to \mathcal{B}$ induces a connection, denoted by $\overline{\nabla}$, on the associated vector bundle (with infinite rank) $\mathcal{E} = \mathcal{A} \times_{\mathcal{G}} \Omega^1(\mathrm{Ad}(P))$. Since $\mathrm{Coker}\, D_A = 0$ for $[A] \in \widehat{\mathcal{M}}_-$, $\mathrm{Ker}\, D_A$ forms a subbundle of the restriction of $\mathcal{E}$ to $\widehat{\mathcal{M}}_-$. Moreover $\mathrm{Ker}\, D_A$ is isomorphic to the tangent bundle $T\widehat{\mathcal{M}}_-$. Let $\nabla$ be the induced connection on $\mathrm{Ker}\, D_A$ and $\sigma$ be the second fundamental form, namely

$$\overline{\nabla}_\alpha \xi = \nabla_\alpha \xi + \sigma_\alpha \xi,$$

where $\xi$ is a section of $\mathrm{Ker}\, D_A$ and $\alpha \in T_{[A]}\widehat{\mathcal{M}}_-$. Then we obtain

**Proposition 6.4.**

$$\sigma_\alpha \xi = G_A D_A^*(\alpha \circ \xi),$$

*where* $\alpha \circ \xi = (\{\alpha, \xi\}, [\alpha \wedge \xi]^+) \in (\Omega^0 \oplus \Omega^+)(\mathrm{Ad}(P))$ *and* $G_A$ *is the Green operator on* $\Omega^1(\mathrm{Ad}(P))$.

Using Propositions 6.1 and 6.4, we can compute the curvature $R$ of the connection $\nabla$

**Proposition 6.5.**

$$\begin{aligned}(R(\alpha,\beta)\xi,\eta) &= -2([G_A(\{\alpha,\beta\}),\xi],\eta)\\ &\quad - (G_A D_A^*(\alpha \circ \xi), G_A D_A^*(\beta \circ \eta)) + (G_A D_A^*(\beta \circ \xi), G_A D_A^*(\alpha \circ \eta))\end{aligned}$$

The above formula coincides with [It3, Theorem 5.1] which is obtained by using the Kuranishi map.

## §6.2. Determinant line bundles

In [Do2], [Do4] Donaldson defined a functional on the space of hermitian metrics on a holomorphic vector bundle in order to prove the existence of Einstein-Hermitian metrics on stable bundles. As explained in [Do4, §1] the functional can be viewed as the distortion of the norm on the determinant line bundle caused by the action of the complex gauge group. Such a norm on the determinant line bundle was introduced for Riemann surface by Quillen, in Riemannian geometry by Bismut and Freed [BF], and in complex geometry by Bismut, Gillet and Soulé [BGS]. The distortion of the norm is calculated in [BGS]. In this section we shall explain the relation of the determinant line bundle to the Donaldson's functional using the result of [BGS].

In order to define the Donaldson's functional we recall the notion of Bott-Chern forms [BC]. See [Do2] or [BGS] for the proof. Let $X$ be a complex manifold. We denote by $d^X = \partial^X + \overline{\partial}^X$ the usual exterior differential operator on $\Omega^*_{\mathbb{C}}$. Let $P$ be the subspace of $\Omega^*_{\mathbb{C}}$ spanned by forms of type $(p,p)$, $p \geqq 0$ and $P' \subset P$ the subspace spanned by forms represented as $\partial^X \alpha + \overline{\partial}^X \beta$ for some $C^\infty$ forms $\alpha$, $\beta$. Let $E$ be a $C^\infty$-vector bundle over $X$ and we suppose that $E$ has a holomorphic structure $\overline{\partial}$. Let $\mathcal{H}$ be the set of hermitian metrics on $E$. Each metric $H \in \mathcal{H}$ defines a unique connection $A$ on $E$ such that $H$ is parallel with respect to $A^H$ and $\overline{\partial}_{A^H} = \overline{\partial}$. (As in §5.2 we fix a holomorphic structure and change hermitian metrics instead of changing integrable connections on a vector bundle with a fixed hermitian metric.) By the Chern-Weil theory the cohomology class of the Chern character $\mathrm{ch}(E)$ is represented as

$$\mathrm{tr}(\exp \frac{\sqrt{-1}}{2\pi} R_H)$$

where $R_H$ is the curvature form of the connection $A^H$. The Bott-Chern form $\widetilde{\mathrm{ch}}(H,K)$ is a double transgression of the Chern character forms. In fact, we have the following.

**Proposition 6.6.** *There exists a map* $\widetilde{\mathrm{ch}}\colon \mathcal{H}\times\mathcal{H}\to P/P'$ *which is characterized by the following properties:*

(1) $\widetilde{\mathrm{ch}}(H,H)=0$, $\widetilde{\mathrm{ch}}(H,K)+\widetilde{\mathrm{ch}}(K,L)=\widetilde{\mathrm{ch}}(H,L)$,

(2) *If $H_t$ is a smooth family of metrics, then*
$$\frac{d}{dt}\widetilde{\mathrm{ch}}(H_t,K) = -\operatorname{tr}(H_t^{-1}\frac{d}{dt}H_t \exp\frac{\sqrt{-1}}{2\pi}R_{H_t}),$$

(3) $$\frac{\sqrt{-1}}{2\pi}\bar\partial^X\partial^X\widetilde{\mathrm{ch}}(H,K)=\operatorname{tr}(\exp\frac{\sqrt{-1}}{2\pi}R_H)-\operatorname{tr}(\exp\frac{\sqrt{-1}}{2\pi}R_K).$$

Now we define Donaldson's functional. We assume moreover that $X$ is an $m$-dimensional compact Kähler manifold with the Kähler form $\omega$. Let $Q_p(H,K)$ be the $(p-1,p-1)$-form component of $\widetilde{\mathrm{ch}}(H,K)$. We set
$$F(H,K)=\int_X c_1Q_1(H,K)\omega^m - c_2Q_2(H,K)\wedge\omega^{m-1}$$
where $c_1$, $c_2$ are constants given by
$$c_1=\int_X c_1(E)\wedge\omega^{m-1},\quad c_2=r\int_X\omega^m.$$
Then Proposition 6.6 implies

**Proposition 6.7.** *If $H_t$ is a smooth family of metrics, then*
$$\frac{d}{dt}F(H_t,K)=\frac{1}{2m\pi}\int_X \operatorname{tr}\left(H_t^{-1}\frac{d}{dt}H_t\,(c_2K_{H_t}-2m\pi c_1\,\mathrm{Id})\right)\omega^m$$
*where $K_{H_t}$ is the Einstein-Hermitian tensor of $A^{H_t}$. In particular, $H$ is a critical point of $F(\cdot,K)$ if and only if the integrable connection $A^H$ associated with $H$ is an Einstein-Hermitian connection.*

Suppose that the Kähler manifold $X$ is moreover a projective algebraic manifold with an ample line bundle $L\to X$. We fix a metric on $L$ with curvature $-2\pi\sqrt{-1}\omega$ where $\omega$ is the Kähler form on $X$. Then formally the double transgression of the Chern character forms associated with the virtual vector bundle
$$F=\bigoplus^{c_2}\left((E\oplus E^*)\otimes(L-L^*)^{m-1}\right)\bigoplus^{-c_1}\left(E\otimes(L-L^*)^m\right)$$

is given by

$$\widetilde{\mathrm{ch}}^F(H,K) = -2^m(c_1Q_1(H,K)\omega^m - c_2Q_2(H,K)\wedge\omega^{m-1}). \tag{6.8}$$

Here for a negative integer $m$ and a vector bundle $V$, $\oplus^m V$ means $-(\oplus^{-m}V)$. Remark that $c_1$ and $c_2$ are integers in this case.

Now we change the viewpoint and fix a hermitian metric $H$ on $E$ and move integrable connections. Let $\mathcal{A}^{(1,1)}$ be the space of integrable connections on a hermitian vector bundle $(E,H)$ over $X$. Then $\mathcal{A}^{(1,1)}$ (rigorously we must take a certain open subset to avoid the possible singularities) is a complex submanifold of $\mathcal{A}\cong\Omega^1(\mathrm{Ad}(E))$ and has a Kähler form

$$\Omega(\alpha,\beta) = -\frac{2^{m-2}c_2}{\pi^2}\int_X \mathrm{tr}(\alpha\wedge\beta)\wedge\omega^{m-1}.$$

(We remark that this differs from the Kähler form defined in Chapter 2 in the multiple by a positive constant.) Let $p_2\colon\mathcal{A}^{(1,1)}\times X\to X$ be the projection to the second factor. The universal bundle $\mathbb{E}=p_2^*(E)$ over $\mathcal{A}^{(1,1)}\times X$ has the tautological connection $\mathbb{A}$ which is trivial in the $\mathcal{A}^{(1,1)}$ directions and equal to $p_2^*(A)$ on $\{A\}\times X$. As in §6.1 we have the formulas for curvature form $\mathbb{R}\in\Omega^2(\mathcal{A}\times X;\mathrm{Ad}(E))$ as follows:

$$\begin{cases} \mathbb{R}(v,w) = R_A(v,w) & \text{for } v,\ w\in T_xX, \\ \mathbb{R}(\alpha,v) = \alpha(v) & \text{for } \alpha\in T_A\mathcal{A}^{(1,1)},\ v\in T_xX, \\ \mathbb{R}(\alpha,\beta) = 0 & \text{for } \alpha,\ \beta\in T_A\mathcal{A}, \end{cases} \tag{6.9}$$

where $T_A\mathcal{A}^{(1,1)}$ is considered as a subspace of $\Omega^1(\mathrm{Ad}(E))$ in the second identity.

For each $A\in\mathcal{A}^{(1,1)}$ we have an elliptic complex

$$0\to\Omega^{0,0}(E)\xrightarrow{\bar\partial_A}\Omega^{0,1}(E)\xrightarrow{\bar\partial_A}\cdots\xrightarrow{\bar\partial_A}\Omega^{0,m}(E)\to 0.$$

We denote by $H_A^i$ the $i$-th cohomology group of the above complex. Then we consider the determinant line:

$$\mathcal{L}_A = \bigotimes_i \left(\bigwedge^{\max} H_A^i\right)^{(-1)^{i+1}}.$$

Bismut and Freed [BF] (see also [BGS]) showed that $\mathcal{L}_A$ forms a line bundle as $A$ varies and defined a metric and connection on it using the zêta function regularization. (It is not at all obvious that $\mathcal{L}_A$ forms a line bundle since the dimensions of $H^i_A$ may change as $A$ varies.) The curvature form of the Bismut-Freed connection is given by

$$2\pi\sqrt{-1}\left\{\int_X \mathrm{ch}(\mathbb{E})\,\mathrm{Td}(X)\right\}^{(2)},$$

where $(\cdot)^{(2)}$ means the 2-form component. This is a refinement of the index theorem for families since the cohomology class of the above form is negative of the first Chern class of the index bundle. Moreover the determinant line bundle $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{A}^{(1,1)}$.

We now use the construction of Bismut and Freed to get a line bundle whose curvature form gives the Kähler form $\Omega$ on $\mathcal{A}^{(1,1)}$.

**Proposition 6.10.** *The curvature of the Bismut-Freed connection on the determinant line bundle $\mathcal{P}$ of the virtual bundle*

$$\mathbb{F} = \bigoplus^{c_2}\left((\mathbb{E}\oplus\mathbb{E}^*)\otimes(L-L^*)^{m-1}\right)\bigoplus^{-c_1}\left(\mathbb{E}\otimes(L-L^*)^m\right)$$

*is given by $-2\pi\sqrt{-1}\Omega$ where $\Omega$ is the Kähler form of $\mathcal{A}^{(1,1)}$.*

*Proof.* The curvature of Bismut-Freed connection on $\mathcal{P}$ is given by

$$\begin{aligned}
&2\pi\sqrt{-1}\left(\int_X \mathrm{ch}(\mathbb{F})\,\mathrm{Td}(X)\right)^{(2)}\\
=&2^m\pi\sqrt{-1}\left(\int_X c_2c_1(L)^{m-1}(\mathrm{ch}(\mathbb{E})+\mathrm{ch}(\mathbb{E}^*)\,\mathrm{Td}(X))\right)^{(2)}\\
&-2^{m+1}c_1\pi\sqrt{-1}\left(\int_X c_1(L)^m\,\mathrm{ch}(\mathbb{E})\,\mathrm{Td}(X)\right)^{(2)}\\
=&2^mc_2\pi\sqrt{-1}\left(\int_X \omega^{m-1}\,\mathrm{tr}(\frac{\sqrt{-1}}{2\pi}\mathbb{R}\wedge\frac{\sqrt{-1}}{2\pi}\mathbb{R})\right)^{(2)}\\
&-2^{m+1}c_1\pi\sqrt{-1}\left(\int_X \omega^m\,\mathrm{tr}(\frac{\sqrt{-1}}{2\pi}\mathbb{R})\right)^{(2)}.
\end{aligned}$$

By the curvature formula (6.9) the second term vanishes and

$$\left(\int_X \omega^{m-1}\,\mathrm{tr}(\frac{\sqrt{-1}}{2\pi}\mathbb{R}\wedge\frac{\sqrt{-1}}{2\pi}\mathbb{R})\right)(\alpha,\beta) = \frac{1}{2\pi^2}\int_X \mathrm{tr}(\alpha\wedge\beta)\wedge\omega^{m-1}$$

for $\alpha,\beta\in T\mathcal{A}^{(1,1)}$. Hence the curvature form of $\mathcal{P}$ is given by $-2\pi\sqrt{-1}\Omega$.

Q.E.D.

Since the gauge group $\mathcal{G}$ acts on the bundle $E$ and hence on $F$, the action on $\mathcal{A}^{(1,1)}$ has a lift to $\mathcal{P}$. For $\gamma \in \mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathrm{Ad}(E))$ we decompose the associated vector field $\overline{\gamma}$ on $\mathcal{P}$ into the horizontal part and vertical part with respect to the Bismut-Freed connection:

$$\overline{\gamma}_\xi = \overline{\gamma}^h_\xi + 2\pi\sqrt{-1}\widehat{\mu}(\gamma,\xi)\xi \quad \text{for } \xi \in \mathcal{P}.$$

Then by Proposition 1.14 the map $\mu$ given by

$$\begin{aligned} \mu\colon \mathcal{A}^{(1,1)} &\to \mathrm{Lie}(\mathcal{G})^* \cong \Omega^{2m}(\mathrm{Ad}(E)) \\ \langle \gamma, \mu(A)\rangle &= \widehat{\mu}(\gamma,\xi) \qquad \text{for } \pi(\xi) = A \end{aligned}$$

is a moment map for the action on the symplectic manifold $(\mathcal{A}^{(1,1)}, \Omega)$.

As in §5.2 we complexify the action of the gauge group to get the action of the complex gauge group $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}^{(1,1)}$ as

$$\overline{\partial}_{g(A)} = g^{-1} \circ \overline{\partial}_A \circ g, \quad \partial_{g(A)} = g^* \circ \partial_A \circ g^{*-1},$$

where $g^*$ is the adjoint of $g$ with respect to the fiber metric on $E$. The action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}^{(1,1)}$ also has a lift to an action on $\mathcal{P}$, then we introduce a function $M\colon \mathcal{A}^{(1,1)} \times \mathcal{G}^{\mathbb{C}} \to \mathbb{R}$ measuring the distortion of the norm in $\mathcal{P}$ caused by $\mathcal{G}^{\mathbb{C}}$ (see §1.4):

$$|\xi.g|^2 = e^{M(A,g)}|\xi|^2 \qquad \text{for } g \in \mathcal{G}^{\mathbb{C}}, A = \pi(\xi) \in \mathcal{A}^{(1,1)}.$$

Then we have a formula

$$\left.\frac{d}{dt}\right|_{t=0} M(A, \exp(t\sqrt{-1}\gamma)) = -2\pi\langle \gamma, \mu(A)\rangle$$

for $\gamma \in \mathcal{G}$. Under the identification of $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ with the space $\mathcal{H}$ of hermitian metrics the complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$ corresponds to the metric

$$H'(\xi,\eta) = H(g^{-1}\xi, g^{-1}\eta).$$

The effect on the norm of the determinant line bundle caused by the change of the hermitian metric is calculated in [BGS, III, Theorem 1.23] and given by

$$\begin{aligned} M(A,g) &= \int_X \widetilde{\mathrm{ch}}^{\mathbb{F}}(H,H')\,\mathrm{Td}\,X \\ &= -2^m \int_X (c_1 Q_1(H,H')\omega^m - c_2 Q_2(H,H') \wedge \omega^{m-1}) \\ &= -2^m F(H,H') = 2^m F(H',H), \end{aligned}$$

where we have used the identity (6.8). Hence we have

$$\begin{aligned}\langle\gamma,\mu(A)\rangle &= -\frac{1}{2\pi}\left.\frac{d}{dt}\right|_{t=0} M(A,\exp(t\sqrt{-1}\gamma))\\ &= -\frac{2^{m-1}}{\pi}\left.\frac{d}{dt}\right|_{t=0} F(H_t,H)\\ &= \frac{2^{m-2}}{m\pi^2}\int_X \operatorname{tr}\left(\gamma(c_2K_A-2m\pi c_1\operatorname{Id})\right)\omega^m,\end{aligned}$$

where $H_t$ is the metric corresponding to the complex gauge transformation $\exp(t\sqrt{-1}\gamma)$. This implies that

$$\mu(A) = -\frac{2^{m-2}}{m\pi}(c_2K_A-2m\pi c_1\operatorname{Id})\,\omega^m,$$

which coincides with the moment map defined in §1.4 up to the multiple of a positive constant.

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