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Einstein Kähler Metrics of Negative Ricci Curvature on Open Kähler Manifolds

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Introduction

In 1954 E. Calabi [C1,2,3] posed a conjecture, now so called Calabi's conjecture. It states that if a real closed $(1,1)$-form γ represents 2π times the first Chern class $c_1(X)$ of a compact Kähler manifolds X, then one can find a Kähler metric whose Ricci form coincides with γ. And an altered version of the conjecture says that a compact Kähler manifold X with negative first Chern class $c_1(X) < 0$ admits an Einstein Kähler metric.

In 1976–77 T. Aubin [A1,2] and S.-T. Yau [Y1,2] proved the conjecture and rich applications to algebraic geometry followed. One of the most remarkable consequences is the Miyaoka-Yau inequality (cf. [M1,2] [Y2]).

Theorem 0.1. *Let X be an n-dimensional compact Kähler manifold with negative first Chern class, then the following inequality between Chern numbers holds:*

$$(-1)^n 2(n+1)c_2(X)c_1(X)^{n-2} \geq (-1)^n n c_1(X)^n.$$

Moreover the equality holds if and only if X is covered by a unit ball in $\mathbf{C}^n$.

It is very natural to try to generalize the existence theorem of Einstein Kähler metrics and the Miyaoka-Yau inequality. Actually Yau himself treated degenerated cases in his paper [Y3] and made an announcement of his extended results in a lecture note of Séminaire Palaiseau 1978 [Y4]. Later on, he and S.-Y. Cheng [C-Y2] showed the existence of Einstein Kähler metrics on bounded domains of holomorphy in $\mathbf{C}^n$, where they introduced the useful notion of bounded geometry. Armed

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with their methods, R. Kobayashi [Kb1] proceeded to the study of quasi-projective varieties. Using quasi-coordinate maps, he generalized the notion of bounded geometry, and obtained

Theorem 0.2. *Let X be an n-dimensional nonsingular projective algebraic variety, K its canonical divisor and D an effective reduced divisor on X with simple normal crossings. Assume that $K+D$ is ample. Then $X-D$ admits a complete Einstein Kähler metric of negative Ricci curvature, and the logarithmic version of the Miyaoka-Yau inequality holds:*

$$(*) \qquad 2(n+1)c_2(\Omega_X^1(\log D))c_1(\Omega_X^1(\log D))^{n-2} \geq nc_1(\Omega_X^1(\log D))^n.$$

He considered algebraic surface cases more thoroughly and obtained very satisfying results [Kb2,3,4]. For an account, see his survey in this volume. In the course of the study, he was led to work with V-manifolds. Cheng and Yau [C-Y3] also generalized their results to open V-manifolds including quasi-projective cases. Results in algebraic geometry by Kawamata [Km1,2,3] [KMM] allowed Tsuji [Ts1,2,3,4,5,6] to apply efficiently the methods of V-manifolds and branched coverings. In particular he proved a significant part of the following result (see also Tian and Yau [T-Y1]).

Theorem 0.3. *Let X be an n-dimensional nonsingular projective algebraic variety and D an effective reduced divisor with simple normal crossings. Assume that $K+D$ is nef and that there exists an effective $\mathbf{Q}$-divisor E whose support is contained in D such that $K+D-E$ is ample. Then $K+D$ is big and $X-D$ admits an almost-complete Einstein Kähler metric with negative Ricci curvature. And the logarithmic Miyaoka-Yau inequality $(*)$ holds. Moreover if the equality holds, then $X-D$ is covered by a unit ball in $\mathbf{C}^n$.*

In this chapter we shall restrict our attention to Theorem 0.3 and follow the presentation by Tian and Yau [T-Y1]. In §1, we explain Calabi's construction that gives us explicit examples, which we can not obtain solveing the Monge-Ampère equation. In §2, we recall the definitions and basic properties of V-manifolds. The notion of V-manifolds was introduced by I. Satake [St1,2] in generalizing the notion of manifolds to allow quotient singularities. Basically, a V-manifolds is a Hausdorff space locally isomorphic to a quotient of the unit ball in $\mathbf{R}^n$ by a finite subgroup of $O(n)$. We require a suitable compatibility in patching. Then we can define geometric objects on V-manifolds going up to the local uniformizations. For instance, a smooth V-function means a function whose liftings are smooth and a V-bundle means a space which has equivariant local liftings to smooth bundles. Intuitively these definitions are enough and we deal only with very explicit cases. Thus the reader would not have any trouble without reading exact definitions. But for the sake of completeness, we include the definition of V-manifolds. In this section we also collect basic facts on V-manifolds for later use. In §3 we state the Schauder estimates on linear and nonlinear second order elliptic partial differential equations which we need in §4. We solve in §4 the complex Monge-Ampère equation to obtain Einstein Kähler metrics on Kähler manifolds of bounded geometry after the method of [T-Y1]. In §5 we see the existence theorem applies to the quasi-projective case under the assumption above. In the last section §6 we derive the Miyaoka-Yau inequality from the results in §5.

Here we explain our notations. We identify a Kähler metric $g = 2\operatorname{Re}\sum g_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}}$ and its Kähler form $\omega = \sqrt{-1}\sum g_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$. Its Ricci form γ_ω is given by $\gamma_\omega = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$. The Laplacian $\triangle_\omega$ acting on functions is defined by

$$\triangle_\omega u = \operatorname{tr}_\omega \sqrt{-1}\partial\bar{\partial}u = \sum g^{i\bar{j}}\frac{\partial^2 u}{\partial z^i \partial \bar{z}^{\bar{j}}}.$$

In this chapter we call ω an Einstein Kähler metric if $\gamma_\omega = -\omega$, unless otherwise stated.

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§1. Calabi's construction

In his papers [C5,6,7], E. Calabi constructed many examples of Einstein Kähler metrics on domains of the total spaces of vector bundles. His idea is to reduce the problem of the existence of Einstein Kähler metrics to that of ordinary differential equations. We here give two examples of the construction of Ricci-negative Einstein Kähler metrics.

Let (M, θ) be an $(n-1)$-dimensional complete Einstein Kähler manifold such that $\gamma_\theta = \alpha\theta$, with $\alpha \in \mathbf{R}$. For instance, every compact Kähler manifold M with negative or vanishing first Chern class $c_1(M) < 0$, or $=$ 0 admits such an Einstein Kähler metric by the resolution of Calabi's conjecture by T. Aubin [A1,2] and S.-T. Yau [Y2,3]. See other surveys in this volume for recent results.

Assume that $\alpha < 1$ and that there exists a holomorphic line bundle $L \longrightarrow M$ over M with a fibre metric $\|\cdot\|$ whose curvature is equal to $-\theta$. We consider the metric $\|\cdot\|$ as a function on the total space L. Define a function t by $t = \log\|\cdot\|$. We seek an Einstein Kähler metric ω on a domain of L in the form $\omega = \sqrt{-1}\partial\bar\partial F$, where $F = F(t)$ is a function in t. Putting $f = F' = \dfrac{dF}{dt}$, we have that if f, $f' > 0$, them

$$\begin{aligned}
&\omega = f\sqrt{-1}\partial\bar\partial t + f'\sqrt{-1}\partial t \wedge \bar\partial t,\\
&\omega^n = f^{n-1}f'\theta^{n-1}\sqrt{-1}\partial t \wedge \bar\partial t,\\
&\gamma_\omega = -\sqrt{-1}\partial\bar\partial \log(f^{n-1}f'\theta^{n-1}) = -\sqrt{-1}\partial\bar\partial(\log f^{n-1}f' - \alpha t).
\end{aligned}$$

Thus if $f^{n-1}f' = \text{const}\exp(\alpha t + F)$, then ω is an Einstein Kähler metric with $\gamma_\omega = -\omega$.

Theorem 1.1 ([C5] [Ts3]). *Assume as above.*
(i) *We define the function $f(t)$ by the following formula:*

$$t = -\int_f^\infty \frac{v^{n-1}\,dv}{\displaystyle\int_{1-\alpha}^{v}(\alpha+u)u^{n-1}\,du}.$$

Then the Kähler metric ω constructed above gives a complete Einstein Kähler metric on the total space of the unit disk bundle $X = \{\,\eta \in L \mid \|\eta\| < 1\,\}$.

(ii) *If $\alpha = 0$ i.e. θ is Ricci-flat, then*

$$\omega = \sqrt{-1}\partial\bar\partial F(t) = \sqrt{-1}\partial\bar\partial\log(-\log\|\eta\|^2)^{-(n+1)}$$

gives a complete Einstein Kähler metric on the total space of the punctured unit disk bundle $X = \{\,\eta \in L \mid 0 < \|\eta\|^2 < 1\,\}$.

§2. V-manifolds

In this section we recall the notation of V-manifolds introduced by I. Satake [St1,2]. It is the notion of generalized manifolds with quotient singularities. We follow the formulation by T. Kawasaki [Ks1,2,3].

To define V-manifolds and related notions we start with defining a few categories.

We define a category $\mathcal{EF}_s$ of smooth bundle maps with finite symmetries as follows. An object of $\mathcal{EF}_s$ is a family D of commutative diagrams like

$$\begin{array}{ccc} (E, G^E) & \xrightarrow{\{\bar\phi\}} & (F, G^F) \\ \big\downarrow{\scriptstyle \pi^E} & & \big\downarrow{\scriptstyle \pi^F} \\ (M, G^M) & \xrightarrow{\{\phi\}} & (N, G^N) \\ & D & \end{array} .$$

Here we assume the following conditions.

(i) $\pi^E \colon E \longrightarrow M$ and $\pi^F \colon F \longrightarrow N$ are smooth fibre bundles over connected smooth manifolds M and N, respectively.

(ii) The symmetry groups G^E and G^F are finite groups of smooth bundle maps of E and F, respectively. We assume that their actions are effective and induce the effective actions of the finite groups G^M and G^N on M and N, respectively. In general the surjective homomorphisms $\rho^E \colon G^E \longrightarrow G^M$ and $\rho^F \colon G^F \longrightarrow G^N$ are not injective.

(iii) Each member $\bar\phi \colon E \longrightarrow F$ of $\{\bar\phi\}$ is a smooth bundle map which induces the smooth map $\phi \colon M \longrightarrow N$ of base manifolds. We assume that for each $g^E \in G^E$ there exists $g^F \in G^F$ such that $\bar\phi \circ g^E = g^F \circ \bar\phi$. Note that then for each $g^M \in G^M$ there exists $g^N \in G^N$ such that $\phi \circ g^M = g^N \circ \phi$.

(iv) G^F acts on $\{\bar\phi\}$ transitively by $(g^F \bar\phi)(p) = g^F \bar\phi(p)$, for $g^F \in G^F$ and $p \in E$. Then G^N acts on $\{\phi\}$ transitively.

A morphism $\lambda \colon D_1 \longrightarrow D_2$

$$\begin{array}{ccccccc} (E_1, G^{E_1}) & \xrightarrow{\{\bar\phi_1\}} & (F_1, G^{F_1}) & & (E_2, G^{E_2}) & \xrightarrow{\{\bar\phi_2\}} & (F_2, G^{F_2}) \\ \big\downarrow{\scriptstyle \pi^{E_1}} & & \big\downarrow{\scriptstyle \pi^{F_1}} & \xrightarrow{\lambda} & \big\downarrow{\scriptstyle \pi^{E_2}} & & \big\downarrow{\scriptstyle \pi^{F_2}} \\ (M_1, G^{M_1}) & \xrightarrow{\{\phi_1\}} & (N_1, G^{N_1}) & & (M_2, G^{M_2}) & \xrightarrow{\{\phi_2\}} & (N_2, G^{N_2}) \\ & D_1 & & & & D_2 & \end{array}$$

is a pair $\left(\{\bar\lambda^E\}, \{\bar\lambda^F\}\right)$ of families of smooth bundle maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\{\bar\lambda^E\}} & E_2 \\ \big\downarrow{\scriptstyle \pi^{E_1}} & & \big\downarrow{\scriptstyle \pi^{E_2}} \\ M_1 & \xrightarrow{\{\lambda^M\}} & M_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} F_1 & \xrightarrow{\{\bar\lambda^F\}} & F_2 \\ \big\downarrow{\scriptstyle \pi^{F_1}} & & \big\downarrow{\scriptstyle \pi^{F_2}} \\ N_1 & \xrightarrow{\{\lambda^N\}} & N_2 \end{array} .$$

Here we assume the following conditions.

(i) The induced maps λ^M and λ^N over the base manifolds are diffeomorphisms into domains of M_2 and N_2, respectively.

(ii) The bundle maps $\bar{\lambda}^E$ and $\bar{\lambda}^F$ give bundle isomorphisms $E_1 \xrightarrow{\simeq} (\lambda^M)^* E_2$ and $F_1 \xrightarrow{\simeq} (\lambda^N)^* F_2$, respectively.

(iii) For each $g^{E_1} \in G^{E_1}$ and $g^{F_1} \in G^{F_1}$ there are $g^{E_2} \in G^{E_2}$ and $g^{F_2} \in G^{F_2}$ such that $\bar{\lambda}^E \circ g^{E_1} = g^{E_2} \circ \bar{\lambda}^E$ and $\bar{\lambda}^F \circ g^{F_1} = g^{F_2} \circ \bar{\lambda}^F$, respectively. Then g^{E_2} and g^{F_2} are uniquely determined by g^{E_1} and g^{F_1}, and these correspondences give injective homomorphisms of groups $G^{E_1} \longrightarrow G^{E_2}$ and $G^{F_1} \longrightarrow G^{F_2}$. Note that then a similar statement holds for λ^M and λ^N.

(iv) For each $\bar{\phi}_1 \in \{\bar{\phi}_1\}$ and $\bar{\phi}_2 \in \{\bar{\phi}_2\}$ there are $\bar{\lambda}^E \in \{\bar{\lambda}^E\}$ and $\bar{\lambda}^F \in \{\bar{\lambda}^F\}$ such that $\bar{\lambda}^F \circ \bar{\phi}_1 = \bar{\phi}_2 \circ \bar{\lambda}^E$, and then $\lambda^N \circ \phi_1 = \phi_2 \circ \lambda^M$.

(v) G^{E_2} and G^{F_2} act on $\{\bar{\lambda}^E\}$ and $\{\bar{\lambda}^F\}$ simply transitively by $\left(g^{E_2}\bar{\lambda}^E\right)\left(p^{E_1}\right) = g^{E_2}\bar{\lambda}^E\left(p^{E_1}\right)$ and $\left(g^{F_2}\bar{\lambda}^F\right)\left(p^{F_1}\right) = g^{F_2}\bar{\lambda}^F\left(p^{F_1}\right)$, for $g^{E_2} \in G^{E_2}$, $g^{F_2} \in G^{F_2}$, $p^{E_1} \in E_1$ and $p^{F_1} \in F_1$, respectively. Then a similar statement holds for $\{\lambda^M\}$ and $\{\lambda^N\}$.

We define a category $\mathcal{TF}$ of continuous maps of topological spaces as follows. An object of $\mathcal{TF}$ is a continuous map $f\colon X \longrightarrow Y$ between connected topological spaces X and Y. A morphism $j\colon (f_1\colon X_1 \longrightarrow Y_1) \longrightarrow (f_2\colon X_2 \longrightarrow Y_2)$ is a pair (j^X, j^Y) of homeomorphisms into subdomains $j^X\colon X_1 \longrightarrow X_2$ and $j^Y\colon Y_1 \longrightarrow Y_2$ such that $f_2 \circ j^X = j^Y \circ f_1$.

An object D of $\mathcal{EF}_s$

$$\begin{array}{ccc}
(E, G^E) & \xrightarrow{\{\bar{\phi}\}} & (F, G^F) \\
\Big\downarrow{\scriptstyle \pi^E} & & \Big\downarrow{\scriptstyle \pi^F} \\
(M, G^M) & \xrightarrow{\{\phi\}} & (N, G^N)
\end{array}$$

gives two objects of $\mathcal{TF}$

$$\phi^t(D)\colon E/G^E \longrightarrow F/G^F \qquad \text{and}$$
$$\phi^b(D)\colon M/G^M \longrightarrow N/G^N\ .$$

which are induced by the maps $\{\bar{\lambda}\}$ and $\{\lambda\}$, respectively. Then we get the forgetting functors $\mathcal{F}^t_{\mathcal{TF}}\colon \mathcal{EF}_s \longrightarrow \mathcal{TF}$ and $\mathcal{F}^b_{\mathcal{TF}}\colon \mathcal{EF}_s \longrightarrow \mathcal{TF}$ by $\mathcal{F}^t_{\mathcal{TF}} = \phi^t(D)\colon E/G^E \longrightarrow F/G^F$ and $\mathcal{F}^b_{\mathcal{TF}} = \phi^b(D)\colon M/G^M \longrightarrow N/G^N$.

Definition 2.1. Let $f\colon X \longrightarrow Y$ be a continuous map between paracompact Hausdorff spaces X and Y. Assume we have saturated bases $\mathcal{U}_X$, $\mathcal{U}_Y$ of open sets of X, Y, respectively, consisting of connected open subsets. Here, a basis $\mathcal{U}$ is called saturated if each connected open subset of a member of $\mathcal{U}$ is again a member of $\mathcal{U}$. Then we have a subcategory $\mathcal{TF}(f; \mathcal{U}_X, \mathcal{U}_Y)$ of $\mathcal{TF}$ whose object is $f|_{U_X} : U_X \longrightarrow U_Y$, with $U_X \in \mathcal{U}_X$ and $U_Y \in \mathcal{U}_Y$ such that $f(U_X) \subset U_Y$, and a morphism is a pair of natural inclusions. A representative of a smooth V-bundle map covering f is a functor $\mathcal{VBF}(f; \mathcal{U}_X, \mathcal{U}_Y) : \mathcal{TF}(f; \mathcal{U}_X, \mathcal{U}_Y) \longrightarrow \mathcal{EF}_s$ such that

$$\mathcal{F}^b_{\mathcal{TF}} \circ \mathcal{VBF}(f; \mathcal{U}_X, \mathcal{U}_Y) = \mathrm{id}|_{\mathcal{TF}(f; \mathcal{U}_X, \mathcal{U}_Y)} .$$

Let $\mathcal{U}'_X$, $\mathcal{U}'_Y$ and $\mathcal{VBF}(f; \mathcal{U}'_X, \mathcal{U}'_Y)$ be another bases and representative. We call $\mathcal{VBF}(f; \mathcal{U}_X, \mathcal{U}_Y)$ and $\mathcal{VBF}(f; \mathcal{U}'_X, \mathcal{U}'_Y)$ are equivalent if there are saturated bases $\mathcal{U}''_X$ of X, $\mathcal{U}''_Y$ of Y, and a representative $\mathcal{VBF}(f; \mathcal{U}''_X, \mathcal{U}''_Y)$ such that

$$\begin{aligned}
\mathcal{VBF}(f; \mathcal{U}_X, \mathcal{U}_Y)|_{\mathcal{TF}(f; \mathcal{U}''_X, \mathcal{U}''_Y)} &= \mathcal{VBF}(f; \mathcal{U}''_X, \mathcal{U}''_Y), \\
\mathcal{VBF}(f; \mathcal{U}'_X, \mathcal{U}'_Y)|_{\mathcal{TF}(f; \mathcal{U}''_X, \mathcal{U}''_Y)} &= \mathcal{VBF}(f; \mathcal{U}''_X, \mathcal{U}''_Y).
\end{aligned}$$

We define a smooth V-bundle map covering f as an equivalence class of representatives.

Hereafter we will identify the representatives and their classes. An ordinary smooth bundle map of ordinary smooth fibre bundles is an example of smooth V-bundle maps.

Now we define some subcategories of $\mathcal{EF}_s$. The subcategory $\mathcal{ES}_s$ of smooth sections with finite symmetries is obtained by requiring its objects take the following form.

$$\begin{tikzcd}
(M, G^M) \arrow[r, "\{s\}"] \arrow[d, "\mathrm{id}_M"] & (E, G^E) \arrow[d, "\pi^E"] \\
(M, G^M) \arrow[r, "\{g^M\}"] & (M, G^M)
\end{tikzcd} .$$

For short we write it as

$$\begin{tikzcd}
(E, G^E) \arrow[d, shift left, "\pi^E"] \\
(M, G^M) \arrow[u, shift left, "\{s\}"]
\end{tikzcd} .$$

The subcategory $\mathcal{E}_s$ of smooth fibre bundles with finite symmetries is obtained by requiring its objects have trivial target bundles.

$$\begin{array}{ccc} (E,G^E) & \longrightarrow & (\{*\},\{\mathrm{id}\}) \\ \Big\downarrow{\scriptstyle \pi^E} & & \Big\downarrow \\ (M,G^M) & \longrightarrow & (\{*\},\{\mathrm{id}\}) \end{array} .$$

For short we write it as

$$\begin{array}{c} (E,G^E) \\ \Big\downarrow{\scriptstyle \pi^E} \\ (M,G^M) \end{array} .$$

The subcategory $\mathcal{F}_s$ of smooth maps with finite symmetries is obtained by requiring its objects have point bundles.

$$\begin{array}{ccc} (M,G^M) & \xrightarrow{\{\phi\}} & (N,G^N) \\ \Big\downarrow{\scriptstyle \mathrm{id}_M} & & \Big\downarrow{\scriptstyle \mathrm{id}_N} \\ (M,G^M) & \xrightarrow{\{\phi\}} & (N,G^N) \end{array} , \text{ or } \quad (M,G^M) \xrightarrow{\{\phi\}} (N,G^N) .$$

The subcategory $\mathcal{M}_s$ of connected manifolds with finite symmetries is obtained as the intersection $\mathcal{E}_s \cap \mathcal{F}_s$. Its objects are pairs (M,G^M).

Then we have natural forgetting functors.

$$\underbrace{\begin{array}{ccc} (E,G^E) & \xrightarrow{\{\bar\phi\}} & (F,G^F) \\ \Big\downarrow{\scriptstyle \pi^E} & & \Big\downarrow{\scriptstyle \pi^F} \\ (M,G^M) & \xrightarrow{\{\phi\}} & (N,G^N) \end{array}}_{D} \longmapsto \underbrace{\begin{array}{c} (E,G^E) \\ \Big\downarrow{\scriptstyle \pi^E} \\ (M,G^M) \end{array}}_{\mathcal{F}^d_{\mathcal{E}_s}(D)} , \quad \underbrace{\begin{array}{c} (F,G^F) \\ \Big\downarrow{\scriptstyle \pi^F} \\ (N,G^N) \end{array}}_{\mathcal{F}^t_{\mathcal{E}_s}(D)} ,$$

$$\underbrace{(E,G^E) \xrightarrow{\{\bar\phi\}} (F,G^F)}_{\mathcal{F}^t_{\mathcal{F}_s}(D)} , \quad \underbrace{(M,G^M) \xrightarrow{\{\phi\}} (N,G^N)}_{\mathcal{F}^b_{\mathcal{F}_s}(D)} ,$$

$$\mathcal{F}^{db}_{\mathcal{M}_s}(D) = (M,G^M) , \quad \mathcal{F}^{dt}_{\mathcal{M}_s}(D) = (E,G^E) ,$$

$$\mathcal{F}^{tb}_{\mathcal{M}_s}(D) = (N,G^N) , \quad \mathcal{F}^{tt}_{\mathcal{M}_s}(D) = (F,G^F) .$$

Also the category $\mathcal{TF}$ has the subcategory of topological spaces $\mathcal{T}$ whose objects take the form $X \longrightarrow \{*\}$, X for short, and the forgetting-functors $\mathcal{F}^d_{\mathcal{T}}(f\colon X \longrightarrow Y) = X$ and $\mathcal{F}^t_{\mathcal{T}}(f\colon X \longrightarrow Y) = Y$.

Moreover taking quotients we can define the forgetting functors from $\mathcal{EF}_s$ to $\mathcal{T}$ in a similar manner, for instance $\mathcal{F}^{db}_{\mathcal{T}}(D) = M/G^M$, $\mathcal{F}^{dt}_{\mathcal{T}}(D) = E/G^E$.

Now we can define the notion of smooth V-sections, smooth V-bundles, smooth V-maps and V-manifolds, requiring that the representatives of smooth V-bundle maps take its values in the corresponding subcategories. For instance,

Definition 2.2. Let X be a paracompact Hausdorff space. X is called a V-manifold if there is a saturated base $\mathcal{U}$ of open sets consisting of connected open subsets of X, which gives a subcategory $\mathcal{T}(\mathcal{U})$ of $\mathcal{T}$, and we have a functor $\mathcal{VM}(\mathcal{U})\colon \mathcal{T}(\mathcal{U}) \longrightarrow \mathcal{M}_s$ such that $\mathcal{F}^{db}_{\mathcal{T}} \circ \mathcal{VM}(\mathcal{U}) = \mathrm{id}|_{\mathcal{T}(\mathcal{U})}$.

For $U \in \mathcal{U}$ we denote $\mathcal{VM}(\mathcal{U}) = (\tilde{U}, G^U)$ and call it together with the projection $p_U\colon \tilde{U} \longrightarrow U = \tilde{U}/G^U$ a local uniformizing system, and G^U a local transformation group. A point p in X is called regular if there is a neighborhood U such that $G^U = \{\mathrm{id}\}$, singular otherwise. Locally, X is isomorphic to $\mathbf{R}^n/G$ with a finite subgroup G of the orthogonal group $O(n)$. Then we think of the standard coordinates on $\mathbf{R}^n$ as a local coordinate system of X.

If functors are compatible, we can say, for instance, that a continuous map $\phi\colon X \longrightarrow Y$ of V-manifolds is smooth V-map and that a V-bundle is defined over a V-manifold.

Definition 2.3. Let $\phi\colon X \longrightarrow Y$ be a continuous map of V-manifolds X and Y defined by $\mathcal{VM}(\mathcal{U}_X)$ and $\mathcal{VM}(\mathcal{U}_Y)$, respectively. Then ϕ is called a smooth V-map if there is a functor $\mathcal{VF}(\phi;\, \mathcal{U}_X, \mathcal{U}_Y)$ of a smooth V-map covering ϕ such that $\mathcal{F}^{db}_{\mathcal{M}_s} \circ \mathcal{VF}(\phi;\, \mathcal{U}_X, \mathcal{U}_Y) = \mathcal{VM}(\mathcal{U}_X)$ and $\mathcal{F}^{tb}_{\mathcal{M}_s} \circ \mathcal{VF}(\phi;\, \mathcal{U}_X, \mathcal{U}_Y) = \mathcal{VM}(\mathcal{U}_Y)$.

For $U_X \in \mathcal{U}_X$, $U_Y \in \mathcal{U}_Y$ in the domain of $\mathcal{VF}(\phi;\, \mathcal{U}_X, \mathcal{U}_Y)$, we denote $\{\mathcal{VF}(\phi;\, \mathcal{U}_X, \mathcal{U}_Y)\}(\phi|_{U_X} : U_X \longrightarrow U_Y)$ as follows

$$(\tilde{U}_X, G^{U_X}) \xrightarrow{\tilde{\phi}_{U_X}} (\tilde{U}_Y, G^{U_Y}) \quad .$$

We call $\tilde{\phi}_{U_X}$ a local lifting of ϕ.

Example 2.1. Let N be a smooth manifold with a finite group G acting on it effectively, and M a G-invariant submanifold. Set $G^M = G/G_M$, where G_M is the isotropy subgroup of M. Then the natural inclusion $M/G^M \longrightarrow N/G$ is an example of V-maps between V-manifolds. Note that in general there is no homomorphism $G^M \longrightarrow G$ corresponding to the inclusion.

Definition 2.4. Let X be a V-manifold defined by $\mathcal{VM}(\mathcal{U})$ and has a functor $\mathcal{VE}(\mathcal{U})\colon \mathcal{T}(\mathcal{U}) \longrightarrow \mathcal{E}_s$ defining a V-bundle. We say the V-bundle is defined over the V-manifold X, if $\mathcal{F}^{db}_{\mathcal{M}_s} \circ \mathcal{VE}(\mathcal{U}) = \mathcal{VM}(\mathcal{U})$. If all $\rho^E\colon G^E \longrightarrow G^M$ in the diagram in the image of $\mathcal{VE}(\mathcal{U})$ are isomorphic, we say that V-bundle is proper. We call a V-bundle a vector V-bundle if all fibre bundles in the diagrams in the image of $\mathcal{VE}(\mathcal{U})$ are vector bundles.

Given a V-bundle $\mathcal{VE}(\mathcal{U})$ over a V-manifold X, we can make the total space E of the V-bundle. Set $E_U = \{\mathcal{F}^{dt}_{\mathcal{T}} \circ \mathcal{VE}(\mathcal{U})\}(U)$. We denote the inclusion map of $U_1 \subset U_2$, U_1, $U_2 \in \mathcal{U}$ as j_{U_1,U_2}. Then we have an open embedding $J_{U_1,U_2} = \{\mathcal{F}^{dt}_{\mathcal{T}} \circ \mathcal{VE}(\mathcal{U})\}(j_{U_1,U_2})\colon E_{U_1} \longrightarrow E_{U_2}$. We glue E_U's by J_{U_1,U_2}.

$$E = \left(\bigcup_{U \in \mathcal{U}} E_U \right) \Big/ \text{“} J_{U_1,U_2}(p) \sim p, \quad \text{for} \quad p \in E_{U_1} \text{”} \quad .$$

Then E makes a paracompact Hausdorff space. Making use of the correspondence $E_U \longrightarrow \{\mathcal{F}^{dt}_{\mathcal{M}_s} \circ \mathcal{VE}(\mathcal{U})\}(U)$ one can show E is naturally a V-manifold. And the functor $\mathcal{VE}(\mathcal{U})$ can be considered as a smooth V-map $\pi\colon E \longrightarrow X$ of V-manifolds. Then a section of V-bundle E can be identified with a smooth V-map $s\colon X \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X$.

We can pull back a V-bundle through a smooth V-map by the following construction.

Assume that we are given a compatible pair of objects in $\mathcal{E}_s$ and $\mathcal{F}_s$.

$$\begin{array}{ccc} & & (F, G^F) \\ & & \Big\downarrow{\scriptstyle \pi^F} \\ (M, G^M) & \xrightarrow{\{\phi\}} & (N, G^N) \end{array}$$

We construct an object in $\mathcal{E}_s$ as follows. First we choose $\phi_0 \in \{\phi\}$ and fix it. Set

$$E = {\phi_0}^* F = \{\, (p^M, p^F) \in M \times F \mid \phi_0(p^M) = \pi^F(p^F) \,\},$$
$$G^E = \{\, (g^M, g^F) \in G^M \times G^F \mid \phi_0 \circ g^M = \rho^F(g^F) \circ \phi_0 \,\}.$$

With the natural projection π^E and G^E-action, (E, G^E) over (M, G^M) makes a smooth fibre bundle with finite symmetries. The different choice of $\phi_0 \in \{\phi\}$ gives a bundle isomorphic to (E, G^E). And we get the family D of commutative diagrams.

$$\begin{array}{ccc} (E, G^E) & \xrightarrow{\{\bar{\phi}\}} & (F, G^F) \\ \Big\downarrow{\scriptstyle \pi^E} & & \Big\downarrow{\scriptstyle \pi^F} \\ (M, G^M) & \xrightarrow{\{\phi\}} & (N, G^N) \end{array} .$$

$$D$$

To define tensor V-bundles for V-manifolds we introduce tensor functors $\mathcal{F}^*_*\colon \mathcal{M}_s \longrightarrow \mathcal{E}_s$ by

$$\mathcal{F}^p_q(M, G^M) = \begin{array}{c} (T^p_q M, G^M) \\ \Big\downarrow{\scriptstyle \pi^{T^p_q M}} \\ (M, G^M) \end{array} ,$$

where $T^p_q M$ is the vector bundle of (p, q)-tensors over M, and the action of G^M on $T^p_q M$ is the natural one.

Definition 2.5. The (p, q)-tensor V-bundle $T^p_q X$ of X defined by $\mathcal{VM}(\mathcal{U})$ is a proper vector V-bundle defined by the functor $\mathcal{F}^p_q \circ \mathcal{VM}(\mathcal{U})$.

In the definition of V-bundles we can work with subcategories of $\mathcal{E}_s$ with additional structures such as orientation, fibre metric, connection, and holomorphic structure, requiring the symmetry groups and morphisms to preserve the corresponding structure, and get the notion of V-bundles with structures, in particular, V-manifolds with structures.

Let us denote the space of smooth V-sections of a V-bundle $\pi\colon E \longrightarrow X$ by $\Gamma(X;\, E)$. If E is a proper vector V-bundle $\Gamma(X;\, E)$ makes a vector space. We denote by $\Omega^p(X)$ the space $\Gamma(X;\, \wedge^p X)$, where $\wedge^p X$ is the vector V-bundle of p-forms. On a local uniformizing system $(\tilde{U}, G^U)$, a smooth V-form $\omega \in \Omega^p(U)$ is nothing but a G^U-invariant p-form on $\tilde{U}$.

Thus we can define the exterior differentiation $d\colon \Omega^p(X) \longrightarrow \Omega^{p+1}(X)$. If X is oriented we can integrate the top degree forms with compact supports.

Theorem 2.1 ([St1]). *Let X be a V-manifold. Then there is an isomorphism between the singular cohomology group $\mathrm{H}^*(X;\, \mathbf{R})$ with $\mathbf{R}$-coefficient and the de Rham cohomology group, under which the cup product in $\mathrm{H}^*(X;\, \mathbf{R})$ corresponds to the exterior product in $\Omega^*(X)$.*

$$\mathrm{H}^p(X;\, \mathbf{R}) \cong \frac{\text{kernel}\left[d\colon \Omega^p(X) \longrightarrow \Omega^{p+1}(X)\right]}{\text{image}\left[d\colon \Omega^{p-1}(X) \longrightarrow \Omega^p(X)\right]} \quad .$$

Moreover if X is compact and oriented, we have the Poincaré duality.

For a V-bundle with a connection one can associate characteristic forms by the Chern-Weil homomorphism. Since as is the smooth case their cohomology classes are independent of the choice of the connections, we get well-defined characteristic classes. In particular we have the Euler class, the Chern classes and the Pontrjagin classes for oriented, complex and real vector V-bundles, respectively.

Now we introduce the notion of bounded geometry which gives us a convenient setting to do analysis on V-manifolds.

Definition 2.6 ([Kb1,2]). Let B^n be the unit ball in the Euclidean space $\mathbf{R}^n$ of the same dimension as X. We call a smooth V-map $\phi\colon B^n \longrightarrow X$ a quasi-coordinate map if the local liftings of ϕ are of maximal rank everwhere. We call the image of the origin $\phi(0)$ the center of ϕ. Then we consider the standard coordinate of B^n as a quasi-coordinate system.

Definition 2.7 ([C-Y2,3],[Kb1,2]). A Riemannian V-manifold (X, g) is called of bounded geometry of order $k+\alpha$, $0 < k \in \mathbf{Z}$, $0 < \alpha < 1$, if for each point $p \in X$ there exists a quasi-coordinate map $\phi\colon B^n \longrightarrow X$ centered at p which satisfies the following conditions:

(i) If we write $\phi^* g = \sum g_{ij}(x)\, dx^i dx^j$ in terms of quasi-coordinate system $x = (x^1, x^2, \ldots, x^n)$, then the matrix (g_{ij}) is bounded from below by a constant positive matrix independent of p.

(ii) The $C^{k,\alpha}$-norms of g_{ij} are uniformly bounded as functions in x.

On such a V-manifold we can define the Banach space $C^{k,\alpha}$ of $C^{k,\alpha}$-bounded functions: The norm of a function $u \in C^{k,\alpha}$ is given by the $C^{k,\alpha}$-norms of u with respect to the quasi-coordinates x.

On a Riemannian V-manifold (X, g) of bounded geometry we have the maximum principle due to Yau [Y1] and Cheng-Yau [C-Y1,2,3].

Lemma 2.1. *Let (X, g) be a Riemannian V-manifold of bounded geometry of order $k + \alpha$, with $k \geq 1$, and u a C^2-function on X.*

(i) *If u is bounded from below, then for any positive number $\epsilon > 0$ there exists a point p in X such that*

$$\begin{aligned} &u(p) < \inf u + \epsilon, \\ &|\nabla u| < \epsilon, \\ &\operatorname{Hess} u(p) > -\epsilon g. \end{aligned}$$

(ii) *If u is non-negative and satisfies*

$$\triangle u \geq u^a - C,$$

with constants $a > 1$ and $C > 0$, then u is bounded from above and satisfies

$$\sup u \leq C^{\frac{1}{a}}.$$

Proof. (i) For given $\epsilon > 0$, there is a point $p \in X$ such that $u(p) < \inf u + \epsilon$. Let x be a quasi-coorninate centered at p. Then u lifts to a C^2-function $\tilde{u}$ defined on the unit ball $B^n = \{\, |x| < 1\}$. The function $\tilde{u}_\epsilon(x) = \tilde{u}(x) + 2\epsilon|x|^2$ on B^n clearly takes its minimum somewhere in B^n, say at x_0. Then we have that

$$\begin{aligned} &\tilde{u}(x_0) \leq \tilde{u}_\epsilon(x_0) \leq \tilde{u}_\epsilon(0) < \inf u + \epsilon, \\ &\frac{\partial \tilde{u}}{\partial x^i}(x_0) + 4\epsilon x_0^i = \frac{\partial \tilde{u}_\epsilon}{\partial x^i} = 0 && \text{and} \\ &\left(\frac{\partial^2 \tilde{u}}{\partial x^i \partial x^j} + 4\epsilon\delta_{i,j} \right) = \left(\frac{\partial^2 \tilde{u}_\epsilon}{\partial x^i \partial x^j}(x_0) \right) \geq 0, && \text{as a matrix.} \end{aligned}$$

Since g is of bounded geometry we get the desired result.

(ii) Because of (i) we only have to prove the boundedness from above. We define a C^2-function v as $v = (u+1)^{-b}$, with $b = \frac{1}{2}(a-1) > 0$. If u is not bounded from above, we have $\inf v = 0$. Then there exists a sequence $\{p_i\} \subset X$ such that $\lim v(p_i) = 0$, $\lim |\nabla v|(p_i) = 0$ and $\liminf \triangle v(p_i) \geq 0$. On the other hand, since u satisfies

$$\triangle u \geq \frac{1}{2}(u+1)^a - C',$$

with some $C' > 0$, we have that

$$\begin{aligned}\triangle v &= -b(u+1)^{-b-1}\triangle u + b(b+1)(u+1)^{-b-2}|\nabla u|^2 \\ &\leq -v^{-1}\left(\frac{b}{2} - \frac{b+1}{b}|\nabla v|^2\right) + C'bv^{1+\frac{1}{b}},\end{aligned}$$

which gives a contradiction. Q.E.D.

Like on a Riemannian manifold, we can define the distance function $d\colon X \times X \longrightarrow \mathbf{R}$ on a Riemannian V-manifold (X, g). We call (X, g) complete if (X, d) is complete as a metric space. If one use the V-manifold version of the Laplacian comparision theorem, one can show that ([Y1], [C-Y1])

Lemma 2.2. *If Ricci curvature of a complete Riemannian V-manifold (X, g) is bounded from below by a constant, then* (ii) *of the above lemma holds.*

Hereafter we restrict our attention to complex V-manifolds. Let X be a complex V-manifold, then X naturally acquires a structure of an analytic variety. A closed subset Y is called a subvariety if for any local uniformizing system $p_U\colon \tilde{U} \longrightarrow U$, $p_U^{-1}(Y) \subset \tilde{U}$ is a subvariety. A subvariety is irreducible if it is not a union of two other subvarieties. We call a codimension one subvariety a hypersurface. A ($\mathbf{Q}$-)divisor on X is a formal sum $D = \sum a_i D_i$, where $a_i \in \mathbf{Z}\,(\mathbf{Q})$ and $\{\, D_i \,\}$ a locally finite sequence of irreducible hypersurfaces in X (respectively). Locally finite means that every point has a neighborhood which meets only finitely many D_i's. The divisor D is called effective if all a_i are non-negative and not all zero.

For a holomorphic line V-bundle L with a meromorphic section s, different from zero section, we have a divisor (s) on X, namely its zero divisor minus its polar divisor. Conversely for any divisor we have a proper line V-bundle L_D with a meromorphic V-section s such that $(s) = D$ as follows. On $\tilde{U}$ we have canonically a line bundle $\tilde{L}_{D,U}$ and a meromorphic section s_U with the required property. One can see that G_U naturally acts on $\tilde{L}_{D,U}$ and we can choose $\{\, s_U \,\}$ in a compatible way. Thus we obtain an object of $\mathcal{ES}_s$ (a meromorphic version to be precise). If X is compact, the first Chern class $c_1(L_D)$ of L_D is the Poincaré dual of D.

Given a proper holomorphic vector V-bundle E we have the sheaf $\mathcal{O}(E)$ of germs of holomorphic V-sections. Like on a smooth manifold, we can calculate its cohomology group $\mathrm{H}^*(X;\, \mathcal{O}(E))$ via the $\bar{\partial}$-complex. If X is compact, we have the harmonic theory and the Serre duality. Moreover if X is Kähler, the Hodge theory holds. If we understand the positivity is defined via positive definite $(1,1)$-forms, we get the Kodaira vanishing theorem and embedding theorem for proper holomorphic line V-bundles.

For Riemann-Roch theorem refer to [Ks2].

Remark. A line V-bundle L on a compact V-manifold becomes a line bundle L^m if we mutiply it by an appropriate positive integer m.

Example 2.2 ([C-Y3], [Ts3,4], [T-Y1]. Let X be a complex manifold and $D = \sum \frac{1}{a_i} D_i$, a_i: positive integers, a $\mathbf{Q}$-divisor with simple normal crossings. That means D_i's are all non-singular and each point has a neighborhood U with coordinates $(z^1, z^2, \ldots, z^n)$ such that $U = \{ |z^i| < 1, \text{ for all } i \}$ and $U \cap \big(\cup D_i \big) = \{ z^1 z^2 \cdots z^k = 0 \}$ with some k. Then we can construct a complex V-manifold $X(D)$ in the following way.

(i) As a topological space $X(D)$ is nothing but X.

(ii) For a point p not contained in $\cup D_i$, we take a neighborhood U disjoint from $\cup D_i$. And consider $(U, \{ \mathrm{id}_U \})$ as a local uniformizing system.

(iii) For a point $p \in \cup D_i$, we take that $\tilde{U} = \{ |w^i| < 1 \}$ and $G^U = \mathbf{Z}_{a_{i_1}} \times \mathbf{Z}_{a_{i_2}} \times \cdots \mathbf{Z}_{a_{i_k}}$, where $D_{i_j} = \{ z^{i_j} = 0 \}$, for $1 \le j \le k$. And the projection $p_U : \tilde{U} \longrightarrow U$ is given by

$$p(w^1, w^2, \ldots, w^n) = \big((w^1)^{a_{i_1}}, (w^2)^{a_{i_2}}, \ldots, (w^k)^{a_{i_k}}, w^{k+1}, \ldots, w^n \big).$$

Then the identity map $X(D) \longrightarrow X$ becomes a holomorphic V-map.

§3. Schauder estimates

In this section we recall some results in analysis: the Schauder estimates (cf. [GT], [S2]).

First we consider the linear case. Let L be a second order linear elliptic differential operator defined on the unit ball $B(1)$ in $\mathbf{R}^n$ expressed as

$$L = \sum a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum a^i(x) \frac{\partial}{\partial x^i} + a(x).$$

Assume that there exists a positive constant $\lambda > 0$ such that

$$\lambda |\xi|^2 \le \sum a^{ij}(x) \xi_i \xi_j, \qquad \text{for all } \xi \in \mathbf{R}^n \text{ and } x \in B(1),$$

and that all functions a^{ij}, a^i and a belong to $C^{k,\alpha}$-space. Then we have the following Schauder estimates.

Theorem 3.1. *There is a constant C which depends only on the dimension n, λ, k, α and $C^{k,\alpha}$-norms on $B(1)$ of the coefficients of L $\|a^{ij}\|_{C^{k,\alpha}(B(1))}$, $\|a^i\|_{C^{k,\alpha}(B(1))}$, $\|a\|_{C^{k,\alpha}(B(1))}$ such that for any $C^{(k+2),\alpha}$-function u on $B(1)$, we have that*

$$\|u\|_{C^{(k+2),\alpha}(B(1/2))} \leq C(\,\|u\|_{C^0(B(1))} + \|Lu\|_{C^{k,\alpha}(B(1))}\,).$$

Next we consider the nonlinear case. Let F be a C^2-function in the variables x^i, z, p_i and r_{ij}, which satisfies the following. There exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum F_{r_{ij}}(x,z,p,r)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \text{for all } \xi \in \mathbf{R}^n.$$

Here subscripts mean the partial differentiation with respect to the corresponding variables. Then we have

Theorem 3.2. *Let $u \in C^4(B(1))$ satisfy*

$$F(x, u(x), u_{x^i}(x), u_{x^i x^j}(x)) = 0,$$

and assume that F is concave in r, then we have an estimates

$$\|u\|_{C^{2,\alpha}(B(\frac{1}{2}))} \leq C.$$

Here α depends only on n, Λ and λ, and C depends in addition on $\|u\|_{C^2(B(1))}$ and the first and second derivatives of F other than F_{rr}.

If F has further differentiability, taking derivatives of the equation and applying the linear Schauder estimates, we can deduce higher order estimates.

For the proofs see [G-T], [S2].

§4. Monge-Ampère equations on V-manifolds

In this section we shall consider the existence problem of complete Ricci-negative Einstein Kähler V-metrics on complex V-manifolds. The idea to work on V-manifolds is first carried out by R. Kobayashi [Kb2,3,4] and later developed by S.-Y. Cheng–S.-T. Yau [C-Y3], H. Tsuji [Ts3,4,6] and G. Tian–S.-T. Yau [T-Y1].

Unfortunately it is not yet proved in general the completeness of V-metrics obtained solving complex Monge-Ampère equations. To deal with the situation the concept of almost completeness was introduced by Tian–Yau [T-Y1].

Definition 4.1. An Einstein Kähler V-metric ω on a complex V-manifold X is said to be almost-complete if there is a sequence $\{\omega_i\}$ of complete Kähler V-metrics satisfying

(i) $\gamma_{\omega_i} \geq -t_i\omega_i$, with $t_i \longrightarrow 1$ as $i \longrightarrow \infty$,

(ii) $\omega_i \longrightarrow \omega$ as $i \longrightarrow \infty$ in C^∞ uniformly on copmact sets.

A direct calculation shows the following infinitesimal Schwarz lemma by Chern [Ch], Lu [L] and Yau [Y5].

Lemma 4.1. *Let (X_i, θ_i), $i = 1, 2$, be Kähler V-manifolds and $\phi: X_1 \longrightarrow X_2$ a holomorphic V-map. If $|d\phi|^2 \neq 0$, then*

$$\triangle_{\theta_1} \log |\partial\phi|^2 \geq \frac{\mathrm{Ric}_{\theta_1}(\partial\phi, \overline{\partial\phi})}{|\partial\phi|^2} - \frac{\mathrm{Bisect}_{\theta_2}(\partial\phi, \overline{\partial\phi}, \partial\phi, \overline{\partial\phi})}{|\partial\phi|^2},$$

where $\triangle_\theta$, Ric_θ and Bisect_θ are the complex Laplacian, the Ricci tensor and the sectional curvature tensor of θ, respectively.

Moreover if X_1 and X_2 are of the same dimension n and $\phi^*\theta_2^n \neq 0$, then

$$\triangle_{\theta_1} \log \frac{\phi^*\theta_2^n}{\theta_1^n} = -\operatorname{tr}_{\theta_1} \phi^*\gamma_{\theta_2} + \operatorname{tr}_{\theta_1} \gamma_{\theta_1}.$$

Applying the maximum principle we obtain the Schwarz lemma.

Lemma 4.2. *Let $\phi: (X_1, \theta_1) \longrightarrow (X_2, \theta_2)$ be as above. Assume that (X_1, θ_1) is complete and its Ricci curvature is bounded below.*

(i) *If there are constants K_1 and $K_2 > 0$ such that $\mathrm{Ric}_{\theta_1} \geq -K_1$ and $\mathrm{Bisect}_{\theta_2} \leq -K_2$, and if $|d\phi|^2$ is not identically equal to zero, then $K_1 > 0$ and*

$$|d\phi|^2 \leq \frac{K_1}{K_2}.$$

(ii) *If there are constants K_1 and $K_2 > 0$ such that $\operatorname{tr}_{\theta_1} \gamma_{\theta_1} \geq -nK_1$ and $\gamma_{\theta_2} \leq -K_2\theta_2$, where $n = \dim X_1 = \dim X_2$, and if $\phi^*\theta_2^n$ is not identically equal to zero, then $K_1 > 0$ and*

$$\frac{\phi^*\theta_2^n}{\theta_1^n} \leq \left(\frac{K_1}{K_2}\right)^n.$$

From the Schwarz lemma one can easily see

Theorem 4.1 ([T-Y1]). *A complex V-manifold admits at most one almost-complete Einstein Kähler V-metric.*

Now we state the existence theorem of Einstein Kähler V-metrics.

Theorem 4.2 ([T-Y1]). *Let (X, ω_0) be an n-dimensional Kähler V-manifold of bounded geometry of order $k+\alpha$, $k \geq 4$, with holomorphic quasi-coordinate maps.*

(i) *Assume that there exist a positive constant $\epsilon_0 > 0$ and a bounded C^2-V-function f such that $\gamma_{\omega_0} \leq -\epsilon_0\omega_0 + \sqrt{-1}\partial\bar{\partial} f$. Then X admits an Einstein Kähler V-metric ω of bounded geometry. And ω is comparable to ω_0, i.e. there exists a positive constant $a > 0$ such that $a\omega_0 \leq \omega \leq a^{-1}\omega_0$.*

(ii) *Assume that there exist a positive constant $\epsilon_0 > 0$ and a C^2-V-function f bounded from above such that $\gamma_{\omega_0} \leq -\epsilon_0\omega_0 + \sqrt{-1}\partial\bar{\partial} f$, and that for any positive $\epsilon > 0$ there exists a bounded C^2-V-function f_ϵ such that $\gamma_{\omega_0} \leq \epsilon\omega_0 + \sqrt{-1}\partial\bar{\partial} f_\epsilon$. Then X admits an almost-complete Einstein Kähler V-metric ω such that $\omega^n \leq c\omega_0^n$ and $e^{Af}\omega_0 \leq c\omega$ with constants c, $A > 0$.*

Proof. We prove only (ii). The proof of (i) will be apparent on the way.

We employ the continuity method. Consider the following equation on the Kähler V-metric ω.

$$\gamma_\omega = -t\omega + (t-s)\omega_0 + (1-s)\gamma_{\omega_0},$$

with $t \geq 1 \geq s \geq 0$. Note that if $s = 0$ then we have the trivial solution $\omega = \omega_0$, and a solution ω for $t = s = 1$ is the desired Einstein Kähler V-metric.

We rewrite the equation on V-metrics ω to the equation on V-functions u. To set a stage we take two Babach spaces $B_1 = C^{k,\alpha}$, $B_2 = C^{(k-2),\alpha}$, and define a domain O in B_1 and an operator E from O to B_2. Set $\omega_{t,s} = (1 - \frac{s}{t})\omega_0 - \frac{s}{t}\gamma_{\omega_0}$. ($\omega_{t,s}$ is not necessarily positive definite.)

$$\begin{aligned} \omega &= \omega_{t,s} + \sqrt{-1}\partial\bar{\partial} u, \\ O &= \left\{ u \in B_1 \;\middle|\; \exists a > 0 \quad \text{such that} \quad a\omega_0 \leq \omega \leq a^{-1}\omega_0 \right\}, \\ \mathrm{E}(u) &= \mathrm{E}_{t,s}(u) = \log \frac{\omega^n}{\omega_0^n} - tu. \end{aligned}$$

Then the equation becomes

$$\mathrm{E}(u) = 0 \qquad \text{i.e.} \qquad \omega^n = e^{tu}\omega_0^n.$$

Note that the Kähler V-metric ω corresponding to an element u in O is of bounded geometry of order $(k-2)+\alpha$.

First we show that for a sufficient large t and $1 \geq s \geq 0$ we can solve the equation. We choose a sufficient large constant T and fix t such that $t \geq T$. Since for $s = 0$ there is the trivial solution, the claim is proved if we see that the set $\left\{ s \in [0,1] \;\middle|\; \mathrm{E}_{t,s}(u) = 0 \text{ has a solution} \right\}$ is open and closed.

We consider the openness. The linearization $d\mathrm{E}$ of E at u is nothing but $\triangle_\omega - t$. The standard method shows that $d\mathrm{E}\colon B_1 \longrightarrow B_2$ is invertible. Namely the uniqueness of the solution v of the equation $\triangle_\omega v - tv = 0$ follows from the maximum principle. And we can find a solution v for the equation $\triangle_\omega v - tv = f \in B_2$ as a limit of the solution on relative compact domains with the Dirichlet boundary condition. We bound the sup norm of v by $\frac{1}{t}$ times that of f via the maximum principle. Then the estimates to guarantee $v \in B_1$ are given by the interior Schauder estimates.

The closedness is assured if one gets an apriori B_1-estimate of the solutions u and the bound of the constants a in the definition of the open set O. Note that the interior Schauder estimates again reduces the problem to the question of the estimates of the sup norm and a.

As to the sup norms, we have that

$$\begin{aligned} &\sqrt{-1}\partial\bar{\partial}tu = t\omega - t\omega_{t,s} = t\omega - (t-s)\omega_0 + s\gamma_{\omega_0}, \\ &t\omega - (t-c_1)\omega_0 \geq \sqrt{-1}\partial\bar{\partial}tu \geq t\omega - (t+c_2)\omega_0, \end{aligned}$$

with some positive constants c_1, $c_2 > 0$. Now we get that

$$\begin{aligned} \triangle_{\omega_0} tu &\geq t \operatorname{tr}_{\omega_0}\omega - n(t+c_2) \geq nt\left(\frac{\omega^n}{\omega_0^n}\right)^{\frac{1}{n}} - n(t+c_2) \\ &\geq nte^{\frac{1}{n}tu} - n(t+c_2), \\ \triangle_\omega(-tu) &\geq (t-c_1)\operatorname{tr}_\omega \omega_0 - nt \geq n(t-c_1)\left(\frac{\omega_0^n}{\omega^n}\right)^{\frac{1}{n}} - nt \\ &\geq n(t-c_1)e^{\frac{1}{n}(-tu)} - nt. \end{aligned}$$

Then the maximum principle gives the upper bound estimate of u for $t \geq 1$ and the lower bound estimate of u for $t \geq T > c_1$.

We bound the constant a. Let c_3 be a constant such that

$$\gamma_\omega = -t\omega + (t-s)\omega_0 + (1-s)\gamma_{\omega_0} \geq -t\omega + (t-c_3)\omega_0,$$

and $c_4 \geq 0$ an upper bound of the bisectional curvature of ω_0. We employ the infinitesimal Schwarz lemma for id: $(X,\omega) \longrightarrow (X,\omega_0)$.

$$\triangle_\omega \log \operatorname{tr}_\omega \omega_0 \geq -tn + \left(\frac{t-c_3}{n} - c_4 \right) \operatorname{tr}_\omega \omega_0.$$

If we choose T such that $T > c_3 + nc_4$, then we get the upper bound of $\operatorname{tr}_\omega \omega_0$. Together with the equation $\omega^n = e^{tu}\omega_0^n$ we obtain the desired bound on a. We remark that up to here we used only the assumption of bounded geometry.

Now we have the solution of the equation for $t \geq T$, $s = 1$.

$$\gamma_\omega = -t\omega + (t-1)\omega_0 \geq -t\omega.$$

We next show that the equation has a solution up to $t = 1$. The same argument as before shows the openness of the set $\{ t \in (1,T] \mid \mathrm{E}_{t,1}(u) = 0 \text{ has a solution} \}$. The closedness is proved as follows.

Fix $t_0 > 1$ and take f_ϵ for $\epsilon = \frac{1}{2}(t_0 - 1)$. We work with t such that $t \geq t_0$.

For the sup norm, as we have seen the upper bound of u is given uniformly for $t \geq 1$, we only have to get the lower bound.

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial} tu &= t\omega - (t-1)\omega_0 + \gamma_{\omega_0} \leq t\omega - \epsilon\omega_0 + \sqrt{-1}\partial\bar{\partial} f_\epsilon, \\ \triangle_\omega(-tu + f_\epsilon) &\geq \epsilon \operatorname{tr}_\omega \omega_0 - nt \geq \epsilon n \left(\frac{\omega_0^n}{\omega^n} \right)^{\frac{1}{n}} - nt \\ &\geq \epsilon n e^{\frac{1}{n}\{-\sup f_\epsilon + (-tu + f_\epsilon)\}} - nt. \end{aligned}$$

The maximum principle implies that there is a constant c_5 such that $(-tu + f_\epsilon) \leq \sup f_\epsilon + c_5$. Thus we get that

$$-tu \leq \sup f_\epsilon - \inf f_\epsilon + c_5.$$

Note that if we use f instead of f_ϵ we get that with a constant c_6, uniformly in $t > 1$,

$$-tu + f \leq \sup f + c_6.$$

We estimate a. With a positive constant A, we get that

$$\triangle_\omega \{ \log \operatorname{tr}_\omega \omega_0 + A(-tu + f_\epsilon) \} \geq -tn(1+A) + \left(\frac{t-c_3}{n} - c_4 + \epsilon A \right) \operatorname{tr}_\omega \omega_0.$$

Taking A large enough we get that with positive constants c_7 and c_8,

$$\begin{aligned} &\log \operatorname{tr}_\omega \omega_0 + A(-tu + f_\epsilon) \leq A \sup(-tu + f_\epsilon) + c_7, \\ &\operatorname{tr}_\omega \omega_0 \leq c_8. \end{aligned}$$

Then combining it with the equation we get the estimate of a. Similarly if we use f instead of f_ϵ we get that with constants c_9 and c_{10}, uniformly in $t > 1$,

$$\log \operatorname{tr}_\omega \omega_0 + A(-tu + f) \le A \sup(-tu + f) + c_9 \le c_{10}.$$

Thus we have a solution $u \in O$ for $t > 1$. Moreover since we have uniform estimates in $t > 1$ on compact domains, we can subtract a sequence $\{ u_{t_i} \}$ which convergents in $C^{k,\beta}$-topology on any compact domains, with $\alpha > \beta > 0$. Then the limit V-metric is the desired almost-complete Einstein Kähler V-metric. Q.E.D.

Remark. In the case of (ii) if f admits an estimate $f = o(\log r)$, where r is a distance function form a fixed point in X with respect to the metric ω_0, then the Einstein Kähler metric is complete.

Corollary 4.1 ([C-Y2], [M-Y]). *Let X be a smooth strictly pseudo-convex bounded domain in $\mathbf{C}^n$. Then X admits an Einstein Kähler metric of bounded geometry. Even if we drop the assumption of strictness, we get a complete Einstein Kähler metric.*

Proof. For a strictly pseudo-convex domain X we have a smooth strictly pseudo-convex function ϕ defined on a neighborhood of X such that $X = \{ \phi < 0 \}$ and $d\phi$ never vanishes on ∂X. Set

$$\omega_0 = \sqrt{-1}\partial\bar\partial \log(-\phi)^{-(n+1)} = (n+1)\{ \frac{\sqrt{-1}\partial\bar\partial\phi}{-\phi} + \frac{\sqrt{-1}\partial\phi \wedge \bar\partial\phi}{(-\phi)^2} \}.$$

Then ω_0 satisfies the assumption of Theorem 4.2 (i). For a weakly pseudo-convex domain, we approximate it by strictly pseudo-convex domains from inside and get a sequence of complete Einstein Kähler metrics. Then we can show that the sequence converges uniformly on compact domains. The proof of completeness is done through the Schwarz lemma. For details see [C-Y2] and [M-Y]. Q.E.D.

§5. On quasi-projective manifolds

Let X be a nonsingular n-dimensional projective algebraic variety and D be a $\mathbf{Q}$-divisor on X. We denote by K the canonical divisor of X.

Definition 5.1.

(i) D is called numerically effective, nef in short, if for any curve C in X the intersection number $D \cdot C$ is non-negative.

(ii) A nef $\mathbf{Q}$-divisor D is called big if $D^n > 0$.

(iii) D is called ample if mD is ample integral divisor for some positive integer m.

The following lemmas by Y. Kawamata suggests the natural assumption to impose for the logarithmic version of Calabi's conjecture (cf. [Km1,2,3] [KMM]).

Lemma 5.1. *Let L be a nef and big divisor on X. Then we have the following.*

(i) *There exists an effective $\mathbf{Q}$-divisor E such that $L - E$ is ample.*

(ii) *There exists a birational morphism $\mu : Y \longrightarrow X$ from a nonsingular variety Y and a family of divisors $\{F_j\}$ on Y such that the union of supports of $\mu^* L$ and $\bigcup F_j$ is a divisor with only simple normal crossings and such that $\mu^* L - \sum \delta_j F_j$ is ample for some $\delta_j \in \mathbf{Q}$ with $0 < \delta_j \ll 1$.*

Lemma 5.2. *Let D be a nef $\mathbf{Q}$-divisor and H an ample divisor. Then $D + \delta H$ is ample for a positive rational number δ.*

Theorem 5.1 ([Ko1,2,3,4], [CY3], [Ts3], [TY1]). *Let X be an n-dimensional nonsingular projective algebraic variety and $D = \sum D_i$ a divisor with simple normal crossings.*

(i) *If $K + D$ is ample, then there exists an Einstein Kähler metric on $X - D$ of bounded geometry. Its Kähler class belongs to the Poincaré dual of $2\pi(K + D)$.*

(ii) *Assume that $K + D$ is nef and there exists an effective $\mathbf{Q}$-divisor E whose support is contained in D such that $K + D - E$ is ample. Then there exists an almost-complete Einstein Kähler metric ω on $X - D$ such that $\int_{X-D} \omega^n = 2\pi(K + D)^n$. In particular $K + D$ is big.*

(iii) *If a $\mathbf{Q}$-divisor $K + \sum(1 - \frac{1}{m_i})D_i$, with positive integers m_i, is ample, then there exists an Einstein Kähler V-metric on $X\left(\sum \frac{1}{m_i} D_i\right)$. Its Kähler class belongs to the Poincaré dual of $2\pi\{K + \sum(1 - \frac{1}{m_i})D_i\}$.*

Here $X\left(\sum \frac{1}{m_i} D_i\right)$ is the V-manifold constructed in the example 2.2 in §2.

Proof. Let us consider (iii) first. Because of the assumption there exist a smooth volume form Ω and fibre metics $\|\cdot\|$ on L_{D_i} such that $\sqrt{-1}\partial\bar{\partial} \log \frac{\Omega}{\prod \|\sigma_i\|^{2(1-\frac{1}{m_i})}}$ is a smooth positive definite $(1,1)$-form. Here σ_i is a holomorphic section of L_{D_i} such that $(\sigma_i) = D_i$, $\|\sigma_i\|^2 < 1$. For a sufficiently small fixed $\epsilon > 0$ we define smooth V-forms Ω_0 and ω_0 on $X\left(\sum \frac{1}{m_i} D_i\right)$ as

$$\Omega_0 = \frac{\Omega}{\prod \|\sigma_i\|^{2(1-\frac{1}{m_i})}\left(1 - \epsilon\|\sigma_i\|^{\frac{2}{m_i}}\right)^2},$$

$$\omega_0 = \sqrt{-1}\partial\bar{\partial}\log\Omega_0.$$

Since

$$\sqrt{-1}\partial\bar{\partial}\log\left(1 - \epsilon e^{-t}\right)^{-1} = -\frac{\epsilon e^{-t}\sqrt{-1}\partial\bar{\partial}t}{1 - \epsilon e^{-t}} + \frac{\epsilon e^{-t}\sqrt{-1}\partial t \wedge \bar{\partial}t}{(1 - \epsilon e^{-t})^2},$$

ω_0 is positive definite and defines a Kähler V-metric. Moreover there is a smooth V-function f satisfying $\omega_0^n = e^{-f}\Omega_0$. Then we have $\gamma_{\omega_0} = -\omega_0 + \sqrt{-1}\partial\bar{\partial}f$. Thus applying the existence theorem 4.2 (i) we get an Einstein Kähler metric.

Next we consider (i). We have a smooth volume form Ω and fibre metrics $\|\cdot\|$ of L_{D_i} such that $\sqrt{-1}\partial\bar{\partial}\log\frac{\Omega}{\prod\|\sigma_i\|^2}$ is positive definite. We define smooth forms Ω_0 and ω_0 on $X - D$ as follows, with a sufficiently small fixed $\epsilon > 0$,

$$\Omega_0 = \frac{\Omega}{\prod \|\sigma_i\|^2\left(1 - \epsilon\log\|\sigma_i\|^2\right)^2},$$

$$\omega_0 = \sqrt{-1}\partial\bar{\partial}\log\Omega_0.$$

Since

$$\sqrt{-1}\partial\bar{\partial}\log(1 + \epsilon t)^{-1} = -\epsilon\frac{\sqrt{-1}\partial\bar{\partial}t}{1 + \epsilon t} + \epsilon^2\frac{\sqrt{-1}\partial t \wedge \bar{\partial}t}{(1 + \epsilon t)^2},$$

ω_0 is positive definite and defines a Kähler metric on $X - D$. Moreover ω_0 is of bounded geometry with the following quasi-coordinate maps. For a point p near D we have a local coordinate system $z = (z^1, z^2, \dots, z^n)$ such that $D = (z^1 z^2 \cdots z^k)$. Then take a quasi-coordinate map $w = (w^1, w^2, \dots, w^n) \longmapsto z$ centered at p as

$$z^i = \begin{cases} z^i(p)e^{(-\log|z^i(p)|)w^i} & \text{for } 1 \le i \le k, \\ z^i(p) + w^i & \text{for } k < i \le n. \end{cases}$$

The function f defined by $\omega_0^n = e^{-f}\Omega_0$, is bounded. Thus by the existence theorem 4.2 (i) we get an Einstein Kähler metric of bounded geometry. In particular its Kähler class belongs to $2\pi(K + D)$.

In the case (ii) there are rational numbers δ_i, $0 < \delta_i < 1$, a smooth volume form Ω and fibre metrics $\|\cdot\|$ on L_{D_i} such that if we set

$$\omega. = \sqrt{-1}\partial\bar\partial \log \frac{\Omega}{\prod \|\sigma_i\|^2},$$

which corresponds to $K + D$, then

$$\omega. - \sum \delta_i \theta_i = \sqrt{-1}\partial\bar\partial \log \frac{\Omega}{\prod \|\sigma_i\|^{2(1-\delta_i)}}$$

is positive difinite, where $\theta_i = -\sqrt{-1}\partial\bar\partial \log \|\sigma_i\|^2$. We define smooth forms $\Omega(\epsilon)$, for positive ϵ, Ω_0 and ω_0 on $X - D$ as follows, with a sufficiently small fixed $\epsilon_0 > 0$,

$$\Omega(\epsilon) = \frac{\Omega}{\prod \|\sigma_i\|^2 \left(1 - \epsilon \log \|\sigma_i\|^2\right)^2},$$

$$\Omega_0 = \frac{\Omega}{\prod \|\sigma_i\|^{2(1-\delta_i)} \left(1 - \epsilon_0 \log \|\sigma_i\|^2\right)^2},$$

$$\omega_0 = \sqrt{-1}\partial\bar\partial \log \Omega_0.$$

Then ω_0 defines a Kähler metric of bounded geometry and we have a bounded smooth function f such that $\omega_0^n = e^{-f}\Omega(\epsilon_0)$. Then we have that $\gamma_{\omega_0} = -\omega_0 + \sqrt{-1}\partial\bar\partial(f + \sum \delta_i \log \|\sigma_i\|^2)$. This means the half of the assumption of the existence theorem (ii) is satisfied. Let ϵ be any sufficiently small positive number and θ be a Kähler form corresponding to an ample divisor H. There exist a bounded function f'_ϵ such that $\omega_0^n = e^{-f'_\epsilon}\Omega(\epsilon)$ and a constant c'' such that $\theta \le c''\omega_0$. Then we have that with a constant c',

$$\begin{aligned}\gamma_{\omega_0} &\le -\omega. + 2\epsilon \sum \frac{\theta_i}{1 - \epsilon \log \|\sigma_i\|^2} + \sqrt{-1}\partial\bar\partial f'_\epsilon \\ &\le -\omega. + c'\epsilon\omega_0 + \sqrt{-1}\partial\bar\partial f'_\epsilon \\ &\le -(\omega. + \epsilon\theta) + (c' + c'')\epsilon\omega_0 + \sqrt{-1}\partial\bar\partial f'_\epsilon.\end{aligned}$$

By Lemma 5.2 there exists a smooth function f''_ϵ on X such that $\omega. + \epsilon\theta + \sqrt{-1}\partial\bar\partial f''_\epsilon$ is positive definite. Then we get that

$$\gamma_{\omega_0} \le (c' + c'')\epsilon\omega_0 + \sqrt{-1}\partial\bar\partial(f'_\epsilon + f''_\epsilon).$$

Now we can apply the existence theorem 4.2 (ii) and get the desired Einstein Kähler metric ω on $X - D$. And

$$\int_{X-D} \omega^n = \{2\pi(K+D)\}^n,$$

by the Lebesgue convergence theorem.

The statements on the Kähler classes in the cases (i) and (iii) are standard.

Remark. We remark that the case (ii) is a log-canonical analogue of the case of compact manifolds with ample canonical bundles. Thus we conjecture that the almost-complete Einstein Kähler metrics are actually complete. We have a characterization of the Einstein Kähler volume form ω^n.

$$\omega^n = \inf\{\, \theta^n \mid \theta \text{ is a complete Kähler form such that } \gamma_\theta \geq -\theta \bigr\}.$$

In the case (ii) of the theorem, there are positive integers m_i such that $K + \sum\bigl(1 - \frac{1}{jm_i}\bigr)D_i$ are ample for all positive integer j. Then we have the Einstein Kähler V-metrics ω_j on the V-manifolds $X\bigl(\sum \frac{1}{jm_i} D_i\bigr)$. Outside the support of D each ω_j defines a smooth Einstein Kähler metric.

Theorem 5.2 ([Ts3], [TY1]). *As j tends to infinity, the sequence $\{\omega_j\}$ converges to the almost-complete Einstein Kähler metric ω on $X - D$ smoothly on compact sets.*

Proof. We use the notation in the proof of theorem (ii).

The Schwarz lemma gives that $\omega_j^n \leq \omega^n \leq c\omega_0^n$. Set $\omega_j^n = e^u \omega_0^n$, $u = u_j$. Then u is a smooth function on $X - D$ and bounded from above.

$$\begin{aligned}\omega_j = -\gamma_{\omega_j} &= -\gamma_{\omega_0} + \sqrt{-1}\partial\bar\partial u\\ &= \omega_0 + \sqrt{-1}\partial\bar\partial\bigl(u - f - \textstyle\sum \delta_i \log\|\sigma_i\|^2\bigr).\end{aligned}$$

For large j considering the order of divergence we see that the function $v = u - f - \sum \delta_i \log\|\sigma_i\|^2$ tends to ∞ near D. So it takes its mimimum somewhere in $X - D$. At that point $\omega_j \geq \omega_0$ and we get a uniform lower bound of v. Now let $c_1 \geq 0$ be an upper bound of the bisectional curvature of ω_0, and A a constant large enough.

$$\triangle_{\omega_j}(\log \operatorname{tr}_{\omega_j}\omega_0 - Av) \geq -(1 + nA) + (A - c_1)\operatorname{tr}_{\omega_j}\omega_0.$$

Again considering the order of divergence we get a constant c_2 independent of large j, such that $\log \mathrm{tr}_{\omega_j} \omega_0 - Av \leq c_2$. Thus we obtain a uniform estimate: with a constant c_3 independent of large j,

$$\log \mathrm{tr}_{\omega_j} \omega_0 + A \sum \delta_i \log \|\sigma_i\|^2 \leq c_3.$$

Then the interior Schauder estimates give a uniform C^∞-estimates on compact sets. So we can take a subsequence which converges to an Einstein Kähler metric, say ω', on $X - D$. We have $\omega'^n \leq \omega^n$, and by the Lebesgue convergence theorem

$$\int_{X-D} \omega'^n = \lim \int_{X-D} \omega_j^n = \{2\pi(K+D)\}^n = \int_{X-D} \omega^n.$$

Thus $\omega'^n = \omega^n$ and $\omega' = \omega$, since both Einstein.

Instead of a reduced divisor D with simple normal crossings we can work with non-reduced one. Namely, R. Kobayashi and the author get

Theorem 5.3. *Let X be an n-dimensional nonsingular projective algebraic variety and $D = \sum a_i D_i$ a $\mathbf{Q}$-divisor with simple normal crossings and $a_i \leq 1$. Assume that $K + D$ is nef and there exists an effective $\mathbf{Q}$-divisor E whose support is contained in D such that $K + D - E$ is ample. Then we have an Einstein Kähler metric ω on $X - D$ whose Kähler class belongs to the Poincaré dual of $2\pi(K + D)$ and $\int_{X-D} \omega^n = \{2\pi(K + D)\}^n$. In particular $K + D$ is big. ω is unique in an appropriate class.*

We remark that the situation like above naturally arises in the study of weak log-terminal general type. The idea of the proof is to solve the following equation and take the limit $\epsilon \longrightarrow 0$.

$$\omega = \omega_\epsilon + \sqrt{-1}\partial\bar{\partial} u,$$

$$\omega^n = e^u \frac{\prod(\|\sigma_i\|^2 + \epsilon)^{\alpha_i} \Omega}{\prod \|\sigma_i\|^2 \left(1 - \epsilon \log \|\sigma_i\|^2\right)^2},$$

where $\epsilon > 0$, $\alpha_i = 1 - a_i$ and

$$\omega_\epsilon = \sqrt{-1}\partial\bar{\partial} \log \frac{\Omega}{\prod \|\sigma_i\|^{2a_i} \left(1 - \epsilon \log \|\sigma_i\|^2\right)^2} + \epsilon\theta,$$

with a suitable choice of a volume form Ω, fibre metrics $\|\cdot\|$ and a Kähler metric θ on X. To prove the statement of the Kähler class we use a lemma of the following type.

Lemma 5.3. *Let $\omega_i = \omega_0 + \sqrt{-1}\partial\bar{\partial}u_i$ be a sequence of smooth Kähler forms on a compact Kähler manifold X. Assume that it smoothly converges a singular Kähler form ω outside of an effective divisor D keeping the following estimates. For any $\epsilon > 0$ there exists a constant C_ϵ such that $|u_i| \le -\epsilon \log \|\sigma\|^2 + C_\epsilon$, for all i, where σ is a holomorphic section of L_D such that $(\sigma) = D$. Then the current T defined by $T(\theta) = \int_{X-D} \omega \wedge \theta$ is d-closed positive $(1,1)$-current and its cohomology class is that of ω_0.*

§6. Miyaoka-Yau inequality

First we state the Miyaoka-Yau inequality for Einstein Kähler V-manifolds.

Theorem 6.1 ([Kb2], [C-Y3], [t-Y1]). *For an n-dimensional compact Einstein Kähler V-manifold X, we have the following inequality of the Chern numbers.*

$$2(n+1)c_2(\Omega^1_X)c_1(\Omega^1_X)^{n-2} \ge nc_1(\Omega^1_X)^n,$$

where Ω^1_X is the holomorphic cotangent V-bundle of X. And if the equality holds, X is a quotient of the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$ with a properly discontinuous group, which has the fixed point locus corresponding to the singular locus of X.

Proof. Let $c_1(X,\omega)$ and $c_2(X,\omega)$ be the first and second Chern class forms given by the Einstein Kähler metric ω, respectively. Define a tensor T which measures the deviation of (X,ω) from being of constant holomorphic sectional curvature, with the curvature tensor R and the metric tensor g,

$$T_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + \frac{1}{n+1}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}).$$

Then a calculation shows that

$$\{2(n+1)c_2(X,\omega) - nc_1(X,\omega)^2\} \wedge \omega^{n-2} = \frac{n+1}{4\pi^2 n(n-1)}\|T\|^2\omega^n.$$

Since the Chern Weil homomorphism holds on V-manifolds, integrating the above equality we get the desired inequality. And if the equality holds ω is of constant holomorphic sectional curvature. The developing map can be used also in V-manifolds situation and we get the last assertion.

Let X be an n-dimensional nonsingular projective algebraic manifold, K its canonical divisor and $D = \sum D_i$ an effective reduced divisor on X with simple normal crossings. Assume that $K + D$ is nef and that there exists an effective $\mathbf{Q}$-divisor E whose support is contained in D such that $K + D - E$ is ample.

Theorem 6.2 ([Kb1,3,4],[C-Y3], [Ts3], [T-Y1]). *We assume as above, then we have the following inequality between the Chern numbers.*

$$2(n+1)c_2(\Omega_X^1(\log D))c_1(\Omega_X^1(\log D))^{n-2} \geq nc_1(\Omega_X^1(\log D))^n,$$

where $\Omega_X^1(\log D)$ is the holomorphic cotangent bundle with logarithmic poles along D. And if the equality holds, $X - D$ is covered by the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$.

Proof. By the assumptions we have a sequence of V-manifolds $X_j = X(\sum \frac{1}{jm_i} Di)$ and the Einstein Kähler metrics ω_j on them, such that ω_j converges to the almost-complete Einstein Kähler metric ω on $X - D$.

Let $D'_j = \sum D'_{ij}$ be the reduced divisor on X_j corresponding to the singular locus of X_j, and $L_{D'_{ij}}$ the line V-bundle associated to D'_{ij}. If we pull back the line bundle L_{D_i} associated to D_i via the V-map $f_j : X_j \longrightarrow X$ induced by the identity map, we get that $f_j^*(L_{D_i}) = (L_{D_{ij}})^{\otimes jm_i}$. Thus $c_1(L_{D'_{ij}}) = \frac{1}{jm_i} c_1(L_{D_i}) \in H^2(X, \mathbf{R})$. (Note that X_j and X are the same as topological spaces.) Just like in the smooth case, considering the Poincaré residue maps, we can get that for the total Chern classes

$$c(\Omega_{X_j}^1) = c(\Omega_{X_j}(\log D'_j)) \prod c((L_{D'_{ij}})^{-1}).$$

Since $\Omega_{X_j}(\log D'_j) = f_j^*(\Omega_X(\log D))$, we get that

$$c_1(\Omega_{X_j}^1) = c_1(\Omega_X^1(\log D)) - \sum \frac{1}{jm_i} c_1(L_{D_i}),$$

$$c_2(\Omega_{X_j}^1) = c_2(\Omega_X^1(\log D)) - c_1(\Omega_X^1(\log D)) \sum \frac{1}{jm_i} c_1(L_{D_i})$$

$$+ \sum_{i_1 \neq i_2} \frac{1}{jm_{i_1}} c_1(L_{D_{i_1}}) \frac{1}{jm_{i_2}} c_1(L_{D_{i_2}}).$$

Now substituting the above equalities into the Miyaoka-Yau inequality for X_j,

$$2(n+1)c_2(\Omega_{X_j}^1)c_1(\Omega_{X_j}^1)^{n-2} \geq nc_1(\Omega_{X_j})^n,$$

and taking the limit $j \longrightarrow \infty$, we get the desired inequality. Moreover by the proof and Fubini's theorem we have that, for the limiting almost-complete Einstein Kähler metric ω on $X - D$

$$\begin{aligned}2(n+1)c_2(\Omega^1_X(\log D))c_1(\Omega^1_X(\log D))^{n-2} - nc_1(\Omega^1_X(\log D))^n \\ \geq \frac{n+1}{4\pi^2 n(n-1)} \int_{X-D} \|T\|^2 \omega^n.\end{aligned}$$

Thus if the equality holds T vanishies. It means that ω is of constant holomorphic sectional curvature. If we know that ω is complete, we get the last assertion of the theorem. For the proof of completeness, we efficiently use the Schwarz lemma. See [T-Y1] for details.

Although we also have an Einstein Kähler metric for non-reduced D, we do not have the corresponding Miyaoka-Yau inequality. It would be an interesting problem.

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