Moduli of Einstein Metrics on a K3 Surface
and Degeneration of Type I

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Abstract.

   This is an expository paper on the moduli space of K3 surfaces with Kähler-Einstein metric. We construct the universal family of Kähler-Einstein metrics on a K3 surface and show that it necessarily includes singular metrics such that some embedded 2-spheres have zero volume. We analyze this degeneration of Kähler-Einstein metrics in detail.

§0. Preliminaries

   The purpose of this expository article is to describe the geometrical picture of the universal family of Kähler-Einstein K3 surfaces with a special emphasis on the mild degeneration of Ricci-flat Kähler metrics. Many branches of mathematics such as complex analysis, algebraic geometry, lattice theory and differential geometry meet in the study of K3 surfaces. This is especially evident when we treat periods or Kähler-Einstein metrics on K3 surfaces. We hope that our article is an introduction to this beautiful unification of algebraic geometry and differential geometry via non-linear PDE. We begin with the definition of a K3 surface.

0.1. Definition of a K3 surface

   The definition of a K3 surface is a generalization of that of an elliptic curve. All algebraic curves are classified into three classes according to the sign of the Chern class of the canonical bundle (defined to be the sign of the minus of the Euler number). Note that the Euler number is the difference of the number of zeros and poles of a non-zero meromorphic 1 form. We list here the following four equivalent definitions for an elliptic curve: (1) an elliptic curve is an algebraic curve C with trivial canonical bundle, i.e., there is a holomorphic 1 form without zeros; (2) an elliptic

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curve is a smooth cubic curve in $P_2(C)$; (3) an elliptic curve is a double cover of $P_1(C)$ branched over distinct 4 points (a flat torus is a branched double covering of a regular tetrahedron with the metric induced from R^3); (4) an elliptic curve is $C/Z+Z\omega$ where ω is a complex number in the upper half plane. A direct generalization of the definition (4) to complex dimension 2 gives a complex 2-torus $Y = C^2/\Gamma$, where Γ is a lattice in C^2 generated by four vectors linearly independent over R. However, this generalization is too straightforward, so that we further combine this with the definition (3), and then we encounter a new phenomenon. Namely, while the involution $\sigma : x \mapsto -x$ induces a branched double covering $C \to P_1(C)$, a similar involution produces a compact complex surface $Y/ < \sigma >$ with 16 conical singularities corresponding to 16 isolated fixpoints. It is clear that all of these singularities are of type C^2/σ for σ as above. To resolve this singularity, we blow up the origin to consider C^2 as a cone over $P_1(C)$, or to consider the tautological line bundle $L \to P_1(C)$. Since the involution σ induces the reflection $\sigma' : x \mapsto -x$ on L with respect to the 0-section, the quotient map $C^2 \to C^2/\sigma$ is lifted to the double covering $L \to L^2$, $x \mapsto x^2$. This implies that the singularity is resolved by the 0-section ($\cong P_1(C)$) of L^2 which is taken as the cotangent bundle $T^*P_1(C)$. This 0-section has self-intersection number -2 in $T^*P_1(C)$. Note also that there exists a canonical non-vanishing holomorphic 2-form (the holomorphic symplectic structure) on $T^*P_1(C)$. We call this exceptional curve a (-2)-curve, i.e., a (-2)-curve is a smooth rational curve with self-intersection number -2. Let X be the smooth surface obtained by resolving 16 singularities in the above way (this is the *minimal resolution*). Then X is a compact complex smooth surface with the following two properties:

(a) X is simply connected,

(b) the canonical bundle K_X is trivial, i.e., there exists a non-vanishing holomorphic 2-form on X.

The property (a) comes from the following fact due to Armstrong [Ar]: *Let M be a connected and simply connected Hausdorff topological space and Γ a group of homeomorphisms of M acting properly discontinuously which is generated by a finite number of elements all of which have fixpoints in M. Then the quotient space M/Γ with the quotient topology is also simply connected.*

A direct computation shows that a non-vanishing holomorphic 2-form on the 2-torus projects down to a non-vanishing holomorphic 2-form on X. We thus get the property (b). We call X the *Kummer surface* associated to Y and write $X = \mathrm{Km}(Y)$.

Generalizing the definition (3), we take the double covering of $P_2(C)$

branched over a sextic curve D. It turns out that if D has at worst simple singularities (see [BPV] for the definition), the minimal resolution X of the resulting surface $X' = P_2(C)(\sqrt{D})$ has the properties (a) and (b). There exists a double meromorphic 2-form, i.e., a meromorphic section of twice the canonical bundle, on $P_2(C)$ with a pole of order one which lifts to X and becomes a double holomorphic two form without zeros. Since X is simply connected, one can take a well-defined square root to get a non-vanishing holomorphic 2-form on X. Double coverings of $P_1(C) \times P_1(C)$ branched over a curve of bidegree (4,4) with at worst simple singularities also have these properties. Smooth quartic surfaces in $P_2(C)$ generalizes the definition (2). These fulfills the properties (a) and (b). Indeed, the Lefschetz hyperplane section theorem (for a proof, see [Mil]) implies that these are simply connected and since the anticanonical bundle of $P_2(C)$ is defined by a quartic surface, the adjunction formula (see, for example [GH]) implies that smooth quartic surfaces have trivial canonical bundle. The minimal resolutions of quartic surfaces with at worst simple singularities (rational double points) also have the same properties (See [Br1]).

Definition. A *K3 surface* is a connected and simply connected non-singular compact complex surface X which has trivial canonical bundle K_X.

The above mentioned surfaces with properties (a) and (b) are examples of K3 surfaces. Kodaira proved that any two K3 surfaces are deformations of each other (see [Kod] and [BPV]). In particular, every K3 surface is diffeomorphic to a smooth quartic surface in $P_3(C)$. So, we mean by a *K3 manifold* the underlying differentiable manifold of a K3 surface. It follows from Kodaira's classification of compact complex minimal surfaces that a K3 surface is a compact complex surface diffeomorphic to a non-singular quartic surface in $P_3(C)$, which is Weil's original definition of a K3 surface. From the definition, the first Chern class $c_1(X)$ vanishes in $H^2(X; Z)$. This implies that the second Stiefel-Whitney class $w_2(X) \in H^2(X; Z_2)$ vanishes, i.e., X is a spin manifold. We recall that a compact oriented 4-manifold is a spin manifold if and only if the integral second cohomology group with the cup bilinear form is an even lattice. In particular, X is a minimal surface, i.e., X contains no exceptional curve of the first kind. The definition and Noether's formula $1 - h^{0,1}(X) + h^{0,2}(X) = (c_1^2(X) + c_2(X))/12$ (for a short proof, see, for instance, [Ro]) directly imply $e = 24$, where e stand for the Euler number. Combining this with $b_1(X) = 0$, we have $b_2 = 22$. Then, since $h^{0,2}(X) = h^{2,0}(X) = 1$, Hodge's index theorem $b^+ = 2h^{0,2} + 1$ for

compact smooth surfaces with even b_1 (see [BPV]) implies $b^+ = 3$ and $\sigma = -16$, where b^+ is the number of positive eigenvalues of the cup bilinear form of $H^2(X; R)$ and σ is the Hirzebruch signature. Recall that any even unimodular indefinite lattice is up to isometry determined by its rank and signature ([BPV]). So the definition of a K3 surface determines a unique lattice which we call the *K3 lattice*:

$$(\oplus^2 - E_8) \oplus (\oplus^3 H),$$

where E_8 is the Cartan matrix of the Dynkin diagram of the exceptional simple Lie algebra E_8 and H is the hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One computes further numerical invariants using the special features of the Hodge theory on complex surfaces (namely, the Hodge decomposition holds on the second cohomology without Kähler condition and on the first cohomology if b_1 is even) and Serre's duality, to get $h^{0,1}(X) = h^{1,0}(X) = h^{2,1}(X) = h^{1,2}(X) = 0$, $h^{1,1} = 20$. See [Chapt. IV, BPV] for the Hodge theory on compact surfaces. Note that we have used simply-connectedness in the definition only in a modest form of $b_1 = 0$ in the above computations.

0.2.   K3 surfaces in differential geometry

Now we explain as what kind of Riemannian manifolds K3 surfaces appeared in differential geometry. We start with the following result of Hitchin [Hi1] and Thorpe [Th]:

Theorem 1 ([Hi1], [Th]).   *Let (M, g) be an Einstein manifold of dimension 4. Then we have the following inequality:*

$$2e(g) \geq -P_1(g),$$

where $e(g)$ (resp. $P_1(g)$) denotes the Euler form (resp. the first Pontrjagin form or three times the signature form) with respect to the Einstein metric g. The equality occurs if and only if the Einstein metric g is Ricci-flat and anti-self-dual (half conformally flat) in the sense that the curvature tensor is anti-self-dual as an $\mathrm{End}(TM)$-valued 2-form.

Proof.   We review some standard facts of 4-dimensional Riemannian geometry ([AHS], [Bes1]) and apply these facts to prove Theorem 1. Let (N, h) be a 4-dimensional Riemannian manifold with a metric h and $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ the decomposition of 2-forms into self-dual and anti-self-dual parts. The Riemann curvature tensor defines a self-adjoint transformation $R : \Lambda^2 \to \Lambda^2$ expressed as $R(e_i \wedge e_j) = (\frac{1}{2}) \sum_{i,j,k,l} R_{ijkl} e_i \wedge e_j$,

where $\{e_i\}$ is a local orthonormal basis of 1-forms. If we write

$$R = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

relative to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, the decomposition of the curvature tensor into irreducible pieces under the $SO(4)$ action is given by

$$R \longrightarrow (\operatorname{tr}(A),\, B,\, W_+,\, W_-),$$

where

$$\operatorname{tr}(A) = \operatorname{tr}(C) = \frac{1}{4}(\text{scalar curvature}) = \frac{s}{4},$$

$$B = \text{the traceless Ricci tensor},$$

$$W_+ = A - \frac{1}{3}\operatorname{tr}(A),$$

$$W_- = C - \frac{1}{3}\operatorname{tr}(C).$$

We have here $W = W^+ + W^-$, the decomposition of the Weyl curvature tensor into self-dual and anti-self-dual parts. Let F be a curvature form, considered as an $\operatorname{End}(TN)$-valued 2-form. In terms of orthonormal basis, F is a 2-form with values in the Lie algebra of $SO(4)$. The (i,j)-entry F_{ij} of F is $R(e_i \wedge e_j)$. Then the Pontrjagin form is defined by taking the following invariant polynomial of $GL(4,R)$ as a generating function:

$$p(N,h) = \det(I - \frac{F}{2\pi}) = 1 + p_1(N,h) + p_2(N,h) + \cdots$$

Since F is skew-symmetric, the only non-zero polynomials are even degree in F. Explicitly, we have

$$p_1(N,h) = -\frac{1}{8\pi^2} \sum_{i,j} F_{ij} \wedge F_{ji}$$

$$= -\frac{1}{8\pi^2} \operatorname{tr}(F \wedge F)$$

$$= \frac{1}{4\pi^2}(\|W_+\|^2 - \|W_-\|^2).$$

The Euler form is defined by taking the Pfaffian invariant polynomial of $SO(4)$ as a generating function. Writing $F = \frac{1}{2}\sum_{i,j} F_{ij} e_i \wedge e_j$, where we consider $e_i \wedge e_j (i < j)$ as a basis for the Lie algebra of $SO(4)$, we then define the Euler form $e(N,h)$ by the Pfaffian $e(F)$:

$$(\frac{F}{2\pi})^2 = 4! e(F) e_1 \wedge \cdots \wedge e_4.$$

It is a general property of the Pfaffian that $e(F)^2 = \det(\frac{F}{2\pi})$ formally. It then follows that

$$e(N, h) = [\det(\frac{F}{2\pi})]^{\frac{1}{2}} \quad \text{formally}$$

$$= \frac{1}{32\pi^2} \sum_{i,j,k,l} \epsilon_{ijkl} F_{ij} \wedge F_{kl}$$

$$= \frac{1}{8\pi^2}(\|W_+\|^2 + \|W_-\|^2 + \frac{s^2}{24} - 2\|B\|^2).$$

It then follows that

$$2e(N, h) + p_1(N, h) = \frac{1}{4\pi^2}(2\|W_+\|^2 + \frac{s^2}{24} - 2\|B\|^2).$$

(N, h) is an Einstein manifold (M, g) if and only if $B = 0$. We thus get the inequality in Theorem 1. The equality holds if and only if $W_+ = 0$ and $s = 0$, i.e., the metric g is anti-self-dual and Ricci-flat. The global version of Theorem 1 is the following in which the equality case was treated by Hitchin [Hi1].

Theorem 1 ([Hi1], [Th]).   *Let (M, g) be a compact Einstein manifold of dimension 4. Then we have the following inequality:*

$$2e(M) \geq -P_1(M).$$

The equality occurs if and only if (M, g) is one of the following four cases: (1) (M, g) is flat, i.e., covered by a flat 4-torus, (2) a Kähler-Einstein K3 surface, (3) a Kähler-Einstein Enriques surface, (4) the quotient of a Kähler-Einstein Enriques surface by a free antiholomorphic isometric involution with $\pi_1(M) = Z_2 \times Z_2$.

To see this, we first remark that Cheeger-Gromoll's theorem [CG] implies that the fundamental group of a Ricci-flat non-flat compact 4-manifold (M, g) is a finite group. Assume (M, g) is non-flat and satisfies the equality. Let $\tilde{M}$ be the universal covering of M. It follows from Fact 1 and the remark below that $(\tilde{M}, g)$ becomes a Ricci-flat compact Kähler surface with a non-vanishing holomorphic 2-form, and hence is a Kähler-Einstein K3 surface. The possible actions of finite groups on $\tilde{M}$ is classified in [Hi1] using topological arguments.

Fact 1 ([Hi1]). *For any Ricci-flat anti-self-dual 4-dimensional Riemannian manifold (M, g), the connection induced from the Levi-Civita connection on the bundle of self-dual 2-forms is a flat connection.*

Proof. Identifying $\Lambda^2(R^4)$ with the Lie algebra of $SO(4)$, we see that the curvature of the induced connection on $\Lambda^2(M)$ is given by the formula

$$\omega \longmapsto [F(e_i \wedge e_j), \omega].$$

This implies that the curvature on Λ^+ is $A + B^*$. It is self-dual if and only if g is Einstein and vanishes if and only if g is Ricci-flat and anti-self-dual.

For a flat 4-torus (M, g), this flat connection is *trivial* because there are three linearly independent self-dual 2-forms on it. Therefore the unit sphere S in the 3-dimensional R-vector space of self-dual 2-forms can be identified with the 2-sphere $SO(4)/U(2)$ which parametrizes all compatible complex structures of R^4 each of which in turn defines an isometric Kähler structure of g. The union of these flat Kähler real 4-dimensional tori forms a non-Kähler complex 3-fold, which is now known to us as the *twistor space* for the flat 4-torus (M, g), fibered holomorphically over $P_1(C)$ identified with S. See [Hi2]. The fiber over $s \in S$ is a flat 4-torus with the Kähler structure corresponding to s. This was first remarked by Atiyah [At] in 1957. For a K3 manifold with a Ricci-flat metric, we can do the same thing. Indeed, the equality in Theorem 1 holds for a K3 manifold. Theorem 1 then implies that the induced Levi-Civita connection of the bundle of self-dual 2-forms is flat. This must be a flat trivial connection since a K3 manifold is simply connected. So one can find a globally defined self-dual 2-form κ on M. Raising one index of κ using the metric followed by a suitable normalization gives a parallel complex structure tensor J making κ the Kähler form of g. Let (a, b) be other two self-dual 2-forms such that (κ, a, b) forms an orthonormal basis for the space of parallel self-dual 2-forms on M. Then $a + ib$ is a globally defined holomorphic 2-form on M. By definition, (M, J) is a K3 surface. A K3 manifold thus appears as a simply-connected compact Ricci-flat anti-self-dual Riemannian 4-manifold (M, g). We note that there is real two dimensional freedom in the choice of a complex structure J. The three complex structures (I, J, K) corresponding to the orthonormal basis for the space of parallel self-dual 2-forms form a *hyper-Kähler* structure (see, for example, [C]) on the tangent bundle of M, i.e., three complex structures (I, J, K) satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\text{id}, IJ = K, JK = I, KI = J$. The twistor space for a K3 manifold with a Ricci-flat metric (M, g) is then a complex 3-fold Z by [AHS] and there is a holomorphic fibration $p : Z \to P_1(C)$

which parametrizes all isometric Kähler structures of (M, g). The fiber
over $(a, b, c) \in S^2 \cong P_1(C)$ with $a^2 + b^2 + c^2 = 1$ is a Kähler-Einstein K3
surface $(M, g, J(a, b, c))$ where $J(a, b, c) = aI + bJ + cK$. Hitchin [Hi3]
showed that this is not Kähler. Hitchin [Hi2] moreover showed

Corollary to Theorem 1 ([Hi1]).   *A compact Ricci-flat Rieman-
nian spin 4-manifold (M, g) with non-zero $\hat{A}$-genus is self-dual or anti-
self-dual according to the orientation.*

Indeed, the Atiyah-Singer Index Theorem for the Dirac operator
[AS] implies the existence of a harmonic spinor and it must be parallel
by the Lichnérowitz vanishing Theorem [Lic](see also [Ro]). Since the
spin representation of Spin(4) splits into Δ^+ and Δ^- according to the de-
composition Spin(4) $= SU(2) \times SU(2)$, the existence of a parallel spinor
implies the reduction of the holonomy group to the isotropy subgroup of
Δ^+ or Δ^-, which is $SU(2)$. Therefore (M, g) is a Ricci-flat Kähler mani-
fold relative to the proper choice of the orientation ([KN]). Writing ϕ for
the Kähler form, we thus have $F \wedge \phi = 0$. Clearly $F \wedge \omega = 0$ for locally
defined holomorphic 2-form ω since F is of type $(1, 1)$. These imply that
the self-dual part $A + B$ of the curvature vanishes. The Corollary implies
that a compact simply connected Ricci-flat Riemannian spin 4-manifold
with non-zero $\hat{A}$-genus is a Kähler-Einstein K3 surface.

Now we recall celebrated results of Siu [Si] (see also [Be] and [BPV])
and Yau [Y1], [Y2] (see also [Bou] and [Au]). We shall treat Yau's
existence theorem in a certain degenerate case in Section 2.

Theorem 2 ([Si]).   *Every K3 surface admits a Kähler metric.*

This theorem allows us to relax the Kähler assumption on every
theorem on K3 surfaces. So in this article we state theorems on period
maps originally proved by many authors (for instance, [BR], [LP], [T1])
with Kähler assumption as general theorems for K3 surfaces. The proof
of Siu's theorem is quite involved and uses many results on the period
maps for Kähler K3 surfaces (almost all results on period maps for K3
surfaces were first proved for Kähler K3 surfaces) obtained so far and
Yau's result on the existence of a Ricci-flat Kähler metric on a Kähler
K3 surface. There is Beauville's version of Siu's proof [Si] in [Be] and
[BPV].

Theorem 3 ([Y1], [Y2]).   *Every Kähler class of a K3 surface con-
tains a unique Ricci-flat Kähler form.*

This is a special case of Yau's general existence theorem [Y1], [Y2]:
Every Kähler class of a compact Kähler manifold with vanishing real first

Chern class contains a unique Ricci-flat Kähler form. A K3 surface with a Ricci-flat Kähler metric considered as a Riemannian 4-manifold is a first example of a compact *simply connected* Ricci-flat Riemannian manifold, since in 2 and 3 dimensions Ricci-flatness implies flatness. There is no known example of a simply connected compact 4-dimensional Ricci-flat Riemannian manifold which is not a K3 manifold. The following is an open question:

Question. Let (M, g) be a simply connected compact Ricci-flat Riemannian 4-manifold. Is M necessarily a K3 manifold?

If M is a spin manifold with $\hat{A} \neq 0$, then the question is true by the Corollary to Theorem 1 ([Hil]).

In the rest of this paper we explain how these fundamental theorems play essential roles in the theory of moduli of K3 surfaces. In Section 2, we look at the *holes* of the moduli of Ricci-flat metrics on a K3 manifold through the argument of Yau's proof of Theorem 3.

§1. The moduli of Kähler-Einstein K3 surfaces

We first fix some notations, conventions and some basic definitions. We write L for the *K3 lattice* $\oplus^2(-E_8) \oplus \oplus^3 H$. Let $L_R = L \otimes_Z R$ and $L_C = L \otimes_Z C$. The *classical period domain* is

$$\Omega = \{\omega \in L_C : <\omega, \omega> = 0, <\omega, \bar{\omega}> > 0\}/C^*.$$

This is an open subset of a smooth quadric in $P_{21}(C)$. We note that Ω is identified with a homogeneous complex manifold $O(3, 19)/SO(2) \times O(1, 19)$. Since $O(a, b)$ has the homotopy type of $O(a) \times O(b)$, the homotopy exact sequence of the fibration $SO(2) \times O(1, 19) \to O(3, 19) \to \Omega$ implies that Ω is connected and simply connected. A point $x \in \Omega$ corresponds to a *Hodge structure* of weight 2 on L_C in the following way: if $x = [\omega]$, i.e., if x is represented by $\omega \in L_C$, then

$$H^{2,0}(x) = C\omega \in L_C$$
$$H^{0,2}(x) = C\bar{\omega} \in L_C$$
$$H^{1,1}(x) = (H^{2,0}(x) \oplus H^{0,2}(x))^{\perp} \in L_C.$$

Let $x \in \Omega$. We define

$$\Delta = \{\delta \in \mathrm{L} : <\delta, \delta> = -2\}$$

and

$$\Delta(x) = \{\delta \in H^{1,1}(x) \cap L : <\delta, \delta> = -2\}.$$

Note that $H^{1,1}(x) \cap L_R$ is a 20-dimensional R-vector space with an inner-product of sign (1,19). We then define a 19-dimensional hyperboloid with two sheets

$$V^+(x) = \{\kappa \in H^{1,1}(x) \cap L_R :< \kappa, \kappa >= 1\}.$$

This generates an open convex cone in L_R. Since Ω is simply connected, we can find a continuous choice of a connected component with respect to $x \in \Omega$. We further set

$$V^+_\Delta(x) = \{\kappa \in V^+(x) :< \kappa, \delta >\neq 0 \text{ for all } \delta \in \Delta\}$$
$$K\Omega = \{(\kappa, x) \in L_R \times \Omega : \kappa \in V^+(x)\}$$
$$K\Omega^0 = \{(\kappa, x) \in K\Omega : \kappa \in V^+_\Delta(x)\}.$$

We call $K\Omega$ (resp. $K\Omega^0$) the *weakly polarized period domain* (resp. *polarized period domain*). We introduce an equivalence relation $(\kappa, x) \sim (\kappa', x')$ on $K\Omega^0$ which means that $x = x'$ and κ, κ' are in the same connected component of $V^+_\Delta(x)$. We then set

$$\widetilde{\Omega} = K\Omega^0 / \sim$$

and we call this the *Burns-Rapoport period domain*. For $\delta \in \Delta$ we define the reflection s_δ by

$$s_\delta : L \ni x \longmapsto x+ < x, \delta > \delta \in L$$

and set

$$W = \text{the group generated by } \{s_\delta : \delta \in \Delta\} \subset \text{Aut}(L)$$

and

$$W(x) = \{s_\delta \in W : \delta \in H^{1,1}(x)\}.$$

The reflecting hyperplane of s_δ is $H_\delta = \{\delta\}^\perp$. The set of fundamental domains in $V^+(x)$ with respect to the operation of $W(x)$ over $V^+(x)$ is the set of (completed) connected components of $V^+_\Delta(x)$ and is in one to one correspondence with the set of partitions of $\Delta(x)$ into $\Delta^+(x)$ and $-\Delta^+(x)$ with the property that if $\delta_1, \ldots, \delta_k \in \Delta^+(x)$ and $\delta = \sum_{i=1}^k n_i \delta_i$ with $Z \ni n_i \geq 0$ then $\delta \in \Delta^+(x)$. For a partition $P = \Delta^+(x) \cup -\Delta^+(x)$, the corresponding fundamental domain $V^+_P(x)$ is

$$V^+_P(x) = \{x \in V^+(x) :< x, \delta >> 0 \text{ for all } \delta \in \Delta^+(x)\}$$

The group W operates over $K\Omega$ in properly discontinuous fashion. The set $K\Omega^W$ of points of $K\Omega$ fixed by an element of W is a locally finite union of submanifolds of codimension 3 and is

$$K\Omega^W = K\Omega - K\Omega^0.$$

Let W_ω be the subgroup of W which fix $\omega \in \Omega$. Then $\widetilde{\Omega}$ is an *analytical sheaf* over Ω and for any $\omega \in \Omega$ the group W_ω operates in a simple transitive manner. Thus $\widetilde{\Omega}$ is a non-Hausdorff smooth analytic space. We set

$$N = \{\text{positive definite oriented 3-planes in } L_R\} \ = \ G_3^+(L_R).$$

Note that N is the noncompact dual of the real oriented Grassmannian $G_3(L_R)$ of dimension $\dim G_3^+(L_R) = 57$. There are natural fibrations of homogeneous spaces

$$K\Omega \ \xrightarrow{\ \pi_2\ } \ N$$

$$\pi_1 \Big\downarrow$$

$$\Omega$$

defined by

$$\pi_1(\kappa, x) = x$$

and

$$\pi_2(\kappa, x) = R \cdot \kappa + P(x) \text{ with the orientation of this order}$$

where $P(x)$ stands for the oriented positive 2-plane determined by $x \in \Omega$. Let $E \in N$. Then $\pi_2^{-1}(E)$ projects down to a smooth rational curve $C(E)$ in Ω defined by

$$C(E) = \{\omega \in E \otimes_R C\}/C^* \subset \Omega.$$

The fiber of π_1 (resp. π_2) is the hyperbolic 19-space (resp. the 2-sphere S^2). The fiber of π_2 projects down to some $C(E)$ in Ω. The group W operates over N in properly discontinuous fashion. We then set

$$N^0 = N - N^W = \pi_2(K\Omega^0).$$

The fixed point set N^W is a configuration of the noncompact dual of the oriented Grassmannians $G_3(R^{21})$ of dimension 54. This configuration is determined by how the set of vectors of square length -2 is imbedded in L.

What we have defined so far were built over the K3 lattice L. Now we relate these objects to geometric K3 surfaces. So let X be a K3 surface. The computation of numerical invariants of a K3 surface together with a theorem on the structure of indefinite unimodular lattices imply that the second integral cohomology group $H^2(X; Z)$ equipped with the cupproduct is isomorphic to the K3 lattice L (see [BPV, p.14 and p.241]). For a K3 surface X, we define

$$\Delta(X) = \{\delta \in H^{1,1}(X) \cap H^2(X; Z) : < \delta, \delta > = -2\}$$

and

$$\Delta^+(X) = \{\text{all effective } \delta \in \Delta(X),$$
$$\text{i.e., } \delta \text{ corresponds to a curve } D \text{ in } X\}.$$

The Riemann-Roch theorem implies that δ or $-\delta$ is effective for all $\delta \in \Delta(X)$. Hence $\Delta(X) = \Delta^+(X) \cup -\Delta^+(X)$ and if $\delta_1, \ldots, \delta_k \in \Delta^+(X)$ and $\delta = \sum_{i=1}^k n_i \delta_k$ with $Z \ni n_i \geq 0$ then $\delta \in \Delta^+(X)$. We thus define

$$V(X) = \{\kappa \in H^{1,1}(X) \cap H^2(X; R) : < \kappa, \kappa > = 1\}$$

$V^+(X)$ = the connected component of $V(X)$ containing a Kähler form

and

$$V_P^+(X) = \{\kappa \in V^+(X) : < \kappa, \kappa > = 1, < \kappa, \delta >> 0 \text{ for all } \delta \in \Delta^+(X)\}.$$

We call the open convex cone generated by $V^+(X)$ the positive cone, $V_P^+(X)$ the *Kähler chamber* and the open convex subcone of the positive cone generated by the Kähler chamber the *Kähler cone*. We call an element in $V_P^+(X)$ a *polarization*. A *Kähler polarization* is a polarization which comes from a Kähler metric. Theorem 9 in Section 1.2 asserts that every polarization is Kähler. There is a remarkable property enjoyed by the set of classes of effective divisors on a Kähler K3 surface X, which we will use later. Let D be an irreducible curve. Since $H^2(X; Z)$ is even and K_X is trivial, we have from the adjunction formula (see, for instance, [BPV]) that $D \cdot D \geq -2$ and the equality holds if and only if D is a smooth rational curve with self-intersection number -2 which we call a (-2)-*curve*. Moreover, if an irreducible curve D satisfies $D \cdot D \neq -2$, then $D \cdot D \geq 0$. Since $(\kappa, D) > 0$ for any Kähler class κ, the class of D is contained in the closure of the positive cone. Hodge's index theorem [p.120, BVP] then implies that if D is a class in the positive cone and C an irreducible curve with $D \cdot C = 0$, then C is a (-2)-curve. In particular, a class D in the positive cone satisfies $C \cdot D > 0$ for any

effective curve C if and only if $D \cdot C > 0$ for any (-2)-curve, i.e., D is in the Kähler cone.

1.1. Moduli of unpolarized K3 surfaces

Let X be the underlying differentiable manifold of a K3 surface, i.e., the differentiable manifold diffeomorphic to a smooth quartic surface in $P_3(C)$. The second cohomology group $H^2(X; Z)$ with the cup product is isomorphic to the K3 lattice L. A *marking* α for x is a choice of the isometry $\alpha : H^2(X; Z) \to L$. Let (X, J, α) be a *marked K3 surface* with a complex structure J and a marking α. The isometry α determines the subspace $H^{2,0}(X) \subset H^2(X; C) \to L_C$ where $H^{2,0}(X)$ is a 1-dimensional C-vector space generated by the class of a non-vanishing holomorphic 2-form ω_J on (X, J). We thus associate a point in Ω to a marked K3 surface. We next look at the behavior of the period map under deformations of complex structures of a K3 manifold. For a K3 surface X we write T_X for the sheaf of holomorphic vector fields. Since $H^0(X, T_X) = H^2(X, T_X) = 0$, Kuranishi's criterion [Kur] implies that there exists a local universal deformation over a non-singular base space whose dimension is equal to $\dim H^1(X, T_X) = \dim H^{1,1}(X, C) = h^{1,1} = 20$ by the Serre duality. Let $X_0 = (X, J_0, \alpha)$ be a marked K3 surface and $p : (\mathcal{X}, X_0) \mapsto (S, 0)$ a local universal deformation of X_0 and $X_t = (X, J_t)$ the fiber over $t \in S$. The marking α defines the *period map* τ for the family:

$$\tau : S \ni t \mapsto \tau(t) = [\alpha_C(\omega_{J_t})] \in \Omega.$$

The period map τ is also obtained by considering the subbundle $\bigcup_{t \in S} H^{2,0}(X_t)$ with a natural holomorphic bundle structure of $p_{*2} C_\mathcal{X}$. The following theorem is due to Andreotti-Weil and Kodaira [Kod].

Theorem 4 (Local Torelli Theorem) ([Kod]). *The period map $\tau : S \to \Omega$ is holomorphic and is a local isomorphism at 0.*

This theorem predicts the existence of beautiful global theories concerning the moduli of K3 surfaces. In fact, Andreotti-Weil and Kodaira ([W], [Kod]) conjectured the Global Torelli Theorem, the Surjectivity Theorem, the existence of the connected analytic deformation of all K3 surfaces and the Kählerian property of all K3 surfaces. To construct a fine moduli space of marked K3 surfaces, we glue up local universal deformations. The following theorem makes the gluing procedure possible:

Theorem 5 (The Global Torelli Theorem) ([PS], [BR], [LP], see also [BPV]). *Let X and X' be two K3 surfaces. If there is an effective*

Hodge isometry ϕ, i.e., $\phi : H^2(X', Z) \to H^2(X, Z)$ such that ϕ_C respects Hodge structures and $\phi_R(V_P^+(X')) = V_P^+(X)$, then there is a unique isomorphism $\Phi : X \to X'$ with $\Phi^* = \phi$.

The uniqueness directly follows from the Holomorphic Lefschetz fixed point formula ([LP, Proposition 7.5], [BPV, Proposition 11.3]). Theorem 5 was first proved in the algebraic case by Piateckii-Shapiro and Šafarevič [PS] and extended to the Kähler case by Burns-Rapoport [BR] and then simplified by Looijenga-Peters [LP]. We associate to a marked K3 surface (X, α) a point in $\widetilde{\Omega}$ determined by $H^{2,0}(X) \subset H^2(X; C)$ and $V_P^+(X)$, i.e., the Hodge structure and the Kähler cone. The Global Torelli Theorem says that two marked K3 surfaces defining the same point in $\widetilde{\Omega}$ are isomorphic in a *unique* way. We can thus glue up local universal deformations

$$(\mathcal{X}, (X, J_0)) \longrightarrow (S, 0) \text{ with a marking } \alpha : H^2(X, Z) \to L$$

through the injective period maps

$$\tau : S \ni t \longmapsto \left\{ \begin{matrix} \text{a point in } \widetilde{\Omega} \text{ deter-} \\ \text{mined by } [\alpha_C(\omega_{J_t})] \\ \text{and } \alpha_R(V_P^+(X, J_t)) \end{matrix} \right\}$$

to obtain an analytic space $\mathcal{M}_1$ such that every point has a neighborhood isomorphic to a parameter space for a local universal deformation. In particular, $\mathcal{M}_1$ is a smooth analytic space (but possibly not a Hausdorff space). So far, $\mathcal{M}_1$ is an *open* set of $\widetilde{\Omega}$. But we have

Theorem 6 (Surjectivity Theorem) ([To1], see also [BPV]). *For every $\widetilde{x} \in \widetilde{\Omega}$, there is a marked K3 surface which defines the point $\widetilde{x}$. In particular, $\mathcal{M}_1 = \widetilde{\Omega}$.*

Hence we conclude

Theorem 7 ([BR]+[T], [LP], [Mor1] see also [BPV]). *There exists a universal marked deformation of K3 surfaces over a smooth analytic space $\widetilde{\Omega}$ of dimension 20.*

This family is universal among marked deformations of K3 surfaces. A marked deformation of K3 surfaces consists of a deformation $p : \mathcal{X} \to B$ and an isomorphism $p_{*2}(Z) \to L_B$ of local systems. The period map $B \to \widetilde{\Omega}$ induces a deformation $\mathcal{X} \to B$. We remark that $\widetilde{\Omega}$ is *not* a Hausdorff space. We will come back to this point later. Of course

Theorem 6 logically implies that any two K3 surfaces are deformations of each other. In particular, any two K3 surfaces are diffeomorphic. But it is important to notice that these are consequences of the following important Density Theorem ([PS] [BR] [LP] [BPV]) which Piatečkii-Shapiro and Šhafarevič [PS], Burns-Rapoport [BR] and Looijenga-Peters [LP] used in the proof of the Global Torelli Theorem and Looijenga used in his version of the proof of Theorem 6 [Lo].

Theorem 8 (Density Theorem) ([PS], [BR], [LP], [BPV]).
(1) *Let T be an oriented positive definite primitive sublattice of rank 2 of L such that $(x,x) \in 4Z$. Then there exists an exceptional Kummer surface X with an isometry $\phi : H^2(X,Z) \to L$ which sends T_X to T, where $T_X = (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,Z)$.* (2) *Rationally defined 2-planes $P \subset L_R$ such that $(x,x) \in 4Z$ for all $x \in P \cap L$ forms a dense subset of the Grasmannian $Gr_2(L_R)$.* (3) *The period points of exceptional Kummer surfaces are dense in Ω.*

A smooth compact complex surface X is said to be *exceptional* if $H^{2,0}(X) \oplus H^{0,2}(X)$ is defined over Q, i.e., the rank of $H^{1,1}(X;R) \cap H^2(X;Z)$ is equal to $h^{1,1}$. The geometric assertion (1) in Theorem 8 is shown by constructing an exceptional abelian surface Y such that $(H^{2,0}(Y) \oplus H^{0,2}(Y)) \cap H^2(Y;Z) \cong \frac{1}{2}T$ in the following way. Let Γ be the first cohomology $\sum_{i=1}^4 e_i$ of a complex 2-torus and $\bigwedge^2 \Gamma$ be the second cohomology which is naturally isometric to $H \oplus H \oplus H$. If $\{v_1, v_2\}$ is a basis for $\frac{1}{2}T$ and $\{e_i, f_i\}_{i=1}^3$ a standard basis for $\oplus_{i=1}^3 H$, then $i(v_1) = e_1 + \frac{1}{2}(v_1, v_1)f_1$ and $i(v_2) = e_2 + \frac{1}{2}(v_2, v_2)f_2 + (v_1, v_2)f_1$ defines a primitive isometric embedding of $\frac{1}{2}T$ into $\bigwedge^2 \Gamma$. We have now specified the 2-dimensional subspace in $H^2 \otimes R$ spanned by the real and imaginary parts of a holomorphic 2-form. We naturally consider this as a complex line in $H^2 \otimes C$. The rest of the construction may be easily guessed from the proof of the following Andreotti-Weil Remark:

Andreotti-Weil Remark ([W]). Let Y be an oriented (possibly non-compact) differentiable manifold of dimension 4. If there is a C-valued 2-form ρ on Y such that (a) $\rho \wedge \rho = 0$, (b) $\rho \wedge \bar{\rho} > 0$ everywhere, and (c) $d\rho = 0$, then Y admits a unique complex structure such that ρ is a holomorphic 2-form.

Outline of proof. The condition (a) and (b) imply that the decomposable 2-form ρ defines at each point x a complex 2-dimensional subspace N_x in $T_xY \otimes C$. We thus get an almost complex structure J by putting N_x the $\sqrt{-1}$-eigenspace and $\bar{N}_x$ the $-\sqrt{-1}$-eigenspace.

The condition (c) is the integrability condition of this almost complex structure.

The Density Theorem implies that any two K3 surfaces are deformations of each other, since any small open set in any local universal deformation contains an exceptional Kummer surface. This fact was first proved by Kodaira [Kod] using the Density Theorem of different type for the period of elliptic K3 surfaces.[1] In particular, we have the following equivalent definition of a K3 surface:

Definition. A K3 surface is a connected compact complex smooth surface X with trivial K_X and $b_1 = 0$.

We will later exhibit how we use the Density Theorem in the study of period maps of K3 surfaces, through the proof sketch of the Surjectivity Theorems ([Lo], [KT]). We should note that the Density Theorem is also important in the proof of Siu's theorem [Si]. In this paragraph, we indicate the proof of the Global Torelli Theorem [PS][BR][LP] which is based on the Density Theorem. Let X and X' be Kähler K3 surfaces and $\phi : H^2(X', Z) \to H^2(X, Z)$ an effective Hodge isometry. We extend this situation to local universal deformations. So let $(\mathcal{X}, X) \to (S, 0)$ and $(\mathcal{X}', X') \to (S', 0')$ be local universal deformations of X and X', respectively. By the Local Torelli Theorem, we may assume that $S = S' \subset \Omega$ and $0 = 0'$. The isomorphism ϕ then uniquely extends to the automorphism of the local systems $p_{*2}(Z_S)$ which induces a Hodge isometry on each fiber. The stability of Kähler metrics [KS] implies that if S is sufficiently small, the nearby fibers are all Kählerian and the fiberwise Hodge isometry is effective. From the Density Theorem, there exists a sequence $\{s_n\}$ of S which consists of period points of exceptional Kummer surfaces X_n which converges to 0. The surface X'_n is also a Kummer surface. Indeed, since effective Hodge isometry respects effectiveness of divisors and the class of a (-2)-curve is represented by only one (-2)-curve as an effective divisor, we infer that the K3 surface X'_n has mutually disjoint sixteen (-2)-curves. It follows from the double covering construction [LP, Lemma 4.1] [BPV, Proposition 6.1] or from the numerical characterization of flat orbifolds [KT,Theorem 9](see Theorem in the next section) that X'_n is a Kummer surface. We now use the Torelli Theorem for Kummer surfaces (cf. [LP, Theorem 4.2] [BPV, Theorem 6.2]):

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[1]There are deep works of Shiga [Sg] and Naruki [N] on the period map for families of elliptic K3 surfaces with large Picard number. These are investigations toward understanding K3 modular functions.

Fact 2 (Torelli Theorem for Kummer surfaces). *Let X' be a K3 surface and X a Kummer surface. Then for any effective Hodge isometry $\phi : H^2(X, Z) \to H^2(X', Z)$, there is a unique biholomorphic map $f : X' \to X$ with $\phi = f^*$.*

Remark. Combining the arguments in the next section, i.e., the description of the Kähler cone of a Kummer surface, and the proof of the Torelli Theorem for projective Kummer surfaces [LP][BPV], we get the Torelli Theorem for general Kummer surfaces (Fact 2).

We are now given the following data. Namely, $p : (\mathcal{X}, X) \to (S, 0)$ and $p' : (\mathcal{X}', X') \to (S, 0)$, i.e., two sufficiently small local universal deformations, an isomorphism

$$(\Phi, \phi) : (p_{*2}(Z), H^2(X, Z)) \to (p'_{*2}(Z), H^2(X', Z)),$$

a sequence $\{s_n\}$ of S which converges to 0, a sequence $\{X_n\} \subset \mathcal{X}$, $\{X'_n\} \subset \mathcal{X}'$ of Kummer surfaces over $\{s_n\}$, a sequence $\{f_n : X_n \to X'_n\}$ of biholomorphic maps. We are now ready to apply the Burns-Rapoport Main Lemma [BR][LP, Theorem 7.1] [BPV, Theorem 10.6] on the degeneration of biholomorphic maps of Kähler K3 surfaces to conclude that f_n converges uniformly to a biholomorphic map $f_0 : X' \to X$ which satisfies the desired property: $\phi = f_0^* : H^2(X, Z) \to H^2(X', Z)$. To prove Fact 2, we must construct an isomorphism of K3 surfaces from a given effective Hodge isometry of the second cohomology groups. It suffices to recover the Hodge isomorphism of the first cohomologies which induces the given Hodge isometry on the second cohomologies as the second exterior product representation. The outline of the proof [LP][BPV] is as follows.

(0) First of all we note that X' is also a Kummer surface.

(1) Let $A = C^2/\Gamma$, resp. $A' = C^2/\Gamma'$ be a complex 2-torus and $X = \mathrm{Km}(A)$, resp. $X' = \mathrm{Km}(A')$. Let V, resp. V' be the set of sixteen 2-division points of A, resp. A' equipped with a natural structure of 4-dimensional affine space over F_2 and W, resp. W' the set of (-2)-curves in X, resp. X' corresponding to V, resp. V'. In the followings, we do the same constructions for both X and X'.

(2) Let $\hat{A}$ be the blowing up over the 16 2-division points. Then the natural map $\hat{\alpha} : H_2(\hat{A}, Z) \to H^2(X, Z)$ multiplies the intersection form by 2. Hence the same is true for the Poincaré dual $\alpha : H^2(A; Z) \to H^2(X; Z)$.

(3) Let M be the smallest primitive sublattice of $H^2(X; Z)$ containing W. Then $\mathrm{Im}(\alpha)$ is determined by M as its orthogonal complement: $\mathrm{Im}(\alpha) = M^{\perp}$. This most delicate part of the proof consists of identifying the set of *integral* classes in $(\frac{1}{2}Z)^W$ (linear combinations of (-2)-curves

with coefficients in *half integers*) as the set of affine linear functions on V. This is to show that the image of M under the natural map $r : (\frac{1}{2}Z)^W \to F_2^V$ consists of affine linear functions of V.

(4) (3) implies that ϕ induces an effective Hodge isometry

$$\phi : H^2(A; Z) \to H^2(A'; Z).$$

(5) There are natural isomorphisms

$$\beta : \operatorname{Hom}_Z(\Gamma, Z) \to H^1(A, Z),$$

its mod 2 reduction

$$\beta_2 : \operatorname{Hom}(T, F_2) \to H^1(A, F_2),$$

where T is the vector space of the affine space V and

$$\gamma : \bigwedge^2 H^1(A, F_2) \cong H^2(A, F_2) \to r(M^*)/r(M).$$

The last isomorphism follows from the string of natural isomorphisms

$$H^2(A, F_2) \cong \operatorname{Im}(\alpha)/2\operatorname{Im}(\alpha) \cong \frac{1}{2}\operatorname{Im}(\alpha)/\operatorname{Im}(\alpha)$$
$$= (M^\perp)^*/M^\perp \cong M^*/M \cong r(M^*)/r(M).$$

(6) $\phi : H^2(X; Z) \to H^2(X'; Z)$ induces a map $\nu : V \to V'$. (3) implies that ν respects affine linear functions and so ν is an affine linear map of V and V' inducing a natural isomorphism

$$\nu^0 : \operatorname{Hom}(T', F_2) \to \operatorname{Hom}(T, F_2).$$

Let ϕ^0 be the map induced on F_2^V which is contravariant relative to ϕ.

(7) There is a commutative diagram

$$
\begin{array}{ccc}
\bigwedge^2 \operatorname{Hom}(T, F_2) & \xleftarrow{\;\nu^0 \wedge \nu^0\;} & \bigwedge^2 \operatorname{Hom}(T', F_2) \\
\Big\downarrow{\scriptstyle \beta_2 \wedge \beta_2} & & \Big\downarrow{\scriptstyle \beta_2' \wedge \beta_2'} \\
H^2(A; F_2) & \xrightarrow{\;\phi_2\;} & H^2(A'; Z) \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \gamma'} \\
r(M^*)/r(M) & \xleftarrow{\;\phi^0\;} & r(M'^*)/r(M')
\end{array}
$$

(8) We want to show that $\phi : H^2(A; Z) \to H^2(A'; Z)$ is the exterior product representation $\psi \wedge \psi$ of some $\psi : H^1(A; Z) \to H^1(A'; Z)$. To see this we extend everything to rationals and projectivize ϕ and examine the induced isomorphism of the Klein quadric $G(1, P(H^1(A; Q)))$ and the Klein quadric is the moduli space of all lines in

$$P(H^2(A; Q)) \cong P(\wedge^2 H^1(A; Q)).$$

There are exactly two types of two-dimensional Schubert cycles on the Klein quadric. Type I is the pencil of lines in $P(H^1(A; Q))$ through a fixed point and Type II is the pencil of lines in $P(H^1(A; Q))$ contained in a fixed plain. We then reduce the problem to elementary projective geometry. If ϕ respects these types, then we are done, because we naturally find a map of $P(H^1(A; Q))$ to $P(H^1(A'; Q))$. But the commutative diagram in (7) implies that these types are preserved under the mod 2 reduction ϕ_2. Hence these types must be preserved also in the rational case.

In the original proof of Theorem 5, Todorov [To1] uses Kulikov's surjectivity theorem ([Kul], [PP]) for algebraic K3 surfaces (see also [Ho and [Sh]) and the twistor space, i.e., the isometric family of Ricci-flat Kähler structures ([AHS], [C]). It is Todorov's observation that if $(\kappa, [\omega]) \in K\Omega$ is a period point of a marked Kähler-Einstein K3 surface X, then the rational curve $C = \pi_1(\pi_2^{-1}(\pi_2(\kappa, [\omega]))) \subset \Omega$ carries the twistor space, i.e., the analytic family $Z \to C$ of isometric Kähler structures of the original Kähler-Einstein K3 surface. We say that the rational curve C parametrizes the isometric deformation of X. Now an algebraic period domain (see for instance [p.322, Mor1]) together with rational curves arising from the isometric deformations of these Kähler-Einstein algebraic K3 surfaces form an open set of Ω of period points of Kähler K3 surfaces. Considering various algebraic period domains, we see that every point in Ω appears as a period point of a Kähler K3 surface.

1.2. Moduli of polarized K3 surfaces

Yau [Y1] proved that any Kähler class of a K3 surface contains a unique Ricci-flat Kähler metric. So there is a bijective correspondence between Kähler polarizations and Ricci-flat K"ahler metrics on a K3 surface. For the universal marked family of K3 surfaces over $\widetilde{\Omega}$, there is no canonical way of specifying a family of Ricci-flat Kähler metrics on the fibers. On the other hand, all complex structures which make a given Ricci-flat Riemannian metric a Kähler metric are parametrized by $P_1(C)$

and form an analytic family of K3 surfaces over $P_1(C)$, which we call the *isometric deformation* or the *isometric families of Kähler structures* (see [At], [C], [To1] and [To2]). This suggests the importance of the space $K\Omega^0$ in the differential geometry of K3 surfaces. To consider $K\Omega$ is to fill the *holes* of $K\Omega^0$ and this corresponds to consider a certain degeneration of Ricci-flat metrics on a K3 manifold. We look at this phenomenon first in algebraic way in this section and then in differential geometric way in the next section. For a K3 surface (X, ϕ, α) with a marking α and a Kähler metric ϕ with $< \phi, \phi >= 1$, we define the period $p(X, \phi.\alpha) \in K\Omega^0$ by $([\alpha_C(\omega_X)], \alpha_R(\phi))$, where ω_X is a non-vanishing holomorphic 2-form on X. Using Theorem 1, we associate to a marked K3 manifold (X, g, α) with a Ricci-flat Riemannian metric g on X the period $p(X, g, \alpha) \in N^0$ by the α_R-image of the oriented 3-dimensional R-vector space of the self-dual 2-forms on X (recall that these are parallel). Here, if $\{e_1, e_2, e_3, e_4\}$ is an oriented basis for R^4 then $\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2, e_1 \wedge e_4 + e_2 \wedge e_3\}$ gives the oriented basis for $\bigwedge_2^+(R^4) \in \bigwedge^2(R^4)$ with the induced orientation (cf. the map $\pi_2 : K\Omega \to N$). For these period domains $K\Omega^0$ and N^0, we have

Theorem 9 ([Lo]). *For any element (κ, x) in $K\Omega^0$, there exists a marked Kähler-Einstein K3 surface (X, ϕ, α) such that $p(X, \phi, \alpha) = (\kappa, x)$. In particular, if X is a K3 surface, every element in $V_P^+(X)$ contains a Kähler form.*

In Looijenga's proof of Theorem 9, he uses the isometric deformation of Kähler-Einstein structure on a K3 manifold ([C], [To1,2]) in the key Lemma which states that the periods in $K\Omega$ of all marked Kähler-Einstein K3 surfaces consist of fibers of $K\Omega \to G_3^+(L_R)$. Looijenga uses this Lemma combined with the density argument (see [Lo] and [BPV, Chapter VIII]) to show that every polarization of a K3 surface comes from a Kähler metric. He then uses Theorem 3 (Yau's existence theorem) to obtain Theorem 9. The Global Torelli Theorem and Theorem 9 imply that $K\Omega^0$ is isomorphic to the moduli space of the equivalence classes of marked Kähler-Einstein K3 surfaces under the period map. This is a fine moduli space:

Theorem 10 (cf. [Lo] and [Mor2]). *There is a universal marked family $\mathcal{M}^0 \to K\Omega^0$ of Kähler-Einstein K3 surfaces over a real analytic manifold $K\Omega^0$ of dimension 59. The point (κ, x) is the period of the fiber (a marked Kähler-Einstein K3 surface).*

Proof. We follow the construction of the universal marked family of unpolarized K3 surfaces. First, it is easy to modify the global Torelli

theorem for the polarized period map for marked Kähler-Einstein K3 surfaces. To formulate the local Torelli theorem, we introduce the notion of the local universal marked deformation of Kähler-Einstein K3 surfaces which is defined in the following way. Let (X, κ_0, J_0) be a Kähler-Einstein K3 surface and $(\mathcal{X}, (X, J_0)) \to (S, 0)$ a local universal deformation. A marking α for (X, J_0) canonically induces a marking, also denoted by α, for the local universal deformation. An assignment $\kappa : S \ni t \mapsto \kappa_t \in H^{1,1}(X, J_t) \cap H^2(X; R)$ is a *polarization* for the family if each κ_t is a Kähler class of (X, J_t) and the map $\alpha_R : S \ni t \mapsto \alpha_R(\kappa_t) \in L_R$ is smooth. A natural way to get a polarization for a local universal family is this: Let (X, J_0, κ_0) be a polarized Kähler-Einstein K3 surface and $C \subset S$ the locus of the isometric deformation of it. Each point $c \in C$ corresponds to a Kähler-Einstein K3 surface (X, J_c, κ_c). Let $S_c \subset S$ be the locus on which the class κ_c is of type (1,1). The complex one-parameter family $\bigcup_{c \in C} S_c$ of complex hypersurfaces in S sweeps a 20-dimensional region and we thus get a polarization for a local universal deformation. In this case the polarization over S_c is a section of the sheaf $L_S \otimes R$ over S. This is a usual definition of the polarization for a family (cf. [Mor1, p.309]). A small tubular neighborhood KS of the image $\{\alpha_R(\kappa_t) : t \in S\} \in K\Omega^0$ parametrizes a local universal marked deformation of Kähler-Einstein K3 surfaces. Namely this family is universal among marked differentiable families $\mathcal{M} \to S$ of Kähler-Einstein K3 surfaces. The local Torelli Theorem in this case is now clear. In a differentiable family $\mathcal{M} \to S$ Kähler-Einstein structures (J_t, ϕ_t) vary differentiably with respect to $t \in S$, where S is a differentiable manifold. This is equivalent to saying that both complex structure and polarization varies differentiably with respect to t. We then form the universal family of polarized marked Kähler-Einstein K3 surfaces over $K\Omega^0$ in the completely same gluing procedure as before using the Global Torelli Theorem and the uniqueness of the Kähler-Einstein form in a given Kähler class.

Theorem 11 (cf. [To2]). *Any element in N appears as a period of a marked Ricci-flat K3 manifold.*

Theorem 11 and the Global Torelli Theorem imply that the period map gives an isomorphism of N and the moduli space of the equivalence classes of marked Ricci-flat K3 manifolds. This moduli space is a fine moduli space in the following sense.

Theorem 12 (cf. [To2]). *There is a universal family $\mathcal{N}^0 \to N^0$ of Ricci-flat K3 manifolds over N^0.*

Proof. For $E \in N^0$, the family of Kähler-Einstein K3 surfaces over

the 2-sphere $\pi_2{}^{-1}(E) \in K\Omega^0$ is the *isometric* deformation (cf. Theorem 1). We can thus form a family of Ricci-flat K3 manifold over N^0 by forgetting the complex structure from the universal marked family of Kähler-Einstein K3 surfaces in Theorem 8. The universal property of the family is now clear.

It follows from Theorems 10 and 12 that

1. If U is an open set in $K\Omega^0$ in the usual topology, the induced family $\mathcal{M}^0{}_U \to U$ is a smooth family of Kähler-Einstein K3 surfaces. The complex structure depends only on $\pi_1(x)$ and the Ricci-flat metric depends only on $\pi_2(x)$ for $x \in U$.

2. If U is an open set in N^0 in the usual topology, the induced family $\mathcal{N}^0{}_0 \to U$ is a smooth family of Ricci-flat K3 manifolds.

Now we fill the it holes of $K\Omega^0$ by introducing generalized K3 surfaces, the notion of weak polarization, K3 orbifolds and Ricci-flat orbifold metrics. The following definitions are due to Morrison [Mor1].

Definition.   A generalized K3 surface is a compact complex surface X with at worst rational double points, whose minimal resolution $Y \to X$ is a K3 surface.

Definition.   Let X be a generalized K3 surface and $\rho : Y \to X$ is the minimal resolution. Let $\delta_1, \ldots, \delta_k \in H^{1,1}(Y) \cap H^2(Y; Z)$ be the class of all (-2)-curves in the exceptional set for ρ. The *root system* $R(X)$ and the *Weyl group* $W(X)$ of X are defined by

$$R(X) = \{\delta \in H^2(Y;Z) : \delta = \sum_{i=1}^{k} a_i\delta_i,\ a_i \in Z,\ <\delta,\delta> = -2\}$$

and

$$W(X) = \text{the group generated by } \{s_\delta : <\delta,\delta> = -2\}$$
$$\subset \text{Aut}(H^2(Y;Z)) \cong \text{Aut}(L).$$

The following definition heavily depends on Theorem 2: Every K3 surface is Kähler [Si].

Definition.   Let X and Y be as above. A marking α for X is a metric injection $\alpha : H^2(Y;Z)^{W(X)} \to L$ which is extendable to an isometry of $H^2(Y;Z)$ to L also denoted by α. A pair (X,α) is a *marked generalized K3 surface*.

Definition. For a generalized K3 surface X we set

$$V_P^+(X) = \{\kappa \in \overline{V_P^+(X)} : \text{for all } \delta \in H^{1,1}(Y) \cap H^2(Y; Z) \text{with}$$
$$<\delta, \delta> = -2, \ <\kappa, \delta> = 0 \text{ if and only if } \delta \in R(X)\}$$

An element $\kappa \in V_P^+(X)$ is a polarization of X and a weak polarization of Y with respect to X. A polarization is Kähler iff it comes from a Kähler metric in the sense of Fujiki-Moišezon. For a marked polarized generalized K3 surface (X, κ, α) we define the period $p(X, \kappa, \alpha) \in K\Omega$ by $([\alpha_C(\omega_Y)], \alpha_R(\kappa))$.

A Kähler metric in the sense of Fujiki-Moišezon on a complex space X is a Kähler metric in the regular part of X and is the restriction of a Kähler metric of some ambient Euclidean space in which a neighborhood of a singular point of X is imbedded. Theorem 14 in Section 1.2 asserts that every polarization for a generalized K3 surface is Kähler.

Definition. We set $I^2(X) = H^2(Y; Z)^{W(X)}$, i.e., the set of all classes orthogonal to $R(X)$. Since $I^2(X)_C$ contains $H^{2,0}(Y)$ this defines a Hodge structure on $H^2(Y; Z)$.

The notions of marking and polarization for a family of K3 surfaces were clear. We need to modify these notions for families of generalized K3 surfaces to construct the global fine moduli space for marked generalized K3 surfaces. This is made possible by the following fundamental result on deformation of rational double points:
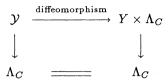
Fact 3 ([Br1], [Br2], [Pi], [Sl]). *Let (X, P) pe a germ of a surface with a rational double point, and let $\rho : (Y, C) \to (X, P)$ be the minimal resolution. We write $\rho^{-1}(P) = C \sum C_i$ for the irreducible decomposition and let $\Lambda = \Lambda(X)$ be the free abelian group generated by C_i. We write $<,>$ for the quadratic form on Λ induced by the intersection pairing. We define*

$$R = R(X) = \{\delta \in \Lambda : <\delta, \delta> = -2\}$$

Then there are representatives of the versal deformations $(\mathcal{Y}, Y) \to$ $(\text{Def}_Y, 0)$, $(\mathcal{X}, X) \to (\text{Def}_X, 0)$ such that
(1) $\text{Def}_Y = \Lambda_C$, $\text{Def}_X = \Lambda_C/W$, and $\text{Def}_Y \to \text{Def}_X$ (induced by the contraction) is the natural quotient by the operation of the Weyl group W of R.

(2) *If $q : \mathcal{Y} \to \Lambda_C$ is the versal family, there exists a trivialization*

$$\mathcal{Y} \xrightarrow{\text{diffeomorphism}} Y \times \Lambda_C$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\Lambda_C \;\; =\!=\!=\!= \;\; \Lambda_C$$

which induces an isomorphism $i : H^2(Y_t; Z) \cong H^2(Y; Z) \cong \Lambda$ such that for each $t \in \Lambda_C$ the root system for Y_t is given by (under the isomorphism above)

$$i(R(Y_t)) = \{\delta \in R :< \delta, t >= 0\}$$

We consider the space Def_X (which is biholomorphically a ball) as the orbifold Def_Y / W with branch loci (which are the image of $\{\{\delta\}^\perp : \delta \in R\}$) of branch index 2 whose fundamental group as the orbifold is W. Let X be a generalized K3 surface and $Y \to X$ the minimal resolution. Let $p : \mathcal{Y} \to S$ be a local universal deformation of Y. By [Ri, p.121, Theorem 2], there is a contraction $\mathcal{Y} \to \mathcal{X}$ such that $\mathcal{X} \to S$ is a deformation of X and for each $t \in S$ the map $Y_t \to X_t$ is the minimal resolution. We define a *marking* for a family of generalized K3 surfaces in such a way that a marking for $\mathcal{Y} \to S$ is one of the markings for $\mathcal{X} \to S$ and that this is universal among marked families of generalized K3 surfaces. Let $q : \mathcal{X}_B \to B$ be any deformation of a generalized K3 surface X where B is a locally contractible complex space and let $Y_t \to X_t$ be the minimal resolution. A *marking* for this family is defined in the following way. We first define a sheaf $I^2(\mathcal{X}_B/B)$ using Fact 1 (for more details, see [Mor1]). For each $t \in B$ any sufficiently small (contractible) neighborhood $U(t)$ has a representative $i : U(t) \to D_t$ where $D_t = \prod_{P \in \text{Sing } X_t} \text{Def}_{(X,P)}$ stands for the versal family for the germs (X, P) of the singularities of X_t (cf. Fact 1). We consider, via Fact 1, D_t as an orbifold with the fundamental group W (the product of the Weyl groups for individual singular points of X_t) and we write $p : \tilde{D}_t \to D_t$ for the universal branched covering. We can thus form the fibered product $\widetilde{U(t)} = U(t) \times_p \tilde{D}_t$. We can then form a family $\tilde{q} : \mathcal{Y}_{\widetilde{U(t)}} \to \widetilde{U(t)}$ of K3 surfaces such that $\tilde{q}^{-1}(r)$ is the minimal resolution of $q^{-1}(t)$ where r lies over t (simultaneous resolution). Since $\widetilde{U(t)}$ is again simply connected, we get a trivialization $\tilde{q}_{*2}(Z) \cong H^2(Y_t; Z)_{\widetilde{U(t)}}$. To each small neighborhood $U(t)$ we associate an abelian group $I^2(X_t)$ (t varies over B). It then follows from Fact 1 that the usual sheafication procedure of a presheaf applied to the collection $\{\widetilde{U(t)}\}_{t \in B}$ (the *orbifold-version of*

the sheafication) yields the sheaf $I^2(\mathcal{X}_B/B)$, which is a generalization of $p_{*2}(Z) \cong L_B$ in the case of a family of smooth K3 surfaces. The stalk over $t \in B$ is $I^2(X_t)$.

Definition. A marking for $\mathcal{X}_B \to B$ is a metric injection

$$\alpha : I^2(\mathcal{X}_B/B) \longrightarrow L_B$$

such that for each $t \in B$, α_t extends to an isometry of $H^2(Y_t; Z)$ to L, where Y_t is the minimal resolution of X_t.

Remark. While a marking exists for any local smooth deformation of a K3 surface, a marking does not always exist for a given local deformation of a generalized K3 surface. Indeed, any marked deformation $\mathcal{X}_B \to B$ of a generalized K3 surface admits a simultaneous resolution $\mathcal{Y}_B \to B$ and by [Ri], there is a contraction $\mathcal{Y}_B \to \mathcal{X}_B$. See Example 1 below.

Let $\mathcal{Y} \to S$ and $\mathcal{X}_B \to B$ be as above with markings extending a marking $\alpha : I^2(X) \to L$ for X. Using the classical period maps for these marked families, we get a holomorphic map $B \to S$. The induced family is then a *simultaneous resolution* of singularities of each fiber of $\mathcal{X} \to B$. This implies that the family of generalized K3 surfaces $\mathcal{X} \to S$, which is obtained by the contraction $\mathcal{Y} \to \mathcal{X}$, and with a marking induced from that of $\mathcal{Y} \to S$, is a local universal *marked* deformation for the generalized K3 surface X. This can be understood in terms of *orbifolds* in the following way. Let X, Y and W be as above and let α be a marking of Y. Let O be the image in Ω under the period map of a sufficiently small local universal deformation of Y. We may assume that the Weyl group W acts on O as a reflection group generated by the reflections with respect to the hypersurfaces $H_\delta = \{\omega \in L_C :< \omega, \delta >= 0$ for $\delta \in R(X)\}$. We write $p : \mathcal{Y}$ (resp. $p' : \mathcal{X}) \to O$ for the induced local universal deformation (resp. the induced deformation of X). The Weyl group acts on $p_{*2}(Z) \cong H^2(Y; Z)_O$. It follows from the Global Torelli Theorem that this action is lifted to the action on $\mathcal{X}$ rather than $\mathcal{Y}$. We explain this. Let $s_\delta \in W$ and set $x' = s_\delta(x) \in O$. Now let x be in a disk D in O which cuts $\{\delta\}^\perp$ transversally at a point u and which is invariant under the action of the involution s_δ. Write D^* for $D - \{u\}$. If $x \in D^*$, i.e., $x \neq x'$, there exists an isomorphism $\phi_{\delta,x} : Y_{x'} \to Y_x$ which induces s_δ on $H^2(Y; Z)$ via the isomorphism $H^($Y; Z) \cong H^2(Y_x; Z)$ for any x. Note that this is just the reason why $\tilde{\Omega}$ is not a Hausdorff space. Now we let x tend to the point u in $\{\delta\}^\perp$ in O. Then, in $\mathcal{X}$, the isomorphisms $\phi_{\delta,x}$ converge to the identity of X_u (cf. [BR, p.262, paragraph 7]). The

argument is as follows. By Bishop's theorem [Bi] (see also [BPV, p.259]) the limit is an isomorphism of X_u and induces the identity on $I^2(X_u)$, which implies that this isomorphism itself must be the identity. We consider the graph Γ_x of the isomorphism $\Phi_{\delta,x}$ in $(\mathcal{X}, p') \times_{D^*} (\mathcal{X}, s_\delta \circ p')$. The map $s_\delta : H^2(X_{x'}; Z) \to H^2(X_x; Z)$ is induced by the Poincaré duality $H^2(X_{x'}; Z) \cong H_2(X_{x'}; Z)$ and the slant product $H_2(X_{x'}; Z) \to H^2(X_x; Z)$, $\alpha \mapsto \Gamma_x/\alpha$. Boundedness of the volume implies that the limit Γ_u exists as a positive analytic integral current ([Bi]) and is in the form of $\Delta + \sum C_i \times D_i$, where Δ is a graph of an isomorphism of Y_u, C_i and D_i are curves in Y_u ([BR], [LP], [BPV]). Since the induced map on the cohomology is s_δ, we see that Γ_u = the diagonal $+ C \times C$, where C is the (-2)-curve corresponding to δ. Arguing inductively, we can thus form the quotient family $\mathcal{X}/W \to O/W$ which is a local universal deformation of X (not a marked family)(cf. [BW, Theorem 3.7] and [Mor1, p.320]). This family is universal among local deformations (with a locally contractible base space) of the generalized K3 surface X. This can be seen in exactly the same way as the argument in the construction of the sheaf $I^2(\mathcal{X}_B/B)$.

Example 1. Let $\mathcal{K}$ be a smooth complex 3-fold in $P_3(C) \times \Delta$ defined by

$$x^4 + y^4 + z^4 + x^2 + y^2 + z^2 - t = 0$$

where (x, y, z) is the affine coordinate of $P_3(C)$ and t runs over a sufficiently small disk Δ of radius r in C. Let $p : \mathcal{K} \to \Delta$ be the projection onto Δ. Then $\mathcal{K} \to \Delta$ is a deformation of X_0, a quartic surface with a rational double point of type A_1. To get a *marked* deformation of X_0, we must consider the double covering of $\mathcal{K}$ branched exactly over X_0, namely the family $\mathcal{K}' \to \Delta$ induced by the map $\Delta \ni t \mapsto t^2 \in \Delta$. The total space $\mathcal{K}'$ is given by the equation

$$x^4 + y^4 + z^4 + x^2 + y^2 + z^2 - t^2 = 0$$

in $P_3(C) \times \Delta$. We may consider the map $(x, y, z, t) \to t$ as a deformation of generalized K3 surfaces. The origin $(0, 0, 0, 0)$ is a double point of type A_1 of the 3-fold $\mathcal{K}'$ (cf. the above remark). A small resolution [HW] yields a deformation $\widetilde{p} : \widetilde{\mathcal{K}} \to \Delta$ of the minimal resolution Y_0 of X_0 which is a K3 surface (cf. [Mor1, p.320], [BR, p.238], [BPV, p.263]). There are exactly two non isomorphic small resolutions $\widetilde{\mathcal{K}}_i$ $(i = 1, 2)$ for the 3-fold double point of type A_1. These two small resolutions give rise to two non isomorphic deformations which are isomorphic over Δ^*. One can see this intuitively by observing that no isomorphism of the family $(p, \mathcal{K}', \Delta)$ extends to a holomorphic map of $\mathcal{K}'_i s$. This is only

possible when the moduli space is not a Hausdorff space. The restricted deformation $\mathcal{K} \times_\Delta \Delta^*$ over Δ^* is not differentiably trivial and this implies the absence of markings for the family $(p, \mathcal{K}, \Delta)$. We see this through the following local model.

Example 2. Let Y_t be an affine quadric

$$Y_t : x^2 + y^2 + z^2 - t = 0$$

in C^3. We consider a deformation $\bigcup Y_t$ of Y_t, where t runs over the unit disk Δ. This deformation is not differentiably trivial over Δ^*. Indeed, the intersection of Y_t and the sphere $S^5(\sqrt{|t|})$ of radius $\sqrt{|t|}$ in C^3 is a 2-sphere. For instance, if $t = \varepsilon^2$ with $\varepsilon > 0$, then $(x, y, z) \in R^3 \subset C^3$ with $x^2 + y^2 + z^2 = \varepsilon^2$ is a 2-sphere. It is easy to see that Y_t for $t \neq 0$ is diffeomorphic to the tangent bundle of this 2-sphere. If we put $t = t(\theta) = \varepsilon^2 e^{2i\theta}$, then $t(0 \leq \theta \leq \pi)$ surrounds the origin. Lifting this circle to the family induces the map $(x, y, z) \mapsto (-x, -y, -z)$ on Y_{ε^2} which restricts to the anti-podal map of the 2-sphere.

Example 3. We use the same symbols as in Example 1. Let δ be the Poincaré dual of the 2-sphere in K_t sitting near the origin whose existence is shown in Example 2. A simple loop in Δ^* with base point t induces the reflection s_δ of $H^2(K_t, Z)$. This is the holonomy group of the flat vector bundle defined by the sheaf $p_{*2}(R_{\Delta^*})$. Indeed, if we look at K_t in C^3, we see that the family $\bigcup(K_t \cap B^6(r'))^c$ is differentiably trivial over Δ for a suitable r' and that the map of $K_t \cap S^5(r') \cong P_3(R) \cong SO(3)$ induced by the lifting of the loop to the family is homotopic to the left translation of $SO(3)$ by an involutive element g. In particular, the diffeomorphism S_δ of K_t which is equal to $(x, y, z) \mapsto (-x, -y, -z)$ in a small tubular neighborhood U of the 2-sphere and is equal to the identity outside another small tubular neighborhood V and on the neck region $V - U \cong P_3(R) \times I$ (I is an interval) gives an isotopy between the identity and the left translation by the involutive element g on $SO(3)$ on this neck region. Thus the marked family $\widetilde{\mathcal{K}} \to \Delta$ is induced by the *injective* period map $\Delta \to \widetilde{\Omega}$ and the image is a disk which is transversal to the hypersurface defined by the condition $< \delta, \omega > = 0$ which means that the class δ is of type (1,1) and is invariant under the operation of the reflection s_δ in the Weyl group. Letting the Weyl group (isomorphic to Z_2) of the generalized K3 surface X_0 act on the image in Ω of the family $\widetilde{\mathcal{K}} \to \Delta$, we get another deformation $\widetilde{\mathcal{K}'} \to \Delta$ with a birational map $e : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}'}$ which is the lifting of the involution. This is the other small resolution of the 3-fold double point. We call this operation the

elementary operation. Identifying these families differentiably, we see
that the map induced on the fibers over $t(\neq 0)$ is homotopic to S_δ. The
closure in $\widetilde{\mathcal{K}} \times_\Delta (\widetilde{\mathcal{K}})'$ of the union of the graphs of these holomorphic
maps is the graph of the birational map e (see [Mor1, p.312] and [BR,
Section 7]). In particular, e induces s_δ on $H^2(Y_0; Z)$, where δ is the class
of the exceptional (-2)-curve for $Y_0 \to X_0$.

We are now ready to discuss the moduli problem of Kähler-Einstein
generalized K3 surfaces. The existence of a Kähler-Einstein metric on a
generalized K3 surface was remarked in [KT].

Theorem 13 (cf. [KT, p.348]). *Let X be a generalized K3 sur-
face, Y its minimal resolution and κ a Kähler polarization for X. Then
there exists a unique Kähler-Einstein orbifold metric (which is necessar-
ily Ricci-flat) on X which is a Kähler-Einstein form on the regular part
of Y and defines a closed current on Y whose cohomology class is κ.*

For a proof, see [KT]. We examine the quantitative aspect of Theo-
rem 13 in the next section. This theorem gives a geometric meaning to
the deformation of K3 surfaces parametrized be $P_1(C) \cong P((P_\omega + R\kappa) \otimes
C) \in \Omega$ for $(\kappa, [\omega]) \in K\Omega - K\Omega^0$. This is the isometric deformation with
respect to a Ricci-flat orbifold-metric of the corresponding generalized
K3 manifold (K3 orbifold).

Theorem 14 ([KT, p.350–352]). *Suppose X is a generalized K3
surface with a orbifold-Kähler-Einstein metric form ϕ. Write g for the
Riemannian metric on X determined by ϕ and the complex structure
of X. Then (1) $X \times S^2$ has a complex structure $\mathcal{X}$ such that (a) the
projection $\pi : \mathcal{X} \to S^2 \cong P_1(C)$ is a holomorphic map and fibers are
generalized K3 surfaces. From $\mathcal{X} \to P_1(C)$, we obtain a deformation
of K3 surfaces $\tilde{\mathcal{X}} \to P_1(C)$; (b) if (X, α) is a marked generalized K3
surface, then α induces an isomorphism of local systems (in fact trivial
systems) $\alpha : \pi_{*2}(Z_\mathcal{X}) \cong L_{P_1(C)}$; (c) for each $t \in P_1(C)$ the period in Ω of
$X_t = \pi^{-1}(t)$ is an oriented 2-plane in the 3-dimensional space $E \subset L_R$
spanned by $(\mathrm{Re}\, \omega_X, \mathrm{Im}\, \omega_X, \phi)$ or more intrinsically three linearly inde-
pendent parallel self-dual 2-forms with respect to the Ricci-flat orbifold
metric. (2) For each $t \in P_1(C)$ the Ricci-flat Riemannian orbifold metric
g on X_t is orbifold-Kählerian with respect to the corresponding complex
structure. (3) The base space $P_1(C)$ parametrizes all complex structures
with respect to which g is orbifold-Kählerian.*

Proof. All oriented 2-planes in 3-dimensional $E \subset L_R$ spanned by
$(\mathrm{Re}\, \omega_X, \mathrm{Im}\, \omega_X, \phi)$ are parametrized by S^2. We may assume that these

are orthonormal basis for self-dual 2-forms. Let E_t be any oriented 2-plane in E and let α and β be orthonormal basis in E_t. Then we define $\omega_t = \alpha + \sqrt{-1}\beta$. This is a C-valued 2-form which satisfies the conditions (a), (b) and (c) in Andreotti-Weil Remark. Hence ω_t defines a new complex structure J_t on the smooth part of X. If x is a simple singular point in X and U is a pseudo-convex neighborhood of x, then $U = V/G$ where $V \subset C^2 \cong R^4$ and $G \subset SU(2) \subset SO(4)$. Let $\pi : V - \{0\} \to U - \{x\}$ be the natural projection. Then $\pi^*(\omega_t|V - \{x\})$ can be prolonged to a smooth C-valued 2-form on V invariant under the action of G. Andreotti-Weil Remark then implies that the complex structure J_t extends to an orbifold-complex structure at singular points of X. We thus get the 3-fold $\mathcal{X}$ which is nothing but the orbifold-twistor space for an anti-self-dual Ricci-flat 4-orbifold [AHS]. The rest of the proof is almost the same as the arguments in Introduction. But here we take a more group theoretical point of view. Let $\phi_t \in E$ be a unit vector orthogonal to E_t. We may suppose that $(\operatorname{Re}\omega_X, \operatorname{Im}\omega_X, \phi)$ and $(\operatorname{Re}\omega_t, \operatorname{Im}\omega_t, \phi_t)$ defines the same orientation of E. Then we can find $A \in SO(4)$ such that $A^+\phi A^{+*} = \phi_t$, where A^+ is the homomorphism of $SO(4) \to SO(3)$ determined by the decomposition of the second exterior product representation of $SO(4)$ into irreducible subspaces. This implies that ϕ_t is a *positive definite* real closed (1,1)-form on the smooth part of X which defines the Riemannian metric g. Since every form in E is orbifold-smooth, ϕ_t defines a Kähler-Einstein orbifold-metric relative to the new complex structure J_t. We can resolve the singularities in the family $\mathcal{X} \to P_1(C)$ by successive blow-ups (see [At]) to get a family $\tilde{\mathcal{X}} \to P_1(C)$ of smooth K3 surfaces. In the case of A_1-singularity, it is understood as follows. Let $p : P_3(C) \to P_1(H) \cong S^4 \cong R^4 \cup \{\infty\}$ be the natural projection sending a complex line in C^4 to the quaternionic line it generates. The fiber is $P_1(C)$ which parametrizes complex lines in the corresponding quaternionic line. The moduli space of all hyperplanes in $P_3(C)$ containing the fiber l_∞ is parametrized by another fiber $\cong P_1(C)$ and the natural projection p induces complex structures on R^4 compatible with the flat metric which are parametrized by $P_1(C)$. The local situation around the $P_1(C)$-family of A_1-singularities is exactly the same as the quotient of the normal bundle $\nu = O(1) \oplus O(1)$ of a fiber under the involution -1. If we blow up the 0-section of ν, the involution becomes the reflection with respect to the exceptional set which is $P_1(C) \times P_1(C)$. This implies that the union of each (-2)-curve on a fiber forms a trivial $P_1(C)$-bundle. Since ν is not trivial and $P_1(C)$ is simply connected, the image in $\nu/(-1)$ of the unit sphere bundle of ν is non-trivial. All the above imply that the family $\mathcal{X} \to$

$P_1(C)$ is not trivial. A marking $\alpha : I^2(X) \hookrightarrow L$ extends uniquely to a metric injection $\alpha : I^2(\mathcal{X}/P_1(C)) \cong I^2(X)_{P_1(C)} \hookrightarrow L_{P_1(C)}$. Since $P_1(C)$ is simply-connected, this extends to an isomorphism of local systems $\pi_{*2}(Z_{\tilde{\mathcal{X}}}) \to L_{P_1(C)}$.

To construct the universal marked family of Kähler-Einstein generalized K3 surfaces over $K\Omega$, we need the Global Torelli Theorem for marked polarized generalized K3 surfaces.

Theorem 15 (Weakly Polarized Global Torelli Theorem) ([M1, p.319]). *Let (X, κ) and (X', κ') be polarized generalized K3 surfaces and let Y and Y' be the minimal resolutions. Suppose $\phi : I^2(X') \to I^2(X)$ is a Hodge isometry which respects polarizations and extends to an isometry $\psi : H^2(Y'; Z) \to H^2(Y; Z)$. Then there is a unique isomorphism $\Phi : X \to X'$ such that $\Phi^* = \phi$.*

The proof is immediate from the Global Torelli Theorem. This theorem implies that two marked generalized K3 surfaces with the same periods in $K\Omega$ are isomorphic in a unique way. If these polarizations come from Kähler-Einstein orbifold-metrics, the isomorphism is a holomorphic isometry of orbifolds. The condition that $\phi : I^2(X') \to I^2(X)$ extends to an isometry $\psi : H^2(Y'; Z) \to H^2(Y; Z)$ cannot be relaxed ([Mor1]). Namely this extendability essentially provides the full information that the singularities of X' and X are isomorphic. Morrison [Mor1, p.319 and pp.328-330] constructed an example of two non isomorphic polarized generalized K3 surfaces $(X_i, \kappa_i)(i = 1, 2)$ which have different singularities but isomorphic $I^2(X_i)$. These are polarized generalized K3 surfaces one with D_{12}-singularity and the other with D_4 and E_8-singularities. So the extendability is not relaxed.

Theorem 16 (Strong Surjectivity Theorem) ([KT, p.353]). *Any element in $K\Omega$ appears as the period of a marked generalized Kähler-Einstein K3 surface. In particular, any polarization of a generalized K3 surface is a Kähler polarization.*

The proof is based on Looijenga's method [Lo] in the proof of Theorem 9 and we use Theorems 13 and 14 (the existence and the isometric deformation of Kähler-Einstein orbifold structure) and the weak version of the surjectivity Theorem [Mor1, p.326] in a crucial way (see [KT, pp.353–355]). For reader's convenience we give firstly Looijenga's proof Theorem 9 and then the proof of Theorem 15.

Proof of Theorems 9 and 16. The proofs of these theorems are divided into parallel three steps. We start with the proof of Theorem 9.

Step 1. Suppose that for an element $(\kappa, [\omega]) \in K\Omega^0$ the 3-plain $(P_\omega + R \cdot \kappa) \cap L$ contains a primitive rank-2 lattice M all of whose vectors have square length in $4N$. Then $(\kappa, [\omega])$ belongs to image of the period map. Indeed, we may assume that $M \subset P_\omega$ by the isometric deformation for the Ricci-flat metric on a K3 manifold. The Density Theorem then implies that there exists a marked exceptional Kummer surface (X, α) such that $\alpha_C(H^{2,0}(X)) = [\omega]$. By composing α with some elements of the Weyl group $W([\omega])$ if necessary, we may assume that there exists a polarization ϕ of X such that $\alpha_R(\phi) = \kappa$. Since $H^{1,1}(X)$ is defined over Q, rational elements form a dense subset of the Kähler cone and the boundary of it. In particular, the Kähler cone contains a dense subset of rationally defined classes. Suppose ϕ itself is rational. Then there exists a unique (up to a constant multiple) holomorphic line bundle E with Chern class proportional to ϕ. We note that for an irreducible curve C, $E \cdot C = 0$ only if C is a (-2)-curve. Since ϕ is in $K\Omega$, $E \cdot C > 0$ for any effective curve C. Nakai's criterion [p.127, BPV] then implies that E is ample. In particular, ϕ contains a Kähler form. Since any ϕ can be expressed as a linear combination of rational elements in the Kähler cone with positive coefficients, ϕ contains a Kähler metric of the K3 surface X. Theorem 13 then implies that $(\kappa, [\omega])$ is a period point of a marked Kähler-Einstein generalized K3 surface.

Step 2. Suppose $(\kappa, [\omega])$ is such that $(P_\omega + R \cdot \kappa) \cap L$ contains a primitive vector x with $(x, x) \equiv 4 \pmod 8$. By isometric deformation, we may assume as in Step 1 that $x \in P_\omega$. $V^+([\omega])$ is divided into fundamental domains by reflection hyperplanes H_δ for the group $W([\omega])$, where δ are (-2)-vectors in $\Delta([\omega])$. Let K be the fundamental domain in $K\Omega^0 \times R$ (we get this space by forgetting the volume normalization) which contains κ. If $\eta \in K$ is such that $(P_\omega + R \cdot \eta) \cap L$ contains a rank-2 lattice all of whose vectors have square length in $4N$, then, by Step 1, $(\eta, [\omega])$ is a period point of a marked Kähler-Einstein K3 surface $(X_\eta, \phi_\eta, \alpha)$.[2] The isomorphism type of the K3 surface X_η is independent of the choice of η. Remark(8.4) and the proof of Proposition (8.2) in [p.256, BPV] directly implies that such η are dense in K. As in Step 1, we infer that the original $(\kappa, [\omega])$ is the period point of a marked Kähler-Einstein K3 surface.

Step 3. Let $(\kappa, [\omega]) \in K\Omega^0$ and K the fundamental domain with respect to the action of $W([\omega])$ which contains κ. If $\eta \in K$ is such that $P_\omega + R \cdot \eta$ contains a primitive vector of square length 4 (mod 8), then Step 2 implies that $(\eta, [\omega])$ is the period point of a marked Kähler-

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[2] X_η is a K3 surface, ϕ_η is the class of the corresponding Kähler-Einstein metric and α is a marking.

Einstein K3 surface. Applying Lemma (8.3) in [p.255, BPV] to the K3 lattice, we see that such η are dense in K. As in the preceding two steps, we see that $(\kappa, [\omega])$ is the period point of a marked Kähler-Einstein K3 surface. This completes the proof of Theorem 9.

Now we proceed to the proof of Theorem 16.

Step 1. Suppose $(\kappa, [\omega]) \in K\Omega$ is such that $(P_\omega + R\kappa) \cap L$ contains a primitive rank-2 lattice M. By the isometric deformation with respect to a Ricci-flat orbifold metric on a K3 orbifold, we may assume that $M \subset P_\omega$. We pick a point $(\kappa', [\omega]) \in K\Omega^0$ in a small neighborhood of $(\kappa, [\omega])$. Theorem 9 implies that there exists a marked Kähler-Einstein K3 surface (X, ϕ', α) which has the period point $(\kappa', [\omega])$. The isomorphism type of X is independent of the choice of κ'. There exists a class ϕ in the closure of the Kähler cone of X with $\alpha_R(\phi) = \kappa$. The class ϕ is contained in the intersection of finite number of reflection hyperplanes $\bigcap H_\delta$ and the positive cone of X. This is a part of the boundary of the Kähler cone and and forms a closed convex subcone of the positive cone. If we omit the intersection with other reflection hyperplanes, this is again divided into open convex subcones. Let K be the corresponding component in the fiber $(K\Omega - K\Omega^0)_\omega$ over $\omega \in \Omega$ containing κ. Suppose ϕ and hence κ are rational. Then the Lefschetz Theorem on (1,1)-Classes [GH, p.163] implies that there exists a holomorphic line bundle L on X with its Chern class proportional to ϕ. Then the line bundle L has the property that $L \cdot L > 0$ and $L \cdot C \geq 0$ for every irreducible curve C and the equality holds only if C is a (-2)-curve. Mayer's theorem [Ma] then implies that for a sufficiently large number n the linear system $|nL|$ has no base point and defines a holomorphic map $f_n : X \rightarrow P(H^0(X, O(nL)))$ which is biholomorphic onto its image outside (-2)-curves which do not meet L and which contracts these (-2)-curves connected-component-wise to points. In particular, ϕ contains a Fujiki-Moišezon-Kähler form. It follows from Theorem 13 that $(\kappa, [\omega])$ is the period point of a marked Kähler-Einstein generalized K3 surface (X, ϕ, α). Since P_ω is defined over Q, such κ are dense K. Hence the original $(\kappa, [\omega])$ is the period point of a marked Kähler-Einstein K3 surface.

Step 2. Suppose $(\kappa, [\omega]) \in K\Omega$ is such that $(P_\omega + R\kappa) \cap L$ contains a primitive rank one lattice M. As above, we may assume that $M \subset P_\omega$. Write $I^2(\kappa, [\omega])$ for the orthogonal complement in L_C of the set $\Delta(\kappa, [\omega])$ of all (-2)-vectors $\delta \in L$ with $(\kappa, \delta) = 0$ and $V^+(\kappa, [\omega])$ for the closed convex subcone of $V^+([\omega])$ defined by the equations $(\delta, \cdot) = 0$ for $\delta \in \Delta(\kappa, [\omega])$. $V^+(\kappa, [\omega])$ is divided into chambers by reflection hyperplanes H_δ for $\delta \in \Delta([\omega])$. Let K be the component containing κ. If $\eta \in K$ is such that $(P_\omega + R\eta) \cap L$ contains a primitive rank-2 lattice, then

there exists a marked Kähler-Einstein generalized K3 surface (X, ϕ_η, α) whose period is $(\eta, [\omega])$. The isomorphism type of X is independent of the choice of η. Such η with the property as above are dense in a convex cone K, since $I^2(\kappa, [\omega])$ is defined over Q and we have assumed that $M \subset P_\omega$. Now we use the following fact due to Varouchas [V].

Fact 4 ([V]). *Let X be an analytic variety admitting an open covering $\{U_i\}$ and a family of functions $\psi_i : U_i \to R$ which are continuous and strictly plurisubharmonic such that $\psi_i - \psi_j$ is pluriharmonic on $U_i \cap U_j$. Then X is Kählerian in the sense of Fujiki-Moišezon.*

Continuation of Proof of Theorem 16. It follows from Varouchas' theorem and Thecrem 13 that (X, ϕ_η) admits a Kähler form in the sense of Fujiki-Moišezon. Hence the original (X, ϕ, α) admits a Fujiki-Moišezon-Kähler form. Theorem 13 implies that $(\kappa, [\omega])$ is its period.

Step 3. Let $(\kappa, [\omega])$ be an arbitrary point in $K\Omega$ and K the component of $V^+(\kappa, [\omega])$ containing κ as in the Step 2. Since $I^2(\kappa, [\omega])$ is defined over Q, the η such that $(P_\omega + R\kappa) \cap L$ contains a rank-1 lattice form a dense subset in K. It follows from Step 2 that for such η there exists a Kähler-Einstein generalized K3 surface (X, ϕ_η, α) with $(\eta, [\omega])$ its period. The isomorphism type of X is independent of η. Varouchas' theorem implies that the class ϕ_η contains not only a Kähler-Einstein orbifold-form but also a Fujiki-Moišezon-Kähler form. Since K is a convex cone, $(\kappa, [\omega])$ is the period of a generalized K3 surface X with a Fujiki-Moišezon-Kähler form whose class is ϕ and from Theorem 13 it is the period of the Kähler-Einstein generalized K3 surface corresponding to (X, ϕ). This completes the proof of Theorem 16.

Theorem 15 and Theorem 16 imply that $K\Omega$ is isomorphic to the moduli space of the equivalence classes of marked generalized Kähler-Einstein K3 surfaces and the isomorphism is given by the period map. Before constructing the universal marked family of Kähler-Einstein generalized K3 surfaces, we look at examples to observe what occurs around the *holes* $K\Omega - K\Omega^0$ of the moduli space.

Example 4. Let (X, J_0) be a K3 surface with a (-2)-curve whose class in $H^2(X; Z)$ we denote by δ. We write $V^+(X)$ for $V^+(X, J_0)$, etc. We consider a smooth curve $a(t)$, $t \in [-\varepsilon, \varepsilon]$ in $V^+(X)$ such that $a([-\varepsilon, 0)) \subset V_P^+(X)$, $a(0) \in H_\delta)$ and $a((0, \varepsilon]) \subset s_\delta(V_P^+(X))$. Here $V_P^+(X)$ is the Kähler cone of X, H_δ is the hyperplane in L_R orthogonal to a (-2)-vector $\delta \in L$ and $s_\delta(V_P^+(X))$ is the reflection of $V_P^+(X)$ with respect to H_δ. Let α be a marking for X. The smooth curve $b(t) = \pi_2(\alpha(a(t), \omega_X))$ in N is such that $b(0) \in N - N^0$ and $b(t) \in N^0$ if $t \neq 0$. Then there is a differentiable real one-parameter family of marked

weakly polarized generalized K3 surfaces (X, J_t, κ_t) such that the period in N is the curve $b(t)$, κ_t is a polarization for (X, J_t) for $t \neq 0$ and $\kappa_0 \in H_\delta \cap V_P^{+-}(X)$ is a weak polarization for (X, J_0) (we can choose representatives J_t of the diffeomorphism-orbits of complex structures on X in a smooth way). The Kähler-Einstein metrics g_t of (X, J_t, κ_t), which are smooth if $t \neq 0$ and an orbifold metric if $t = 0$, vary smoothly with respect to t outside the (-2)-curve in (X, J_0) (for a proof see Section 2 of this paper). After performing a suitable (uniquely defined) isometric deformations parametrized by t (which vary smoothly with respect to t) to the family in such a way that we get a new family (X, I_t, λ_t) of Kähler-Einstein generalized K3 surfaces such that the period in $K\Omega$ is the α-image of $a(t)$ and that the complex structure tensors I_t and the Kähler-Einstein forms $h_t \in \lambda_t$ vary smoothly outside the (-2)-curve in $(X, I_0) = (X, J_0)$. All complex structures I_t for $t \in [-\varepsilon, 0)$ lie in the same diffeomorphism-orbit as J_0 and those for $t \in (0, \varepsilon]$ lie in the same diffeomorphism-orbit as $S_{\delta*}(J_0)$, where S_δ is a diffeomorphism of X which induces the reflection s_δ (see Example 3 for the definition).

Example 5.   There is another description for the one-parameter family (X, I_t, λ_t) in Example 4. We recall the universal marked family of Kähler-Einstein K3 surfaces over $K\Omega^0$. We get two families of Kähler-Einstein K3 surfaces with the same marking. One is over $\alpha(a(t))$, $t \in (-\varepsilon, 0)$, and the other is over $\alpha(a(t))$, $t \in (0, \varepsilon)$. The Global Torelli Theorem implies that there is a unique isomorphism of the fibers over $a(t)$ and $s_\delta(a(t))$. This induces an isomorphism of the limit generalized Kähler-Einstein K3 surfaces over $a(-0)$ and $a(+0)$. We glue these two families by this isomorphism. We thus get a family $\mathcal{X}$ of generalized Kähler-Einstein K3 surfaces over $B = (-\varepsilon, \varepsilon)$. The minimal resolution over 0 yields a family $\mathcal{Y}$ of K3 surfaces which is differentiably trivial. The relevant marking for the family of Kähler-Einstein generalized K3 surfaces $\mathcal{X}$ is not a marking for the family $\mathcal{Y}$ but a marking for $\mathcal{X}$ defined by using the sheaf $I^2(\mathcal{X}/B)$ (the similar definition is possible in this case) and the action of the reflection s_δ.

The inductive use of the argument in Example 5 gives the following

Theorem 17.   *There exists a universal marked family $\mathcal{M}$ of generalized Kähler-Einstein K3 surfaces over $K\Omega$ such that the restriction over $K\Omega^0$ coincides with the universal marked family $\mathcal{M}^0$ in Theorem 9 and the restriction over $K\Omega - K\Omega^0$ is the universal marked family of generalized Kähler-Einstein K3 surfaces with orbifold-singularities.*

§2. Ricci-flat K3 surfaces with concentrated curvature

The purpose of this section is to derive the *a priori* estimates for Ricci-flat Kähler metrics on a K3 surface if its curvature concentrates along some distinguished (-2)-curves. The result is that a concentrated Ricci-flat Kähler metric on a K3 surface looks like a superposition of ALE gravitational instantons (see [Kr]), with an Ricci-flat Kähler orbifold-metric on the corresponding generalized K3 surface as a background metric. This was observed implicitly in [KT]. Bando-Kasue-Nakajima [BKN] showed that this phenomenon of curvature concentration characterizes the limit behavior for families of Einstein metrics on a 4-manifold with a lower bound of volumes and an upper bound of diameters. This is well understood if one pays attention to the case of a Kummer surface. So we first look at a Kummer surface with concentrated curvature. The following arguments remain valid in other flat background case. We carry out the estimates as in [Y2] by solving the Monge-Ampère equation. We first piece together a flat orbifold-metric on a Kummer surface and 16 Eguchi-Hanson spaces [EH] with concentrated curvature to construct a family of almost Ricci-flat metrics as in [Pa] and [GP](see also [Hi3]). We then want to get such estimates that measure the deviation of the approximate Ricci-flat metrics with concentrated curvature thus obtained from the nearest Ricci-flat metric in terms of the strength of the curvature concentration.

Let (M, g) be a compact Riemannian manifold. We denote L^p-norms with respect to the metric g by $\| \cdot \|_p$. For $p = \infty$ this gives the C^0-norm, i.e., the maximum norm. We use Li's Sobolev inequality [Li] on a compact Riemannian manifold (M, g) of dimension $m \geq 3$

$$(1) \qquad \frac{c(n)}{c_1(g)} \|df\|_2 + v(g)^{\frac{1}{n}} \|f\|^2 \geq \|f\|_{\frac{2m}{m-2}},$$

for L_1^2-functions[3](see [Li] and [Bom]). In (1), $c_1(g)$ stands for the conformally invariant isoperimetric constant of (M, g)

$$c_1(g) = \inf\{ \frac{\mathrm{Vol}(\partial \Omega)}{\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}} \}$$

where inf is taken over domains Ω in M with smooth boundary and $\mathrm{Vol}(\Omega) \leq \frac{1}{2} \mathrm{Vol}(M)$.

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[3] $L_k^p(M, g)$ denotes the Sobolev space of functions with L^p-derivatives up to order k

The Sobolev inequality plays a central role in deriving the *a priori* estimates for the solution of the Monge-Ampère equation. We need to control the constants $c_1(g)$ for such a family of metrics g that has the property (P) : there exist positive constants $k_i(i = 1,\ldots,5)$ with $|\operatorname{Ric}| \le k_1$, $k_2 \le d(g) \le k_3$ and $k_4 \le v(g) \le k_5$, where $d(g)$ and $v(g)$ denote the diameter and the volume of (M,g), respectively. For such a family of metrics, there is a powerful isoperimetric inequality due to Gallot [Ga] which gives the control on $c_1(g)$ in the Sobolev inequality (1). Let (M,g) be as above. We then introduce a scale invariant quantity a_g defined by

$$a_g = d(g)^2 \inf_x \{\operatorname{Ric}(x,x)/(m-1); |x|_g = 1\},$$

where x runs over all tangent vectors to M with length 1. Gallot's isoperimetric inequality is stated as follows.

Fact 5 ([Ga]). *Let (M,g) be as above. Then there exists a constant $\kappa(m, a_g)$ depending only on m and a_g such that the scale-invariant number*

$$\bar{c}_1(g) = \kappa(m, a_g)v(g)^{\frac{1}{m}} d(g)^{-1}$$

satisfies

(2) $\|df\|_1 \ge \bar{c}_1(g)\|f\|_{\frac{m}{m-1}},$

for every $f \in L_1^2(M,g)$ with $\int_M f\,dv_g = 0$.

Applying Hölder's inequality to Gallot's isoperimetric inequality, we have

(3) $\|df\|_2 \ge \dfrac{(m-2)\bar{c}_1(g)}{2(m-1)}\|f\|_{\frac{2m}{m-2}}$

for every $f \in L_1^2(M,g)$ with $\int_M f\,dv_g = 0$. Using Li's Sobolev inequality (1) and Gallot's inequality (2), we now study Ricci-flat metrics on a Kummer surface quantitatively. Let $X' = Y/< 1, i >$ be a generalized K3 surface constructed from a complex 2-torus Y acted on by the involution $i : z \mapsto -z$. Y has 16 simple singularities of type A_1 and its minimal resolution is the Kummer surface $X = \operatorname{Km}(Y)$. Since each of the 16 singularities of X' is analytically a quadratic cone, it is surrounded by $P_3(R)$. On the other hand, there is a family of asymptotically locally Euclidean complete Ricci-flat Kähler metrics with boundary $P_3(R)$ at infinity (see [EH] and [C1]), which is called the *Eguchi-Hanson metric*.

Roughly speaking, Eguchi-Hanson metric is a Ricci-flat complete Kähler metric on the total space of the holomorphic cotangent bundle $T^*P_1(C)$, which is diffeomorphic to the real tangent bundle TS^2, and is asymptotically a canonical flat metric of the quadratic cone and is determined by a single parameter which measures the strength of the curvature concentration. We glue 16 Eguchi-Hanson spaces with sufficiently concentrated curvature to an orbifold-flat background metric on X' along the boundaries of small neighborhoods of 16 singular points which are 16 copies of $P_3(R)$. From algebro-geometric point of view, this surgery is nothing but the minimal resolution of a simple singularity of type A_1.

We now look at the Eguchi-Hanson metric in detail. The Eguchi-Hanson space ([EH], [C1]) is the minimal resolution $q : E \to C^2/<1, i>$ equipped with a $SO(3)$-invariant Ricci-flat complete asymptotically locally Euclidean Kähler metric. Such metrics are solutions of Einstein's equation with an $SO(3)$-symmetry which respects both complex structure and metric. It follows that the Einstein equation for such metrics is reduced to an ordinary differential equation for the *Kähler potential* which is a scalar function. Therefore we seek a Ricci-flat Kähler metric on $C^2 : (z_1, z_2)$ in the form $\sqrt{-1}\partial\bar{\partial}f$, where $f = f(u)$ and $u = |z_1|^2 + |z_2|^2$. The equation for a Ricci-flat Kähler metric is

$$(4) \qquad (\sqrt{-1}\partial\bar{\partial}f)^2 = \omega_0^2,$$

where $\omega_0 = \sqrt{-1}\partial\bar{\partial}u$ is the standard flat metric of C^2. Since

$$\sqrt{-1}\partial\bar{\partial}f = f'(u)\sqrt{-1}\partial\bar{\partial}u + f''(u)\sqrt{-1}\partial u \wedge \bar{\partial}u,$$

the equation (4) reduces to

$$(5) \qquad f'(u)^2 + uf'(u)f''(u) = 1.$$

Putting $f'(u) = g(u)$ reduces (5) to

$$(6) \qquad \frac{1}{2u}\frac{d(u^2g(u)^2)}{du} = 1.$$

We therefore have from (6) that

$$(7) \qquad f'(u) = g(u) = \sqrt{1 + \frac{a^2}{u^2}}.$$

It is elementary to integrate (7) getting

$$(8) \qquad f_a(u) = \text{const.} + u\sqrt{1 + \frac{a^2}{u^2}} + a \log \frac{u}{\sqrt{u^2 + a^2} + a}.$$

If $a > 0$ then this function projects down to a strictly plurisubharmonic function on the complement of the vertex of the quadratic cone $C^2/ < 1, i >$ or equivalently on the complement of the zero-section E of $T^*P_1(C)$. We therefore assume in (8) that $a > 0$. The function f_a behaves asymptotically like u at infinity and like $a \log u$ in the limit of $u \to 0$, i.e., near the vertex of the quadratic cone. The Kähler form $\sqrt{-1}\partial\bar{\partial}f_a$ extends smoothly across the zero-section of $T^*P_1(C)$ and defines a complete Ricci-flat asymptotically locally Euclidean Kähler metric on the whole $T^*P_1(C)$. This is the *Eguchi-Hanson metric* g_a and the resulting space is the Eguchi-Hanson space M_a ([EH], [C]). Let $g_{i\bar{j}}$ be the components of this metric in the standard coordinate system z_i ($i = 1, 2$) of C^2. From the explicit solution (8), we get the following estimates:

$$(9) \qquad f_a(u) = u + O(\frac{1}{u}) \quad \text{as } u \to \infty,$$

$$(10) \qquad g_{i\bar{j}} = \delta_{ij} + O(\frac{1}{u^2}) \quad \text{as } u \to \infty,$$

$$(11) \qquad \partial^k g_{i\bar{j}} = O(((\frac{1}{u})^{\frac{1}{2}})^{4+k}) \quad \text{as } u \to \infty.$$

In particular, the metric is asymptotically locally Euclidean in the sense that, writing R for the curvature tensor,

$$(12) \qquad |R| = O(\frac{1}{u^3}) \quad \text{as } u \to \infty.$$

We now examine the behavior of the metric near the zero-section. Let E be the (-2)-curve defined by the zero-section and L_E the holomorphic line bundle associated to a divisor E. There is a unique (up to a constant multiple) holomorphic section σ for L_E with E as its zero locus. Let $\|\cdot\|$ denote the Hermitian metric for L_E induced from the flat metric of C^2. Near the zero-section E the Kähler potential f_a of the Eguchi-Hanson metric g_a can be written, modulo a smooth function which is a constant on the zero-section, as

(13)
$$a \log \frac{\|\sigma\|^2}{\sqrt{\|\sigma\|^4 + a^2} + a}.$$

Hence the Kähler form of the Eguchi-Hanson metric g_a defines a cohomology class $-2\pi a[E]$ in $H^2_{\text{comp}}(T^*P_1(C); Z)$ where $[E]$ is the Poincaré dual of the class of E in $H_2(T^*P_1(C); Z)$. This together with the $SO(3)$-symmetry implies that the metric induced from g_a on E is a metric of constant Gaussian curvature a^{-1} with volume equal to $4\pi a$. We note that the diagonal C^*-action on C^2 given by $(z, w) \mapsto (\zeta z, \zeta w)$ for $\zeta \in C^*$ projects down to the canonical C^*-action on $T^*P_1(C)$. The S^1-part of this action is isometric relative to g_a. Since the (-2)-curve E is the fixpoint set of this C^*-action, the 2-sphere E is a totally geodesic submanifold. The maximum of the absolute value of sectional curvature of g_a is a^{-1} and is attained at any tangent 2-plane of E and its orthogonal complement. The metric spheres in the Eguchi-Hanson space are $P_3(R)$ defined by $u = d$ for $d > 0$ and the induced metric is that of the Berger sphere, i.e., the size of the fiber in the Hopf fibration $P_3(R) \to S^2$ goes to zero as $d \to 0$. We sum up: for small positive number a, the Kähler manifold $(T^*P_1(C), \sqrt{-1}\partial\bar{\partial}f_a)$ contains a highly positively curved totally geodesic 2-sphere E of curvature $\frac{1}{a}$. Since in the limit of $a \to 0$ the Kähler potential f_a approaches the flat Kähler potential $f_0 = u$ in the C^∞-topology on any compact subset of the complement of E, for very small a the curvature of g_a is highly concentrated in a very small neighborhood of E and the Eguchi-Hanson space M_a becomes "undistinguishable" from the flat quadratic cone in a very small distance from E. The limit metric is the flat orbifold metric g_0 of the quadratic cone. Writing ϕ_t for the biholomorphism of $T^*P_1(C)$ corresponding to $t \in C^*$, we express this process of curvature concentration in the form, with $t \in R_{>0}$, of:

(14)
$$\phi_t^*\left(\frac{g_a}{t^2}\right) = g_{\frac{a}{t^2}}.$$

This is the approximation of the tangent cone at infinity.

Let us now return to K3 surfaces. Let $X' = Y/ <1, i>$ and $X = \text{Km}(Y)$ be as above. Let κ be any class of flat orbifold-Kähler form contained in the Kähler cone $V_P^+(X') \subset V_P^+(X) \subset H^{1,1}(X) \cap H^2(X; R)$ of the generalized K3 surface X'. Such Kähler form comes from a flat Kähler form on the torus Y. Let E_i $(i = 1, \ldots, 16)$ be the distinguished (-2)-curves in X and e_i $(i = 1, \ldots, 16)$ their classes in $H^2(X; Z)$. For sufficiently small 16 positive numbers $a = \{a_i \ (i = 1, \ldots, 16)\}$, the class

(15) $$\kappa(a) = \kappa - \sum_{i=1}^{16} a_i e_i$$

is contained in the cone generated by $V_P^+(X)$, the Kähler cone of X. It follows from Yau's existence theorem [Y2] that the class (15) is represented by a unique Ricci-flat metric. The degree of freedom in the definition of (15) is

$$\dim_R(H^{1,1}(Y) \cap H^2(Y; R)) + 16 = 4 + 16 = 20,$$

which is equal to $\dim_R(H^{1,1}(X) \cap H^2(X; R)) = 20$ the dimension of the Kähler cone of a K3 surface. This implies that all Kähler classes sufficiently close to $\kappa \in V_P^+(X\prime)$ have the form of (15). We construct an approximate Ricci-flat Kähler metric on X whose cohomology class is of the form (15) in the following way (cf. [H3], [Pa] and [GP]). We want to get a *scale-invariant a priori* estimate for the perturbation which we must make to reach the Ricci-flat Kähler form in the same cohomology class. So, we multiply the class (15) by some positive constant if necessary, and we assume that for each singular point $p_i \in X'$ ($i = 1, \ldots, 16$) there exists a pseudoconvex neighborhood U_i which is isomorphic to $B_{1+\delta}/ < 1, i >$ for some fixed positive number $\delta \ll 1$, where B_t denotes a metric ball in C^2 of radius t. Here the isomorphism is concerned with both complex structure and metric structure induced from the complex Euclidean space C^2. Assume moreover $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $r = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$ be the distance function in C^2 and $t(r)$ a real valued function on R such that $t(r) = 1$ for $r \leq 1 - \delta$ and $t(r) \equiv 0$ for $r \geq 1$ and $|t'(r)| \leq \frac{2}{\delta}$. If a positive number a_i is sufficiently small, then the function $F_i(r) = F_{a_i}(r)$ defined by

(16) $$F_i(r) = (1 - t(r))r^2 + t(r)f_i(r)$$

$$= r^2 + t(r)\left\{ r^2 \left(\sqrt{1 + \frac{a_i^2}{r^4}} - 1 \right) + a_i \log \frac{r^2}{a_i + \sqrt{a_i^2 + r^4}} \right\},$$

where $f_i(r) = f_{a_i}(u)$, as appeared in the Kähler potential of the Eguchi-Hanson metric, is a Kähler potential on the minimal resolution $\hat{U}_i$ of U_i over the simple singularity $p_i \in U_i$. The kähler metric $\sqrt{-1}\partial\bar{\partial}F_i$ coincides with the Eguchi-Hanson metric on the minimal resolution $\hat{U}_{1-\delta'}$ of $B_{1-\delta'}/ < 1, i >$ where $\delta < \delta' \ll 1$ and coincides with the flat metric

on $(B_{1+\delta} \backslash B_1)/ < 1, i >$. We can thus glue 16 Eguchi-Hanson metrics $\sqrt{-1} \partial \bar{\partial} F_i$ defined in a small neighborhood of p_i $(i = 1, \ldots, 16)$ with an appropriate flat orbifold Kähler metric (torus-metric) ω_0 on X' to construct an approximate Ricci-flat Kähler metric form ω_a, where $a = \{a_i \ (i = 1, \ldots, 16)\}$ is a 16-ple of sufficiently small positive numbers. The quantity a_i and $\frac{1}{a_i}$ respectively measures the smallness of the volume of the (-2)-curves E_i and the strength of the localization of curvature of ω_a along E_i. The class of the Kähler class ω_a is of the form (15) in which κ corresponds to the torus metric ω_0. The Kähler metric ω_a is not Ricci-flat only in the 16 neck regions N_i given by $t_i(r)(1 - t_i(r)) \neq 0$ where $t_i(r)$ are interpolation functions used in the definitions of F_i $(i = 1, \ldots, 16)$. It is easy to see that if a is sufficiently small, the sectional curvatures in the neck regions N_i are bounded by $O(a_i^2)$ as a_i goes to zero. Yau's existence theorem [Y2] tells us that there exists a smooth function u_a on X such that $\tilde{\omega}_a = \omega_a + \sqrt{-1} \partial \bar{\partial} u_a$ is a Ricci-flat Kähler form and such u_a is unique up to an additive constant. Let $\eta = \frac{1}{\sqrt{2}} dz_1 \wedge dz_2$ be a holomorphic 2-form in C^2. Any Ricci-flat volume form on X is a constant multiple of $\eta \wedge \bar{\eta}$. Yau's existence theorem is then equivalent to saying that the following Monge-Ampère equation (17) with a normalization condition (18) has a unique solution:

$$(17) \qquad (\omega_a + \sqrt{-1} \partial \bar{\partial} u_a)^2 = \exp(g_a) \omega_a^2$$

$$(18) \qquad \int_X u_a \, \omega_a^2 = 0$$

where a real valued function g_a is defined by the following relation:

$$(19) \qquad g_a = \log(\frac{\eta \wedge \bar{\eta}}{\omega_a^2}).$$

A direct computation shows that the function g_a is a_i^2-times a smooth function h_i on in a neighborhood V_i of E_i containing the closure of the neck region N_i and moreover h_i and their derivatives converge uniformly on V_i. We note that u_a is identically zero outside of the 16 neck regions. In particular, we have

$$(20) \qquad \|g_a\|_\infty \leq C |a|^2,$$

for some constant C independent of a, where $|a|^2 = \sum_{i=1}^{16} a_i^2$. Hereafter C always means a positive constant independent of a which may differ

in different occurrences. Following [Y2] (see also [Bou], [Au] and [BK]), we derive the a priori estimates for the solution u_a of (17) and (18). We begin with the C^0-estimate.

$$(21) \qquad (1 - e^{g_a})\omega_a^2 = \tilde{\omega}_a^2 - \omega_a^2$$
$$= \sqrt{-1}\partial\bar{\partial}u_a \wedge (\tilde{\omega}_a - \omega_a)$$

Multiplying (21) by $|u_a|^{p-2}u_a$ with $p > 1$ and applying integration by parts give

$$(22) \quad \int_X (1 - e^{g_a})|u_a|^{p-2}u_a\,\omega_a^2$$
$$= \frac{4(p-1)}{p^2} \int_X \sqrt{-1}\partial|u_a|^{\frac{p}{2}} \wedge \bar{\partial}|u_a|^{\frac{p}{2}} \wedge (\tilde{\omega}_a + \omega_a)$$
$$\geq \frac{4(p-1)}{p^2} \int_X \sqrt{-1}\partial|u_a|^{\frac{p}{2}} \wedge \bar{\partial}|u_a|^{\frac{p}{2}} \wedge \omega_a.$$

Hereafter the volume element in integrations over X is always ω_a^2. Since g_a is very small, we have from (22)

$$(23) \qquad \int_X |d|u_a|^{\frac{p}{2}}|^2 \leq Cp \int_X |g_a||u_a|^{p-1}.$$

We now put $p = 2$. Then we can apply (3) (the Sobolev inequality) to the left side of (23). Applying Hölder's inequality to the right side, we have

$$\left(\int_X |u_a|^4\right)^{\frac{1}{2}} \leq C\left(\int_X |g_a|^q\right)^{\frac{1}{q}}\left(\int_X |u_a|^{q'}\right)^{\frac{1}{q'}},$$

where $q, q' > 0$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Putting $q' = 4$ ($q = \frac{4}{3}$), we have

$$(24) \qquad \left(\int_X |u_a|^4\right)^{\frac{1}{4}} \leq C\left(\int_X |g_a|^q\right)^{\frac{1}{q}}.$$

Combining (24) and (20) gives an *a priori* L^4-estimate for u_a:

$$(25) \qquad \|u_a\|_4 \leq C|a|^2.$$

Applying again the a-independent Sobolev inequality (1) and the Hölder inequality to (23), we have

$$\|u_a\|_{2p}^p \leq C\|d|u_a|^{\frac{p}{2}}\|_2^2 + C'\||u_a|^{\frac{p}{2}}\|_2^2$$

$$\leq Cp\|g_a\|_p\|u_a\|_p^{p-1} + C'\|u_a\|_p^p$$

(26)
$$\leq (A + \frac{B|a|^2p}{\|u_a\|_p})\|u_a\|_p^p.$$

or

(27)
$$\|u_a\|_{2p}^p \leq (A + \|u_a\|_p + B|a|^2p)\|u_a\|_p^{p-1}.$$

Put $p = p_n = 2^n p_0$ with $p_0 = 4$. Suppose

$$\|u_a\|_{p_n} \leq C_n|a|^2.$$

It follows from (25) that this is the case when $n = 0$. Moreover we have from (26) and (27) that $\|u_a\|_{p_{n+1}} \leq C_{n+1}|a|^2$ with a constant C_{n+1} which obeys the following inequalities:

(28) $\qquad C_{n+1} \leq (A + 2^n B)^{\frac{1}{2^n}} C_n \quad$ if $C_n \geq 1$

(29) $\qquad C_{n+1} \leq (A + 2^n B)^{\frac{1}{2^n}} \quad$ if $C_n \leq 1$.

It is easy to show applying induction to (28) and (29) that $C_n \leq C$ for some constant C independent of a and n. This implies the *a priori* C_0-estimate for the solution u_a of (17) and (18):

(30) $\qquad \|u_a\|_\infty \leq C|a|^2.$

We now proceed to deriving the C^2-estimate. Let R_a be the maximum of the absolute value of holomorphic bisectional curvature [KN] of the Kähler metric ω_a. Then $R_a = \max_{1 \leq i \leq 16}\{a_1^{-1}, \ldots, a_{16}^{-1}\}$. We set $c_a = 2R_a$. Let $\triangle_a$ and $\tilde{\triangle}_a$ be respectively the Laplacians of the Kähler metrics ω_a and $\tilde{\omega}_a = \omega_a + \sqrt{-1}\partial\bar{\partial}u_a$, e.g., $\triangle_a = \sum_{\alpha,\beta} g^{\alpha\bar{\beta}}\nabla_\alpha\nabla_{\bar{\beta}}$, where $g_{\alpha\bar{\beta}}$ and ∇ are respectively the components and the Levi-Civita connection of the metric ω_a. The inequality in [Au1] and (2.22) in [Y2] reads in our case as follows

(31) $e^{c_a u_a} \tilde{\triangle}_a (e^{-c_a u_a}(2 + \triangle_a u_a))(x)$

$$\geq A(x) + B(x)(2 + \triangle_a u_a) + C(x)(2 + \triangle_a u_a)^2,$$

where $A(x)$, $B(x)$ and $C(x)$ are

$$A(x) = \triangle_a g_a(x) - 4 \inf_{i \neq j}(-R_{i\bar{i}j\bar{j}})(x),$$

$$B(x) = -2c_a,$$

$$C(x) = (c_a + \inf_{i \neq j}(-R_{i\bar{i}j\bar{j}})(x))e^{g_a(x)}.$$

Here we have used the same convention as in [KN] on the curvature tensor and have taken the subscripts i and j with respect to any local unitary frame for the holomorphic tangent bundle of X. We define a continuous function $k(x)$ by

$$k(x) = - \inf_{i \neq j}(-R_{i\bar{i}j\bar{j}})(x)/R_a,$$

where inf is taken for all unitary frames at x. It is clear from the definition that $|k(x)| \leq 1$. Using (31), we obtain an *a priori* estimate for $\triangle_a u_a$. Suppose that $e - c_a u_a(2 + \triangle_a u_a)$ assumes its minimum at $x \in X$. Then we have from (31)

$$0 \geq e^{c_a u_a} \tilde{\triangle}_a (e^{-c_a u_a}(2 + \triangle_a u_a))(x)$$

$$= (\triangle_a g_a(x) + 4k(x)R_a) - 4R_a(2 + \triangle_a u_a)(x)$$

$$+ (2 - k(x))R_a e^{g_a}(2 + \triangle_a u_a)^2(x)$$

$$= e^{g_a}(2 - k(x))R_a \Big\{ \big((2 + \triangle_a u_a)(x) - \frac{2e^{-g_a(x)}}{2 - k(x)}\big)^2$$

$$- \big(\frac{2e^{-g_a(x)}}{2 - k(x)}\big)^2 + \frac{e^{-g_a(x)}(\triangle_a g_a + 4R_a k)(x)}{(2 - k(x))R_a} \Big\}.$$

Hence we have

(32) $\left| (2 + \triangle_a u_a)(x) - \dfrac{2e^{-g_a(x)}}{2 - k(x)} \right|$

$$\leq \left| \left(\frac{2e^{-g_a(x)}}{2 - k(x)}\right)^2 - \frac{e^{-g_a(x)}(\triangle_a g_a + 4R_a k)(x)}{(2 - k(x))R_a} \right|^{\frac{1}{2}}.$$

If x lies outside of the neck regions $\bigcup U_i$, then $g_a(x) = 0$ and $\triangle_a g_a(x) = 0$. In this case (32) becomes

$$(33) \qquad 2 + \triangle_a u_a(x) \leq \frac{2}{2 - k(x)}$$

$$+ \left| \left(\frac{2}{2 - k(x)} \right)^2 - \frac{4k(x)}{2 - k(x)} \right|^{\frac{1}{2}} = 2.$$

If x is in the neck regions, we have

$$|g_a(x)| \leq C |a|^2,$$

$$|\text{sectional curvature at } x| \leq C |a|^2,$$

$$|\triangle_a g_a(x)| \leq C|a|^2$$

and hence $|k(x)| \leq C |a|^3.$

These estimates together with $\frac{1}{R_a} = \min_{1 \leq i \leq 16}\{a_i\} \leq |a|$ give

$$(34) \quad (2 + \triangle_a u_a)(x) \leq \frac{2(1 + C|a|^2)}{2 - C|a|^3}$$

$$+ \left| \left(\frac{2(1 + C|a|^2)}{2 - C|a|^3} \right)^2 - \frac{C(1 + C|a|^2)|a|^3}{2 - C|a|^3} \right|^{\frac{1}{2}}$$

$$\leq 2 + C|a|^2.$$

It follows from (33) and (34) that

$$(35) \qquad 2 + \triangle_a u_a \leq e^{c_a u_a} e^{-c_a u_a(x)}(2 + \triangle_a u_a)(x)$$

$$\leq e^{c_a(\sup u_a - \inf u_a)}(2 + C|a|^2).$$

Putting $r_a = \frac{\max\{a_i\}}{\min\{a_i\}}$, we have

$$c_a = \frac{r_a}{\max\{a_i\}} \leq \frac{16 r_a}{|a|}.$$

We have then from (35)

$$(36) \qquad 2 + \triangle_a u_a \leq e^{C r_a |a|}(2 + C|a|^2)$$

$$\leq 2 + C r_a |a|.$$

The equation (17) and the estimate (20) imply

$$(37) \qquad 2 + \triangle_a u_a = \sum_i (1 + u_{i\bar{i}})$$

$$\geq 2\sqrt{\prod_i (1 + u_{i\bar{i}})} = 2e^{\frac{g_a}{2}}$$

$$\geq 2(1 - C|a|^2).$$

It follows from the estimates (36),(37),(20) and the equation (17) that

$$(38) \qquad (1 - C|a|^{\frac{1}{2}})\omega_a \leq \tilde{\omega}_a \leq (1 + C|r_a a|^{\frac{1}{2}})\omega_a.$$

This gives an *a priori* C^2-estimate for u_a. By the Hölder estimates for second derivatives [GT, Theorem 17.14] for fully nonlinear elliptic equations, we have an *a priori* $C^{2,\alpha}$-estimate of u_a for some $0 < \alpha < 1$:

$$(39) \qquad \|u_a\|_{C^{2,\alpha}} \leq C$$

which is independent of a. We now consider the following equation containing a parameter $t \in [0,1]$ with a normalization condition:

$$(40) \qquad (\omega_a + \sqrt{-1}\partial\bar{\partial}u_{a,t})^2 = (1 + t(e^{g_a} - 1))\omega_a^2,$$

$$(41) \qquad \int_X u_{a,t}\omega_a^2 = 0.$$

The case $t = 1$ is the equation (17) for a Ricci-flat Kähler metric. By Yau's solution to Calabi's conjecture [Y2], the proof of which we have followed to examine the behavior of the solution u_a of (17) in the limit $a \to \infty$, the equation (40) (with normalization (41)) has a unique solution. So we can argue in the following way to get a better C^2 estimate and estimates for higher derivatives of u_a. Since $\log\{1 + t(e^{g_a} - 1)\}$ has the same properties as g_a, it follows from the above discussion in the C^0 estimate that there are C^0 and C^2 estimates (30) and (38) for $u_{a,t}$ which are independent of $t \in [0,1]$ as well as a. Differentiating the equation (40) with respect to t, we have

$$(42) \qquad \triangle_{a,t}\frac{\partial u_{a,t}}{\partial t} = \frac{e^{g_a} - 1}{1 + t(e^{g_a} - 1)}.$$

From the above discussion, we have an *a priori* estimate

$$(43) \qquad \mathrm{Osc}(\frac{\partial u_{a,t}}{\partial t}) \le C|a|^2$$

where Osc means the oscillation. The estimate (43) is independent of a and $t \in [0,1]$. It follows from the a priori C^0 estimate (30) for u_a, (42), (43) and the interior Schauder estimates (see [Au, p.88] and [GT, Chapter 6]) that there is an *a priori* $C^{2,\alpha}$ estimate

$$(44) \qquad \|\frac{\partial u_{a,t}}{\partial t}\|_{C^{2,\alpha}} \le C|a|^2$$

independent of a and $t \in [0,1]$ on any relatively compact subdomain in the complement of the sixteen (-2)-curves. Integrating the estimate (44), we infer that, on any compact subdomain disjoint from (-2)-curves, there is an *a priori* $C^{2,\alpha}$ estimate:

$$(45) \qquad \|u_a\|_{C^{2,\alpha}} \le C|a|^2.$$

By using bootstrapping argument, we have $C^{k,\alpha}$ estimates

$$(46) \qquad \|u_a\|_{C^{k,\alpha}} \le C_k|a|^2.$$

In a neighborhood of a (-2)-curve, the metric ω_a has concentrated curvature and we must consider the scale change (cf. (14)) to get a proper C^2 estimate of u_a. As a result, we have in a neighborhood of a distinguished (-2)-curve

$$(47) \qquad \|u_a\|_{C^2} \le C|a|$$
$$(48) \qquad \|u_a\|_{C^4} \le C.$$

Estimates (45)–(48) imply that ω_a converges to the orbifold metric ω_0 and the curvature concentrates along distinguished (-2)-curves and the metric looks like the Eguchi-Hanson metric up to the curvature level.

Next we consider the case of general background. We start with a generalized K3 surface with an orbifold Ricci-flat (generally non-flat) Kähler metric. By [Kr], there exists an ALE gravitational instanton corresponding to each singularity. At a singularity, we take holomorphic

normal coordinates (x, y) such that the background metric is of the form $\delta_{ij} + [2]$ ($[2]$ begins with a quadratic term) and the holomorphic 2-form is $dx \wedge dy$. The Kähler potential of any ALE gravitational instanton has the same asymptotic properties as that of the Eguchi-Hanson metric. We choose an ALE Ricci-flat Kähler metric with sufficiently strong curvature concentration so that, for example, the are of any (-2)-curve is grater than $\frac{1}{2}a$ and smaller than $\frac{3}{2}a$ where a is a small positive number ($\frac{1}{a}$ measures the strength of the curvature concentration). We glue two Kähler potentials, one for the background metric $r^2 + [4]$ and the other for the ALE gravitational instanton $r^2 + \frac{a^2}{r^2}$ (for $r \approx a^\beta$ for $\beta < \frac{1}{2} < 1$), by applying the construction (16). Note that the diameter of the standard 2-sphere of area a is $O(\sqrt{a})$. If we set $\beta = \frac{3}{8} < \frac{1}{2}$, we get an approximate Ricci-flat metric ω_a with

$$(49) \qquad g_a = \log \frac{\eta \wedge \bar{\eta}}{\omega_a^2} = O(a^{\frac{1}{2}}).$$

Consider the equations (17) and (18). Arguing in the same way as in the Kummer surface case (cf. (25)), we have

$$(50) \qquad \|u_a\|_\infty \leq C a^{\frac{13}{8}}$$

$$(51) \qquad \|u_a\|_{C^{2,\alpha}} \leq C a^{\frac{1}{4}}.$$

From the Interior Schauder estimates we get

$$(52) \qquad \|u_a\|_{C^{k,\alpha}} \leq C_k a^{\frac{13}{8}}$$

on any relatively compact subdomain in the complement of distinguished (-2)-curves and

$$(53) \qquad \|u_a\|_{C^2} \leq C a^{\frac{5}{8}}$$

$$(54) \qquad \|\frac{u_a}{a}\|_{C^4(\frac{\omega_a}{a})} \leq C a^{\frac{5}{8}}$$

in a neighborhood of a distinguished (-2)-curve. These imply that ω_a converges to the orbifold-metric ω_0 and the curvature concentrates along the distinguished (-2)-curves and after the scale change

$$\omega_a \to \frac{\omega_a}{a}$$

we capture the corresponding bubble of ALE gravitational instanton.

We thus get the following convergence theorems:

Theorem 18. *Let X be a generalized K3 surface with a flat orbifold Kähler metric ω_0 and $Y \to X$ be the minimal resolution. Let ω_a be a Kähler metric on Y as in (16) (with $r \approx 1$), i.e., ω_a is constructed by gluing the flat metric ω_0 with ALE gravitational instanton metrics corresponding to the singularities with the areas of the distinguished (-2)-curves pinched between ca and $c^{-1}a$ for some positive number c (so the curvature is pinched between ac^{-1} and $c^{-1}a^{-1}$). We consider the limit $a \to 0$. Then, for any relatively compact subdomain Ω in the regular part of X, there exist positive numbers C, $\alpha \in (0, 1)$ and $C_k(c, \Omega)$ such that*

$$(30) \qquad\qquad \|u_a\|_\infty \le Ca^2$$

on the whole Y and

$$(46) \qquad\qquad \|u_a\|_{C^{k,\alpha}} \le C_k(c, \Omega)a^2$$

for all $k \in N$ and in a neighborhood of the exceptional set, there exists a positive number $C = C(c)$ such that

$$(47) \qquad\qquad \|u_a\|_{C^2} \le Ca$$
$$(48) \qquad\qquad \|u_a\|_{C^4} \le C.$$

Theorem 19. *Let X be a generalized K3 surface with a non-flat Ricci-flat orbifold Kähler metric ω_0 and Y its minimal resolution. Let ω_a be a Kähler metric constructed by gluing ω_0 and ALE gravitational instanton metrics corresponding to the singularities with the area of the distinguished (-2)-curves pinched between ca and ca^{-1} by using (16) with $r \approx a^{\frac{3}{8}}$. We consider the limit $a \to 0$. Then the same conclusion holds provided we replace a^2 in (30) and (46) by $a^{\frac{13}{8}}$ (cf. (50) and (52)), a in (47) by $a^{\frac{5}{8}}$ (cf. (53)) and C in (48) by $Ca^{-\frac{3}{8}}$ (cf. (54)).*

Let X and ω_0 be as in Theorem 19. Let $\Omega(\omega_0)$ be the moduli space of polarized K3 surfaces determined by requiring that the class $[\omega_0]$ is of type (1,1). The space $\Omega(\omega_0)$ is isomorphic to the Hermitian symmetric domain $SO(2, 19)/SO(2) \times SO(19)$ of type IV of dimension 19. From Theorem 13, there is a family of Kähler-Einstein generalized K3 surfaces over $\Omega(\omega_0)$. Let us call this deformation a polarized deformation. We know that in the family *restricted* on $K\Omega_0$, the canonical Kähler-Einstein metrics depend smoothly on parameters in polarized deformations. If we (locally) combine polarized deformations and their isometric deformations, we get a smooth (local) section of the fibration $K\Omega \to \Omega$. It follows from this and Theorem 19 that

Theorem 20. *Let (X, ω_0, Y) be as in Theorem 19. In the polarized deformation of $(Y, [\omega_0])$, the canonical Kähler-Einstein metrics depend "smoothly" on the parameters in the whole $\Omega(\omega_0)$. To be more precise, for any sequence $p_i \in \Omega(\omega_0)$ of Kähler polarizations such that $p = \lim_{i\to\infty} p_i$ annihilates some (-2)-vectors, the canonical Kähler-Einstein metrics corresponding to $[p_i]$ converge smoothly to the orbifold Kähler metric corresponding to $[p]$ outside singularities. In particular, the canonical Ricci-flat Kähler metrics converges to an orbifold Kähler-Einstein metric in any polarized degeneration of type I.*

Theorem 21. *The moduli space of all Einstein metrics including orbifold-metrics on a K3 manifold with volume 1 is isomorphic to*

$$N = SO(3, 19)/SO(3) \times SO(19).$$

The singular metrics appear as those metrics corresponding to the fixed point set N^W with respect to the action of the Weyl group W. There exists a universal marked family of Ricci-flat K3 manifolds including Ricci-flat K3 orbifolds (on N^W) in which the Ricci-flat (orbifold-)metrics depend "smoothly" (outside quotient singularities) on the parameters in the whole N. To be more precise, suppose g_i is a sequence in $N_0 = N - N^W$ with $g = \lim_{i\to\infty} g_i \in N^W$. Then
(i) for large i, the curvature of g_i concentrates near some Dynkin diagram configurations E of distinguished 2-spheres of self-intersection number -2. If we rescale the metrics g_i by the local maximum values of the curvature (recall that we have put $\beta = \frac{3}{8}$ just before (49)), then we can capture the ALE gravitational instantons corresponding to the simple singularities obtained by contracting E.
(ii) outside the singularities the metrics g_i converge smoothly to the orbifold-metric g.

Remark. In [KT], a uniformization theorem for a K3 surface is proved using Ricci-flat metrics: if X is a generalized K3 surface then X is holomorphically covered by a complex 2-torus if and only if the Euler number of X *in the sense of orbifolds*

$$e_{\mathrm{orb}}(X) = e(Y) - \sum_{p \in \mathrm{Sing}(X)} \left(e(E_p) - \frac{1}{|G_p|}\right)$$

vanishes, where E_p is the exceptional set of the minimal resolution over p and G_p is the local fundamental group of p. There is a lattice theoretical approach to the uniformization of K3 surfaces [M]. Riemannian geometrically this means that X is flat if and only if the sum of the $\frac{1}{8\pi^2}$ times

the L^2 norm of the curvature tensor $\int \|R\|^2$ of the ALE gravitational instantons corresponding to $\mathrm{Sing}(X)$ is 24.

Theorems 18–21 fill the "holes" (fix points of W) of the moduli space $\Gamma \backslash K\Omega$ (resp. $\Gamma \backslash N$) of Kähler-Einstein (resp. Einstein) metrics on a marked K3 surface. If we fix a polarization and consider only integral isometries fixing this polarization, we get the coarse moduli space of Kähler-Einstein K3 surfaces with a fixed polarization. This is a Hermitian locally symmetric V-manifold with cusps. We recall that there is a natural compactification due to Satake, Baily-Borel and Mumford. It is then natural to ask what Riemannian geometric objects correspond to quotient singularities and cusps. The object corresponding to a quotient singularity is a Kähler-Einstein orbifold-metric. In [Kob], we make an attempt understanding the Riemannian geometric objects corresponding to cusps.

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