Uniformization of Complex Surfaces

Ryoichi Kobayashi

Abstract.

This is an expository paper on the structure of complex surfaces which have the Hirzebruch proportionality $3c_2 = c_1^2$ or $2c_2 = c_1^2$ between their Chern numbers. We characterize surfaces with $3c_2 = c_1^2$ as ball quotients in the category of normal surfaces with branch loci. We discuss the uniformization problem for surfaces with $2c_2 = c_1^2$ from the point of view of Kähler-Einstein metrics and holomorphic conformal structures.

§0. Introduction

The purpose of this expository paper is to discuss some aspects of the uniformization problem for complex surfaces. Let us first of all fix the notion of uniformization.

Definition. A complex orbi-surface is a pair (X, D) of a quasi-projective complex surface X and a Q-divisor D with the following properties:
(1) There exists a compact complex normal surface $\bar{X}$ containing X as a Zariski open set and $\bar{X} - X$ consists of a finite number of points.
(2) If D_i are the irreducible components of D, then $D = \sum_{i=1}^{r} (1 - \frac{1}{b_i}) D_i$ where each b_i is either infinity or an integer greater than one.
(3) For each point $p \in X - \cup_{b_i=\infty} D_i$ there exists a neighborhood $U(p) \subset X$ of p and a holomorphic (branched) Galois covering $B^2 \to U(p)$ (B^2 is the unit open ball in C^2) with covering transformation group $\Gamma \subset U(2)$ and $\cup_i (D_i \cap U(p))$ is the branch locus.
(4) For each point $p \in \cup_{b_i=\infty} D_i$ there exists a neighborhood $U(p) \subset X$ of p and a holomorphic (branched) Galois covering $\Delta \times \Delta^* \to U(p) - \cup_{b_i=\infty} D_i$ or $\Delta^* \times \Delta^* \to U(p) - \cup_{b_i=\infty} D_i$ with covering transformation group $\Gamma \subset U(2)$ and $\cup_{b_i<\infty} D_i$ is the branch locus.

(5) For each point $p \in \bar{X} - X$ there exists a neighborhood $U(p) \subset \bar{X}$ which is uniformized by
(i) a horoball of B^2 at a boundary point ∞ with covering transformation group $\Gamma \subset PSU(2,1)$ a discrete parabolic subgroup at ∞,
or by
(ii) a horoball of $\Delta \times \Delta$ at the boundary point (∞, ∞) with covering transformation group $\Gamma \subset \mathrm{Aut}(\Delta \times \Delta)$ a discrete parabolic subgroup at (∞, ∞) which does not project down to operate $\Delta^* \times \Delta^*$.
In both (i) and (ii), $\cup_{b_i < \infty}(D_i \cap U(p))$ is the branch locus of the (branched) Galois covering.

The integers b_i are branch indices. If $b_i = \infty$, we can delete the corresponding D_i. Namely, writing D_∞ for the union of D_i with $b_i = \infty$, we put the following

Definition. A complex orbi-surface (X, D) is uniformizable if there exists a holomorphic branched Galois covering $Y \to X - D_\infty$ with branch locus $\mathrm{Supp}(D - D_\infty)$ of indices b_i along D_i.

These definitions refine the definition of *V-manifolds* in [Sat] in some respects. There will be many ways to generalize the definition of orbi-surfaces in higher dimensions, because various bounded symmetric domains come in in higher dimensions. But it may be possible to consider a restricted class of orbifolds, for instance compact n-dimensional orbifolds (X, D) $(D = \sum_i (1 - \frac{1}{b_i})D_i$ is a Q-divisor as above) with all $b_i < \infty$.

There are several approaches to the uniformizability problem for orbi-surfaces and orbifolds.

The most direct approach is looking at the relation of the global fundamental group $\pi_1(X - \mathrm{Supp}(D))$ and the local fundamental groups at points in D_i. This approach is due to Kato [Kat] and Namba [Nam]. They proved theorems which give sufficient conditions for the uniformizability in terms of the fundamental group $\pi_1(X - \mathrm{Supp}(D))$.

Secondly, there is a differential geometric approach to the uniformization problem. This is to characterize Hermitian symmetric spaces by curvature conditions. Mok studied this problem extensively ([Mo]). There are also works by Yang [Yan], Mostow-Siu [MS], Siu-Yang [Si-Y] and Wong [Wo]. Siu and Yang [Si-Y] developed differential geometry of Kähler-Einstein surfaces. In general, developing differential geometry

of Kähler-Einstein manifolds would be very important, if one wants to apply Kähler-Einstein manifolds to problems in algebraic geometry.

Thirdly, seeking numerical characterizations for ball quotients is also a uniformization problem. This is initiated by Yau [Yau1]. In this approach, there is so far a big difference between dimension 2 and higher dimensions. This difference arises from the fact that 2-dimensional log-canonical singularities are all quotient singularities (possibly with an infinite group) but this is not the case in higher dimensions (see Sugiyama's survey [Su] in this volume). This causes the difficulty in analyzing the canonical Kähler-Einstein metric (see [Su]) around singularities. In higher dimensions, there are partial results by Tsuji [T1] and Tian and Yau [T-Y].

The fourth approach is to look at the holomorphic geometric structures modeled after Hermitian symmetric spaces. Partially based on Gunning's earlier work ([Gu]), Kobayashi and Ochiai developed a theory of G-structures modeled after Hermitian symmetric spaces. We refer to papers of Inoue-Kobayashi-Ochiai [IKO], Kobayashi-Ochiai [KO1,2,3]. In the case of complex dimension two, these structures are those of affine C^2, $P_2(C)$ and $Q_2(C)$. The infinitesimal version of these structures are respectively the *holomorphic affine connection, holomorphic projective connection* and the *holomorphic conformal structure*. All compact complex smooth surfaces admitting such structures are completely classified in [IKO] and [KO1,2]. In particular, uniformization for such surfaces are also given. Their approach is to study the properties of surfaces admitting such structures.

We should mention here that there is a new approach due to Simpson [Sim] to the higher dimensional uniformization problem based on the construction of the variations of Hodge structures using Yang-Mills theory.

In this paper, we make some contributions to the third and fourth approaches to the uniformization problem of complex surfaces. Now we state our results with some comments on future developments of the theories.

In Section 3, we prove the best possible form of the numerical characterization of ball quotients in dimension 2. This general result heavily

depends on the fact that *every two dimensional log-canonical singular-
ity is uniformizable by a bounded symmetric domain Ω with local cover-
ing transformation group an appropriate parabolic discrete subgroup of
$Aut(\Omega)$* (see Subsection 3.1). It would be interesting to understand this
fact differential geometrically. For instance, Bando, Kasue and Naka-
jima [BKN] are able to characterize log-terminal surface singularities
as those singularities which appear in the limit of Kähler-Einstein sur-
faces with a fixed Einstein constant, a fixed volume and *bounded diam-
eters*. By putting Kobayashi-Todorov's example (see, for instance [K4,
Section 3]) into a general theory, they captured ALE gravitational in-
stantons bubbling off in the limit. In the known examples ([KT](see
[K4]) and [T2] (see also [Su])), the bubbling out ALE gravitational in-
stanton corresponds to the simple surface singularity appearing in the
limit. It is desirable to generalize Bando-Kasue-Nakajima's result in the
cases of log-canonical surface singularities. To do so, one must aban-
don the hypothesis of bounded diameter and must capture other kinds
of (non-ALE) gravitational instantons bubbling off *at infinity*. Natu-
ral candidates for these are complete Ricci-flat Kähler surfaces which
are compactified by adding an anti-canonical divisor (e.g., an elliptic
curve, a cycle of rational curves or two rational curves with an ordinary
contact point). The corresponding surface singularities are essentially
smoothable simple elliptic singularities, smoothable cusp singularities.
The multi-Taub-NUT gravitational instantons (see, for example [E-G-
H]) are examples of such Ricci-flat Kähler surfaces. Some existence
theorems for such Ricci-flat Kähler surfaces are established in [K5].

The fourth approach we take in Section 4 of this paper is based
on an idea which is complementary to Kobayashi-Ochiai's point of view
[KO1,2,3]. Namely, our approach is to seek a method of constructing a
flat G-structure modelled after a standard Hermitian symmetric spaces
M on an orbifold which we want to uniformize. If there exists such a
flat G-structure which is compatible with the given orbifold structure,
then the *developing map* of (X, D) to an open subset of M for this flat
G-structure will give a multivalued map $(X - D_\infty, D) \to M$ which is the
inverse of the universal branched covering map $M \to (X - D_\infty, D)$. This
point of view opens a way to constructing a differential equation satisfied
by the period map of a family of algebraic varieties, i.e., finding geomet-
rically a new transcendental function ("period map") characterized by
a simple uniformizing differential equation on a locally Hermitian sym-
metric orbifold. We mention briefly some examples due to [Sa-Y] and
[Sato] in Section 4. See [Nar2] for the construction of families of K3
surfaces over a quotient space of domains of type IV.

We now explain the geometric meaning of our results. Let X be the canonical model of a minimal algebraic surface X' of general type. We set

$$(0.1) \qquad \bar{c}_2(X) = c_2(X') - \sum_{p \in \text{Sing}(X)} \left(e(E(p)) - \frac{1}{|G(p)|} \right),$$

and

$$(0.2) \qquad \bar{c}_1(X)^2 = c_1(X')^2,$$

where $E(p)$ is the exceptional set in X sitting over p in the minimal resolution and $G(p)$ is the local fundamental group of p. Note that $\bar{c}_2(X)$ is the *Euler number of the orbifold* X. The space X is a complex orbifold (in an extended sense) and there exists a unique Kähler-Einstein orbifold-metric [Ko1]. Using this Kähler-Einstein metric, we can show that

$$(0.3) \qquad 3\bar{c}_2(X) - \bar{c}_1(X)^2 = \frac{3}{8\pi^2} \int_X |W_-|^2,$$

where W_- is the anti-self-dual part of the Weyl conformal curvature tensor. Since W_- vanishes if and only if the holomorphic sectional curvatures are constant for a Kähler-Einstein metric on a complex surface, we have the following three equivalent conditions for a compact complex smooth surface X of general type (see for example [KNr, p.486]).
(a) $3c_2(X) = c_1(X)^2$,
(b) X admits a holomorphic projective connection,
(c) X is uniformized by an open unit ball B^2 in C^2.
This is the reason why ball quotients are characterized simply by a numerical condition of Chern numbers. Namely it suffices to show only the existence of a Kähler-Einstein metric on surfaces with the extremal equality (a) between Chern numbers without examining the existence of a holomorphic projective connection. Examples of orbi-surfaces uniformized by the open unit ball are found in [BHH] and uniformizing differential equations for these orbi-surfaces are extensively studied in [Yo]. These differential equations are fulfilled by the developing map of holomorphic projective structures and are expressed in terms of holomorphic projective connections with appropriate singularities along branch loci. We shall not discuss these examples but instead shall examine explicitly some examples of uniformization of orbifolds $D \times D/\Gamma$ in Section 4.

Another class of surfaces we are interested in is that of normal surfaces with Hirzebruch proportionality $2c_2 = c_1^2$ in some modified sense. It turns out to be the case that such surfaces are not necessarily uniformized by the bidisk. Examples are constructed by [MT] (a simply connected compact smooth surface of general type with $2c_2 = c_1^2$) and [KNr] (orbifold example, see Section 4). In this paper, we treat such surfaces from a rather special point of view. As is noted by Kobayashi and Ochiai [KO2], holomorphic conformal structure together with Ricci-negative Kähler-Einstein structure characterizes bidisk quotients. But assuming the existence of only one of these seems to be insufficient for the characterization of bidisk quotients. To clarify the reason why this is insufficient, we attempt to present a geometric method of the construction of holomorphic conformal structures. Roughly speaking, we construct a singular holomorphic conformal structure on $P_2(C)$ with prescribed singularities and then desingularize it by using a covering trick. The construction reduces to solve the algebraic equations determined canonically by the singularities. This construction leads us to an explicit construction of the holomorphic conformal structure on $P_2(C)$ as Hilbert modular orbifolds ([KNr], [Sato]). An interesting by-product are two examples of compact complex surfaces, with simple singularities and ample canonical bundle, which fulfill the equality $2c_2 = c_1^2$, but are *not* covered by the bidisk [KNr]. These are Kähler-Einstein surfaces with negative Ricci-curvature, one being rigid and the other being deformed into a certain degenerate member (see Subsection 4.4). Sasaki and Yoshida [Sa-Y] employed such a singular holomorphic conformal structure to regard $P_2(C)$ as the Hilbert modular orbifold of $Q(\sqrt{2})$ and relate the structure with classical projective geometry to determine explicitly the uniformizing differential equation for that orbifold. Subsequently, Sato [Sato] improved their methods in some technical points to get uniformizing equations for other polyhedral modular groups (see Section 4). These differential equations come up via the developing maps of the holomorphic quadric structures and are expressed partially in terms of holomorphic conformal structures (one need more global consideration including integrability conditions to determine all the coefficients in the differential equations). We notice that there are no geometric method to construct explicitly examples of Kähler-Einstein metrics which are not half-conformally flat. These objects are not reduced to holomorphic ones and hence can be very difficult to construct geometrically. On the other hand, the Penrose transform and its inverse [Hit] (see also [Bes, Chapter 13] allows us to construct explicitly examples of Ricci-flat Kähler-Einstein complete open manifolds. For holomorphic conformal structure, there is a way to use singularities for the construction. Gen-

erally speaking, one can often reach explicit construction of geometric objects by putting the problem into holomorphic category and even by introducing singularities.

Finally, the author would like to express hearty thanks to Professors I. Naruki, F. Sakai, Doctors S. Nakamura, T. Sato for encouragement and stimulus discussions. Without collaborations with them, this paper would have never come out.

§1. Kähler-Einstein metrics and uniformization

The uniformization of compact algebraic curves[1] is classical. Any compact algebraic curve has either $P_1(C)$, C or H as its universal covering complex manifold. Here, H denotes the upper-half-plane. These are known as the complete list of simply connected Riemannian 2-manifold of constant curvature. Let X be a compact algebraic curve. Its genus and its Euler number are denoted by $g(X)$ and $e(X)$, respectively. The number $g(X)$ is the dimension of the complex vector space of holomorphic 1-forms on X and is related to $e(X)$ by the Riemann-Roch formula $e(X) = 2 - 2g(X)$. If we introduce a Kähler metric h on X, then the Gauss-Bonnet theorem says that $e(X) = \frac{1}{2\pi} \int_X K(h) dv_h$, where $K(h)$ is the Gaussian curvature of h. This suggests that the negative curvature property of h is reflected in the ampleness of $g(X)$ and *vice-versa*. Indeed, if $T^*(X)$, the bundle of holomorphic 1-forms, is generated by global sections, there exists a holomorphic surjective homomorphism $f : X \times C^n \to T^*(X)$, where $n = g(X)$. Then we apply the Gauss-Codazzi equation (cf. [GH], [Kob2]) to the flat metric for the trivial bundle $X \times C^n$ and the projection f to see that $T^*(X)$ admits a Hermitian metric with nonnegative curvature. This implies that X admits a Kähler metric with nonpositive curvature. Now suppose $e(X) < 0 \rightleftharpoons g(X) > 1$. Since $b_2(X) = 1$, X admits a Kähler metric of negative curvature. In fact, this correspondence holds in more precise sense. Namely, X admits a Kähler metric of constant negative curvature if and only if $g(X) \geq 2$. So the uniformization of compact algebraic curves is to find a metric of constant curvature. Let g be a Riemannian metric on X. The existence of isothermal coordinates implies that g determines a unique complex structure such that g is a Kähler metric. Let ω be a Kähler form of g. We seek a new Kähler metric of the form $\omega + \sqrt{-1}\partial\bar{\partial}u$ of constant negative curvature. Let γ be the curvature form of g. If g is so chosen

---

[1] Any 1-dimensional compact complex manifold is algebraic.

that $\gamma = -\omega - \sqrt{-1}\partial\bar{\partial}f$ for a real valued function f (since $b_2 = 1$, it becomes the case after a scale change if necessary), then u must be the solution to the non-linear equation

$$(1.1) \qquad\qquad\qquad 1 + \triangle u = e^{f+u}.$$

Hereafter we follow the convention $\triangle = $ trace of ∇^2 for the Laplacian. We apply the maximum principle to the equation (1.1) to get an *a priori* estimate $\|u\|_\infty \leq C(f)$, where $C(f)$ is a constant which depends only on $\|f\|_\infty$. The maximum principle implies also the uniqueness of the solution. If $p > 2$, the Sobolev imbedding theorem [GT, Chapt.7] gives the estimate for du: $\|du\|_\infty \leq C\|u\|_{2,p}$. Since u and $\triangle u$ have *a priori* bounds, the L^p-estimate for linear elliptic equations [GT, Theorem 9.11] gives the estimate of $\|u\|_{2,p}$ in terms of $\|u\|_\infty$. We thus get an *a priori* estimate for $\|du\|_\infty$. This then gives local Hölder estimates for u and $\triangle u = -1 + e^{f+u}$. Applying the interior Schauder estimates [GT, Theorem 6.2] we get an *a priori* estimate for the Hölder norm $\|u\|_{2,\alpha}$. Differentiating the equation (1.1) and applying the interior Schauder estimates inductively, we get *a priori* estimates for all Hölder norms of u. This allows us to use the continuity method (see, for example [GT]) to show the existence of the solution. So the uniformization problem for Riemann surfaces of genus ≥ 2 is reduced to the equation (0.1). It is well known that this uniformization theorem is generalized to the equivariant version in the following way. Let (X, D) be a pair of a compact algebraic curve and a Q-divisor $D = \sum_{i=1}^r (1 - \frac{1}{b_i})D_i$, where b_i are positive integers possibly ∞. We say (X, D) a *complex 1-dimensional orbifold* or shortly an *orbi-curve*. A finite b_i is the branch index at a point D_i and $b_i = \infty$ means the point D_i is deleted. We say (X, D) is *uniformizable* if there exists a pair (Y, Γ) of a curve Y (possibly non-compact, or a punctured Riemann surface) and a discrete group Γ of automorphisms of Y such that (X, D) is the quotient space Y/Γ. A simply connected Y is uniquely determined. We denote this $\hat{X}$ and call this the *universal branched covering*. Suppose (X, D) is uniformizable. By a theorem of Fenchel-Fox [Fo], there exists a finite uniformization $(Y - D, \Gamma)$, where Y is a compact curve, D is a possibly empty finite set of points of Y and Γ is a finite group of automorphisms of Y which leaves D stable. We want to define the Euler number of (X, D) by $e(Y - D) = |\Gamma|\, e(X, D)$. This definition agrees with the expression of the Euler number as a curvature integration, if we consider an orbifold-metric on (X, D). Therefore we define the Euler number of (X, D) in the following way.

Definition. The Euler number of (X, D) is

$$(1.2) \qquad e(X, D) = e(X) + \sum_{i=1}^{r} (\frac{1}{b_i} - 1).$$

The generalized uniformization theorem is well-known.

Theorem 1.1. *The orbifold (X, D) is not uniformizable in the following two cases: (1) $r = 1$, (2) $r = 2$ and $b_1 \neq b_2$. Otherwise, (X, D) is uniformizable. If (X, D) is uniformizable, then $\hat{X}$ is $P_1(C)$, C or H according as $e(X, D) > 0$, $= 0$, < 0.*

We call (X, D) with $e(X, D) < 0$ a hyperbolic 1-dimensional complex orbifold or a hyperbolic orbi-curve. Any hyperbolic orbi-curve (X, D) is uniformizable and $\hat{X} = H$. This assertion is proved by finding a constant negative curvature orbifold-metric on (X, D). To find such a metric, we first note that $K_X + \sum_{i=1}^{r} (1 - \frac{1}{b_i})D_i$ is an ample Q-divisor on X. Hence there exists a volume form Ω, a Hermitian metric $\| \cdot \|^2$ for $O_X(\sum_{i=1}^{r} D_i)$, holomorphic sections σ_i for $O_X(D_i)$ with zeros at D_i, such that $\|\sigma_i\| < 1$ and the minus of the Ricci-form of the singular volume form

$$(1.3) \qquad \Psi = \frac{\Omega}{\prod_{i=1}^{r} b_i^2 \|\sigma_i\|^{2(1 - \frac{1}{b_i})} (1 - \|\sigma_i\|^{\frac{2}{b_i}})^2}$$

defines a complete orbifold Kähler form $\omega = \sqrt{-1}\partial\bar{\partial} \log \Psi$ on the orbifold (X, D). Here, we interpret $b_i(1 - \|\sigma_i\|^{\frac{2}{b_i}}) = \log \frac{1}{\|\sigma_i\|^2}$ if $b_i = \infty$. Of course this agrees with taking the limit $b_i \to \infty$. This metric looks like an orbifold metric $\frac{|dz^{\frac{1}{n}}|^2}{(1 - |z|^{\frac{2}{n}})^2}$ around a point with $b_i = n$ and like a Poincaré metric $\frac{|dz|^2}{|z|^2 (\log \frac{1}{|z|^2})^2}$ of the punctured disk around a point of $b_i = \infty$. We again solve the equation (1.1). It is easy to see that the function $f = \log \frac{\Psi}{\omega}$ is an orbifold-smooth bounded function on (X, D). The maximum principle then implies the uniqueness of the solution and gives an *a priori* estimate $\|u\|_\infty < C(f)$ as before. We consider each D_i a quotient singularity (if $b_i = \infty$, the group is a discrete parabolic group of H) and locally uniformize the metric ω to get a smooth metric on each local uniformization. Using the homogeneity of H with the invariant metric, we can construct a system of quasi-coordinate neighborhoods

and define the Banach space of $C^{k,\alpha}$-functions on (X, D). See Section 3
for precise definitions. We then apply the Sobolev imbedding theorem
$\|du\|_\infty \leq C\|u\|_{2,p}$, the L^p-estimates and the interior Schauder estimates
in these quasi-coordinate neighborhoods to get *a priori* estimates for the
derivatives of u. The continuity method then gives the existence of a
solution for (1.1). We thus get an orbifold-metric on (X, D) of constant
negative curvature. Since there is a simply connected space form H, we
can develop (X, D) isometrically over H by analytic continuation along
geodesics. This shows that $\hat{X} = H$.

It is then natural to ask whether it is possible to generalize this
simple criterion for uniformization of Riemann surfaces to higher di-
mensions. Hereafter we consider the case of dimension two. First of all
we have no simple uniformization theorem which is applicable in any
wide class of manifolds of dimension ≥ 2. Indeed there are infinitely
many topological types in simply connected compact complex surfaces.
For instance, the Lefschetz hyperplane section theorem [Mil] implies that
any smooth surface in $P_3(C)$ of degree ≥ 5 is simply connected. As in
one-dimensional case, the uniformization theorem should be understood
in the classification theory of varieties. Higher dimensional algebraic va-
rieties are roughly classified according to their Kodaira dimensions. Let
X be an n-dimensional Kähler manifold. The Kodaira dimension of X
is the dimension of the image of X under the rational map Φ_m defined
by the pluricanonical system $|mK_X|$ of X for a sufficiently large m. The
canonical class in the de Rham cohomology is represented by the neg-
ative of the Ricci form $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega$ for any volume form Ω. It follows
from the Calabi/Yau theorem [Y1] (to find a Kähler metric with the
prescribed volume form) that if the pluricanonical system has no base
points, then X admits a Kähler metric whose Ricci form is the pull back
of the negative of a Kähler metric in the canonical class of the image
$\Phi_m(X)$. Thus the Kodaira dimension m of X corresponds to the nega-
tivity of Ricci curvature in the sense that X admits a Kähler metric with
nonpositive Ricci form with rank m. So, whereas 1-dimensional varieties
are classified according to their Gaussian curvatures, higher dimensional
ones are roughly classified according to their Ricci curvatures and there
is no *general* uniformization theorem for Kähler manifolds with con-
stant Ricci curvature, i.e., Kähler-Einstein manifolds. For spaces with
constant holomorphic sectional curvature, the following uniformization
theorem is well-known [KNm, pp.169–171]:

Fact 1.1 ([KNm]). *A simply connected n-dimensional complete
Kähler manifold with constant holomorphic sectional curvature c is one
of the following three spaces with canonical Kähler-Einstein metric:*

(1) $P_n(C)$, *if* $c > 0$, (2) C^n, *if* $c = 0$, *or* (3) B^n, *the open unit ball in* C^n, *if* $c < 0$.

A special uniformization theorem due to Chen-Ogiue for higher dimensional Kähler-Einstein manifolds is:

Theorem 1.2 ([CO]). *Let* (X, ω) *be an n-dimensional compact Kähler-Einstein manifold. Then*

$$(1.4) \qquad (2(n+1)c_2(X) - n\,c_1(X)^2) \cup [\omega]^{n-2}([X]) \geq 0,$$

and the equality holds if and only if (X, ω) *is of constant holomorphic sectional curvature.*

The left hand side is the deviation of the Kähler-Einstein form ω to be of constant holomorphic sectional curvature. Suppose $\gamma_\omega = \frac{(n+1)c}{2}\,\omega$, where γ_ω is the Ricci form of ω and c is a constant, the holomorphic sectional curvature. Set[2]

$$T_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + \frac{c}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{\bar{k}l}).$$

Then a simple tensor calculation shows that for a Kähler-Einstein (X, ω),

$$(1.5) \quad (2(n+1)c_2(X,\omega) - nc_1(X,\omega)^2) \wedge \omega^{n-2} = \frac{(n+1)\|T\|^2}{4\pi^2 n(n-1)}\omega^n \geq 0.$$

Integrating (1.5) over X gives (1.4) and the equality holds if and only if $T = 0$, i.e., ω is of constant holomorphic sectional curvature. The next problem is then to find an algebro-geometric condition for the existence of a Kähler-Einstein metric. This is the solution to Calabi's conjecture due to Aubin [Au1] and Yau [Yau1].

Theorem 1.3 ([Au1], [Yau1]). *Let* X^n *be a compact Kähler manifold whose real first Chern class* $c_1(X)_{\mathbb{R}}$ *vanishes or is negative, i.e., is represented by a negative definite real closed* (1,1)-*form. Then every Kähler class of* X *contains a unique Ricci-flat Kähler form if* $c_1(X)_{\mathbb{R}} = 0$, *or there exists a unique (up to a constant multiple) Kähler-Einstein metric if* $c_1(X)_{\mathbb{R}} < 0$.

The essential point in Aubin and Yau's proof of this theorem in the case of $c_1(X)_{\mathbb{R}} < 0$ is to derive *a priori* estimates for the equation

---

[2]We use the same convention as in [KNm] for the curvature tensor. Namely, $R_{i\bar{i}i\bar{i}}$ is negative in the positive curvature case.

(1.6) $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^{u+f}\omega^n,$

which replaces (1.1) in one-dimensional case. The geometric meaning of (1.6) is completely the same as (1.1). As in the case of (1.1), we get a uniform estimate for u via maximum principle. The non-linearity in (1.6) requires geometric arguments in getting *a priori* C^2-estimate for u. In fact, a variant of Bochner-type argument is used for this purpose. We recall the following infinitesimal Schwarz lemma:

Fact 1.2 ([C], [Yau2]). *Let (M, g) be a Kähler manifold and (N, h) a Hermitian manifold. If f is a non-constant holomorphic map of M to N, then*

$$\triangle_g \log |\partial f|^2 \geq \frac{\mathrm{Ric}_g(\partial f, \overline{\partial f})}{|\partial f|^2} - \frac{\mathrm{Bisect}_h(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})}{|\partial f|^2}.$$

Since this lemma is typical in differential geometry, we outline its proof. The Bochner formula for holomorphic maps due to Chern [C] is

$$\triangle_g |\partial f|^2 = \mathrm{Ric}_g(\partial f, \overline{\partial f}) - \mathrm{Bisect}_h(\partial f, \overline{\partial f}, \partial f, \overline{\partial f}) + |\nabla^{(1,0)}\partial f|^2.$$

Now we compute

$$\triangle_g \log |\partial f|^2 = \frac{\triangle_g |\partial f|^2}{|\partial f|^2} - |\partial |\partial f|^2|^2$$

and use the following Schwarz type inequality valid for holomorphic maps:

$$|\partial |\partial f|^2|^2 \leq |\partial f|^2 |\nabla^{(1,0)}\partial f|^2$$

to get the result. Since f is holomorphic, we have $\nabla^{(1,0)}\overline{\partial f} = 0$. The above Schwarz type inequality then follows:

$$LHS = |(\nabla^{(1,0)}\partial f, \overline{\partial f}) + (\partial f, \nabla^{(1,0)}\overline{\partial f})|^2$$
$$= |(\nabla^{(1,0)}\partial f, \overline{\partial f})|^2$$
$$\leq |\partial f|^2 |\nabla^{(1,0)}\partial f|^2$$

We apply this to the identity map of $(X, \tilde{\omega})$ to (X, ω), where $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$. Since we use the continuity method to the equations (E_t) obtained by replacing f by tf for $0 \leq t \leq 1$, we may assume $\tilde{\omega} = \tilde{\omega}_t$. In the following argument, all constants are independent of t. Since

$$\mathrm{Ric}(\tilde{\omega}_t) = -\tilde{\omega}_t + (1-t)\sqrt{-1}\partial\bar{\partial}f$$
$$\geq (1-t)\sqrt{-1}\partial\bar{\partial}f$$
$$\geq -c\omega,$$

where c is a positive constant depending only on ω and f, and the bisectional curvature of ω is bounded, we have, for some positive constants C_1 and C_2,

$$\triangle_{\tilde{\omega}} \log \mathrm{tr}_{\tilde{\omega}}\,\omega \geq -C_1 - C_2\,\mathrm{tr}_{\tilde{\omega}}\,\omega.$$

Since $\triangle_{\tilde{\omega}} u = n - \mathrm{tr}_{\tilde{\omega}}\,\omega$, if we choose a sufficiently large constant A such that $A - C_2 > 0$, we have

$$(1.7) \qquad \triangle_{\tilde{\omega}}(\log \mathrm{tr}_{\tilde{\omega}}\,\omega - Au) \geq -An - C_1 + (A - C_2)\,\mathrm{tr}_{\tilde{\omega}}\,\omega.$$

Applying the maximum principle to (1.7), we get, from the uniform estimate of u, the C^2-estimate for u. Namely

$$C\omega < \tilde{\omega} < C^{-1}\omega,$$

for some positive constant C. We then apply the Hölder estimates for second derivatives [GT, Theorem 17.14] to get the $C^{2,\alpha}$-estimates for u with some $0 < \alpha < 1$. Interior Schauder estimates then give the C^k-estimates for $k \geq 3$. Using the same strategy as in [Au1], [Yau1] and [CY], Kobayashi [Ko2,3] proved an equivariant version of Theorem 1.3 for some class of normal surfaces with logarithmic Kodaira dimension 2. Theorem 1.3 has a direct application to the uniformization problem for compact complex manifolds with negative first Chern class.

Theorem 1.4 ([Yau1]). *Let X be an n-dimensional compact Kähler manifold with negative first Chern class. Then the inequality (1.4) holds and the equality occurs if and only if X is covered by the open unit ball in C^n.*

Miyaoka [Miy1] proved the inequality (1.4) for wider class of compact complex surfaces by means of algebraic geometry (without mentioning the equality case). This is the class of algebraic surfaces of general type. A compact complex surface X is of general type if the Kodaira dimension of X is two. The Kodaira imbedding theorem implies that a surface with negative first Chern class is of general type. There is a unique minimal model in the birational class of surfaces of general type.

The difference between the negativity of the first Chern class and being of general type was clarified by Kodaira [Kod].

Fact 1.3 ([Mu1], [Kod], [Bom], [BPV]).   *Let X be a minimal algebraic surface of general type. Then for $n \geq 5$, the pluricanonical system $|nK_X|$ has no base point and the pluricanonical map f_n is biholomorphic modulo $\mathcal{E}$, where $\mathcal{E}$ is the union of all curves E such that $K_X \cdot E = 0$. In particular, $c_1(X) < 0$ if and only if X has no rational curve with self-intersection number -2 ((-2)-curve).*

The image $X_1 = f_n(X)$ is uniquely determined in the birational class and is called the *canonical model* of X. The canonical model X_1 has at worst simple singularities and has ample canonical bundle. The Miyaoka inequality for minimal algebraic surfaces of general type with (-2)-curves is also understood in the Kähler-Einstein context [Ko1] using a singular metric. Namely the canonical model X_1 is an orbifold and one can establish the equivariant version of Theorem 1.3 which asserts that there exists a unique (up to a constant multiple) Kähler-Einstein orbifold-metric on X_1. The refinement of the inequality (1.4) was obtained in [Miy2] and [Ko2,3]. Although a more general form is proved in [Miy2] and [Ko3], we state their result in a restricted form for simplicity.

Theorem 1.5 ([Miy2], [Ko2,3]).   *Let X be a minimal algebraic surface of general type and X_1 its canonical model. For $p \in \mathrm{Sing}(X_1)$ we set $E(p)$ the exceptional set for the minimal resolution of p and $G(p)$ the local fundamental group for p. Then*

$$(1.8) \qquad 3(c_2(X) - \sum_{p \in \mathrm{Sing}(X_1)} (e(E(p)) - \frac{1}{|G(p)|})) \geq c_1(X)^2,$$

and the equality holds if and only if the orbifold X_1 is uniformizable by the open unit ball in C^2.

An equivariant version [Ko2,3] of Theorem 1.3 implies that the canonical model X has a unique Kähler-Einstein orbifold-metric. Using this metric, we have an inequality (0.3), which in fact holds in pointwise level. In Section 3, we generalize these results in the category of log canonical normal surfaces with branch loci whose logarithmic Kodaira dimension is two. As a corollary, we get a numerical characterization for ball quotients (possibly with cusps, quotient singularities and branch loci) among log canonical normal surfaces with branch loci.

§2. Holomorphic G-structures and uniformization

In this section we discuss the structures of surfaces with the Hirze-bruch proportionality of the quadric. We are mainly interested in holomorphic conformal structures on Kähler-Einstein surfaces. We shall introduce a certain kind of degeneration of holomorphic conformal structures together with a method of construction. The informations arising from singularities are effective to construct explicit examples on $P_2(C)$ ([KNr], [Sat]). We begin with the definition of holomorphic conformal structures and some results of Kobayashi-Ochiai [KO2] which have motivated recent research ([KNr], [SaY1,2] and [Sat]). Let M be an n-dimensional complex manifold, $L(M)$ the holomorphic $GL(n, C)$-principal bundle of frames of the holomorphic tangent bundle $T^{(1,0)}M$.

Definition ([KO2]). A holomorphic conformal structure on M is a holomorphic $CO(n, C)$-subbundle of $L(M)$, where $CO(n, C) = \{cU; c \in C^*, U \in O(n, C)\}$.

As in Riemannian geometry, the degeneration of holomorphic conformal structures is best understood by introducing a tensor field and look at degenerations of it. The following is equivalent to the above:

Definition ([KO2]). A holomorphic conformal structure on M is a pair $\{U_\alpha, g_\alpha\}$ such that (i) $\{U_\alpha\}$ is an open covering of M by holomorphic coordinate neighborhoods, (ii) g_α is a holomorphic non-degenerate symmetric covariant 2-tensor field on U_α with the compatibility condition

$$(2.1) \qquad g_\alpha = f_{\alpha\beta} g_\beta \text{ on } U_\alpha \cap U_\beta,$$

where $f_{\alpha\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$.

Let $g_\alpha = \sum_{i,j=1}^n g_{\alpha ij} dz_{\alpha i} dz_{\alpha j}$ locally. By taking the determinant of (2.1) with respect to the basis $\{dz_{\beta i}\}$, we see that

$$(2.2) \qquad O_M = F^n \otimes K^{-2},$$

where F^n is the holomorphic line bundle defined by the transition functions $\{f_{\alpha\beta}\}$. Note that a holomorphic conformal structure on M is a holomorphic section of $F \otimes S^2 T^*(M)$.

Holomorphic conformal structure in 2-dimension is exceptional in the point that it is equivalent (modulo passing to a double covering) to

the holomorphic splitting of the holomorphic tangent bundle. The two directions at each point is given by the two projective solutions of the quadratic equation $g_\alpha(X, X) = 0$.

Example 2.1. C^n has a $CO(n)$-invariant holomorphic conformal structure $\sum_{i=1}^n (dz_i)^2$.

Example 2.2 ([KO2]). Set $G = O(n + 2, C)$. Then $G = \text{Aut}(Q_n(C))$. $Q_n(C)$ has a G-invariant holomorphic conformal structure. Indeed, if we write

$$Q_n(C) = \{[\zeta^0 : \cdots : \zeta^{n+1}] \in P_{n+1}(C);$$
$$-2\zeta^0\zeta^{n+1} + (\zeta^1)^2 + \cdots + (\zeta^n)^2 = 0\},$$

then

$$-2d\zeta^0 d\zeta^{n+1} + (d\zeta^1)^2 + \cdots + (d\zeta^n)^2$$

gives an G-invariant holomorphic conformal structure on $Q_n(C)$. The group G acts naturally on $L(Q_n(C))$. The subbundle P corresponding to the above holomorphic conformal structure is given by $P = G \cdot e$, where $e = (\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n})$ is the frame at $p_0 = [1 : 0 : \cdots : 0] \in Q_n(C)$ with $z^i = \frac{\zeta^i}{\zeta^0}$.

Example 2.3 ([KO2]). The noncompact dual $\check{M}$ of $M = Q_n(C)$ is naturally embedded in M and $\check{G} = \text{Aut}(\check{M})$ is the subgroup of $G = \text{Aut}(M)$ which preserves $\check{M}$ invariant. It follows that $\check{M}$ has a G-invariant holomorphic conformal structure. If Γ is a discrete subgroup of $\check{G}$, then $\Gamma \backslash \check{M}$ has a holomorphic conformal structure. If Γ has fixed points, then the holomorphic conformal structure on the quotient space will have singularities along branch loci.

The holomorphic conformal structures in Examples 2.2 and 2.3 comes from the quadric structure.

Definition ([KO2]). A quadric structure in M is a pair (U_α, ϕ_α) such that (i) ϕ_α is a biholomorphic map into an open set of $Q_n(C)$, (ii) the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is induced from an element of G.

If M has a quadric structure, then it naturally has a holomorphic conformal structure. A quadric structure is called a *flat* holomorphic

conformal structure. This definition may be reasonable, since, for a quadric structure, the *developing map* is canonically defined in the following way. Let (U_α, ϕ_α) be a quadric structure on M. Pick a point $p \in U_\alpha$. For another point q, we draw a curve joining p and q and choose a sequence $(\alpha_0 = \alpha, \alpha_1, \cdots, \alpha_k)$ such that $\{U_{\alpha_i}\}$ is a chain of open sets joining p and q along c. The analytic continuation along c of ϕ_α is well defined from the above definition. Indeed, if $\phi_{\alpha_0} = f_{\alpha_0 \alpha_1} \circ \phi_{\alpha_1}$ on $U_{\alpha_0} \cap U_{\alpha_1}$, where $f_{\alpha_0 \alpha_1} \in \mathrm{Aut}(Q_n(C))$ is as above, then ϕ_{α_0} is analytically continued to $f_{\alpha_0 \alpha_1} \circ \phi_{\alpha_1}$ on U_{α_1}. We can thus analytically continue ϕ_{α_0} along c from U_{α_0} to U_{α_1}. We can do this along any curve joining p and q. The result depends only on the homotopy class of the path c. A developing map for a quadric structure is a multi-valued holomorphic map of M to $Q_n(C)$ thus constructed. This construction is very important in our examples in Section 4.

The existence of a holomorphic conformal structure causes a strong restriction on the analytic structure of a compact complex manifold M. Among them, (2.2) is of fundamental importance. For Chern numbers, Kobayashi-Ochiai [KO2] proved the following theorems.

Theorem 2.1 ([KO2]). *Let $c^{(i,i)}$ be the (i,i)-component of the real i-th Chern class. Then*

$$c^{(r,r)} = a_r n^{-r} (c^{1,1})^r, \quad 1 \le r \le n,$$

where a_i are positive integers subject to the relation

$$\sum_{q=0}^{m} (1+h)^{n-2q} h^{2q} = 1 + a_1 h + a_2 h^2 + \cdots + a_n h^n.$$

Corollary 2.2 ([KO2]). *If M is Kähler, then*

(2.3) $$c_r = a_r n^{-r} c_1^r.$$

Corollary 2.3 ([KO2]). *If M is a surface, whether Kähler or not, then*

$$2c_2 = c_1^2.$$

Theorem 2.1 is proved by constructing a certain affine connection from a holomorphic conformal metric tensor and computing its curvature form. Corollary 2.3 motivates our interest in surfaces with $2c_2 = c_1^2$. It is generally not known which manifold admits a holomorphic conformal structure. For Kähler-Einstein manifolds, Kobayashi-Ochiai [KO2] proved

Theorem 2.4 ([KO2]).   *Let M be a compact Kähler-Einstein manifold of dimension n. If M admits a holomorphic conformal structure, then M is either* (i) $Q_n(C)$, (ii) *flat, i.e., M is covered by a torus, or* (iii) *the universal covering is the noncompact dual of $Q_n(C)$, according as the scalar curvature of M is positive, zero and negative.*

Since the differential geometric argument Kobayashi-Ochiai gave in [KO2, pp.597–600] is typical in concluding that the given metric is locally symmetric (the origin of this argument goes back to Berger's holonomy theorem (see [Bes]) which singles out the possibilities of holonomy groups of non symmetric spaces), we give here an outline of their proof of Theorem 2.4. For simplicity, we assume $\text{Ric} \neq 0$. Since the given holomorphic conformal structure is non-degenerate, we have a non trivial holomorphic section σ of

$$(2.4) \qquad\qquad K^{-2} \otimes S^4 T^*(X) \subset T_4^4(T(X))$$

by symmetrizing $g^2 = g \otimes g$. A version of Bochner's vanishing theorem [Kob1] implies that σ is covariant constant with respect to the Kähler-Einstein metric. The existence of this parallel object implies the existence of a non trivial invariant subspace in the corresponding representation space (2.4) of the holonomy group. Berger's theorem [Bes] then causes reduction of the holonomy group to the one of a symmetric space. Using the special properties between Chern numbers enjoyed by manifolds admitting a holomorphic conformal structure, we infer that M is as in Theorem 2.4. Kobayashi-Ochiai [KO2] posed the question whether a compact Kähler manifold with $c_1 > 0$ admitting a holomorphic conformal structure is biholomorphic to $Q_n(C)$. It is proved in [KO2] that if n is odd the question is true. In general, the question is still open. In 2-dimensional case, the question is true, as a consequence of the complete classification ([KO2]) of smooth compact complex surfaces with a holomorphic conformal structure. There is in particular a complete list of the uniformization of smooth compact complex surfaces admitting holomorphic conformal structure. In fact, each of these surfaces admits a flat holomorphic conformal structure ([KO2]). Kobayashi-Ochiai's list is as follows.

Theorem 2.5 ([KO2]).   *The class of compact complex surfaces admitting holomorphic conformal structure is as follows:*
(1) *the quadric $P_1(C) \times P_1(C)$;*
(2) *flat holomorphic fiber bundles over a compact Riemann surface with fiber $P_1(C)$;*

(3) *hyperelliptic surfaces;*

(4) *complex tori;*

(5) *minimal elliptic surfaces with $c_2 = 0$ and even first Betti number* (*these consist of flat orbifold principal bundles with group an elliptic curve T and base an orbi-curve; such objects are always uniformizable*);

(6) *algebraic surfaces uniformized by the bidisk $D \times D$;*

(7) *Hopf surfaces $(C^2 - \{0\})/\Gamma$, where Γ consists of linear transformations of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$;*

(8) *Inoue surfaces S_U associated with $U \in SL(3; Z)$;*

These surfaces in fact admit a quadric structure.

For non-Kähler cases (7) and (8), the uniformization is explicitly given in [KO2]. Any surface in cases (3) and (4) is uniformized by the Euclidean space C^2 with a group of Euclidean motions. The cases (2) (surfaces uniformized by $\tilde{\Delta} \times P_1(C)$ where $\tilde{\Delta} = D$ or C) and (5) (surfaces uniformized by $\tilde{\Delta} \times T$, where $\tilde{\Delta} = P_1(C)$ or C or D and T is an elliptic curve) are hybrid cases for which Kähler geometry is not yet fully developed. The case (6) are treated in Theorem 2.4 but not in a constructive way. It is then natural to ask whether there is a method of construction of explicit examples. For this purpose, we introduce the notion of *generalized holomorphic conformal structures* (we write GHCS for abbreviation). Hereafter X will denotes a smooth compact complex surface.

Definition ([KNr]). A generalized holomorphic conformal structure defined by a holomorphic line bundle L over a compact complex surface X is a primitive holomorphic section τ of $L \otimes S^2 T^*(X)$.

Here the primitiveness means that, at the germ level, τ is not divisible by any non-units in the structure sheaf of X. We give a local expression of GHCS in the following way. Let U_α be an open Stein covering of X by coordinate neighborhoods and assume $H^2(U_\alpha; Z) = \{0\}$ for all α. For a GHCS, we can find a system of holomorphic sections $\tau_\alpha \in \Gamma(U_\alpha, S^2 T^*(X))$ such that if $\tau_\alpha = u\tau'_\alpha$ for $u \in \Gamma(U_\alpha, O)$ and $\tau'_\alpha \in \Gamma(U_\alpha, S^2 T^*(X))$, then $u \in \Gamma(U_\alpha, O^*)$, and $\tau_\alpha = h_{\alpha\beta}\tau_\beta$ where $h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, O^*)$ and the cocycle $\{h_{\alpha\beta}\}$ defines the holomorphic line bundle L.

Definition ([KNr]). The discriminant divisor D for a GHCS $\{\tau_\alpha\}$ is the divisor on X defined by $\{det(\tau_\alpha) = 0\}$. The divisor D does not depend on the choice of $\{\tau_\alpha\}$.

By taking the determinant of $\tau_\alpha = h_{\alpha\beta}\tau_\beta$, we obtain

(2.5) $[D] = 2(L + K_X),$

which replaces (2.2). If the discriminant divisor $[D]$ is empty, then a
GHCS reduces to a non-singular holomorphic conformal structure and
(2.5) becomes (2.2). The relation (2.5) says that we can consider the
double covering $\tilde{X}$ of X branching exactly over D, which one can realize
as an analytic subspace of the total space of the line bundle $L + K_X$ (see
[BPV, p.42]). We note that there are $2^{b_1+\delta}$ such coverings where b_1 is
the first Betti number and δ is the number of independent 1-cycles whose
doubles are zero. Then the number of homomorphisms of $H_1(X; Z) \rightarrow$
Z_2 is $2^{b_1+\delta}$. If one takes the double $\tilde{X}$ associated with $L + K_X$, then
$\tilde{X}$ depends functorially on (X, D). We now introduce a special class
of GHCS's which describes the behavior near cusp singularities of the
standard holomorphic conformal structure inherited from $D \times D$.

   Definition ([KNr]). *Let X be a smooth surface and D a reduced
divisor with normal crossings. A GHCS τ behaves logarithmically near
D if τ is locally given by*

(2.6) $$y^2 P(dx)^2 + 2xy Q(dx)(dy) + x^2 R(dy)^2$$

*with $Q^2 - PR \neq 0$ in coordinates (x, y) such that D is locally given by
$xy = 0$. The discriminant divisor is $2D$.*

   Example 2.4.   For this example we refer to Hirzebruch's theory
of Hilbert modular surfaces [Hir2] (see also [BPV]). Let K be a totally
real quadratic field over Q and M a free abelian subgroup of rank 2 and
V a totally positive multiplicative group of rank 1 such that $VM = M$.
Let $M = Z + Zw$ with $0 < w^1 < 1 < w$, where w^1 means the non-trivial
Galois action in K. The semi-direct product

(2.7) $$G(M, V) = \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & 1 \end{pmatrix} \mid \epsilon \in V, \mu \in M \right\}$$

acts on $H^2 = \{(z_1, z_2) \in C^2 \mid \operatorname{Im}(z_1) > 0, \operatorname{Im}(z_2) > 0\}$ by

$$(z_1, z_2) \rightarrow (\epsilon z_1 + \mu, \epsilon^1 z_2 + \mu^1).$$

This action is free and properly discontinuous. The (*Hilbert modular*)
cusp singularity is obtained by adjoining a point ∞ to the complex man-
ifold $H^2/G(M, V)$ with its neighborhood system the image of $y_1 y_2 >$
$d\,(d > 0)$, where $z_i = x_i + \sqrt{-1} y_i\,(i = 1, 2)$. Of course the point ∞

is regarded as the image of the boundary point $(i\infty, i\infty)$ of H^2. The Hilbert modular cusp singularity is desingularized if we replace ∞ by a cycle of $P_1(C)$'s, say $E = \sum_{i=0}^{r-1} E_i$, with self intersection numbers $-q_i$, $(0 \leq i \leq r-1$, $q_i \geq 2$ and some $q_i > 2)$. The r numbers q_i are determined by expanding the quadratic irrational number w into the periodic continued fraction: $w = q_0 - (q_1 - (q_2 - \cdots - (q_{r-1} - (q_r - (q_{r+1} - (\cdots)^{-1})^{-1})^{-1})^{-1}) \cdots)^{-1})^{-1}$ with period r, where $q_r = q_0$ and $q_{r+1} = q_1$, etc. Let P_k and Q_k be positive integers such that $\frac{P_k}{Q_k}$ is equal to the finite continued fraction obtained by cutting w at q_{k-1}. Define $R_k = P_k - Q_k w$. Then $q_k R_k = R_{k-1} + R_{k+1}$. This relation with the initial conditions determines R_k for all integers and R_r is a generator for $V \cong Z$ under the isomorphism $R_r^n \to n$. The equation

$$2\pi\sqrt{-1}(z_1, z_2) = (\log u_k, \log v_k) \begin{pmatrix} R_{k-1} & R_{k-1}^1 \\ R_k & R_k^1 \end{pmatrix}$$

determines a canonical local coordinates (u_k, v_k) around the intersection $E_{k-1} \cap E_k$, where (z_1, z_2) are the standard coordinates of H^2 and E_k are considered to be periodic with period r. It is then clear that the standard holomorphic conformal structure $dz_1 dz_2$ projects down to a GHCS in coordinates (u_k, v_k) which behaves logarithmically along E.

There are many ways in formulating logarithmic versions of the case (iii) in Theorem 2.4. The following is the simplest one.

Theorem 2.6 ([KNr]). *Let $\bar{X}$ be a smooth compact complex surface with at worst quotient singularities and E a divisor lying over the regular part of $\bar{X}$ and set $X = \bar{X} - E$. Assume $K_{\bar{X}} + E$ is numerically ample modulo E and big. If X admits an orbifold-holomorphic conformal structure which extends across E to a GHCS on $\bar{X}$ which behaves logarithmically along E, then X is uniformized by the bidisk.*

Proof. [Ko3, Theorem 1] implies that there exists a unique complete Kähler-Einstein orbifold metric on X. Just as in [KO2], we can construct from the given logarithmic GHCS a holomorphic section s of

$$(K_{\bar{X}} \otimes [E])^{-2} \otimes S^4 T^* \bar{X}(\log E).$$

Examining the behavior near E of the complete Kähler-Einstein metric (see [Ko2]), we see that if U is the boundary of small a neighborhood of E,

$$\int_U \frac{\partial |s|^2}{\partial n} \longrightarrow 0$$

as U goes smaller and smaller. We can thus use the Stokes' Theorem to get

$$\int_X \triangle |s|^2 = 0.$$

Bochner's formula [Kb] tells us that

$$\triangle |s|^2 = |\nabla s|^2.$$

Therefore s is covariant constant. As in [KO2], by the holonomy theorem, the canonical Kähler-Einstein metric on X is locally symmetric and modeled after bidisk. This implies the uniformization.

There is a desingularization procedure for a certain class of GHCS. This is simply a local uniformization procedure for orbifold holomorphic conformal structures with branch loci. Instead of formulating holomorphic conformal structures on orbifolds in full generality, we give a characterization for a certain class of them. Let X be a smooth complex surface and $\tau = \{\tau_\alpha\}$ be a GHCS on X with the defining line bundle L and with the discriminant divisor D. We consider the following:

(I) the discriminant divisor D is reduced,

(II) τ_α is of rank 1, i.e., $\mathrm{rank}(g_{\alpha ij}) = 1$, along the regular part $\mathrm{Reg}(D)$ of D,

(III) for every $p \in \mathrm{Reg}(D)$, the one dimensional null space N_p (multiplicity 2) of τ at p coincides with $T_p(D)$.

Definition ([KNr]).   *A GHCS is called tangential if it satisfies the condition (III) along* $\mathrm{Reg}(D)$.

Let X be compact. Since $[D]$ is divisible by 2 in the Picard group, we may consider a double covering $\tilde{X}$ of X branching exactly over D. Let $\tilde{X} = X(\sqrt{D})$ be the double covering associated to $L + K_X$ (recall (2.5)). It is well known ([BPV, III-7]) that $Sing(\tilde{X}) = Sing(D)$ and any quotient singularity of $\tilde{X}$ is a *rational double point* (corresponding to a *simple singularity* of D). Set $D^* = \mathrm{Reg}(D) \cup \{\text{simple singularities of } D\}$. Then we can formulate a desingularization procedure as follows.

Theorem 2.7 ([KNr]).   *Assume that the conditions (I), (II), (III) are fulfilled. Then the holomorphic conformal structure induced on $\tilde{X} - D$ uniquely extends to a non-degenerate holomorphic conformal structure on $\mathrm{Reg}(\tilde{X}) = (\tilde{X} - D) \cup \mathrm{Reg}(D)$. Moreover, it extends automatically to an orbifold holomorphic conformal structure on $(\tilde{X} - D) \cup D^*$.*

For a proof, see [KNr, pp.491–492]. This theorem characterizes orbifold holomorphic conformal structures on quasi-regular orbi-surfaces X

with branch loci D with $O_X(D)$ divisible by 2 in the Picard group with only simple singularities. Here, an orbi-surface X is called *quasi-regular* if X is smooth simply as a complex surface. We shall use Theorems 2.6 and 2.7 to construct explicitly examples and counter examples for uniformization problem in Section 4.

§3. Numerical characterization of ball quotients for normal surfaces with branch loci

3.0. Kähler-Einstein geometry for normal surfaces

The aim of this section is to develop Kähler-Einstein geometry in the category of log-canonical normal surfaces with branch loci (for definitions, see Section 3.2.1). Note that a normal surface has at worst isolated singularities. For basic results on normal surfaces, we refer to [Sak1], [Sak2]. In particular, we use the intersection theory defined by Mumford [Mu2]. As is stated in Introduction, the canonical Kähler-Einstein metric on a compact complex smooth surface X of general type is naturally defined on its canonical model X', in other words, the canonical Kähler-Einstein metric singles out the obstruction $((-2)$-curves) for the canonical bundle to be ample and sees X as X' whose canonical divisor $K_{X'}$ is ample. We shall generalize this in the category of normal surfaces. Let (X, D) be a pair of a compact complex normal Moishezon surface X and a Q-divisor[3] $D = \sum_i (1 - \frac{1}{b_i}) D_i$ $(b_i = 2, 3, \ldots, \infty)$. We call such a pair (X, D) a *normal surface pair*.

Definition. Let (X, D) be a normal surface pair. We say a point $p \in X$ is a singularity of (X, D) if p is a singular point of a surface X or if p is a smooth point of X and a curve Supp(D) has a singularity at this smooth point $p \in X$. Otherwise, p is a regular point. Furthermore, p is quasi-regular if p is a smooth point of V.

Remark. We can also deal with non-Moishezon surfaces and get the same results. For simplicity's sake, we treat Moishezon case. For necessary modifications, see [Sak1,2].

The classification theory for normal surfaces is studied by Sakai [Sak1,2]. The rough classification starts with the log-Kodaira dimension. Define the *log-canonical ring* (as a normal surface pair):

$$\bar{R}(X, D) = \oplus_{m \geq 0} H^0(X, O(m(K_X + D))),$$

---

[3] By a divisor, we mean a *Weil divisor*, i.e., a linear combination of irreducible curves with integral coefficients, unless otherwise specified. A Q-divisor is a linear combination of irreducible curves with rational coefficients.

where $m(K_X + D)$ is understood to be the integral part. Then the log-Kodaira dimension $\bar{\kappa}$ is defined by:

$$\bar{\kappa}(X, D) = \begin{cases} \operatorname{tr.deg.} \bar{R}(X, D) - 1 \\ -\infty \quad \text{if } \bar{R}(X, D) = C. \end{cases}$$

Then $\bar{R}(X, D) \leq 2$. These are not bimeromorphic invariants in a general sense. Normal surfaces (the case of $D = \emptyset$) with $\kappa(X) \leq 1$ ($\kappa(X) = \bar{\kappa}(X, \emptyset)$) are classified in [Sak2]. For normal surface pairs, something should still be done. It is an interesting open question whether the Hirzebruch proportionality between orbifold Chern numbers (see Introduction and Section 3.0) characterizes quotients of $P_2(C)$. For the case of maximum (log-)Kodaira dimension, very little is known (cf. [Sak2]). We want to develop Kähler-Einstein geometry for this case. To do this, we first look at singularities of normal surface pairs.

Let (V, D, p) be a germ of a normal surface pair, i.e., (V, p) is a germ of a normal surface and D is a finite union of branch loci $D = \sum_i (1 - \frac{1}{b_i})D_i$ (b_i are integers ≥ 2 or ∞) (each component D_i passes through the point p). A *resolution* $(\tilde{V}, \tilde{D}, \tilde{E})$ of (V, D, p) consists of

(i) a resolution $\mu : (\tilde{V}, \tilde{E}) \to (V, p)$ of (V, p) such that the proper transform $\cup_i \tilde{D}_i$ of $\cup_i D_i$ is non-singular,

(ii) $\tilde{D} = \sum_i (1 - \frac{1}{b_i})\tilde{D}_i$,

and

(iii) the exceptional set $\tilde{E} = \mu^{-1}(p)$.

A resolution of (V, D, p) is *good* if the union of the proper transform of $\operatorname{Supp}(D)$ and the exceptional set $\tilde{E}$ has only simple normal crossings. Suppose the blowing down of a (-1)-curve $C \subset \tilde{E}$ preserves goodness. Then, by blowing down such (-1)-curves successively, we arrive at a *minimal good resolution*. For a given (V, D, p), there exists a unique minimal good resolution. Let (V, D, p) be a germ of a normal surface pair and $\mu : (\tilde{V}, \tilde{D}, \tilde{E}) \to (V, D, p)$ a minimal good resolution and $\tilde{E} = \sum_\alpha E_\alpha$ the decomposition of $\tilde{E}$ into irreducible components. Let K_V denote the canonical divisor of V. We call the sum $K_V + D$ a *log-canonical divisor*.

We want to know how the log-canonical divisor $K_V + D$ looks like in a resolution $(\tilde{V}, \tilde{D})$. We can write as

$$(3.1) \qquad \mu^*(K_V + D) = K_{\tilde{V}} + \tilde{D} + \Delta \qquad (\operatorname{Supp}(\Delta) \subset \tilde{E}).$$

Define

$$\Delta = \sum_{\alpha} a_{\alpha} E_{\alpha}.$$

The intersection theory of Mumford ([Mu2], [Sak2], see Section 3.2.1) uniquely defines rational numbers a_{α} by the equations

$$(3.2) \qquad (K_{\tilde{V}} + \tilde{D} + \sum_{\alpha} a_{\alpha} E_{\alpha}) \cdot E_{\beta} = 0, \qquad \text{for each } \beta.$$

We have thus defined the inverse image of a log-canonical divisor. This definition is natural from a differential geometric point of view (see [Ko2]), since any Kähler metric of (V, p) induced from the Euclidean space in which (V, p) is imbedded sees exceptional set $\tilde{E}$ as a point p. We then introduce the notion of log-canonical and log-terminal singularities (see [Kaw1], [Kaw2], [Sak2], [Wa] and [Na]) using the inverse image of a log-canonical divisor.

Definition. A germ of a normal surface pair (V, D, p) is a log-canonical (resp. log-terminal) singularity if there exists a good resolution such that $a_{\alpha} \leq 1$ for all α (resp. $a_{\alpha} < 1$ for all α and $b_i < \infty$ for all i).

Here the coefficients a_{α} have the same meaning as in (3.2). Clearly a smooth point of (X, D) is log-canonical (see the classification in Section 3.1). We note that if (V, D, p) is a log-terminal (resp. log-canonical) singularity, then the property in the definition for log-terminal (resp. log-canonical) singularities holds for all good resolutions. And if (V, D, p) is a log-terminal (resp. log-canonical) singularity, then for any resolution $\mu :$ $(V', D', E') \to (V, D, p)$ the coefficients in $D' + E' = \mu^*(K_V + D) - K_{V'}$ is smaller than (resp. not greater than) one. Any singularities which appear in the compactifications of orbit spaces of finite volume of 2-dimensional Hermitian symmetric domains is of log-terminal and log-canonical type. The converse of this is also true in a certain sense, namely, the 2-dimensional log-canonical singularities are characterized as normal surface singularities locally uniformizable by symmetric domains, or an extended unification of quotient singularities, simple elliptic singularities and cusp singularities. This was conjectured and partially proved by F. Sakai and R. Kobayashi. A complete proof is given by S.Nakamura [Na]. We explain this fact in Section 3.1 with a table of classification. For a normal surface pair with at worst log-canonical singularities, we write $LCS(X, D)$ for all log-canonical singularities of (X, D) which are not log-terminal. We shall show in Section 3.1 that essentially all LCS singularities are finitely uniformized by simple elliptic

singularities and cusp singularities. The Kähler-Einstein geometry we shall develop for normal surface pairs (X, D) is stated as follows (basic definitions will be given in Section 3.2.1).

Theorem 1. *Let (X, D) be a normal surface pair with $\bar{\kappa}(X, D) = 2$. Suppose (X, D) has at worst log-canonical singularities. Let (X'', D'') be the log-canonical model for (X, D) which again has at worst log-canonical singularities. Then*

(i) *$X_0'' = X'' - \cup_{b_i = \infty} D_i'' - LCS(X'', D'')$ with $D_0'' = D'' \cap X_0''$ is an orbifold with branch loci $\mathrm{Supp}(D_0'')$ with branch indices $\{b_i\}$. In particular, any singular point of X'' outside $\cup_{b_i = \infty} D_i'' \cup LCS(X'', D'')$ is an isolated quotient singularity,*

(ii) *there exists a unique complete Kähler-Einstein orbifold metric with negative scalar curvature on the orbifold (X_0'', D_0'') whose Kähler form ω defines a closed current on any resolution $\mu : Y'' \to X''$ and satisfies $[\mu^* \omega] = 2\pi c_1(\mu^*(K_{X''} + D''))$.*

The unique complete Kähler-Einstein orbifold metric on the log-canonical model (X'', D'') is called the canonical Kähler-Einstein metric of (X, D). If we integrate the Chern forms for the canonical Kähler-Einstein metric of (X, D) in Theorem 1, we have the following inequality for Chern numbers.

Theorem 2. *Let (X, D) and (X'', D'') be as in Theorem 1. Then we have*

$$(3.3) \quad (K_{X''} + D'')^2 \leq 3\{e(X_0'')$$
$$+ \sum_i (\frac{1}{b_i} - 1)(e(D_{0i}'') - d_i) + \sum_p (\frac{1}{|\Gamma(p)|} - 1)\},$$

where $e(X_0'')$ means the Euler number of X_0'' etc., $D_{0i}'' = D_i'' \cap X_0''$, d_i is the number of singularities of (X'', D'') lying over D_{0i}'', and $|\Gamma(p)|$ is the order of the local fundamental group $\Gamma(p)$ (which is a finite subgroup of $U(2)$) of a log-terminal singular point p of (X'', D'') in the sense of orbifolds. The equality holds if and only if the orbifold (X_0'', D_0'') is biholomorphic to the ball quotient $\Gamma \backslash B^2$ with Γ a discrete subgroup of $PSL(2, 1)$ and $\cup_{b_i \neq \infty} \mathrm{Supp}(D_{0i}'')$ is the branch loci with branch indices $\{b_i\}$.

In (3.3), we write $\bar{c}_1(X'', D'')^2$ (resp. $\bar{c}_2(X'', D'')$) for the left (resp. right) hand side and call them the logarithmic Chern numbers for an open orbifold (X_0'', D_0''). The proof of Theorem 1 goes as follows. We

start with a normal surface pair (X, D) with $\bar{\kappa}(X, D) = 2$. It is known [Sak1] that the log-canonical model (X'', D'') exists and is unique. If (X, D) has at worst log-canonical singularities, then so does (X'', D'') and it follows that the log-canonical divisor $K_{X''} + D''$ is ample, i.e., some positive multiple of $K_{X''} + D''$ is an ample line bundle (every log-canonical surface singularity turns out to be Q-Gorenstein). Thus X'' is realized in the projective space $P_N(C)$ of the space $H^0(X'', O(m(K_{X''} + D'')))$ of holomorphic sections of a very ample line bundle $m(K_{X''} + D'')$ for some positive integer m (see Section 3.2.1). The space X'' is precisely the space on which the canonical Kähler-Einstein metric lives and the ample divisor $K_{X''} + D''$ gives its cohomology class. Let $\mu : Y'' \to X''$ be a resolution of singularities of (X'', D''). If we restrict the Fubini-Study Kähler form on X'', we get a Kähler metric on X'' whose Kähler form ω'' satisfies $[\mu^*\omega''] = 2\pi c_1(\mu^*(K_{X''} + D''))$. We say this simply $[\omega''] = 2\pi c_1(K_{X''} + D'')$. Intuitively, the Kähler metric ω'' is represented by a generic hyperplane section of X''. We seek a Kähler-Einstein metric in $c_1(K_{X''} + D'')$ by a Kähler deformation $\omega'' \to \omega = \omega'' + \sqrt{-1}\partial\bar{\partial}u$, where u is contained in a certain function space constructed using the local uniformization of log-canonical singularities (the Bergmann metric of symmetric domains), and may neither be smooth nor be bounded on X'' (see Section 3.2.2). To find u with ω Kähler-Einstein, we first glue ω'' with Bergmann metrics (which are Kähler-Einstein) defined only near log-canonical singularities to construct a complete Kähler metric on X_0'' whose Kähler form ω_0 is cohomologous to ω'' in the sense of current. This procedure concerns the unbounded part of u and we may reach a Kähler-Einstein metric after a slight perturbation.

3.1. Classification and uniformization of 2-dimensional log-canonical singularities

In this section we present a table of log-terminal and LCS surface singularities together with examples of arguments for the classification and uniformization. For details we refer to [Na]. First, we recall the definition of log-canonical singularities. Let (V, D, p) be a germ of a normal surface V with a normal point p and $D = \sum_i (1 - \frac{1}{b_i})D_i$ a Q-divisor (branch loci) passing through p, where $b_i = 2, 3, \cdots, \infty$. We take a good resolution $\mu^* : (\tilde{V}, \tilde{D}, \tilde{E}) \to (V, D, p)$ and consider the inverse image of the log-canonical divisor:

$$\mu^*(K_V + D) = K_{\tilde{V}} + \tilde{D} + \Delta.$$

We call the singularity (V, D, p) a log-terminal (resp. log-canonical) singularity if the coefficients in the Q-divisor $\tilde{D} + \Delta$ are smaller than

(resp. not greater than) one. This definition does not depend on the choice of a good resolution and any resolution of a log-terminal (resp. log-canonical) singularity has the same property stated in the definition. A log-canonical singularity is LCS if it is not log-terminal. If p is a regular point of V, we call p *quasi-regular*. Quasi-regular log-canonical singularities are very important in the classification. Next, we examine quotient and covering operations on log-canonical singularities.

Lemma 1. *Let (V, D, p) be a germ of a normal surface pair. Given a ramified covering $f : \hat{V} \to V$ which ramifies only over p, we define a new pair $(\hat{V}, \hat{D}, \hat{p})$ where $\hat{D}$ is the strict transform of D by f and $\hat{p}$ is the point over p. Then, (V, D, p) is a log-terminal (resp. LCS) singularity if and only if so is $(\hat{V}, \hat{D}, \hat{p})$.*

Proof. Let $\mu : (\tilde{V}, \tilde{D}, \tilde{E}) \to (V, D, p)$ be a good resolution. Then we can find a good resolution $\hat{\mu} : (V^*, D^*, E^*) \to (\hat{V}, \hat{D}, \hat{p})$ which fits into the following commutative diagram:

$$
\begin{array}{ccc}
(V^*, D^*, E^*) & \xrightarrow{\ \tilde{f}\ } & (\tilde{V}, \tilde{D}, \tilde{E}) \\
\hat{\mu} \downarrow & & \downarrow \mu \\
(\hat{V}, \hat{D}, \hat{p}) & \xrightarrow[\ f\ ]{} & (V, D, p)
\end{array}
$$

in which f and $\tilde{f}$ are holomorphic maps. We claim that every coefficient in $\hat{\Delta}$ in

$$\hat{\mu}^*(K_{\hat{V}} + \hat{D}) = K_{V^*} + D^* + \hat{\Delta}$$

is ≤ 1 (resp. some coefficient is $= 1$) iff every coefficient in Δ is so. But from the above commutative diagram we have:

$$
\begin{aligned}
K_{V^*} + D^* + \hat{\Delta} &= \tilde{f}^* \mu^*(K_V + D) \\
&= \tilde{f}^*(K_{\tilde{V}} + \tilde{D} + \Delta).
\end{aligned}
$$

We apply the ramification formula for the canonical divisor to $\tilde{f}$. Set $\mathrm{Supp}(D^*) = \cup D_i^*$ and $\mathrm{Supp}(E^*) = \cup E_\alpha^*$. Then we have

$$\tilde{f}^*(K_{\tilde{V}} + \tilde{D} + \Delta) \leq K_{V^*} + \sum D_i^* + \sum E_\alpha^*.$$

Since the property $a > 1$ (resp. $a = 1$, $a < 1$) in $\frac{dz}{z^a}$ is invariant under the covering and quotient operation, we get the claim.

The following lemma combined with Lemma 1 tells us that the uniformization problem for log-canonical singularities reduces (up to rigidity) to that for quasi-regular ones.

Lemma 2 ([Kaw3, Lemma 9.2]). *For every log-canonical singularity (V, D, p) with $D \neq \emptyset$, the isolated singularity $(V, \emptyset, p)$ is a log-terminal singularity.*

Proof. Let $\mu : (\tilde{V}, \tilde{D}, \tilde{E}) \to (V, D, p)$ be a good resolution and $\mu_0 : (\tilde{V}_0, \emptyset, \tilde{E}_0) \to (V, \emptyset, p)$ the minimal resolution. Then there is a birational morphism $\phi : \tilde{V} \to \tilde{V}_0$ which consists of a finite number of blow ups. Set

$$\mu^*(K_V + D) = K_{\tilde{V}} + \tilde{D} + \Delta,$$

$$\mu_0^* K_V = K_{\tilde{V}_0} + \Delta_0,$$

$$\mu^* D = \tilde{D} + \Delta_D.$$

We then have

$$\begin{aligned} K_{\tilde{V}_0} + \Delta_0 &= \phi_*(\phi^*(K_{\tilde{V}_0} + \Delta_0)) \\ &= \phi_*(\mu^* K_V) \\ &= \phi_*(K_{\tilde{V}} + \Delta - \Delta_D) \\ &= K_{\tilde{V}_0} + \phi_*(\Delta - \Delta_D). \end{aligned}$$

Since $0 = \mu^* D \cdot E_\alpha = \tilde{D} \cdot E_\alpha + \Delta_D \cdot E_\alpha$, we have $\Delta_D \cdot E_\alpha \leq 0$. It follows from Zariski's Lemma [Z, Lemma 7.1] that $\Delta_D > 0$ and $\mathrm{Supp}(\Delta_D) = \tilde{E}$. Thus every coefficient in Δ_0 is < 1.

The above proof using Zariski's Lemma is ubiquitous in the classification of log-canonical singularities. For reader's convenience, we state Zariski's Lemma.

Zariski's Lemma ([Z, Lemma 7.1]). *Let (V, p) be a germ of a normal surface and $(\tilde{V}, E) \to (V, p)$ be a resolution of a singularity. Let Δ be a Q-divisor supported in E, i.e., $\Delta = \sum_\alpha a_\alpha E_\alpha$, $a_\alpha \in Q$. If $\Delta \cdot E_\beta \leq 0$ for all $E_\beta \subset E$, then $\Delta \geq 0$.*

We now proceed to present the classification and uniformization of log-canonical singularities. We fix the notation. For a sequence of positive integers $(a_1, \cdots, a_n)$ with $a_i \geq 2$, define the continued fraction:

$$\frac{d}{e} = [a_1, \cdots, a_n] = a_1 - (a_2 - \cdots - (a_{n-1} - a_n^{-1})^{-1} \cdots)^{-1},$$

where $(d, e) = 1$. We denote by $< d, e; b >$ a chain of $P_1(C)$'s with self-intersections $-a_1, \cdots, -a_n$ and a segmental curve with b being the index of ramification. We write this by a weighted dual graph:

$$< d, e; b >$$
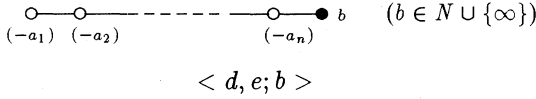
As conventions, we also use $< 1, 0; b >$ and $< d, e; 1 >$ to indicate the extremal cases:

$$< 1, 0; b > \text{ and } < d, e; 1 >$$
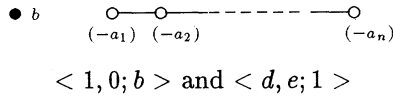
We denote by $< a_0, < d_1, e_1; b_1 >, \cdots, < d_k, e_k; b_k >>$ a star shaped dual graph:

$$< a_0, < d_1, e_1; b_1 >, \cdots, < d_k, e_k; b_k >>$$
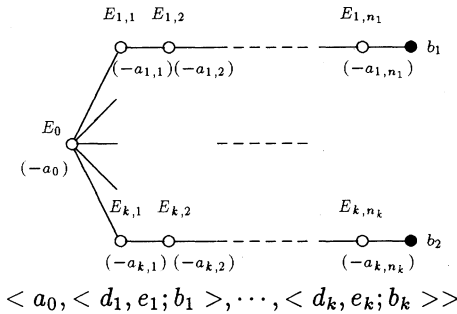
We also use the notations $<< n_1, \cdots, n_k >>$ to indicate the set of weighted dual graphs with the invariants: $d_1 b_1 = n_1, \cdots, d_k b_k = n_k$.

Theorem 3.1 ([Na]). *Log-canonical surface singularities are classified into the following twelve classes* (i),(i)*,(i)**, $i = 1, 2, 3, 4$.

(1) Every regular point (V, D, p) in which $D = \emptyset$ or $D = (1 - \frac{1}{b_1})D_1$ with $b_1 < \infty$ is log-terminal. In the latter case, it is uniformized by the former via $(z_1, z_2) \to (z_1^{b_1}, z_2)$ with the covering transformation group $Z_{b_1} \times \{id\}$.

(1)* An isolated singularity $(V, \emptyset, p)$ is log-terminal if and only if it is a quotient singularity [Br]. The exceptional set in the minimal resolution is a configuration of $P_1(C)'s$ of ADE-type (see [Br]). Every

quotient singularity is rigid, i.e., its analytic structure is determined by its dual graph. It is uniformized by C^2 and the covering transformation group is a finite subgroup of $U(2)$ free of reflections.

(1)** The singularities given by the group (1)** of dual graphs in Appendix exhaust *log-terminal singularities* with $D \neq \emptyset$. These singularities are rigid. Each of these singularities is obtained by taking the factor space of C^2 with respect to a finite subgroup $G(V, D, p)$ of $U(2)$ which is an extension of a unitary reflection group. In fact, the set of analytic types of the log-terminal singularities and the conjugacy classes of the finite subgroups of $U(2)$ is in 1-1-correspondence. Among these singularities, we give the list of quasi-regular ones $(QR)_1$ and their uniformization. It turns out to be the case that quasi-regular log-terminal singularities and 2-dimensional unitary reflection groups are in 1-1 correspondence under $(V, D, p) \leftrightarrow G(V, D, p)$. In the table (1)**, $((n))$, $n = 4, 5, \cdots, 22$ stand for the number used in Shephard-Todd's classification of unitary reflection groups [ST].

(2) Every cusp singularity is LCS. The exceptional set in the minimal resolution is a cycle of $P_1(C)'s$ or a rational curve with a node and every component appears in $\tilde{E}$ with its coefficient one. Every cusp singularity is rigid. It is uniformized by $H \times H$ with the covering transformation group $G(M, V)$ (see [Hir2] or Section 3 of this paper) which is a reflection-free discrete subgroup of $\mathrm{Aut}(H \times H)$ fixing the point (∞, ∞) in the boundary. It is easy to see from the construction (see Example 2.4) that the boundary of the tubular neighborhood of a cusp singularity is a torus bundle over a circle.

(2)* Every surface singularity $(V, \emptyset, p)$ whose minimal resolution is given by one of the dual graphs in group (2)* in Appendix is LCS ([Sak1]). Each of these singularities is rigid and obtained by taking the factor space of a cusp singularity with respect to an involution (cf. [Hir2]). Hence it is uniformized by $H \times H$ with the covering transformation group a discrete subgroup $\tilde{G}(M, V)$ of $\mathrm{Aut}(H \times H)$ fixing (∞, ∞) which is an extension of some $G(M, V)$ by Z_2. It is easy to find a symmetric cycle of $P_1(C)'s$ sitting over the above $P_1(C)$-configuration by which we determine $G(M, V)$. For instance, the dual graph (a) sits over the dual graph (b).
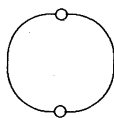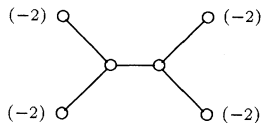
(a) (b)

We can then determine the action of the involution on this $P_1(C)$-cycle with four isolated fixed points corresponding to four (-2)-curves (see [Hir2, Section 1]). Then we determine the group $\tilde{G}(M, V)$ via the formula in Example 2.4.

(2)** Each of the singularities given by one of the dual graphs in the group (2)** in Appendix is LCS. Since each of these $P_1(C)$-configurations is rigid, its uniformization is reduced to that for the quasi-regular ones $(QR)_2$. Karras [Kar] classified $(QR)_2's$. We list the dual graphs and uniformizations [Kar] for $(QR)_2's$. Every LCS singularity of this type is uniformized by $H \times H$ and the covering transformation group is a discrete subgroup $\tilde{G}(M, V)$ of Aut$(H \times H)$ fixing (∞, ∞) which is an extension of some $G(M, V)$ by the transposition $\tau : (z_1, z_2) \to (z_2, z_1)$. It is easy to recover the symmetric $P_1(C)$-cycle sitting over a given dual graph, by which we determine $G(M, V)$. Several examples of this process are found in [Hir2, Section 1]. The parabolic subgroup P of Aut$(H \times H)$ corresponding to $(i\infty, i\infty)$ consists of automorphisms $(z_1, z_2) \to (az_1 + b, a'z_2 + b')$ with $a > 0$, $aa' = 1$ and $b, b' \in R$. Considering the developing map for the holomorphic conformal structures, we have a 1-1-correspondence between $H \times H$-cusp singularities and the conjugacy classes (in P) of the discrete subgroups of P with a finite co-volume (w.r.to the Bergman metric) in P.

(3) Every simple elliptic singularity (cf. [S]) is LCS. A simple elliptic singularity is resolved by an elliptic curve C of self-intersection number $-b$. Grauert's criterion [Gr] implies that any simple elliptic singularity is obtained from a line bundle over an elliptic curve C of negative degree $-b$ by blowing down the zero section. Hence a simple elliptic singularity involves a parameter $\tau \in H$ of analytic structures of C and a discrete parameter b, so it is determined by (τ, b). It is uniformized by the 2-dimensional open ball B^2 and the covering transformation group Γ is isomorphic to the Heisenberg group which is a discrete parabolic subgroup of Aut(B^2) fixing a boundary point p. To see these explicitly, we realize B^2 as a Siegel domain:

$$\mathcal{S} = \{(u, v, 1) \in P_2(C) | \operatorname{Im}(u) - |v|^2 > 0\}$$

via $z_1 = \frac{u-i}{u+i}$ and $z_2 = \frac{2v}{u+i}$. Then the parabolic subgroup P fixing the boundary point $p = (1, 0) \in \partial B^2$ is written in the Siegel domain expression as follows (using the terminology in [Yo]):

$$P = \{((\mu, \gamma, r)) \, | \, \mu \in U(1), \, \gamma \in C, \, r \in R\},$$

where $((\mu, \gamma, r))$ stands for the automorphism of $\mathcal{S}$ given by

$$\begin{pmatrix} 1 & 2i\mu\bar{\gamma} & r + i|\gamma|^2 \\ 0 & \mu & \gamma \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$((\mu, \gamma, r))((\mu', \gamma', r')) = ((\mu\mu', \mu\gamma' + \gamma, r + r' - 2\operatorname{Im}(\mu\bar{\gamma}\gamma'))).$$

Suppose $\mu = 1$, $\gamma \in L$ where $L = \{m + n\omega \mid \operatorname{Im}(\omega) > 0\}$ is the lattice defining C and $r = -2h(\gamma)$, where $h(m + n\omega) = mna$ (mod $\frac{2a}{b}Z$). Then we get a discrete subgroup $\Gamma \in P$ (and its conjugates in P) which fits into the exact sequence

(3.4) $1 \to Z \to \Gamma \to L \to 1$

where Z is identified with the center $Z(\Gamma) = ((1, 0, \frac{2a}{b}Z))$ and $\Gamma \to L$ is defined by $((1, \gamma, r)) \mapsto \gamma$. Note that the new coordinates (w, z) defined by $w = \exp(\frac{b\pi i u}{2a})$ and $z = v$ exactly describe the factor space $Z\backslash\mathcal{S}$. Conversely, starting from a simple elliptic singularity, we again arrive at Γ (cf. [Ko2]).

   (3)* Every surface singularity $(V, \emptyset, p)$ given by one of dual graphs in the group (3)** in Appendix is LCS. These singularities are called the *ball cusp singularities* (see [Ho] and [Sak1]). Each of these singularities is uniformized by a simple elliptic singularity. All of these graphs are star shaped and the central curve is the image of the elliptic curve C resolving the simple elliptic singularity sitting over $(V, \emptyset, p)$. Such C has a non-trivial point group G, i.e., the corresponding lattice is invariant under the action of a non-trivial finite subgroup of $U(1)$. The central curve is an orbifold defined over $P_1(C)$ described by $(b_1, \cdots)$ where $b_1, \cdots$ are branch indices. The possible triads $(C, G, (b_1, \cdots))$ are (i) $L = Z + Z\omega$ (general lattice), $G = <-1>$, $(2, 2, 2, 2)$; (ii) $L = Z + Zi$ (square lattice), $G = <i>$, $(2, 4, 4)$; (iii) $L = Z + Z\alpha$ (hexagonal lattice), $G = <\alpha>$ $(\alpha = e^{\frac{2\pi i}{6}})$, $(2, 3, 6)$; (iv) $L = Z + Z\alpha$, $G = <\alpha^2>$, $(3, 3, 3)$. It is not difficult to construct discrete parabolic groups Γ corresponding to these triads, which fit into the exact sequence

(3.5) $1 \to Z \to \Gamma \to E \to 1,$

where Γ consists of automorphisms $((\mu, \gamma, r))$ with $\mu \in G$ (a finite subgroup of $U(1)$), $\gamma \in L$ (a lattice with a non-trivial point group G) and $r = r(\mu, \gamma) \in R$ modulo $\frac{4a}{b}Z$ obeying

$$r(\mu\mu', \mu\gamma' + \gamma) = r(\mu, \gamma) + r(\mu', \gamma') - 2Im(\mu\bar{\gamma}\gamma') \quad \text{mod} \ \frac{4a}{b}Z,$$

and E is a discrete Euclidean motion group generated by L and G. The map $\Gamma \to E$ is defined by $((\mu,\gamma,r)) \mapsto \begin{pmatrix} \mu & \gamma \\ 0 & 1 \end{pmatrix}$. For instance, we can construct the discrete group Γ consisting of the automorphisms $((\mu,\gamma,r))$ if we require $\mu \in <-1>$, $\gamma \in L$ (a general lattice)) and $r(-1,0) = \frac{2a}{b}$ mod $\frac{4a}{b}Z$, $r(1,m+n\omega) = -2mna$ mod $\frac{4a}{b}Z$. The last condition on r fixes the conjugacy class of Γ in P. The group Γ' consisting of elements with $\mu = 1$ is a normal subgroup of Γ and the factor space $\Gamma'\backslash S$ is compactified to the normal bundle of C on which $\Gamma'\backslash\Gamma$ acts with four isolated fixed points on C of order 2. We thus get $\Gamma's$ corresponding to the above dual graphs.

(3)** Every surface singularity (V, D, p) with $D \neq \emptyset$ defined by one of the dual graphs in the group (3)** in Appendix is LCS. The quasi-regular ones $(QR)_3$ turn out to be identical to Yoshida-Hattori's classification [YH] of 2-dimensional parabolic reflection groups. All of these singularities are uniformized by B^2 and the universal branched covering transformation group is Γ with the exact sequence (3.5). The same procedure as in (3)* enables us to find the explicit form of Γ by choosing $r(\mu, \gamma)$ in a suitable way. For instance, take the hexagonal lattice L and $\alpha = e^{\frac{2\pi i}{6}}$ and consider the group Γ consisting of $((\mu,\gamma,r))$ such that $\mu \in <\alpha>$, $\gamma \in L$ and $r(\alpha,0) = \frac{2a}{3b}$ mod $\frac{4a}{b}Z$, $r(1,m+n\alpha) = -2mna$ mod $\frac{4a}{b}Z$. Suppose $b \equiv 2$ mod(6). Then we uniformize the singularity given by the dual graph:
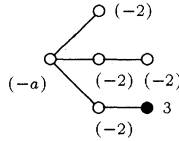
a ball cusp singularity with a branch locus

Indeed, if $\Gamma' \subset \Gamma$ is the normal subgroup defined by $\mu = 1$, then the factor space $\Gamma'\backslash S$ is compactified to a line bundle N of degree $-b < 0$ over the elliptic curve $C(L)$ of the lattice L. The group $\Gamma/\Gamma' \cong Z_6$ then acts on N with three special orbits O_1, O_2 and O_3 of fix points on $C(L)$ with the isotropy of order 6, 3 and 2. At these points on $C(L)$, the action is locally $(x,y) \mapsto (-x,-y)$, $\mapsto (\alpha^{-2}x, \alpha^2 y)$ or $\mapsto (-x, \alpha y)$, where x is a base coordinate and y a fiber coordinate. The above dual graph is obtained by resolving quotient singularities of the factor space of N with respect to the action of Γ/Γ'. In particular, the dual graph is star-shaped with the image of $C(L)$ as the central curve E. The self-intersection number of E is $\frac{-b-10}{6}$, since a C^∞ section of

N having zeros at points in O_1 (resp. O_2 and O_3) of order $+3$ (resp. $+2$ and $+1$) and simple zeros (with sign ± 1) outside special orbits is mapped to a C^∞ multisection of multiplicity 6 of the normal bundle of E with no zeros at intersection points of three branches. The number 10 is then $3 + 2 \times 2 + 1 \times 3$. Since $b \equiv 2 \mod (6)$, possible numbers $E \cdot E$ are all integers ≤ -2. Considering the developing map for the G-structure characterizing B^2 (the holomorphic projective structure or the Kähler structure with constant holomorphic sectional curvature), we get a one to one correspondence of the ball-cusp singularities (dual graphs in $(3),(3)^*,(3)^{**}$) and the conjugacy classes (in P) of the discrete subgroups of P with a finite co-volume (w.r.to the Bergman metric).

(4) A regular point (V, D, p) with $D = D_1$, i.e., $b_1 = \infty$, is LCS. It is uniformized by $H \times H$ via the map $(z_1, z_2) \to (e^{2\pi i z_1}, z_2)$ with the covering transformation group Z acting on $H \times H$ by $(z_1, z_2) \to (z_1 + n, z_2)$. This map is a infinitely cyclic branched covering branching along D_1 and the condition $b_1 = \infty$ means to delete D_1. So $D^* \times D$ is a (deleted) neighborhood of p. A surface singularity (V, D, p) with $D = D_1 + D_2$ where D_1 and D_2 intersect transversely is LCS. It is uniformized by $H \times H$ via $(z_1, z_2) \to (e^{2\pi i z_1}, e^{2\pi i z_2})$ with the group $Z \times Z$ acting on $H \times H$ by $(z_1, z_2) \to (z_1 + n, z_2 + m)$ and p has $D^* \times D^*$ as a (deleted) neighborhood.

$(4)^*$ Each surface singularity (V, D, p) with non-empty branch loci D_i only with $b_i = \infty$ defined by one of the dual graphs in the group $(4)^*$ in Appendix is LCS. Note that only types A and D occur in these dual graphs. Each of these is uniformized by $D^* \times D$ or $D^* \times D^*$ and the covering transformation group is a finite subgroup of $U(2)$ acting linearly on $D^* \times D(D^*) \subset C^2$ without reflections. The singularity with the dual graph of type A is uniformized by $D^* \times D$ (resp. $D^* \times D^*$) if it has one (resp. two) •, and that with the dual graph of type D is always uniformized by $D^* \times D^*$.

$(4)^{**}$ Each surface singularity (V, D, p) with mixed branch loci D_i in which both $b_i < \infty$ and $b_i = \infty$ appear and whose dual graph is given by one of those in the group $(4)^{**}$ in Appendix is LCS. Note that only types A and D occur. Each of these is uniformized by $D^* \times D$ (if the dual graph is of type A and contains only one •) or $D^* \times D^*$ (in the case of type A with two • or of type D) and the covering transformation group is a finite subgroup of $U(2)$ which is an extension of a reflection group. Finding explicit forms of uniformizations is reduced to examining the action of finite subgroups of $U(2)$ of types A and D, so we omit this.

It is now easy to calculate rational numbers a_α such that

$$(K_{\tilde{V}} + \sum_i (1 - \frac{1}{b_i})\tilde{D}_i + \sum_\alpha a_\alpha E_\alpha) \cdot E_\beta = 0 \qquad \text{for all } \beta,$$

to see that all singularities in the above list are log-canonical. Conversely, any log-canonical singularity has the minimal good resolution whose dual graph is one of the above list. The proof of the converse is elementary but quite long. Namely, we first classify the dual graphs of log-canonical singularities according to the maximum multiplicity of intersections of $\tilde{E}$ and $\tilde{D}$. Then we use the definition of the log-canonical singularity to bound possible types of dual graphs. Zariski's Lemma is very useful in simplifying the proof. For details, we refer to Nakamura's master thesis [Na]. Once one classifies the possible dual graphs of log-canonical singularities, it is an elementary (up to long computations) task to show the rigidity (except LCS singularities of type $<< 2, 2, 2, 2 >>$) and uniformize these singularities. Each of these exceptional LCS singularities is uniformized by a simple elliptic singularity.

It directly follows from the classification that

Corollary. *All log-canonical surface singularities are Q-Gorenstein.*

3.2. Kähler-Einstein metrics on log-canonical normal surfaces with branch loci

3.2.1. Log-minimal models and log-canonical models

For details of this section, we refer to [Sak1] and [Sak2]. Let X be a normal compact complex surface. According to Mumford [Mu2], we have a nice intersection theory on divisors on X, which is defined in the following way. Let $\mu : \tilde{X} \to X$ be a resolution of singularities and $E = \cup_i E_i$ the exceptional set. Let $\tilde{D}$ be the strict transform of D by μ. Then we define the inverse image $\mu^* D$ of D by the following Q-divisor on $\tilde{X}$:

$$\mu^* D = \tilde{D} + \sum_i a_i E_i,$$

with rational numbers a_i obeying

$$\sum_i a_i (E_i \cdot E_j) = -\tilde{D} \cdot E_j.$$

This is well-defined since the intersection matrix $(E_i \cdot E_j)$ is negative definite.

Definition. For two divisors D and D' on X, the intersection number $D \cdot D'$ is defined as $(\mu^* D)(\mu^* D') \in Q$.

To define log-minimal and log-canonical models, we need to generalize the definition of the inverse image of a divisor to holomorphic mappings of normal surfaces. Let X and X' be normal surfaces and $f : X \to X'$ a holomorphic mapping. There exist resolutions $\mu : \widetilde{X} \to X$ and $\mu' : \widetilde{X}' \to X'$ and a holomorphic mapping $\widetilde{f} : \widetilde{X} \to \widetilde{X}'$ such that $\mu' \circ \widetilde{f} = f \circ \mu$. For a divisor D' on X', we define the inverse image $f^* D'$ to be $\mu_*(\widetilde{f}^* \mu'^* D')$. We can now discuss the log-minimal and log-canonical models. Let (X, D) be a normal surface pair as in Section 3.0. Assume for simplicity that X is a Moishezon surface, i.e., any resolution is a smooth projective algebraic surface.

Definition. An irreducible curve C on X is a log-exceptional curve of the first kind (resp. log-exceptional curve of the second kind) if $C^2 < 0$ and $(K_X + D) \cdot C < 0$ (resp. if $C^2 < 0$ and $(K_X + D) \cdot C = 0$).

Let C be a log-exceptional curve. Since $C^2 < 0$, C is contracted to a normal point (see [Sak2, Theorem 1.1]). Let $f : X \to X'$ be the contraction of C and set $D' = f_* D$. We have from the definition that

$$K_X + D = f^*(K_{X'} + D') + \frac{(K_X + D) \cdot C}{C^2} C.$$

In particular we have $K_{X'} + D' = f_*(K_X + D)$[4]. By successive contractions of log-exceptional curves of the first kind, we arrive at a *log-minimal* normal surface pair (X', D'), i.e., it contains no log-exceptional curves of the first kind (induction on the Picard number). We call (X', D') a *log-minimal model* of (X, D). A log-minimal model (X', D') of (X, D) is characterized by the following two properties:
(i) (X', D') is log-minimal,
(ii) there exists a bimeromorphic holomorphic mapping $f : (X, D) \to (X', D')$ such that $D' = f_* D$ and $K_X + D = f^*(K_{X'} + D') + \sum_i a_i C_i$, $a_i > 0$ for all i, where $C = \cup_i C_i$ is the exceptional set of f, i.e., the set of all curves contracted by f.
If one further contract log-exceptional curves of the second kind in the log-minimal model (X', D'), we arrive at a *log-canonical* normal surface pair (X'', D''), i.e., it contains neither log-exceptional curves of the first kind nor log-exceptional curves of the second kind. We call (X'', D'')

---

[4]More generally, if $f : X \to X'$ is a bimeromorphic mapping of normal compact complex surfaces, then $K'_X = f_* K_X$.

a *log-canonical model* of (X, D). If (X', D') is log-minimal, then a log-canonical model (X'', D'') is characterized by the following two properties:

(i) (X'', D'') is log-canonical,

(ii) there exists a bimeromorphic holomorphic mapping $g : (X', D') \rightarrow (X'', D'')$ with $D'' = g_* D'$ and $g^*(K_{X''} + D'') = K_{X'} + D'$.

To discuss the uniqueness of these models, we need to introduce the following definition.

Definition. Let D be a Q-divisor on a normal surface X. We say that D is numerically effective if $D \cdot C \geq 0$ for all irreducible curves C on X and that D is pseudoeffective if $D \cdot P \geq 0$ for all numerically effective divisor P on X.

Fact 1 (a special case of [Sak1, Theorem 7.4]). *Let (X, D) be a normal surface pair. If $K_X + D$ is pseudoeffective, then the log-minimal model is unique. In this case $K_{X'} + D'$ is numerically effective.*

Suppose now $K_X + D$ is pseudoeffective. This assumption is fulfilled if some positive multiple $m(K_X + D)$ becomes an effective divisor. Let (X', D') be the unique minimal model and $f : (X, D) \rightarrow (X', D')$ the bimeromorphic holomorphic mapping. Set $P = f^*(K_{X'} + D')$ and $N = (K_X + D) - P$. Then:

(i) P is a numerically effective Q-divisor on X and N is either 0 or an effective Q-divisor whose support has negative intersection matrix,

(ii) the intersection of P and each irreducible component of N is zero.

The decomposition $K_X + D = P + N$ with the properties (i) and (ii) is unique (corresponding to Fact 1) and is called the *Zariski decomposition* of pseudoeffective Q-divisor $K_X + D$ (see [Sak1, Corollary 7.5]). The set $Supp(N)$ consists of curves contracted by f.

Fact 2 ([Sak1, p.886], see also [Sak2, Theorem 4.7]). *Let (X, D) be a normal surface pair with pseudoeffective log-canonical divisor, (X', D') its minimal model and $K_X + D = P + N$ the Zariski decomposition. Suppose $P^2 > 0 (\Leftrightarrow \bar{\kappa}(X, D) = 2)$. Then the canonical model (X'', D'') is unique. In this case, $K_{X''} + D''$ is numerically ample, i.e., $(K_{X''} + D'') \cdot C > 0$ for all irreducible curves C on X'' and $(K_X'' + D'')^2 > 0$.*

Assume $P^2 > 0$. Just as in Kodaira's theory [Kod] of canonical models for algebraic surfaces of general type, we directly get the unique log-canonical model by contracting the set A of all curves C with $P \cdot C = 0$ (which turns out to be finite and contractible by the Hodge index theorem). The image in (X', D') of the set $A - N$ consists of log-exceptional

curves of the second kind. *For normal surfaces, even if the log-canonical model X'' is unique, it is not necessarily realized as a projective image of sections of a log-pluricanonical bundle* (compare this with the classical surface theory [Kod]). This problem is quite complicated because log-pluricanonical divisors may never become a line bundle, i.e., any log-pluricanonical divisor may passes through singularities. Sakai [Sak2] proved

Fact 3 ([Sak2]). *For a log-canonical normal surface pair (X'', D'') with $\bar{\kappa}(X'', D'') = 2$, the log-canonical ring $\bar{R}(X'', D'')$ is finitely generated if and only if (X'', D'') is Q-Gorenstein. In this case, $X'' = \mathrm{Proj}(\bar{R}(X'', D''))$.*

We call (a germ of) normal surface pair (X, D) Q-*Gorenstein* if its log-canonical divisor is Q-Cartier. A Q-divisor D on a normal surface X is called Q-Cartier if some positive multiple mD becomes a line bundle which is trivial around singular points of X.

We sketch the proof of Fact 3. We recall that a Q-Cartier divisor is ample if and only if it is numerically ample, which is a consequence of Nakai's criterion of ampleness (see [BPV]). We apply this to the Q-Cartier divisor $K_{X''} + D''$ to prove the if part. Next, suppose $\bar{R}(X, D)$ is finitely generated. For a log-canonical normal surface pair (X'', D''), by taking the inverse image of a log-pluricanonical divisor, we get a line bundle L (proportional to $\mu^*(K_{X''} + D'')$) on a resolution Y of X'', which is numerically effective and $L^2 > 0$. It now follows that L is semiample, i.e., some positive multiple of L is generated by global sections. Indeed, [Z, Corollary 10.3 (or Theorem 10.6) and Theorem 6.1] (see also [Fu, Corollary (6.14)]) implies

Fact 3′. *Let L be a line bundle over a compact complex smooth surface S. Suppose L is numerically effective and $L^2 > 0$. Then $R(S, L)$ is finitely generated if and only if L is semiample.*

Since $L \to Y$ is generated by global sections and $L \cdot E_i = 0$, L is trivial near the exceptional set of μ. Therefore $\mu_* L$ is a line bundle on X'' and is a multiple of $K_{X''} + D''$, i.e., (X'', D'') is Q-Gorenstein.

Let (X'', D'') be a log-canonical model. We want to find a Kähler metric ω on X'' such that for any resolution $\mu : Y \to X''$ we have $[\mu^*\omega] = 2\pi c_1(\mu^*(K_{X''} + D''))$. It is possible if a line bundle defined by a positive multiple of $\mu^*(K_{X''} + D'')$ is generated by global sections. Fact 3 implies that this occurs if and only if (X'', D'') is Q-Gorenstein. According to the classification in Section 3.1, all log-canonical singularities are Q-Gorenstein, because the branch loci pass through only quotient

singularities and other log-canonical singularities free of branch loci are also Q-Gorenstein as is shown in [Sak2, Appendix]. In particular, if (X, D) is a normal surface pair with at worst log-canonical singularities, then (X, D) is Q-Gorenstein. Although Q-Gorensteinness is not preserved in the process of going to log-minimal and log-canonical models (this is Kawamata's observation, see [Sak2]), we have the following lemma which suggests the goodness of the log-canonical singularities.

Lemma 1 (Sakai's observation). *Let (X, D) be a normal surface pair. Then*

(i) *if (X, D) has at worst log terminal singularities, then so does its log-minimal model (X', D').*

(ii) *if (X, D) has at worst log-canonical singularities, then so does its log-canonical model (X'', D'').*

Lemma 1 gives a meaning to the following

Fact 4 (cf. [Sak2]). *For a normal surface pair (X, D) with at worst log-canonical singularities, the following conditions are equivalent:*

(i) $\bar{\kappa}(X, D) = 2$,

(ii) $K_{X''} + D''$ *is numerically ample,*

(iii) $K_{X''} + D''$ *is ample.*

Furthermore, if one of the above conditions is fulfilled, then $\bar{R}(X, D)$ is finitely generated and $X'' = \operatorname{Proj} \bar{R}(X, D)$.

Indeed, if $\bar{\kappa}(X, D) = 2$, then the log-minimal model and the log-canonical model exists uniquely. The implication (i)$\Rightarrow$ (ii) follows from Fact 2. If $K_{X''} + D''$ is numerically ample, then we have a line bundle L (proportional to $\mu^*(K_{X''} + D'')$) on a resolution Y of X'' such that $L \cdot C \geq 0$ for all irreducible curves on Y and $L^2 > 0$. The assertion (i) then follows from the Riemann-Roch inequality. Since (X'', D'') has at worst log-canonical singularities, (X'', D'') is Q-Gorenstein. This implies (ii) $\Leftrightarrow$ (iii).

Let (X, D) be as in Fact 4. Then the log-minimal model (X', D') exists uniquely with at worst log-canonical singularities and $K_{X'} + D'$ is numerically effective and has positive self-intersection number. It then follows from Fact 4 that $K_{X'} + D'$ is semiample in the sense that it is Q-Cartier and the line bundle $m(K_{X'} + D')$ is generated by global sections for $m \gg 0$. Furthermore, the log-canonical model (X'', D'') is the projective image of sections of $m(K_{X'} + D')$ and it has at worst log-canonical singularities.

3.2.2. Kähler-Einstein metrics

In this section we prove Theorem 1. Let (X, D) be a normal surface pair as in Theorem 1, i.e., with at worst log-canonical singularities and $\bar{\kappa}(X, D) = 2$. Then by Lemma 1 and Fact 4 the log-canonical model (X'', D'') has again at worst log-canonical singularities and the log-canonical divisor $K_{X''} + D''$ is ample. Let

$$\mu : (\tilde{X}, \tilde{D}, \tilde{E}) \longrightarrow (X'', D'')$$

be the minimal good resolution of (X, D), where $\tilde{D} = \sum_i (1 - \frac{1}{b_i}) \tilde{D}_i (= \sum_i (1 - \frac{1}{b_i}) D_i$ for simplicity) is the strict transform of $D = \sum_i (1 - \frac{1}{b_i}) D_i$ and $\tilde{E} = \sum_\alpha E_\alpha$ is the exceptional set. Let

$$f : X'' \longrightarrow P_n(C)$$

be the projective embedding defined by the ample line bundle $L = m(K_{X''} + D'')$ for some large m. Let $\omega(1) = \sqrt{-1} \partial \bar{\partial} \log \|Z\|^2$ be the Fubini-Study form of $P_n(C)$. Then $\omega_0 = \frac{1}{m}(f \circ \mu)^* \omega(1)$ is a semipositive real closed $(1, 1)$-form on $\tilde{X}$ which is positive definite outside of the curves contracted by μ. Set

$$\mu^*(K_{X''} + D'') = K_{\tilde{X}} + \tilde{D} + \sum_\alpha a_\alpha E_\alpha,$$

with rational numbers $a_\alpha \leq 1$. Then there exist a smooth volume form Ω on $\tilde{X}$, holomorphic sections σ_i and Hermitian metrics $\|\cdot\|^2$ for $[D_i]$, and holomorphic sections σ_α and Hermitian metrics $\|\cdot\|^2$ for $[E_\alpha]$ such that

(3.6) $$\omega_0 = \text{Ric}\,\Phi$$

where

(3.7) $$\Phi = \frac{\Omega}{\prod_i \|\sigma_i\|^{2(1 - \frac{1}{b_i})} \prod_\alpha \|\sigma_\alpha\|^{2 a_\alpha}},$$

and the Ricci form of a (singular) volume form Ψ is defined by

$$\text{Ric}\,\Phi = -\sqrt{-1} \partial \bar{\partial} \log \Phi.$$

Since we deform ω_0 into a Kähler-Einstein metric in the cohomology class $2\pi c_1(L)$ containing ω_0, we call ω_0 a background metric (for the construction of a Kähler-Einstein metric).

As we have shown in the previous section, all two dimensional log-canonical singularities are "quotient singularities" in a broad sense.[5] We take a full advantage of this special situation to prove the existence of a canonical Kähler-Einstein metric on log-canonical normal surface pairs (X'', D'') (this special situation is characteristic in dimension 2 and this seems to correspond partially to the fact that the L^2-norm of the curvature tensor is scale invariant in real dimension 4 and partially to the fact that the square of the L^2-norm of the curvature tensor is a topological invariant, the Euler number, for Einstein 4-manifolds (cf. [BKN])).

First of all we recall that the Bergman metric of the bounded symmetric domains, which is Kähler-Einstein, is invariant under the action of automorphisms. So, all log-canonical singularities have a neighborhood on which a Kähler-Einstein metric is defined which becomes the Bergman metric on the local uniformizations.

All log-terminal singularities of types (1), (1)* and (1)** are locally the factor space $G \backslash C^2$, where G is a finite subgroup of $U(2)$. The Kähler potential $\log\left(\frac{1}{(1-\|Z\|^2)^2}\right)$ of the ball metric is invariant under $U(2)$. Hence the ball metric is defined near the log-terminal singularities in the Kähler potential level.

All LCS singularities (V, D, p) of types (2), (2)* and (2)** are locally the factor space $G \backslash H \times H$, where G is a discrete subgroup of $Aut(H \times H)$ fixing the boundary point $(i\infty, i\infty)$. The Kähler potential $\log\left(\frac{1}{y_1 y_2}\right)$ for the $H \times H$-metric is invariant under G. Hence the $H \times H$-metric is defined near the LCS-singularities of these types in the Kähler potential level. The Kähler potential assumes $-\infty$ at the singular point p and this corresponds to the pseudoconcavity of the end which arises if p is deleted. Clearly the end has a finite volume and this metric is complete toward the end.

All LCS singularities (V, D, p) of types (3),(3)* and (3)** are locally the factor space $G \backslash \mathcal{S}$ where $\mathcal{S} = \{(u, v, 1) \in P_2(C) | Im(u) - |v|^2 > 0\}$ is the Siegel domain realization of the unit open ball B^2 and G is a discrete subgroup of the parabolic subgroup $P \subset Aut(\mathcal{S})$ corresponding to the boundary point $(1, 0, 0) \in P_2(C)$. The Kähler potential $\log \frac{1}{Im(u)-|v|^2}$ for the ball-metric is invariant under P. Hence the ball-metric descends near the LCS singularities of these types in the Kähler potential level

---

[5]All log-terminal (resp. LCS) surface singularities are uniformizable by bounded symmetric domains with a covering transformation group with an interior point (resp. with a point in the boundary (at infinity)) which is fixed by all elements.

and this is complete toward the end corresponding to p. The end has a finite volume and is pseudoconcave in the sense that the Kähler potential assumes $-\infty$ at p.

For LCS singularities (V, D, p) of types (4), (4)* and (4)**, the product of the Poincaré metrics of D and D^* descends near p in the Kähler potential level (we take the Kähler potential of D^* to be $\log \left(\log \frac{1}{|z|^2} \right)^{-2}$ which goes to $-\infty$ at the end). The resulting metric is complete toward D_i with $b_i = \infty$ and has locally finite volume.

Let (X'', D'') be as above. Then, the classification of log-canonical singularities implies that

$$X_0'' = X'' - \cup_{b_i = \infty} D_i'' - LCS(X'', D'')$$

together with $D_0'' = D'' \cap X_0''$ is an orbifold with branch indices b_i along Supp$(D_i'' \cap X_0'')$. Since orbifolds are the so called $b - spaces$ [Kat], we say a function $g : X'' \to R$ is $b - C^\infty$ if it is C^∞ in the local uniformizations. The same definition is possible for covariant tensor fields. Since the invariant metrics of bounded symmetric domains descend near the singularities of (X'', D'') in the Kähler potential level, there exists a positive $b - C^\infty$ function $g : X_0'' \to R$ (we write g instead of $(f \circ \mu)^* g$.) such that the singular volume form

$$\Psi = g\Phi$$

on $\tilde{X}$ satisfies the following properties:

(i) Ψ is $b - C^\infty$ and has negative Ricci form, i.e., $\omega = - \operatorname{Ric} \Psi$ defines a Kähler orbifold metric on X_0'',

(ii) the Kähler orbifold metric ω is complete on X_0'',

(iii) the $b - C^\infty$ function $f = \log \frac{\Psi}{\omega^2}$ is bounded on X_0''.

The function $\log g$ is strictly plurisubharmonic and goes to $-\infty$ at the end of X_0''. The local behavior of g at quotient singularities and at ends is like the Kähler potential of the Poincaré metric of D (or B^2) and D^*. We look at some examples of its behavior. Around a smooth point where D_1 and D_2 (two branches) with b_1, $b_2 < \infty$ intersect transversely, we have approximately

$$g \approx \prod_{i=1}^{2} \frac{1}{b_i^2 (1 - \|\sigma_i\|^2)^{\frac{2}{b_i}}}$$

with some Hermitian metrics for $[D_1]$ and $[D_2]$. Note that $\lim_{b_i \to \infty} b_i (1-$

$\|\sigma_i\|^{\frac{2}{b_i}}) = \log \frac{1}{\|\sigma\|^2}$. Around a smooth point on D_i with $b_i = \infty$, we have

$$g \approx \frac{1}{(\log \frac{1}{\|\sigma_i\|^2})^2}.$$

with certain Hermitian metrics for $[D_i]$. Around a smooth point where D_1 and D_2 (two branches) with $b_1 = b_2 = \infty$ intersect transversely, we have

$$g \approx \prod_{i=1}^{2} \frac{1}{(\log \frac{1}{\|\sigma_i\|^2})^2}.$$

with some Hermitian metrics for $[D_1]$ and $[D_2]$. Around an elliptic curve E_α resolving a simple elliptic singularity, we have

$$g \approx \frac{1}{(\log \frac{1}{\|\sigma_\alpha\|^2})^3}$$

with a certain Hermitian metric for $[E_\alpha]$. More precisely, if (u, v) are the Siegel domain coordinates, then $w = \exp(\frac{b\pi i u}{2a})$ and $z = v$ provide a system of local coordinates around a point in E_α such that the zero-locus of w is E_α and

$$g = \frac{1}{[\log c \frac{(\exp(-|z|^2))^{\frac{b\pi}{a}}}{|w|^2}]^3}$$

for some positive constant c. Around a cycle E_α $(0 \leq \alpha \leq r-1)$ of $P_1(C)'s$ with self-intersection number $-q_\alpha$ resolving a cusp singularity, we have

$$g = \frac{1}{(R_{\alpha-1} \log \frac{1}{|u_\alpha|} + R_\alpha \log \frac{1}{|v_\alpha|})^2 (R^1_{\alpha-1} \log \frac{1}{|u_\alpha|} + R^1_\alpha \log \frac{1}{|v_\alpha|})^2},$$

where (u_α, v_α), local coordinates around $E_{\alpha-1} \cap E_\alpha$, and R_α are as introduced in Example 2.4.

We now consider a complete Kähler orbifold (X'', ω). Using the same strategy as in [Ko2], we construct a good quasi-coordinate system on (X'', D'', ω) which exhibit (X'', D'') as a complete Kähler orbifold with bounded geometry (cf. [Ko2, Lemma 6]). The above approximations of g at the ends are in C^∞-level in the following sense. Namely two metrics ω and the invariant metric defined at one end are of bounded geometry at the same time, i.e., define the same quasi-coordinate system

in which both of two metrics $g_{\alpha,i\bar{j}}$ and $h_{\alpha,i\bar{j}}$ satisfy

$$c(\delta_{ij}) < g(h)_{\alpha,i\bar{j}} < c^{-1}(\delta_{ij})$$

$$\frac{\partial^{|p|+|q|} g(h)_{\alpha,i\bar{j}}}{\partial v_\alpha^p \partial \bar{v}_\alpha^q} < A_{|p|+|q|}$$

with common positive constants c, *etc.* and (v_α^i) are quasi-coordinates exhibiting bounded geometries of g and h. We then introduce the Banach space $b - C^{k,\alpha}$ of $b - C^{k,\alpha}$-functions on (X'', D'', ω) with the Hölder type norm defined by a good quasi-coordinate system for (X'', D'', ω). Note that this definition is canonical in the sense that the orbifold structure and the metric structure which are canonically introduced on (X'', D'') by the classification and the uniformization of log-canonical singularities are involved. We then have

Lemma 3.5 (cf. [Ko2,Lemma 7]). *The function $f = \log \frac{\Psi}{\omega^2}$ is of class $b - C^{k,\alpha}$ for any nonnegative integer k and $\alpha \in (0,1)$.*

We consider the Monge-Ampère equation:

(3.8) $$(\omega + \sqrt{-1}\partial\bar{\partial}u)^2 = e^{u+f}\omega^2.$$

The solution $u \in \cap_{k,\alpha} b - C^{k,\alpha}$ of (3.8) is a Kähler-Einstein orbifold metric on (X_0'', D_0''). Using again the same strategy as in [Ko2] (the idea using bounded geometry was first introduced by Cheng-Yau [CY] to show the existence of a complete Kähler-Einstein metric on strictly pseudoconvex domains (see also [MY])), it is shown that such a solution u exists *with* a priori estimates which are formally the same as those for the smooth case (cf. [Au1,2], [Y1], see also Section 1 of this paper). In particular, there exists a positive constant c such that

$$c\omega < \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u < c^{-1}\omega.$$

which implies that $\tilde{\omega}$ is a complete Kähler-Einstein orbifold-metric with negative scalar curvature. It follows from (3.6), (3.7), the construction of ω and the a priori estimates for u that the singular differential form $\tilde{\omega}$ defines a real closed (1,1)-current on $\tilde{X}$ whose cohomology class is $c_1((f \circ \mu)^*(K_{X''} + D'')) = \frac{1}{m}c_1((f \circ \mu)^*L)$. The point here is the fact that the Poincaré metric of the punctured disk D^* has the origin as an end with a finite volume. This completes the proof of Theorem 1.

3.2.3. Inequality between Chern numbers and the numerical characterization of ball quotients for normal surfaces with branch loci

Normal surface pairs often appear as ball quotients (cf. [BHH]). Ball quotients are characterized by the existence of the ball metric. From Theorem 1, there exists a unique Kähler-Einstein orbifold-metric $\tilde{\omega}$ with negative scalar curvature on a log-canonical surface (X'', D'') with at worst log-canonical singularities. If γ_1 and γ_2 are the first and the second Chern forms of $\tilde{\omega}$, then $3\gamma_2 - \gamma_1^2$ is equal to the square of the pointwise deviation of the Kähler-Einstein metric $\tilde{\omega}$ being of constant holomorphic sectional curvature (i.e., being the ball metric). Since the metric is an orbifold-metric which is approximately an invariant metric at the ends, we compute as in [Ko2] and [Ko3] the following curvature integra:

$$\int_{X''} \gamma_1^2 = (K_{X''} + D'')^2,$$

$$\int_{X''} \gamma_2 = e(X_0'') + \sum_i (\frac{1}{b_i} - 1)(e(D_{0i}'') - d_i) + \sum_p (\frac{1}{|\Gamma(p)|} - 1)$$

where $X_0'' = X'' - \cup_{b_i = \infty} D_i'' - LCS(X'', D'')$, $e(\cdot)$ is the Euler number of $\cdot$, $D_{0i}'' = D_i'' \cap X_0''$, d_i is the number of singularities of (X'', D'') lying over D_{0i}'' and $|\Gamma(p)|$ is the order of the local fundamental group (in the sense of orbifolds) of $\Gamma(p)$ of a log-terminal singular point p of (X'', D''). We thus have an inequality in Theorem 2 between Chern numbers of (X'', D''). The equality holds if and only if $\tilde{\omega}$ is the ball metric. Then the geodesic developing map of the complete orbifold with the ball metric exhibit $(X'', D'', \tilde{\omega})$ as a ball quotient. This completes the proof of Theorem 2.

Recently, Holzapfel [Ho2] obtained an effective finiteness theorem for ball lattices whose quotient surface has the orbifold structure supported by a given orbital configuration.

The general Miyaoka-Yau inequality in [Miy2] can be applied to the classical problem on how many singularities a plane curve of degree d has. Considering a cyclic covering and applying [Miy] give an estimate. For this application, see [Hir5], [I], [M-S], [Yr]. If we use Theorem 2, then we do not need the existence of a cyclic covering and so do not need any assumption on d. See [Ko6] and [Sak3].

§4.   GHCS and uniformization of complex surfaces

In this section we exhibit some examples and counter examples for uniformization problem for compact complex surfaces with ample canon-

ical bundle and with at worst rational double points satisfying $2\bar{c}_2 = \bar{c}_1^2$ in the sense of (0.1) and (0.2).

4.1. GHCS and OUDE

We begin by establishing a method of the construction of GHCS's on $P_2(C)$ (see [KNr]). Let Ω_α be an affine part of $P_2(C)$ defined by $z_\alpha \neq 0$ and $\pi : C^3 - \{0\} \rightarrow P_2(C)$ be the natural projection. Let $\tau = \{\tau_\alpha\}$, $\tau_\alpha \in \Gamma(\Omega_\alpha, S^2T^*P_2(C))$ be a GHCS on $P_2(C)$ defined by a line bundle L. Let D be the discriminant divisor and assume $[D] = O(2m)$, i.e., D is a curve of degree $2m$. Since $[D] = 2(L + K)$ and $K = O(-3)$, we have that $L = O(m + 3)$. Recall that $O(1)$ is the hyperplane bundle on $P_2(C)$ defined by the cocycle $\{\frac{z_\beta}{z_\alpha}\}$. Hence we may assume after multiplying a suitable non-vanishing holomorphic function that

$$\frac{\pi^*(\tau_\alpha)}{\pi^*(\tau_\beta)} = \left(\frac{z_\beta}{z_\alpha}\right)^{m+3} \quad \text{in } C^3 - \{0\}.$$

So we have compatibility conditions

$$\pi^*(\tau_\alpha)z_\alpha^{m+3} = \pi^*(\tau_\beta)z_\beta^{m+3}$$

and the Hartogs Theorem implies that this extends across the origin to a holomorphic symmetric covariant 2-tensor

$$g = \pi^*(\tau_\alpha)z_\alpha^{m+3}$$

on the whole C^3. Since g is of homogeneous degree $m + 3$, if we write g as

$$g = \sum_{i,j=1}^{3} h_{ij}(z)dz_i dz_j \quad (h_{ij} = h_{ji}),$$

$h_{ij}(z)$ is a homogeneous polynomial of degree $m + 1$. That g defines a GHCS on $P_2(C)$ is equivalent to $i_\xi g = 0$, where $\xi = \sum_{i=1}^3 z_i \frac{\partial}{\partial z_i}$, the vector field generating the action of C^* on C^3. And this is written as

$$(4.1) \qquad \sum_{j=1}^{3} h_{ij}(z)z_j = 0 \quad (i = 1, 2, 3).$$

Using (4.1), we have

$$(4.2) \qquad \tau_\alpha = \frac{g}{z_\alpha} = \sum_{i,j\neq\alpha} \frac{h_{ij}(z)}{z_\alpha^{m+3}} d\left(\frac{z_i}{z_\alpha}\right) d\left(\frac{z_j}{z_\alpha}\right) \quad \text{on } \Omega_\alpha,$$

where $\frac{h_{ij}}{z_\alpha}$ is a polynomial in $\frac{z_i}{z_\alpha}$ ($i \neq \alpha$). It also follows from (4.1) that the following polynomial of degree $2m$

$$\Delta = \frac{\det(h_{ij}(z))_{i,j \neq \alpha}}{z_\alpha^2}$$

does not depend on α. By (4.2), $\Delta = 0$ defines the discriminant divisor of τ. Since we want to construct examples which are desingularizable by Theorem 2.7, we rewrite conditions (I), (II) and (III) in Section 2 (see Theorem 2.7) in this special situation. First, we look at the condition (II). Clearly (II) is equivalent to

(II)′ the rank of τ is one along the regular part of $\Delta = 0$.

We may assume by a suitable linear change of (z_1, z_2, z_3) if necessary that

(i) no prime factor of Δ divides one and the same column or row of $(h_{ij}(z)_{1 \leq i,j \leq 3})$,

(ii) Δ is not divisible by any coordinate z_i.

Assuming these two conditions and (II)′, (III) is equivalent to

(III)′ the homogeneous polynomial $h_{jj}\Delta_k - h_{jk}\Delta_j$ of degree $3m$ is divisible by Δ for every even permutation (i, j, k) of (1,2,3).

Indeed, for $p \in \mathrm{Reg}(D)$ the 1-dimensional null space N_p is spanned by the vector $v_i = h_{jj}(x)\frac{\partial}{\partial x_k} - h_{jk}(x)\frac{\partial}{\partial x_j}$, where $x_j = \frac{z_j}{z_i}$, etc. are inhomogeneous coordinates in the affine part $z_i \neq 0$. The condition $v \in T_p(D)$ is then equivalent to $v_i\Delta(p) = 0$ and this holds for every $p \in \mathrm{Reg}(D)$ which implies (III)′ in the C^3 level. (I) is of course equivalent to

(I)′ no multiple factor occurs in the prime factorization of Δ.

For explicit examples, we want to construct a GHCS for prescribed discriminant divisor D with high symmetry with group G. Therefore we further use the G-invariance (which is translated to some relations among coefficients of polynomials h_{ij}) to determine the GHCS with divisor D. We have thus established, modulo (maybe) involved computations in explicit examples, the method of geometric construction of a GHCS with prescribed discriminant divisor. We remark that the above construction is generalized to higher dimensions by Sasaki-Yoshida [SaY2].

Before proceeding to examples, we briefly discuss *orbifold uniformizing differential equation* (we say OUDE for abbreviation). Let M be a Hermitian symmetric space and Γ a discrete group of automorphisms of M and $\pi : M \to X = \Gamma \backslash M$ the projection. A multivalued inverse map $\psi = \pi^{-1} : \Gamma \backslash M \to M$ is called a developing map of the orbifold $\Gamma \backslash M$.

Definition. A linear differential equation on X of rank N is an orbifold uniformizing differential equation (OUDE) if its projective solution

$$\psi : X \ni x \mapsto (z_1(x) : \cdots : z_N(x)) \in P_{N-1}(C)$$

takes its values in $M \subset P_{N-1}(C)$ and gives a developing map.

Gauss' hypergeometric differential equation (see [Yo]) is the first example of OUDE's. This is the second order ordinary differential equation defined on $P_1(C)$ with singularities at 0, 1 and ∞

$$(4.3) \qquad z(z-1)u'' + \{c - (a+b+1)x\}u' - abu = 0.$$

If we assume $|1-c|^{-1}$, $|c-a-b|^{-1}$ and $|a-b|^{-1}$ are integers (or ∞) b_0, b_1 and b_∞, then the projective solution of (4.3) uniformizes the orbi-curve $(P_1(C), D)$ with $D = (1 - \frac{1}{b_0})\{0\} + (1 - \frac{1}{b_1})\{1\} + (1 - \frac{1}{b_\infty})\{\infty\}$. We are interested in higher dimensional analogue of this. Yoshida [Yo] studied extensively OUDE's on $P_2(C)$ with the open unit ball as the uniformization. Here we discuss how GHCS's are applied in finding new OUDE's, those uniformizing Hilbert modular orbifolds. In fact, although Theorem 2.7 says that a GHCS (with a Kähler-Einstein metric) is sufficient to show the *existence* of the developing map to D^2, it is not sufficient in determining the *explicit* form of the OUDE, i.e., we need another essential information. This is first pointed out by Sasaki-Yoshida [SY1]. Since D^2 is naturally imbedded in $P_1(C) \times P_1(C)$ and $P_1(C) \times P_1(C)$ is a smooth quadric surface in $P_3(C)$, the OUDE we want to find is of *rank four* and the projective solution obeys a *quadratic relation*. This implies that we must know the criterion in terms of local invariants which describes whether a surface segment in $P_3(C)$ is contained in a quadric surface up to projective transformations, i.e., we must know the solution to the equivalence problem of quadric surfaces in projective differential geometry (see [Sas] and [SaY2]).

In fact, as in surface theory in R^3, there is the fundamental theorem which asserts that two surfaces in $P_{n+1}(C)$ are locally equivalent if and only if certain local invariants under the integrable condition (equations of Gauss-Codazzi and Minardi) coincides. But such theorem exists only for dimensions $n \geq 3$. This corresponds to the special feature in 2-dimensional conformal geometry. Namely the 2-dimensional holomorphic conformal structure (up to passing to the double covering) is equivalent to the splitting of the tangent bundle, which corresponds to the local decomposition $SO(4, C) \cong SO(3, C) \times SO(3, C)$. In other words, $Q_n(C)$ is irreducible for $n \geq 3$ and $Q_2(C) = P_1(C) \times P_1(C)$. We now compute local invariants of a complex surface (segment) $i : M \hookrightarrow P_3(C)$

(the following discussion up to Fact 4.3 is valid for a hypersurface segment in $P_{n+1}(C)$). Let $e_0 : M \to C^4 - \{0\}$ be a holomorphic lifting of i. We associate a set $e(p) = \{e_1(p), e_2(p), e_3(p)\}$ of linearly independent vectors at $e_0(p)$ such that $e_i(p)$ $(i = 1, 2)$ are tangent to $e_0(M)$ modulo $e_0(p)$, and we call $\{e_0(p), e_1(p), e_2(p), e_3(p)\}$ a *projective frame* for M at p. We always assume that $\det(e_0, \cdots, e_3) = 1$. We consider e as a holomorphic map of M to $(C^4)^4$ which obeys an infinitesimal equation

$$(4.4) \qquad de = e\omega \quad (de_\alpha = \sum_{\beta=0}^{3} e_\beta \omega_\alpha^\beta),$$

where ω is a holomorphic one form with values in $sl(4, C)$ (differentiate $\det(e_0, \cdots, e_3) = 1$) and is called the *Maurer-Cartan form*. Taking the exterior derivative of (4.4) gives the *integrability condition* for the total differential equation (4.4) for e:

$$(4.5) \qquad d\omega + \omega \wedge \omega = 0, \quad (d\omega_\alpha^\beta + \sum_{\gamma=0}^{3} \omega_\gamma^\beta \wedge \omega_\alpha^\gamma = 0).$$

Since $\omega_0^4 = 0$, we have from (4.5)

$$(4.6) \qquad \sum_{i=1}^{2} \omega_0^i \wedge \omega_i^3 = 0.$$

Set $\omega_0^i = \omega^i$ $(i = 1, 2)$. Then ω^i $(i = 1, 2)$ are coframes along $e_0(M)$. Since ω^i $(i = 1, 2)$ are linearly independent, (4.5) and Cartan's lemma imply that there exists a symmetric form h_{ij} such that

$$(4.7) \qquad \omega_i^3 = \sum_{j=1}^{2} h_{ij}\omega^j \quad (i = 1, 2).$$

Define a holomorphic symmetric covariant 2-tensor

$$(4.8) \qquad \phi_2 = \sum_{i,j=1}^{2} h_{ij}\omega^i\omega^j.$$

Then it is easy to see the following

Fact 4.1.   *The conformal class of ϕ_2 is independent of the choice of a frame.*

Here a frame change is written as

$$\tilde{e} = ge, \quad e_\alpha = \sum_{\beta=0}^{3} g_\alpha^\beta e_\beta$$

with $g_0^i = g_0^3 = g_i^3 = 0$ $(i = 1, 2)$. In this way, degenerate holomorphic conformal structures naturally appear in projective differential geometry. Assume that ϕ_2 is non-degenerate. Then M admits a holomorphic conformal structure. Under this assumption, we may assume after a frame change if necessary that

$$(4.9) \qquad\qquad \det(h_{ij}) = 1, \quad \omega_0^0 + \omega_3^3 = 0.$$

If we set

$$(4.10) \qquad \sum_{k=1}^{2} h_{ijk}\omega^k = dh_{ij} - \sum_{k=1}^{2} h_{ik}\omega_j^k - \sum_{k=1}^{2} h_{jk}\omega_i^k,$$

then (4.9) implies the following

Fact 4.2.   *Set*

$$(4.12) \qquad\qquad \phi_3 = \sum_{i,j,k=1}^{2} h_{ijk}\omega^i\omega^j\omega^k.$$

Then ϕ_3 is a symmetric cubic form and its conformal class is independent of the choice of a frame, i.e., ϕ_3 is uniquely determined up to multiplication of non-vanishing holomorphic functions.

We call ϕ_3 the *Wiczynski-Fubini-Pick cubic form.* Using these invariants, we have a local characterization for quadric surfaces.

Fact 4.3 (Berwelt, Wilczynski; see [Sas]).   *Let M be a holomorphic surface segment in $P_3(C)$ and assume ϕ_2 is non-degenerate. Then one can find a frame with (4.9) and define the Fubini-Pick cubic form ϕ_3. The hypersurface M is locally a quadric if and only if $\phi_3 = 0$.*

We are now ready to construct OUDE for orbifolds $M = \Gamma \backslash H \times H$. Now the special feature of the 2-dimensional conformal geometry comes

in. First of all, it is essential to realize $D^2 \cong H \times H$ as an open domain in a quadric surface in $P_3(C)$. Explicitly, we have

$$H \times H \hookrightarrow P_1(C) \times P_1(C) \cong Q_2(C) \hookrightarrow P_3(C)$$

(4.11) $\qquad (z_1, z_2) \mapsto ([1 : z_1], [1 : z_2]) \mapsto [1 : z_1 : z_2 : z_1 z_2]$

where $[\cdot : \cdot]$ stands for the ratio. Moreover, $Aut(H \times H)$ is a subgroup of $Aut(P_3(C))$ which leaves $H \times H$ invariant. In the regular part of M, we take (z_1, z_2) as local coordinates. The above $Q_2(C)$ is given by $\zeta_0 \zeta_3 = \zeta_1 \zeta_2$ and the standard holomorphic conformal structure is $2d(\frac{\zeta_1}{\zeta_0})d(\frac{\zeta_2}{\zeta_0})$. Then our differential equation in these coordinates is that which is fulfilled by a map covering (4.11). Write $z_1 = u$, $z_2 = v$. Then the differential equation

(4.12)
$$z_{uu} = 0$$
$$z_{vv} = 0$$

is of rank four and the basis of the solution space is given by $\{1, u, v, uv\}$ (note (4.12) is characteristic in dimension 2 since it corresponds to the splitting $Q_2(C) = P_1(C) \times P_1(C)$). Therefore (4.12) is our differential equation in these coordinates and (4.11) is its *projective solution*. There is a $PGL(4, C)$-ambiguity in the choice of a projective solution. The analytic continuation of a locally defined projective solution gives a developing map for M. Set $e_0 = (1, u, v, uv)$, $e_1 = \frac{\partial e_0}{\partial u} = (0, 1, 0, v)$, $e_2 = \frac{\partial e_0}{\partial v} = (0, 0, 1, u)$, $e_3 = \frac{\partial^2 e_0}{\partial u \partial v}$ and $e = \{e_0, \cdots, e_3\}$. Then e forms a projective frame for the imbedding (4.11) with $\det(e_0, \cdots, e_3) = 1$. If we set

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ du & 0 & 0 & 0 \\ dv & 0 & 0 & 0 \\ 0 & dv & du & 0 \end{pmatrix}$$

then e obeys the total differential equation $de = e\omega$, which, as an equation for $e = (z, z_u, z_v, z_{uv})$, is equivalent to (4.12) (cf. (4.4)). Note that the holomorphic conformal structure $2dudv$ is induced from that of $Q_2(C)$ under the map (4.11). If we perform a coordinate change $(u, v) \to (x, y)$, then (4.12) is transformed into the following form.

(4.13)
$$z_{xx} = l z_{xy} + a z_x + b z_y$$
$$z_{yy} = m z_{xy} + c z_x + d z_y$$

with

$$l = \frac{-2y_u y_v}{x_u y_v + y_u x_v} \quad \text{and} \quad m = \frac{-2x_u x_v}{x_u y_v + y_u x_v}.$$

If $M = P_2(C)$, we take an affine coordinate system (x, y) of $P_2(C)$ as new coordinates. If we further perform a change of the unknown $z \to e^\rho z$ to (4.13) with ρ a holomorphic function, we arrive at the general form of linear differential equations with 2 variables and of rank 4.

$$\begin{aligned}
z_{xx} &= l z_{xy} + a z_x + b z_y + p z \\
z_{yy} &= m z_{xy} + c z_x + d z_y + q z
\end{aligned}$$

(4.14)

We note that under the above change of the unknown, the coefficients l and m are unchanged. Moreover we have

Fact 4.4. *The conformal class of the symmetric 2-form $l(dx)^2 + 2(dx)(dy) + m(dy)^2$ is independent of the change of coordinates $(x, y) \to (u, v)$ with $x_u y_v + v_u x_v \neq 0$.*

Indeed, a direct computation shows

$$l(dx)^2 + 2(dx)(dy) + m(dy)^2 = \frac{2(x_u y_v - y_u x_v)^2}{x_u y_v + y_u x_v} du \, dv.$$

Proposition 4.1. *The holomorphic conformal structure $l(dx)^2 + 2(dx)(dy) + m(dy)^2$ coincides with the holomorphic conformal structure obtained by pulling back the canonical one on the quadric surface by a projective solution.*

Indeed, $du \, dv$ is the pull back of the canonical holomorphic conformal structure under the projective solution of (4.12). Thus Fact 4.4 implies the assertion of the proposition.

Let $\{z_0, z_1, z_2, z_3\}$ be a basis for the space of solutions of (4.14). Set $z = (z_0, \cdots, z_3)$ and

$$e^{2\theta} = \det(z, z_x, z_y, z_{xy}).$$

We call this the *normalization factor*. If we fix a coordinate system, the normalization factor varies if we change the unknown z by $e^\rho z$. This fact is used successfully by Sasaki-Yoshida [SaY1] and Sato [Sat]. Thus the surface in $P_3(C)$ defined by the projective solution $[e_0]$ of (4.14) has a projective frame $e = (e_0, e_1, e_3, e_3) = (z, z_x, z_y, e^{-2\theta} z_{xy})$ with $\det(e_0, \cdots, e_3) = 1$. Indeed, a direct computation shows

$$\det(z, z_x, z_y, z_{xy}) = \frac{e^{4\rho}(x_u y_v + y_u x_v)}{(x_u y_v - y_u x_v)^3} \neq 0.$$

hereafter we assume that the orbifold structure M is defined over $P_2(C)$ and (x, y) in (4.14) are affine coordinates. Hirzebruch's descriptions (see [Hir3,4]) for certain Hilbert modular orbifolds give such examples. His descriptions are so explicit that we can get all informations on the orbifold structure. We take a full advantage of this special situation to find the explicit form of OUDE. Proposition 4.3 implies that the first step in finding OUDE for a Hilbert modular orbifold over $P_2(C)$ is to apply the method of construction of GHCS on $P_2(C)$ ([KNr]) explained at the beginning of this section. The next step is to apply Fact 4.2 which guarantees that the surface $e_0(M)$ in $P_3(C)$ defined by the projective solution $[e_0] = (z_0 : \cdots : z_3)$ is a quadric surface. One can compute the Wilczynski-Fubini-Pick cubic form of the surface $e_0(M)$ explicitly, after a certain explicit frame change $(e_0, \cdots, e_3) \to (\tilde{e}_0, \cdots, \tilde{e}_3)$ (see [SaY1]) necessary to achieve the condition (4.9). A direct computation then shows that Fact 4.2 in this case reads as follows.

Proposition 4.2 ([SaY1]). *The surface in $P_3(C)$ defined by the projective solution $[e_0]$ is locally equivalent to a quadric if and only if*

(4.15)
$$a = \frac{\partial}{\partial x}(\frac{1}{4}\xi + \theta) - \frac{l}{2}\frac{\partial}{\partial y}(\log l - \frac{1}{4}\xi + \theta)$$

$$b = \frac{l}{2}\frac{\partial}{\partial x}(\log l - \frac{3}{4}\xi - \theta)$$

$$c = \frac{m}{2}\frac{\partial}{\partial y}(\log m - \frac{3}{4}\xi - \theta)$$

$$d = \frac{\partial}{\partial y}(\frac{1}{4}\xi + \theta) - \frac{m}{2}\frac{\partial}{\partial x}(\log m - \frac{1}{4}\xi + \theta)$$

where
$$\xi = \log(1 - lm).$$

Proposition 4.2 implies that once we fix the normalization factor $e^{2\theta}$, then the coefficients a, b, c and d are determined by the holomorphic conformal structure $\phi_2 = l(dx)^2 + 2(dx)(dy) + m(dy)^2$. Since ϕ_2 is written using rational functions, a, b, c and d are also rational functions provided $e^{2\theta}$ is fixed to be a rational function. We note that p and q do not appear in the condition $\phi_3 = 0$. Therefore we need global considerations (not infinitesimal considerations) to determine p and q. We should compare this with the case of dimension ≥ 3 which is treated in [SaY2].

Sasaki-Yoshida showed that, in higher dimensions, the condition $\phi_3 = 0$ determines all coefficients of OUDE in terms of (the derivatives of) the GHCS and the integrability condition (4.5) is automatically fulfilled if the GHCS is flat (i.e., the quadric structure).

The final step is to determine p and q. For this purpose, we make use of the integrability condition for the equation (4.14). The projective frame $e = (z, z_x, z_y, z_{xy})$ for the surface defined by the projective solution $[z]$ obeys the following total differential equation which is equivalent to (4.14):

$$de = e\omega,$$

with the Maurer-Cartan form ω given by

$$\begin{pmatrix} 0 & p(dx) & q(dy) & e^{-2\theta}(B^0 dx + C^0 dy) \\ dx & a(dx) & c(dy) & e^{-2\theta}(B^1 dx + C^1 dy) \\ dy & b(dx) & d(dy) & e^{-2\theta}(B^2 dx + C^2 dy) \\ 0 & e^{2\theta}(ldx + dy) & e^{2\theta}(mdy + dx) & -a(dx) - d(dy) \end{pmatrix},$$

where

$$B^0 = \frac{p_y + bq + l(q_x + cp)}{1 - lm}, \quad C^0 = \frac{q_x + cp + m(p_y + bq)}{1 - lm}$$

$$B^1 = (A + lq)(1 - lm)^{-1}, \quad C^1 = (C + q)(1 - lm)^{-1}$$

(4.16) $\quad B^2 = (B + q)(1 - lm)^{-1}, \quad C^2 = (D + mp)(1 - lm)^{-1}$

$$A = a_y + bc + l(c_x + ac), \quad B = b_y + bd + l(d_x + bc)$$

$$C = c_x + ac + m(a_y + bc), \quad D = d_x + bc + m(b_y + bd).$$

Proposition 4.2 ([SaY1]). *The integrability condition (4.5) for the equation (4.14) is given by the following equations:*

$$\begin{cases} \text{(IC0) } (a + B^3)_y = (d + C^3)_x \\ \text{(IC1) } lq_y - 2q_x - mp_y - (l\xi_y - \xi_x - 2l_y)q = R^1 \\ \text{(IC2) } mp_x - 2p_y - lq_x - (m\xi_x - \xi_y - 2m_x)p = R^2 \\ \text{(IC3) } p_{yy} - q_{xx} - mp_{xy} + lq_{xy} \\ \qquad = cp_x - bq_y + (d + 2m_x + \xi_y - m\xi_x)p_y \\ \qquad - (a + 2l_y + \xi_x - l\xi_y)q_x \\ \qquad + (ma_y + 2c_x - 2cl_y - lc_y - c(\xi_x - l\xi_y))p \\ \qquad - (ld_x + 2b_y - 2bm_x - mb_x - b(\xi_y - m\xi_x))q, \end{cases}$$

where

$$(4.17) \quad \begin{aligned}
R^1 &= (C^3 + \xi_y)A - (B^3 - a + \xi_x)C - A_y + C_x - cB \\
R^2 &= (B^3 + \xi_x)D - (C^3 - d + \xi_y)B - D_x + B_y - bC \\
B^3 &= (l_y + a + bm) + l(m_x + d + cl))(1 - lm)^{-1} \\
C^3 &= (m_x + d + cl) + m(l_y + a + bm))(1 - lm)^{-1}.
\end{aligned}$$

We note that (IC0) is a condition for l, m, a, b, c and d. For explicit examples, once one finds l and m (also a fixed normalization factor), (IC0) should be examined directly. (IC1) and (IC2) are effectively used to determine p and q. Sasaki-Yoshida established the strategy in determining p and q, which, modulo very involved computations, is as follows ([SaY1], see also [Sat]).

(0) We use the expression of the GHCS ϕ_2 and the already fixed normalization factor $e^{2\theta}$ and a, b, c and d. A clever choice of the normalization factor, for example imposing the invariance under the symmetry group of D, is very important to simplify the computation.

(1) We estimate the poles of p and q outside the discriminant locus D of ϕ_2 (which coincides with the branch locus of the orbifold). To do this, we start with (4.12) and transform (4.12) into the form (4.15) by the coordinate change $(u, v) \to (x, y)$ where (x, y) are the inhomogeneous coordinates and the change of the unknown $z \to e^\rho z$. One can easily write down the new coefficients in terms of the old ones and the derivatives of ρ with respect to (x, y) as well as the derivatives of (x, y) with respect to (u, v). Using the expressions as rational functions of $e^{2\theta}$ and ϕ_2, we can estimate the poles of rho and therefore those of p and q outside D (see [SY1, transformation formula (2.9) in p.103]).

(2) To estimate the poles along D, we need to start with the canonical form

$$(4.18) \quad \begin{aligned}
z_{uu} &= z_{uv} \\
z_{vv} &= (1 - \frac{1}{4v})z_{uv} + \frac{1}{2v}z_u - \frac{1}{2v}z_v
\end{aligned}$$

representing locally the composite map of the (1:2)-map

$$P_2(C) \ni (u, v) = (z_1 + z_2 - (z_1 + z_2)^2, (z_1 - z_2)^2) \mapsto (z_1, z_2) \in P_1(C)^2$$

and the canonical imbedding

$$(P_1(C))^2 \ni (z_1, z_2) \mapsto (1, z_1, z_2, z_1 z_2) \in P_3(C).$$

The branch locus is $v = 0$. In exactly the same way as in (1), we transform (4.18) into the general form and estimate the poles of ρ along D. This gives the estimate of the poles of p, q along D. We thus see that p and q are rational functions whose numerators are to be determined.

(3) Using (IC1), (IC2) and the G-invariance of the equation (4.15), we determine the numerators, where G is the group of symmetries of D. This process requires involved computations and we sometimes pass to the orbifold $G\backslash P_2(C)$ with the branch loci the image of D and the branch locus of the quotient map by G (see [Sat]). Algebraically, passing to the orbit space $G\backslash P_2(C)$ corresponds to introducing new coordinates which are expressed as certain rational functions of G-invariants. We now proceed to explicit examples of uniformizations of Hilbert modular orbifolds of $Q(\sqrt{2})$ [SaY1] and $Q(\sqrt{5})$ [Sat]. For the case of $Q(\sqrt{3})$, see [Sat].

4.2. Hilbert modular orbifold for $Q(\sqrt{2})$ and its OUDE

Let $k = Q(\sqrt{2})$ and o be the ring of integers in k. We denote by $\Gamma(2)$ be the principal congruence subgroup of $SL(2, o)$ associated with the ideal (2) of o:

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, o); \ \alpha \equiv \delta \equiv 1, \ \beta \equiv \gamma \equiv 0 \mod (2) \right\}.$$

Let further $\Gamma(2) \subset \Gamma \subset SL(2, o)$ be the group such that $\Gamma/\Gamma(2)$ is the center of $SL(2, o)/\Gamma(2)$. It is easy to see that $[\Gamma : \Gamma(2)] = 2$ and $SL(2, o)/\Gamma(2) \cong S_4$ (*the symmetric group of degree* 4 *or the symmetry group of the cube*). The group $SL(2, o)$ acts on $H \times H$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z_1, z_2) \mapsto (\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'})$$

where $'$ means the conjugate in $Q(\sqrt{2})$. Let τ be the transposition $(z_1, z_2) \to (z_2, z_1)$ and $\Gamma\tau$ be the group generated by Γ and τ. Hirzebruch [Hir1] showed that the factor space $H \times H/\Gamma\tau$ is isomorphic to $P_2(C)$ minus six points. The branch locus D of the projection $\pi : H \times H \to P_2(C)$ is the curve of degree 10 consisting of the following four lines and three conics:

$$x = \pm 1, \ y = \pm 1, \ xy = \pm 1, \ x^2 + y^2 = 2,$$

provided the affine coordinates x, y are suitably chosen. The double covering branched exactly along D is the compactified quotient $\overline{\Gamma\backslash H \times H}$. The curve D has exactly six singular points and these are quasi-regular

log-canonical singularities of the normal surface pair $(P_2(C), \frac{1}{2}D)$. Indeed these singularities correspond to the cusp singularities in the double covering and each of them is resolved by a the cycle of two rational curves of self-intersection numbers -2 and -4 (see Section 4). Thus there exists a unique *tangential* GHCS with discriminant divisor D. With the aid of the S_4-invariance we can determine the explicit form of this ([KNr]):

$$l(dx)^2 + 2(dx)(dy) + m(dy)^2$$

with

$$l = \frac{2 - y^2 - x^2 y^2}{xy(x^2 - 1)}, \quad m = \frac{2 - x^2 - x^2 y^2}{xy(y^2 - 1)}.$$

Sasaki-Yoshida [SaY1] constructed the OUDE for the orbifold $\Gamma\tau\backslash H \times H$. They fixed the normalization factor for the equation (4.14) to be

$$e^{2\theta} = (1 - lm)^{-\frac{7}{2}}(xy)^{-6},$$

which is S^4-invariant (i.e., if we change the variables (x, y) to (x', y') by an element of S^4, the normalization factor of the new equation has the same form with respect to $(x', y', \cdots, l', m')$). And they carried out the above procedure $(0), \cdots, (3)$. The result is an equation (4.14) with l, m as above, and a, b, c, d as determined by the formulae (4.15), and

$$p = \frac{-2(x^2 - y^2)}{(1 - x^2)^2(1 - y^2)}, \quad q = \frac{-2(y^2 - x^2)}{(1 - x^2)(1 - y^2)^2}.$$

Let $\mathcal{G}$ be the full modular group $PSL(2, o)$ and $\mathcal{G}\tau$ be the group generated by $\mathcal{G}$ and τ. It is shown in [KKN] that $S_4\backslash P_2(C)$ is birational to $P_2(C)$ and the rational map

$$\delta: P_2(C) \to P_2(C) \simeq S_4\backslash P_2(C)$$

is given by $\delta[x : y : z] = [A^3 : B^2 : AC]$ where $A = x^2 + y^2 + 2z^2$, $B = (x^2 - y^2)z$, and $C = (z^2 - x^2)(z^2 - y^2)$. It is an easy task to transform $\phi_2(x, y)$ into $\bar\phi_2(X, Y)$ in terms of the new variables X and Y which are defined by $X = \frac{B^2}{A^3}$ and $Y = \frac{C}{A^2}$. Hence if we perform the change of variables $(x, y) \to (X, Y)$, we get the transformed equation $\overline{OUDE}$ which is the OUDE for the terminal orbifold $\mathcal{G}\tau\backslash H \times H$. The terminal orbifold defined birationally over $P_2(C)$ have the branch locus $\bar D = D_1 \cup D_2$ of branch index 2 where D_1 is the branch locus of δ and D_2 is the image of D (D_1 and D_2 have no common components):

(4.19)
$$D_1 := X\{(1 + 12Y)^3 - (54X + 36Y - 1)^2\} = 0$$
$$D_2 := Y(X + Y) = 0$$

The origin $X = Y = 0$ is a log-canonical singularity of the same type as above which corresponds to the cusp point at ∞. Since the singular locus of D as well as D itself is realized in the real part $P_2(R)$ (see Figure 4.1), we can determine the generators and relations of the fundamental group $\pi_1(P_2(C) - \bar{D})$ using the Zariski-van Kampen's method (see, for example, [KKN]). Analyzing the rational map δ, we get further relations which determine the structure of the fundamental group $\mathcal{G}\tau$ of the terminal orbifold.
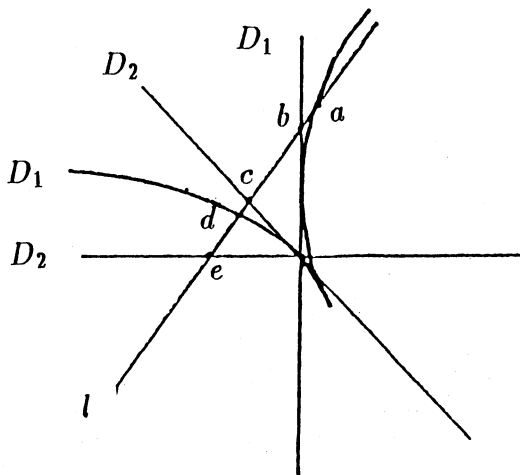
Figure 4.1

Let us use the symbols in Figure 4.1 to explain the result. Let α, β and δ be loops sitting in the complex feature of the line l surrounding D_1 at a, b, and d counterclockwise. Let similarly γ and ϵ be loops in the complex feature of l surrounding D_2 at c and e. Then $\mathcal{G}\tau$ is a reflection group generated by α, β, γ, δ and ϵ with the following relations:

$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \epsilon^2 = 1,$$
$$(\alpha\beta)^2 = (\beta\alpha)^2, \ \alpha\epsilon = \epsilon\alpha, \ \alpha\gamma = \gamma\alpha, \ \alpha\delta\alpha = \delta\alpha\delta,$$
$$\beta\delta\gamma\epsilon = \epsilon\beta\delta\gamma, \ (\beta\delta\gamma\epsilon)\delta\gamma = \gamma(\beta\delta\gamma\epsilon)\delta, \ (\beta\delta\gamma\epsilon)\delta\gamma\delta = \delta(\beta\delta\gamma\epsilon)\delta\gamma,$$
$$(\beta\delta\gamma\epsilon)\beta = \beta(\beta\delta\gamma\epsilon), \ \epsilon = (\alpha\beta\gamma\delta\epsilon)^3.$$

It is not difficult to find the matrix elements in $\mathcal{G}\tau$. For instance we have

$$\alpha = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}, \ \epsilon = \begin{pmatrix} 0 & E \\ E^* & 0 \end{pmatrix},$$

where $*$ means the conjugation in $Q(\sqrt{2})$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix}, \ C = \begin{pmatrix} u & 0 \\ 0 & -u^* \end{pmatrix}$$

$$D = \begin{pmatrix} u & 1 \\ 0 & -u^* \end{pmatrix}, \ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

on $PSL(2, o)$-level, with $u = -\sqrt{2} - 1$ (a fundamental unit of o). This group is isomorphic to the monodromy group of $\overline{OUDE}$. In general we must realize the monodromy group as a discrete subgroup of the automorphisms of the target space. But this is automatically fulfilled in our case, since the projective solution is the developing map of the quadric structure $\bar{\phi}_2(X, Y)$. If one wish to examine directly the monodromy condition, it will be necessary to develop the local theory of Fuchsian systems (see [YH]) of rank four with the *quadratic relation*. It should corresponds to the study of the uniformizing differential equations for quasi-regular log-canonical singularities (whose double coverings are Hilbert modular cusp singularities). Note that these log-canonical singularities have a canonical tangential GHCS. For their classification, see [Kar] (also [Na]) and Section 3 of this paper.

4.3.   Hilbert modular orbifold of $Q(\sqrt{5})$ and its OUDE

Let o be the ring of integers in $Q(\sqrt{5})$ and $\Gamma = \Gamma(\sqrt{5})$ the principal congruence subgroup of $G = PSL(2, o)$ associated with the prime ideal $(\sqrt{5})$ of o. Let $\Gamma\tau$ $(G\tau)$ be the groups of automorphisms of $H \times H$ generated by Γ (resp. G) and the transposition τ. Hirzebruch [Hir4] showed that the Hilbert modular surface $\Gamma\backslash H \times H$ is completed with six cusps and the Hilbert modular orbifold $\Gamma\tau\backslash H \times H$, which is the quotient by the action of the involution induced by the transposition τ, is isomorphic to $P_2(C)$ minus six points with the *Klein curve $C = 0$ of degree ten* as the branch locus C with branch index 2. The equation of C is given by:

$$C = 320x^2y^2 - 160x^3y^3 + 20x^4y^4 + 6x^5y^5$$
$$- 4(x^5 + y^5)(32 - 20xy + 5x^2y^2) + x^{10} + y^{10}.$$
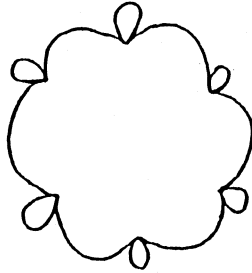
Klein's curve of degree 10

Figure 4.2

The curve C has six singularities each of which is the *double* $(2,3)$-*cusp*. This is locally given by the equation $(x^3 + y^2)(x^2 + y^3) = 0$. It is easy to see that the double covering $z^2 - (x^3 + y^2)(x^2 + y^3) = 0$ is the Hilbert modular cusp singularity resolved by a cycle of two rational curves of self-intersection number -3. It is known [Hir2] that the operation of $G/\Gamma = PSL(2, F_5) \cong A_5$ is the natural action of the icosahedral group ($\cong A_5$) induced from the canonical action on R^3. The Klein curve is invariant under this action and the six double cusps form a unique minimal orbit of A_5 (see [Kl] and [KKN]). It is shown in [KKN] that the completed terminal orbifold $\overline{G_\Gamma \backslash H \times H} = A_5 \backslash P_2(C)$ is birationally isomorphic to $P_2(C)$ and the rational map is given by

$$\delta : P_2(C) \ni [1 : x : y] \mapsto [A^5 : A^2B : C] \in P_2(C)$$

in the inhomogeneous coordinates (x, y), where $A = 1 + xy$, $B = 8xy - 2x^2y^2 + x^3y^3 - x^5 - y^5$ and C is as above. A is the invariant conic, B the unique curve of degree 6 through six double cusps and C is the unique curve of degree 10 through the unique minimal orbit of A_5. The branch locus $\bar{C}$ of the terminal orbifold consists of the image $C_1 = 0$ of C and the branch locus $C_2 = 0$ of δ and is of index 2: $\bar{C} = C_1 \cup C_2$. Explicitly, we have

$$C_1 = Y$$
$$C_2 = 1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3.$$

(-2)

the exceptional curve of the
cusp singularity corresponding
to the double (2.3)-cusp
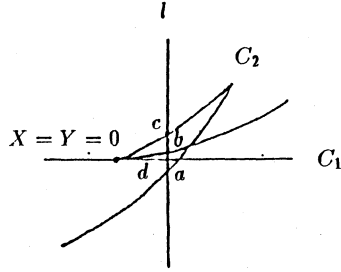
Figure 4.3   Figure 4.4

The origin $X = Y = 0$ is the image of the cusp point which is a quasi-regular log-canonical singularity locally given by $(x + y^2)(x^2 + y^5) = 0$ the double covering of which is the Hilbert modular cusp singularity $z^2 = (x + y^2)(x^2 + y^5)$ resolved by a rational curve with a node of self-intersection number -2 (see Figure 4.4). Sato [Sat] determined the OUDE for these orbifolds. First, the tangential GHCS $\phi_2 = l(dx)^2 + 2(dx)(dy) + m(dy)^2$ was computed by Kobayashi-Naruki [KNr]:

$$
l = \frac{2(8y^2 - 6xy^3 + x^2y^4 - x^4y + 4x^3)}{-24xy + 10x^2y^2 - 2x^3y^3 + x^5 + y^5}
$$
$$
m = \frac{2(8x^2 - 6x^3y + x^4y^2 - xy^4 + 4y^3)}{-24xy + 10x^2y^2 - 2x^3y^3 + x^5 + y^5}.
$$

Changing the variables $(x, y) \rightarrow (X, Y)$, where $X = \frac{B}{A^3}$ and $Y = \frac{C}{A^5}$, gives rise to the transformation of the equation (4.14) with coefficients $l(x, y)$, $m(x, y), \cdots, q(x, y)$ into the equation (4.14) with coefficients $\bar{l}(X, Y)$, $\bar{m}(X, Y), \cdots, \bar{q}(X, Y)$ defined on the terminal orbifold with coordinates X and Y. We have the tangential GHCS $\bar{\phi}_2 = \bar{m}(dX)^2 + 2(dX)(dY) + \bar{m}(dY)^2$ on the terminal orbifold [Sat]:

$$
\bar{l} = \frac{20(-4X^2 + 4Y - 3XY)}{36X^2 - 32X - Y}
$$
$$
\bar{m} = \frac{2(50X^2 - 54X^3 - 2Y + 3XY)}{5Y(36X^2 - 32X - Y)}.
$$

Sato [Sato] fixed the normalization factor for (4.14) to be

$$
e^{2\theta} = (1 - lm)^{-\frac{1}{2}} A^{-3},
$$

which is A_5-invariant in the same sense as above. Changing the variables $(x, y) \to (X, Y)$, we compute the normalization factor for $\overline{(4.14)}$ on the terminal orbifold:

$$e^{2\bar{\theta}} = \frac{(32X - 36X^2 + Y)Y^{\frac{1}{2}}}{(1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3)^{\frac{3}{2}}}.$$

Formulae (4.15) then gives the coefficients $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$:

$$\bar{a} = \frac{5(4X^2 - 9XY + 4Y)}{Y(36X^2 - 32X - Y)}$$

$$\bar{b} = \frac{-10(8X + 3Y)}{36X^2 - 32X - Y}$$

$$\bar{c} = \frac{54X^3 - 50X^2 + 3XY - 2Y}{10Y^2(36X^2 - 32X - Y)}$$

$$\bar{d} = \frac{-216X^2 + 200X + 9Y}{10Y(36X^2 - 32X - Y)}.$$

Estimating the poles of $\bar{p}$ and $\bar{q}$ and using the integrability condition $\overline{(\mathrm{IC1})}$ and $\overline{(\mathrm{IC2})}$ for $\overline{(4.14)}$, one may determine $\bar{p}$ and $\bar{q}$ [Sat]:

$$\bar{p} = \frac{40X + 9Y}{2Y(36X^2 - 32X - Y)}$$

$$\bar{q} = \frac{-540X^2 + 400X - 3Y}{400Y^2(36X^2 - 32X - Y)}.$$

The reflection group $PSL(2, o)\tau$ is isomorphic to the fundamental group of the terminal orbifold. We may compute this explicitly [KKN]. Let l be a line as in Figure 4.4 and α, β, γ loops in the complex feature of l surrounding C_2 counterclockwise at a, b, c and similarly δ a loop in l surrounding C_1 at d. Then the fundamental group of the terminal orbifold is isomorphic to the group generated by α, β, γ and δ with the following relations:

$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1,$$
$$\alpha\beta = \beta\alpha, \quad \alpha\gamma\alpha = \gamma\alpha\gamma, \quad \alpha\delta = \delta\alpha, \quad (\delta\beta\gamma)^2\delta = \delta(\delta\beta\gamma)^2,$$
$$(\delta\beta\gamma)^2\beta\gamma = \beta(\delta\beta\gamma)^2\beta, \quad (\delta\beta\gamma)^2\beta = \gamma(\delta\beta\gamma)^2,$$
$$(\alpha\beta\gamma\delta)^5 = 1.$$

As the matrix representations for the generators α, β, γ, δ of $PSL(2, o)\tau$,

we have for instance

$$\alpha = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}, \ \delta = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix},$$

where $*$ means the conjugation in $Q(\sqrt{5})$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} u & 0 \\ 0 & 1+u \end{pmatrix}, C = \begin{pmatrix} u & 1 \\ 0 & 1+u \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $u = \frac{\sqrt{5}-1}{2}$ and the action of a on $H \times H$ is given by $\alpha(z_1, z_2) = (Az_2, A^* z_1)$, etc.

4.4. Counter examples of uniformization problem

The general uniformization theorem in Section 3 characterizes ball quotients among certain normal surfaces in terms of numerical invariants (i.e., the logarithmic Chern numbers $\bar{c}_1^2$ and $\bar{c}_2$ of open orbifolds). This is the converse of Hirzebruch's proportionality theorem [Hir1] $\bar{c}_1^2 = 3\bar{c}_2$ for ball quotients with finite volume. The other Hirzebruch's proportionality asserts $\bar{c}_1^2 = 2\bar{c}_2$ for $H \times H$-quotients with finite volume. However, the converse of this is not true ([KNr, Examples 6.5 and 6.6]). We briefly describe counter examples. These are double planes for which the proportionality is fulfilled but not uniformized by $H \times H$.

Counter Example 1.   Let C be a non-degenerate conic in $P_2(C)$ and p_1 be a point outside it. We draw two lines l_1, l_2 through p_1 which are tangent to C. Let l_3' be the line connecting two contact points of l_1, l_2 and C. We then take a point p_1' on the line l_3' and from it draw two tangent lines l_1' and l_2' to C. Let l_3 be the line connecting two contact points of l_1, l_2 and C. It is then a simple theorem in projective geometry that p_1 is on the line l_3. Let D be the octic curve consisting of one conic C and six lines l_i and l_i' $(i = 1, 2, 3)$.
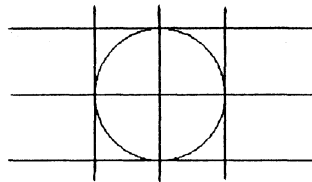
Configuration D

The configuration D is rigid, i.e., unique up to projective transformations. The double covering branching along D is a canonical surface

with 5 A_1, 2 D_4 and 4 D_6 simple singularities (see [BPV, p.87]). In particular, it admits a Kähler-Einstein orbifold metric with negative scalar curvature. We compute from Theorem 2 that

$$\bar{c}_1^2 = 2(K_{P_2(C)} + \frac{1}{2}D)^2 = 2,$$

$$\bar{c}_2 = 2(e(P_2(C)) - e(\tilde{D})) + e(\tilde{D}) - (\text{correction terms}) = 1,$$

where $\tilde{D}$ is a smooth octic curve and the correction term is $e(E_p) - \frac{1}{|\Gamma(p)|}$ for a simple singularity p where E_p is the exceptional set in the minimal good resolution and $\Gamma(p)$ is the local fundamental group of p. We thus have the proportionality $\bar{c}_1^2 = 2\bar{c}_2$. We show that there exists no tangential GHCS with discriminant divisor D. By the rigidity, we may suppose that D is given by $D = 0$ with

$$D = xy(x^2 - 1)(y^2 - 1)(x^2 + y^2 - 1)$$

The configuration D is invariant under the transformations $(x, y) \to (\pm x, \pm y)$ and $(x, y) \to (y, x)$. We seek a GHCS $(ds)^2 = P(dx)^2 + 2Q(dx)(dy) + R(dy)^2$ such that (i) $PR - Q^2 = \pm D$, (ii) P, Q, R are polynomials of degree 5, (iii) $(ds)^2$ is invariant as a GHCS under the above transformations (see Section 4.1). The invariance under $(x, y) \to (-x, y)$ implies that P and R are simultaneously even or odd in x. The invariance under the transposition then implies that P and R are divisible by $x^2 y^2$. Since P and Q are of degree 5, this contradicts to the invariance $P(x, y) = \pm R(y, x)$. This means that there exists no GHCS (tangential or not) with discriminant D. In particular, the double plane $P_2(C)(\sqrt{D})$ is not uniformized by $H \times H$.

Remark. Let τ be an involution $(x, y) \to (s, t) = (x^2, y)$ of the bidisk. Then the invariant holomorphic conformal structure $dxdy$ on the bidisk projects down to a GHCS $\frac{2}{\sqrt{s}}dsdt$ with discriminant $s = 0$. Not only tangential GHCS but also this possibility is excluded in the above counter example.

Counter Example 2. We consider the five conics

$$C_i : \quad z^2 = \frac{x^2}{a_i^2} + xy + a_j a_k y^2 = \frac{1}{a_i^2}(x - a_i a_j y)(x - a_i a_k y)$$

$$c_\pm : \quad z^2 = (xy + +py^2) \pm 2xy,$$

where (x, y, z) are the homogeneous coordinates of $P_2(C)$, a_1, a_2 and a_3 are the parameter of the family obeying $a_1 + a_2 + a_3 = 0$ and $p =$

$a_2 a_3 + a_3 a_1 + a_1 a_2$. The dependence on the parameters is projective and hence we get a family of 5 conics parametrized by $t \in P_1(C)$. Each member has generically 16 ordinary contact points. This family contains singular members (corresponding to $p = 0$) with more ordinary contact points. In approaching a singular member, 2 A_1 singularities become closer and collapse into 1 A_3 singularity in the limit. The existence of such conic arrangements was found by Naruki (see [Nar1]). Pictorially these arrangements look like

Naruki's arrangement of 5 conics and their degeneration

where $\circ$ represents a non-degenerate conic in $P_2(C)$, $\circ\!\!-\!\!-\!\!\circ$ means two conics intersect transversely at two points and contact at one point, and $\circ\!\!=\!\!\circ$ means two conics contact at two points. Let D_t be the arrangement corresponding to t and X_t the double plane branching over D_t. The generic double planes fulfill the proportionality

$$\bar{c}_1^2 = 2\bar{c}_2,$$

and we have

$$2\bar{c}_2 - \bar{c}_1^2 = -\frac{3}{2}$$

for singular members. We note that this family consists of canonical surfaces, i.e., surfaces with at worst simple singularities and with ample canonical divisor. In particular any member admits a unique Kähler-Einstein orbifold-metric with negative scalar curvature. Applying Tsuji's theory [T2] of convergence of Kähler-Einstein metrics under degeneration in the category of canonical surfaces (more generally, in the category of minimal algebraic varieties of general type (see also Sugiyama's survey in this volume [Su])), we infer that Kähler-Einstein metrics of the generic members converge to those defined on the singular members. It follows that the Kähler-Einstein metric for the generic member sufficiently close to the singular member has highly concentrated curvature in a small region. This implies that any general member sufficiently close to a singular member is not uniformized by $H \times H$.

The Euler number defect $\frac{3}{4}$ in the singular member is interpreted, by Bando-Kasue-Nakajima's theory, as the part of the curvature of the Kähler-Einstein metrics of generic members, which pops out in the limit as an ALE gravitational *orbifold-instanton* (cf. [BKN]).

§5. Appendix

In this Appendix, we present tables of log-canonical surface singularities mainly with branch loci. We follow the notations of Section 3.1.

(1)** log-terminal singularities with branch loci.

$<< m_1, m_2 >>$; $2 \le m_1, m_2,\quad 1 \le b_2 \le b_1 < \infty$

$$b_1 d_1 = m_1,\ b_2 d_2 = m_2$$
$$\frac{d_1}{e_1} = [a_{1,1} \cdots, a_{1,n_1}]$$
$$\frac{d_2}{e_2} = [a_{2,1} \cdots, a_{2,n_2}]$$

"$a_0 > \frac{e_1}{d_1} + \frac{e_2}{d_2}$"
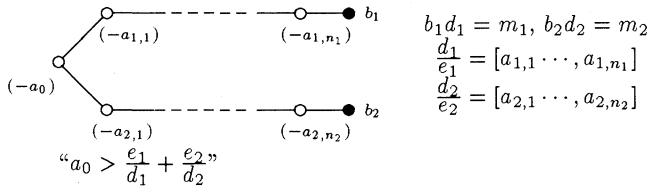
$<< 2, 2, m >>$; $2 \le m < \infty,\quad b_1 d_1 = b_2 d_2 = 2,\quad b_3 d_3 = m$
$$\frac{d_3}{e_3} = [a_1, \cdots, a_n]$$

"$a_0 > 1 + \frac{e_3}{d_3}$"

"$a_0 > \frac{1}{2} + \frac{e_3}{d_3}$"

"$a_0 > \frac{e_3}{d_3}$"

$<< 2, 3, 3 >>$

$"a_0 \geq 1"$ $"a_0 \geq 2"$ $"a_0 \geq 1"$ $"a_0 \geq 1"$
$a_0 = 1 \Leftrightarrow ((4))$ $a_0 = 1 \Leftrightarrow ((5))$

$"a_0 \geq 2"$ $"a_0 \geq 2"$ $"a_0 \geq 1"$ $"a_0 \geq 1"$
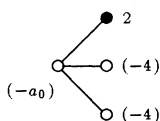 $a_0 = 1 \Leftrightarrow ((6))$

$"a_0 \geq 1"$
$a_0 = 1 \Leftrightarrow ((7))$

$<< 2, 3, 4 >>$

$"a_0 \geq 1"$ $"a_0 \geq 2"$ $"a_0 \geq 2"$ $"a_0 \geq 2"$
$a_0 = 1 \Leftrightarrow ((8))$

$"a_0 \geq 1"$ $"a_0 \geq 2"$ $"a_0 \geq 1"$ $"a_0 \geq 2"$

2 $(-a_0)$ (-2) (-2) (-4)
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((12))$

2 $(-a_0)$ (-2) (-2) (-2) (-2) (-2)
"$a_0 \geq 2$"

(-2) 3 4 $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((10))$

(-2) 3 2 $(-a_0)$ (-2)
"$a_0 \geq 2$"

2 (-3) 4 $(-a_0)$
"$a_0 \geq 1$"

2 (-3) 2 (-2) $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((13))$

2 (-2) (-2) 4 $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((9))$

2 (-2) (-2) 2 (-2) $(-a_0)$
"$a_0 \geq 2$"

2 3 (-4) $(-a_0)$
"$a_0 \geq 1$"

2 3 $(-a_0)$ (-2) (-2) (-2)
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((14))$

2 3 4 $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((11))$

2 3 2 (-2) $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((15))$

$<< 2, 3, 5 >>$

(-2) (-3) 5 $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((16))$

2 (-3) (-2) (-3) $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((22))$

2 (-2) (-2) 5 $(-a_0)$
"$a_0 \geq 2$"

(-2) 3 (-5) $(-a_0)$
"$a_0 \geq 1$"

(-2) 3 (-3) (-2) $(-a_0)$
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((20))$

(-2) 3 (-2) (-3) $(-a_0)$
"$a_0 \geq 2$"

(-2) 3 (-2) (-2) (-2) (-2) $(-a_0)$
"$a_0 \geq 2$"

2 (-3) (-5) $(-a_0)$
"$a_0 \geq 1$"

2 (-3) (-3) (-2) $(-a_0)$
"$a_0 \geq 1$"

2 (-3) (-2) (-2) (-2) (-2) $(-a_0)$
"$a_0 \geq 2$"

2 (-2) (-2) (-5) $(-a_0)$
"$a_0 \geq 1$"

2 (-2) (-2) (-3) (-2) $(-a_0)$
"$a_0 \geq 2$"

R. Kobayashi

2
$(-a_0)$ (-2) (-2)
(-2) (-3)
"$a_0 \geq 2$"

2
$(-a_0)$ (-2) (-2)
(-2) (-2) (-2) (-2)
"$a_0 \geq 2$"

(-2)
3
$(-a_0)$
5
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((18))$

2
$(-a_0)$ (-3)
5
"$a_0 \geq 1$"

2
$(-a_0)$ (-2) (-2)
5
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((17))$

2
$(-a_0)$ 3
(-5)
"$a_0 \geq 1$"

2
$(-a_0)$ 3
(-3) (-2)
"$a_0 \geq 1$"

2
$(-a_0)$ 3
(-2) (-3)
"$a_0 \geq 1$"

2
$(-a_0)$ 3
(-2) (-2) (-2) (-2)
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((21))$

2
$(-a_0)$ 3
5
"$a_0 \geq 1$"
$a_0 = 1 \Leftrightarrow ((19))$

(2)* cusp singularities uniformized by those in (2).
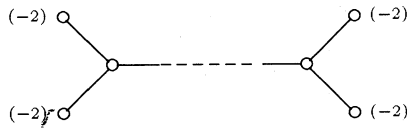
(-2)
(-2)
(-2)
(-2)

(2)** cusp singularities with branch loci.

(-2) 2
(-1) $(-a)$
2 2
$a \geq 3$

(-2) (-2)
(-1) $(-a)$
2 2
$a \geq 5$

(-2) (-2)
(-1) $(-a)$
2 2
$a \geq 3$

(-2) (-2)
(-1) $(-a)$
2 (-2)
$a \geq 4$

(-2) (-2)

$(-a_0)(-a_1)$ $(-a_{n-1})$ $(-a_n-2)$ (-1)

2   2

$$a_0 \geq 2,\ a_n \geq 1$$

(-2) (-2)

$(-a_0)(-a_1)$ $(-a_{n-1})$ $(-a_n-2)$ (-1)

(-2) 2

"$a_0 \geq 3,\ a_n \geq 1$" or "$a_0 \geq 2,\ a_n \geq 2$" or
"$n \geq 2,\ 1 \leq \exists j \leq n-1;\ a_j \geq 3,\ a_0 \geq 2,\ a_n \geq 1$"

2   (-2)

$(-a_0)(-a_1)$ $(-a_{n-1})$ $(-a_n-2)$ (-1)

2   2

"$a_0 \geq 2,\ a_n \geq 1$" or "$a_0 \geq 1,\ a_n \geq 2$"
or $n \geq 2,\ 1 \leq \exists j \leq n;\ a_j \geq 3,\ a_0 \geq 1,\ a_n \geq 1$"

(-2) (-2)

$(-1)(-a_0-2)$ $(-a_{n-1})$ $(-a_n-2)$ (-1)

2   2

"$a_0 \geq 2,\ a_n \geq 1$" or "$n \geq 2,\ 1 \leq \exists j \leq n-1;\ a_j \geq 3,\ a_0 \geq 1,\ a_n \geq 1$"

2   2

$(-a_0)$ $(-a_n)$

2   2

"$a_0 \geq 1,\ a_n \geq 2$" or "$n \geq 2$,
$1 \leq \exists j \leq n-1;\ a_j \geq 3,\ a_0 \geq 1,\ a_n \geq 1$"

(-2) 2

$(-a_0)$ $(-a_n)$

(-2) 2

"$a_0 \geq 2,\ a_n \geq 2$" or "$a_0 \geq 3,\ a_n \geq 1$" or
"$n \geq 2,\ 1 \leq \exists j \leq n-1$;
$a_j \geq 3,\ a_0 \geq 2,\ a_n \geq 1$"

(-2) (-2)

$(-a_0)$ $(-a_n)$

(-2) 2

"$a_0 \geq 2,\ a_n \geq 2$"

(-2) (-2) "$a_0 \geq 2,\ a_n \geq 2$"

$(-a_0)$ $(-a_n)$ 2

(-2) "$a_0 \geq 1,\ a_n \geq 2$"

$(-a_0)$ $(-a_n)$
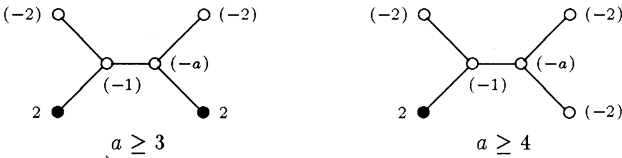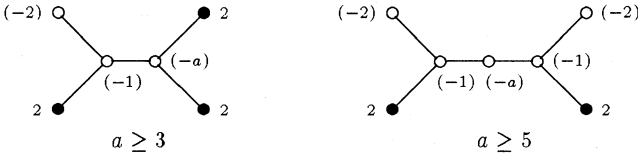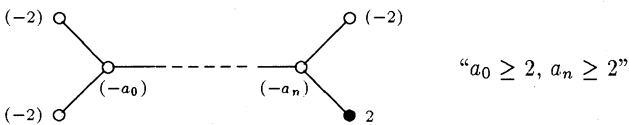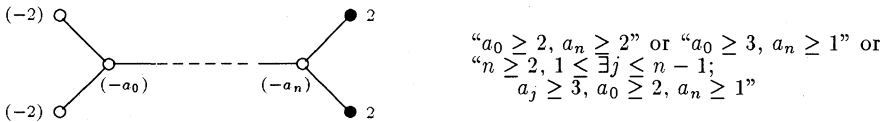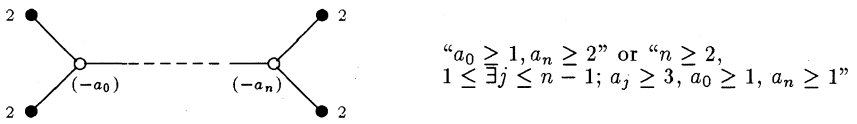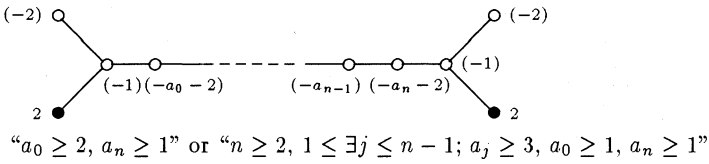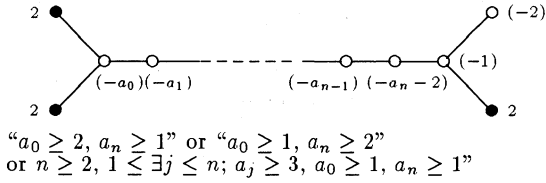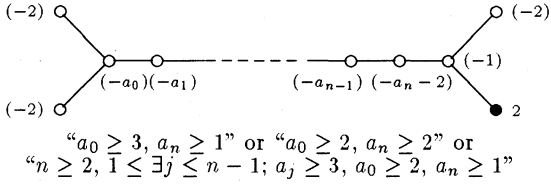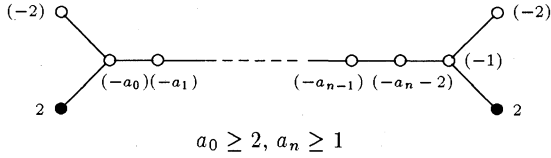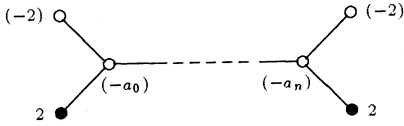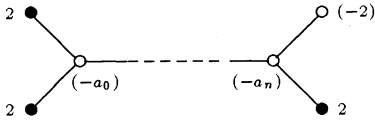
(3)* ball cusp singularities uniformized by simple elliptic singularities (2).
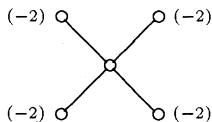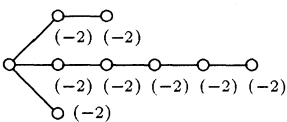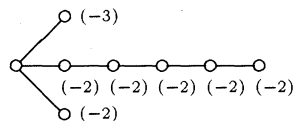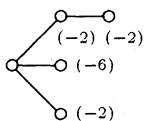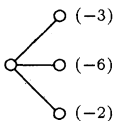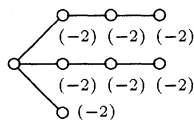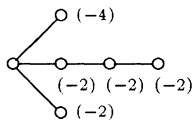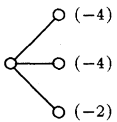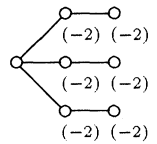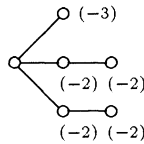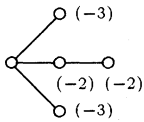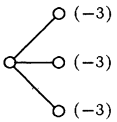
(3)** ball cusp singularities with branch loci.

$<< 3, 3, 3 >>$
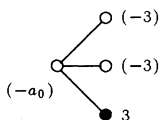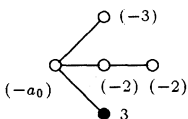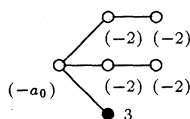
"$a_0 \geq 1$"   "$a_0 \geq 2$"   "$a_0 \geq 2$"
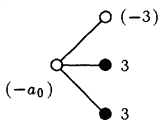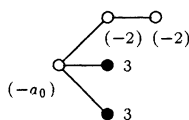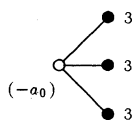
"$a_0 \geq 1$"   "$a_0 \geq 1$"   "$a_0 \geq 1$"

$<< 2, 4, 4 >>$
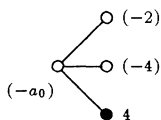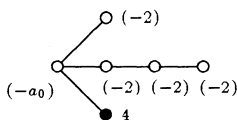
"$a_0 \geq 1$"   "$a_0 \geq 2$"   "$a_0 \geq 2$"
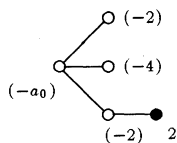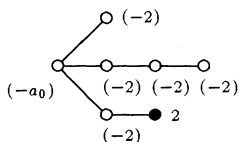
"$a_0 \geq 2$"   "$a_0 \geq 1$"   "$a_0 \geq 2$"

"$a_0 \geq 2$"   "$a_0 \geq 1$"   "$a_0 \geq 2$"

"$a_0 \geq 2$"   "$a_0 \geq 1$"   "$a_0 \geq 1$"

2

$(-a_0)$ (-4)

(-2) 2

"$a_0 \geq 1$"

2

$(-a_0)$ (-2) (-2) (-2)

(-2) 2

"$a_0 \geq 2$"

2

$(-a_0)$ 4

4

"$a_0 \geq 1$"

2

$(-a_0)$ 2

(-2)

4

"$a_0 \geq 1$"

2

$(-a_0)$ 2

(-2) 2

(-2)

"$a_0 \geq 2$"

$<< 2, 3, 6 >>$

(-2)

$(-a_0)$ (-3)

6

"$a_0 \geq 1$"

(-2)

$(-a_0)$ (-3)

(-2) 3

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-3)

(-3) 2

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-3)

(-2) (-2) 2

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-2) (-2)

6

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-2) (-2)

(-2) 3

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-2) (-2)

(-2) 2

"$a_0 \geq 2$"

(-2)

$(-a_0)$ (-2) (-2)

(-2) 2

"$a_0 \geq 2$"

(-2)

$(-a_0)$ 3

(-6)

"$a_0 \geq 1$"

(-2)

$(-a_0)$ 3

(-2) (-2) (-2) (-2) (-2)

"$a_0 \geq 2$"

2

$(-a_0)$ (-3)

(-6)

"$a_0 \geq 1$"

2

$(-a_0)$ (-3)

(-2) (-2) (-2) (-2) (-2)

"$a_0 \geq 2$"

2

$(-a_0)$ (−2) (−2)
(−6)

"$a_0 \geq 1$"

2

$(-a_0)$ (−2) (−2)
(−2) (−2) (−2) (−2) (−2)

"$a_0 \geq 2$"

(−2)

$(-a_0)$ 3
6

"$a_0 \geq 1$"

(−2)

$(-a_0)$ 3
3
(−2)

"$a_0 \geq 2$"

(−2)

$(-a_0)$ 3
2
(−3)

"$a_0 \geq 1$"

(−2)

$(-a_0)$ 3
2
(−2) (−2)

"$a_0 \geq 2$"

2

$(-a_0)$ (−3)
6

"$a_0 \geq 1$"

2

$(-a_0)$ (−3)
3
(−2)

"$a_0 \geq 1$"

2

$(-a_0)$ (−3)
2
(−3)

"$a_0 \geq 1$"

2

$(-a_0)$ (−3)
2
(−2) (−2)

"$a_0 \geq 2$"

2

$(-a_0)$ (−2) (−2)
6

"$a_0 \geq 1$"

2

$(-a_0)$ (−2) (−2)
3
(−2)

"$a_0 \geq 2$"

2

$(-a_0)$ (−2) (−2)
2
(−3)

"$a_0 \geq 2$"

2

$(-a_0)$ (−2) (−2)
3
(−2) (−2)

"$a_0 \geq 2$"

2

$(-a_0)$ 3
(−6)

"$a_0 \geq 1$"

2

$(-a_0)$ 3
(−2) (−2) (−2) (−2) (−2)

"$a_0 \geq 1$"

2

$(-a_0)$ 3
6

"$a_0 \geq 1$"

2

$(-a_0)$ 3
3
(−2)

"$a_0 \geq 1$"

2

$(-a_0)$ 3
2
(−3)

"$a_0 \geq 1$"

2

$(-a_0)$ 3
2
(−2) (−2)

"$a_0 \geq 1$"

$<< 2, 2, 2, 2 >>$

2   2

2   $(-a_0)$ 2

"$a_0 \geq 1$"

2   2

2   $(-a_0)$ (-2)

"$a_0 \geq 1$"

2   2

(-2) $(-a_0)$ (-2)

"$a_0 \geq 2$"

2   (-2)

(-2) $(-a_0)$ (-2)

"$a_0 \geq 2$"

(4)* LCS singularities with only branch loci with indices ∞.

$(-a_{1,1})$ $(-a_{1,n_1})$ ∞

$(-a_0)$

$(-a_{2,1})$ $(-a_{2,n_2})$ ∞

"$a_0 > \dfrac{e_1}{d_1} + \dfrac{e_2}{d_2}$" $\dfrac{d_1}{e_1} = [a_{1,1}, \cdots, a_{1,n_1}]$

$\dfrac{d_2}{e_2} = [a_{2,1}, \cdots, a_{2,n_2}]$

(-2)

(-2)

$(-a_0)$
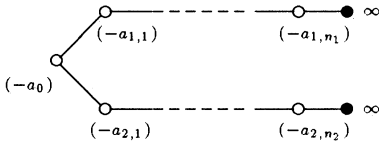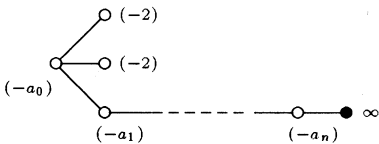
$(-a_1)$ $(-a_n)$ ∞

"$a_0 > 1 + \dfrac{e_3}{d_3}$" $\dfrac{d_3}{e_3} = [a_1, \cdots, a_n]$

(4)** LCS singularities with mixed branch loci.

$(-a_{1,1})$ $(-a_{1,n_1})$ ∞

$(-a_0)$

$(-a_{2,1})$ $(-a_{2,n_2})$ $\infty < b$

"$a_0 > \dfrac{e_1}{d_1} + \dfrac{e_2}{d_2}$" $\dfrac{d_1}{e_1} = [a_{1,1}, \cdots, a_{1,n_1}]$

$\dfrac{d_2}{e_2} = [a_{2,1}, \cdots, a_{2,n_2}]$

(−2)

• 2

(−a_0)

(−a_1) (−a_n) ∞

"$a_0 > \frac{1}{2} + \frac{e_3}{d_3}$" $\quad \frac{d_3}{e_3} = [a_1, \cdots, a_n]$

2

2

(−a_0)
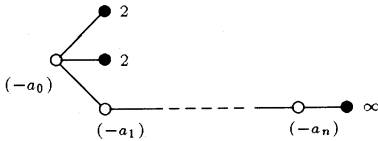
(−a_1) (−a_n) ∞

"$a_0 > \frac{e_3}{d_3}$" $\quad \frac{d_3}{e_3} = [a_1, \cdots, a_n]$

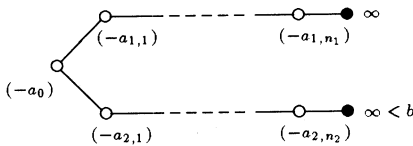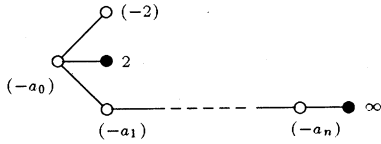Quasi-regular log-canonical singularities are those ones such that their dual graphs contain a (−1)-curve and the successive blowing down of (−1)-curves yields a regular point with branch loci. For the uniformization of quasi-regular log-terminal singularities, we refer to Yoshida [Yo]. For the uniformization of quasi-regular cusp singularities, we refer to Karras [Kar]. Taking the double covering of a quasi-regular cusp singularity gives a cusp singularity in the ordinary sense. For the uniformization of quasi-regular ball cusp singularities, we refer to Yoshida-Hattori's table [YH] (we observed in (3)** an example of the uniformizations of non quasi-regular ball cusp singularities). The point here is that all of these singularities are unified in the notion of log-canonical singularities and all of them have canonical Kähler-Einstein and holomorphic G-structures.

References

[Au1]   T. Aubin, Equation du type Monge-Ampère sur les variété kählér-iennes compactes, C. R. Acad. Sc. Paris, **283** (1976), 119–121.

[Au2]   ———, "Nonlinear analysis on manifolds, Monge-Ampère equations", Springer-Verlag, New York, 1982.

[BKN]   S. Bando, A. Kasue and H. Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume decay, preprint. (Inventionas Math. to appear.)

[BPV]   W. Barth, C. Peters and A. Van de Ven, "Compact Complex Surfaces", Springer-Verlag, New York/Heidelberg/Berlin/Tokyo, 1984.

[BHH]   G. Barthel, F. Hirzebruch and Th. Höfer, "Geradenkonfigurationen und Algebraische Flächen", Aspekte der Mathematik, Vieweg,

1987.

[Bes]   A. Besse, "Einstein manifolds", Springer-Verlag, New York, 1987.

[Bom]   E. Bombieri, The pluricanonical map of a complex surface, in "Several Complex Variables I, Maryland 1970", Lect. Notes Math., vol. 155.

[Br]   E. Brieskorn, Rationale Singularitäten Komplexer Flächen, Inventiones Math., 4 (1968), 336–358.

[C]   S. S. Chern, On holomorphic mappings of Hermitian manifolds, in "Proc. Symp. Pure Math. 11", Amer. Math. Soc. Providence, R.I., 1968, pp. 157–170.

[CO]   B. Y. Chen and K. Ogiue, Some characterization of complex space form in terms of Chern classes, Quart. J. Math., 26 (1975), 459–464.

[CY1]   S. Y. Cheng and S. T. Yau, On the existence of a complete Kähler-Einstein metric on noncompact complex manifolds and regularity of Fefferman's equation, Comm. Pure Appl. Math., 32 (1980), 507–544.

[CY2]   _____, Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of $SU(2, 1)$, in "Complex Differential Geometry and Nonlinear Differential Equations, Proceedings of a Summer Research Conference, 1984", Contemporary Mathematics 49, ed. Y. T. Siu, Amer. Math. Soc., 1986, pp. 31–43.

[Fo]   R. Fox, On Fenchel's conjecture about F-groups, Mat. Tidskrift (1952), 61–65.

[Fu]   T. Fujita, Semipositive line bundles, J. Fac. Sci. Univ. Tokyo Ser. I A, 30 (1983), 353–378.

[GT]   D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order, 2nd ed.", Springer-Verlag, New York, 1983.

[GH]   P. Griffiths and J. Harris, "Principles of Algebraic Geometry", John Willy and Sons, New York, 1978.

[Gr]   H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., 146 (1962), 331–368.

[Gu]   R. Gunning, "On uniformization of complex manifolds", Math. Notes number 22, Princeton Univ. Press, Princeton, 1978.

[Hir1]   F. Hirzebruch, Automorphic Formen und der Satz von Riemann-Roch, Symp. Int. Top. Alg. Univ. de Mexico (1956), 129–144.

[Hir2]   _____, Hilbert modular surfaces, Enseign. Math., 71 (1973).

[Hir3]   _____, The ring of Hilbert modular forms for real quadratic fields of small discriminant, Lect. Notes Math., 627 (1977), 287–323.

[Hir4]   _____, Hilbert modular group of of the field $Q(\sqrt{5})$ and the cubic diagonal surface of Clebsch and Klein, Russ. Math. Surv., 3 (1976), 96–110.

[Hir5]   _____, Singularities of algebraic surfaces and characteristic numbers, Contemp. Math., **58** (1986), 141–155.

[Hit]   N. Hitchin, Twistor construction of Einstein metrics, in "Global Riemannian Geometry", N. Hitchin and T. Willmore (eds.), Ellis Horwood, Chichester, 1984, pp. 115–125.

[Ho1]   R. Holzapfel, Arithmetic curves on ball quotient surfaces, Ann. Glob. Analysis and Geometry, **1-2** (1983), 21–90.

[Ho2]   _____, An effective finiteness theorem for ball lattices, preprint 1990.

[I]   K. Ivinskis, Normale Flächen und die Miyaoka-Kobayashi Ungleichung, Diplomarbeit Bonn (1985).

[IKO]   M. Inoue, S. Kobayashi and T. Ochiai, Holomorphic affine connections on compact complex surfaces, J. Fac. Sci. Univ. Tokyo, **27** (1980), 247–264.

[Kar]   U. Karras, Eigenschaften der lokalen ringe in zweidimensionalen Spitzen, Math. Ann., **215** (1975), 117–129.

[Kat]   M. Kato, On the existence of finite principal uniformization of CP^2 along weighted line configurations, Mem. Fac. Sci. Kyushu Univ. Ser. A, **38** (1984), 127–131.

[Kaw1]   Y. Kawamata, On the classification of non-complete algebraic surfaces, Lect. Notes Math., **732** (1979), 215–232.

[Kaw2]   _____, The cone of curves of algebraic varieties, Ann. Math., **119** (1984), 603–633.

[Kaw3]   _____, Crepant blowing-up of 3-dimensional canonical, Ann. Math., **127** (1988), 93–163.

[Kl]   F. Klein, Weitere Untersuchungen über das Ikosaeder, in "Gesammelte Math. Abhandlungen Bd. 2", Springer-Verlag, Berlin, pp. 321–384.

[Ko1]   R. Kobayashi, A remark on the Ricci curvature of algebraic surfaces of general type, Tohoku Math. J., **36** (1984), 399–418.

[Ko2]   _____, Einstein-Kähler metrics on open algebraic surfaces of general type, Tohoku Math. J., **37** (1985), 43–77.

[Ko3]   _____, Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities, Math. Ann., **272** (1985), 385–398.

[Ko4]   _____, Moduli of Einstein metrics on a K3 surface and degeneration of type I, in this volume.

[Ko5]   _____, Ricci-flat Kähler metrics on affine algebraic manifolds, in this volume.

[Ko6]   _____, An application of Kähler-Einstein metrics to singularities of plane curves, in this volume.

[Kob1]   S. Kobayashi, The first Chern class and holomorphic symmetric tensor fields, Math. Soc. Japan, **32** (1980), 325–329.

[Kob2]   _____, "Differential Geometry of Complex Vector Bundles", Iwanami Shoten Publishers and Princeton University Press, 1987.

[KNr]   R. Kobayashi and I. Naruki, Holomorphic conformal structures and uniformization of complex surfaces, Math. Ann., **279** (1988), 485–500.

[K-N-S] R. Kobayashi, S. Nakamura and F. Sakai, A numerical characterization of ball quotients for normal surfaces with branch loci, Proc. Japan Academy, **65** (1989), 238–241.

[KKN]   R. Kobayashi, K. Kushibiki and I. Naruki, Polygons and Hilbert modular groups, Tohoku Math. J., **41** (1989), 633–646.

[KT]   R. Kobayashi and A. Todorov, Polarized period map for generalized K3 surfaces and the moduli of Einstein metrics, Tohoku Math. J., **39** (1987), 341–363.

[KNm]   S. Kobayashi and K. Nomizu, "Foundations of Differential geometry I,II", John Willy and Sons, New York, 1969.

[KO1]   S. Kobayashi and T. Ochiai, Holomorphic projective structures on complex surfaces, Math. Ann., **249** (1980), 75–94; II, Math. Ann., **255** (1981), 519–522.

[KO2]   _____, Holomorphic structures modeled after hyperquadrics, Tohoku Math. J., **34** (1982), 587–629.

[KO3]   _____, Holomorphic structures modeled after compact Hermitian symmetric spaces, in "Manifolds and Geometry, Papers in Honor of Yozǒ Matsushima", Progress in Mathematics, vol 14, J. Hano et al. (eds.), Birkhäuser, Boston, 1981.

[Kod]   K. Kodaira, Pluricanonical system on algebraic surfaces of general type, J. Math. Soc. Japan, **20** (1968), 170–192.

[M-S]   T. Matsuoka and F. Sakai, The degree of rational cuspidal curves, Math. Ann., **285** (1989), 233–247.

[Mil]   J. Milnor, "Morse theory", Ann. Math. Studies 76, Princeton Univ. Press, Princeton, 1974.

[Miy1]   Y. Miyaoka, On the Chern number of of surfaces of general type, Inventiones Math., **42** (1977), 225–237.

[Miy2]   _____, The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann., **268** (1984), 159–171.

[MT]   B. Moišezon and M. Teicher, Existence of simply connected algebraic surfaces of general type with positive and zero indices, Proc. Nat. Acad. Sci. USA, **83** (1986), 6665–6666.

[Mo]   N. Mok, Uniqueness theorems of Hermitian metrics of seminegative curvature on locally symmetric spaces of negative Ricci curvature, Ann. Math., **125** (1987), 105–152; The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, J. Differential Geometry, **27** (1988), 179–214.

[MY]   N. Mok and S. T. Yau, completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions, in "The Mathematical Her-

itage of Henri Poincaré", Proc. Symp. Pure Math. vol 39, Amer. Math. Soc., 1983, pp. 41–59.

[MS]   G. Mostow and Y. T. Siu, A compact Kähler surface of negative curvature not covered by the ball, Ann. Math., **112** (1980), 321–360.

[Mu1]   D. Mumford, The canonical ring of an algebraic surface, Ann. Math., **76** (1962), 612–615.

[Mu2]   _____, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Inst. Hautes Études Sci. Publ. Math., **9** (1961), 5–22.

[Nak]   S. Nakamura, Classification and uniformization of log-canonical surface singularities in the presence of branch loci (in Japanese), Master's thesis, Saitama Univ. 1989.

[Nam]   M. Namba, Lectures on branched coverings and algebraic functions, preprint, 1984; On branched coverings of projective manifolds, Proc. Jap. Acad., **61** (1985), 121–124.

[Nar1] I. Naruki, Some invariants for conics and their applications, Publ. RIMS Kyoto Univ., **19** (1983), 1139–1151.

[Nar2] _____, K3 surfaces related to root systems in E_8, in "Prospects of Algebraic Analysis", Acad. Press, 1988.

[S]   K. Saito, Einfach-elliptische Singularitäten, Inventiones Math., **23** (1974), 289–325.

[Sak1] F. Sakai, Classification of normal surfaces, in "Algebraic Geometry Bowdoin 1985", Proc. Symp. Pure Math. vol 46, Part 1, Amer. Math. Soc., 1987, pp. 451–465.

[Sak2] _____, Weil divisors on normal surfaces, Duke Math. J., **51** (1984), 877–887.

[Sak3] _____, Singularities of plane curves, preprint 1990.

[Sas]   T. Sasaki, On the projective geometry of hypersurfaces, in "équations différentielles dans les champs complexe (colloque franco-japonais 1985)", ed. R. Gerard and K. Okamoto, Herman, 1987.

[SaY1] T. Sasaki and M. Yoshida, Linear differential equation in two variables of rank four I-II, Math. Ann., **282** (1988), 69–111.

[SaY2] _____, Linear differential equations modeled after hyperquadrics, Tohoku Math. J., to appear.

[Sat]   I. Satake, The Gauss-Bonnet theorem for V-manifolds, J. Math. Soc. Japan, **9** (1957), 464–492.

[Sato] T. Sato, Uniformizing equations of Hilbert modular orbifolds for the real quadratic fields $Q(\sqrt{3})$ and $Q(\sqrt{5})$, preprint, 1989.

[ST]   G. Shepherd and J. Todd, Finite unitary reflection groups, Canad. j. Math., **6** (1954), 274–304.

[Sim]   C. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc., **1-4** (1988), 867–918.

[Si-Y] Y. T. Siu and P. Yang, Compact Kähler-Einstein surfaces of non-negative bisectional curvature, Inventiones Math., **64** (1981),

471–487.

[Su]   K. Sugiyama, in this volume.

[T1]   H. Tsuji, A characterization of ball quotients with smooth bound-
        ary, Duke Math. J., **40** (1988), 537–543.

[T2]   ———, Existence and degeneration of Kähler-Einstein metrics on
        minimal algebraic varieties of general type, Math. Ann., **281**
        (1988), 123–133.

[T-M]   S. Tsunoda and M. Miyanishi, The structure of open algebraic sur-
        faces II, in "Progress in Math. vol 39", Birkhäuser, Bston/Basel/
        Stuttgart, 1983, pp. 499–544.

[TY]   G. Tian and S. T. Yau, Existence of Kähler-Einstein metrics on
        complete Kähler manifolds and their applications to algebraic
        geometry, in "Mathematical Aspects in String Theory", ed. S.
        T. Yau, Adv. Ser. Math. Phys. vol 1, World Scientific, Singa-
        pore/New Jersey/Hong Kong, 1987, pp. 574–628.

[Wa]   K. Watanabe, On plurigenera of normal isolated singularities,
        Math. Ann., **253** (1980), 241–262.

[Wo]   B. Wong, Uniformization of compact Kähler surfaces of negative
        curvature, J. Differential Geometry, **16** (1981), 407–420.

[Yan]   P. Yang, On Kähler manifolds with negative bisectional curvature,
        Duke Math. J., **43** (1976), 871–874.

[Yau1] S. T. Yau, Calabi's conjecture and some new results in algebraic
        geometry, Nat. Acad. Sci. USA, **74** (1977), 1798–1790; On the
        Ricci curvature of a compact Kähler manifolds and the com-
        plex Monge-Ampère equations, I, Comm. pure Appl. Math., **31**
        (1978), 339–411.

[Yau2] ———, A general Schwarz lemma for Kähler manifolds, Amer. J.
        Math., **100** (1978), 197–203.

[Yo]   M. Yoshida, "Fuchsian Differential Equations with Special Empha-
        sis on the Gauss-Schwarz Theory", Vieweg, Braunschweig, 1987.

[YH]   M. Yoshida and S. Hattori, Local theory of Fuchsian systems with
        certain discrete monodromy groups III, Funkcial. Ekvac., **19-2**
        (1976), 175–189.

[Yr]   H. Yoshihara, Plane curves whose singular points are cusps and
        triple coverings of P^2, Manuscripta Math. (1989).

[Z]   O. Zariski, The theorem of Riemann-Roch for high multiples of an
        effective divisor on an algebraic surface, Ann. Math., **76** (1962),
        560–615.

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Komaba, Tokyo 153
Japan