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On Tangent Sheaves of Minimal Varieties

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In this chapter, we shall study the tangent sheaves of minimal varieties, especially concentrating on the stability and an inequality between Chern numbers of the sheaves. To begin with, we shall recall the history of inequality between Chern numbers of minimal varieties.

Van de Ven [V] recognized, for the first time, the inequality $c_1(S)^2 \leq 8c_2(S)$ for a surface of general type S. And he also conjectured that a surface of general type S should satisfy the inequality $c_1(S)^2 \leq 3c_2(S)$. Later, Bogomolov and Gieseker ([Bo], [G]) proved an inequality between Chern numbers:

$$c_1(E)^2 \leq \frac{2r}{r-1}c_2(E)$$

for an H-semistable vector bundle with rank r on a projective surface S, where H is an ample divisor on S. Also Bogomolov obtained an inequality $c_1(S)^2 \leq 4c_2(S)$ for a surface of general type S. Afterward Miyaoka [Mi-1] finally proved Van de Ven's conjecture.

On the other hand, by virtue of Mumford-Mehta-Ramanathan's results ([M-R]), one can easily see that, for an H-semistable vector bundle with rank r on an n dimensional non-singular projective variety M, the following inequality holds:

$$\text{(0.1)} \qquad \left\{(r-1)c_1(E)^2 - 2rc_2(E)\right\} \cdot H^{n-2} \leq 0.$$

Related to this, Lübke [Lü] proved

$$\text{(0.2)} \qquad \int_M \{(r-1)c_1(E) - 2rc_2(E)\} \wedge \Phi^{n-2} \leq 0$$

for an Einstein-Hermitian vector bundle $\{E, h\}$ over an n dimensional Kähler manifold (M, Φ). Now one may ask the relation between stability of vector bundles and Einstein-Hermitian metric. In fact, S. Kobayashi

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conjectured a Φ-semistable vector bundle on a compact Kähler manifold (M, Φ) should admit an Einstein-Hermitian metric. (The converse is rather easy. See [Ko-S].) This conjecture was first solved by Donaldson [D] under the additional assumption that M is projective. Later on, Yau and Uhlenbeck [U-Y] completely solved the conjecture.

Now we go back to the tangent sheaves of varieties. In 1977–1978, Aubin and Yau ([A], [Y]) proved that an n dimensional compact Kähler manifold M whose Ricci curvature is negative admits an Einstein-Kähler metric ω. Therefore in particular, one has the inequality (0.2) for the tangent vector bundle of (M, ω). Combining Guggenheimer-Chen-Ogiue's results ([Ch-O]), Yau obtained a stronger inequality, so called Miyaoka-Yau's inequality:

$$(0.3) \qquad (-1)^n \left\{ nc_1(M)^2 - 2(n+1)c_2(M) \right\} \cdot c_1(M)^{n-2} \leq 0.$$

Moreover he showed that the equality in (0.3) holds if and only if M is covered by the unit ball in $\mathbf{C}^n$. Note that, when $n = 2$, this is nothing but Miyaoka's inequality. (Although Yau's result requires the stronger assumtion.)

In algebraic geometry, there is a very important class of varieties, so-called *minimal varieties of general type.*(See Section 1 for the definition.) All projective varieties whose canonical divisor is ample are contained in this class. So it is desirable to obtain Miyaoka-Yau's inequality for any minimal variety of general type. And it will be very interesting to study the structure of the varieties when the equality holds. Now we want pose an conjecture which we shall discuss in this chapter.

Conjecture 0.4. *Let M be an n dimensional minimal variety of general type and*

$$N \xrightarrow{\mu} M$$

its desingularization. Then

$$\left\{ nc_1(N)^2 - 2(n+1)c_2(N) \right\} \cdot \mu^* K(M)^{n-2} \leq 0,$$

where $K(M)$ is the canonical divisor of M. Moreover if the equality holds, the regular part M_{reg} should be locally isometric to the unit ball in $\mathbf{C}^n$.

Related to this conjecture, many people obtained partial results. In fact, in [Mi-2], for a minimal surface of general type S, Miyaoka showed that if $c_1(S)^2 = 3c_2(S)$, K_S is ample. Therefore, combining Yau's result cited above, we obtain the desired results. Also, in [Mi-4], he obtained a refinement of the inequality. On the other hand, using the notion of bounded geometry, R. Kobayashi [Ko-R] proved such an inequality and he also characterized a quotient of the 2-dimensional unit ball in terms of Chern numbers, which generalize, Yau's results in the surface case. Tsuji [Ts-1] showed that a non-singular minimal variety of general type admits singular Einstein-Kähler metrics. Later on Bando-R. Kobayashi [B-K], and the author [Sg] independently refined Tsuji's results. In fact, the existence of singular Einstein-Kähler metrics on minimal varieties of general type was proved (see Section 2). One may expect that, using these results, it will be possible to obtain stability and Miyaoka-Yau type inequality of Chern numbers as before. But since our metrics have singularities, there are still difficulties to be overcome. In [Ts-2], Tsuji proved Miyaoka-Yau type inequality and *weak stability* for *non-singular* minimal varieties.

Enoki [E] also got *weak stability* of tangent sheaves of *minimal Kähler spaces.* In [Mi-3], Miyaoka initiated a new notion *$\mathcal{B}$-semistability* and in order to obtain an inequality of Chern numbers for strictly minimal varieties different from Conjecture 0.4. Very recently, Tsuji [Ts-3] showed the same inequality as Theorem 4.1 by an analytic method. We shall prove Theorem 4.1 by a geometric method which uses stability of a tangent sheaf of minimal varieties.

The contents of this chapter is as follows. In Section 1, we shall explain terminologies and notations which will be used later. In Section 2, we shall show the existence of singular Einstein-Kähler metrics on minimal varieties of general type, while Section 3 is devoted to overview Enoki's results mentioned above. In Section 4, using the results of Section 3, we shall show an inequality of Bogomolov-Gieseker type for minimal varieties of general type. In Section 5, we shall give an example of minimal varieties whose tangent sheaf is stable.

Although this chapter is written as survey, it contains some results which have not appeared in literature. Complete proofs for these results are given in [Sg] contained in this volume.

§1. Terminologies

Following [K-M-M], we shall explain terminologies of algebraic geometry. Let M be an n dimensional projective variety. For simplicity, we always assume M is *normal.* We begin with the following notations.

$Z_{n-1}(M) :=$ the group of Weil divisors, i.e. the free abelian group generated by prime divisors on M.

$\mathrm{Div}(M) :=$ the group of Cartier divisors on M.

$\mathrm{Pic}(M) :=$ the group of line bundles on M.

Note that we have a natural homomorphism $\mathrm{Div}(M) \to \mathrm{Pic}(M)$. Take a compact curve C on M. For D in $\mathrm{Pic}(M)$, we define the intersection number $(D \cdot C) := \deg_{\bar{C}}(f^*D)$, where $f : \bar{C} \to C$ is the normalization of C. An element of $Z_{n-1}(M) \otimes \mathbf{Q}$ (resp. $\mathrm{Div}(M) \otimes \mathbf{Q}$) is called a $\mathbf{Q}$-divisor (resp. $\mathbf{Q}$-Cartier divisor). D in $\mathrm{Div}(M) \otimes \mathbf{Q}$ is called *nef* if $(D \cdot C) \geq 0$ for any effective curve C.

Fact 1.1 (Kleiman's criterion for ampleness). *Let X be a projective variety. Then H in $\mathrm{Pic}(X)$ is ample if and only if there exists a positive number ϵ such that $(H \cdot C) \geq \epsilon \parallel C \parallel_A$ for any effective curve C, where $\parallel C \parallel_A$ is the volume of C with respect to an ample divisor A on X.*

Therefore if D in $\mathrm{Div}(M) \otimes \mathbf{Q}$ is nef, $D + \epsilon A$ is $\mathbf{Q}$-ample for any positive rational number ϵ and for any ample divisor A. Here H in $\mathrm{Div}(M) \otimes \mathbf{Q}$ is called $\mathbf{Q}$-ample if mH becomes ample for a positive integer m.

Definition 1.2. The *canonical divisor* K_M on M is an element of $Z_{n-1}(M)$ such that $\mathcal{O}_{M_{\mathrm{reg}}}(K_M) = \Omega^n_{M_{\mathrm{reg}}}$, where M_{reg} is the regular part of M. M is called $\mathbf{Q}$-*Gorenstein* if the canonical divisor K_M is a $\mathbf{Q}$-Cartier divisor, i.e. if $\mathcal{O}_M(rK_M)$ becomes invertible for some r in $\mathbf{N}$.

Definition 1.3. M is said to have *only canonical* (resp. *terminal*) *singulariries* if the following two conditions are satisfied.

(1) M is $\mathbf{Q}$-Gorenstein
(2) there exists a desingularization $\mu : N \to M$ such that $K_N = \mu^* K_M + \sum a_i E_i$ for $a_i \in \mathbf{Q}$ with $a_i \geq 0$ (resp, $a_i > 0$) for all i

where the E_i vary among all the prime divisors which are exceptional with respect to μ.

When $\dim M = 2$, M has only canonical (resp. terminal) singularities if and only if M has only rational double or non-singular points (resp. non-singular points).

Definition 1.4. Assume that M has only canonical singularities. M is called *minimal variety* if K_M is nef. Moreover if M has only terminal singularities, M is called a *strictly minimal variety.*

Definition 1.5. We define *the Kodaira dimension* $\kappa(M)$ of M to be $\kappa(M) := \kappa(K_N)$ where N is a desingularization of M. The canonical ring of M is defined to be $R(M) := R(N) := \bigoplus_{m=0}^{\infty} H^0(N, \mathcal{O}_N(mK_N))$. Then

$$\kappa(M) := \begin{cases} \text{trans.deg}_{\mathbf{C}} R(M) - 1, & \text{if } R(M) \neq \mathbf{C} \\ -\infty, & \text{otherwise.} \end{cases}$$

M is said to be of general type if $\kappa(M) = \dim M = n$.

Note that, for any two non-singular projective varieties N_1 and N_2 birational to each other, the canonical rings $R(N_1)$ and $R(N_2)$ are isomorphic, and therefore Kodaira dimension of M is well defined, i.e. $\kappa(M)$ is independent of choice of desingularizations. We shall often use the following two facts.

Fact 1.6 ([K-M-M, Corollary 0-3-5]). *Let M be a minimal variety of general type. Then there exists an effective $\mathbf{Q}$-divisor D_0 such that $K_M - \delta D_0$ is $\mathbf{Q}$-ample for all δ in $\mathbf{Q}$ with $0 < \delta < 1$.*

Fact 1.7 ([K-M-M, Corollary 3-3-2]). *Let M be a minimal variety of general type. Then its canonical ring $R(M)$ is a finitely generated algebra over $\mathbf{C}$. Thus the canonical model $M_{\mathrm{can}} := \mathrm{Proj} R(M)$ of M exists. $\Phi_{|mK_M|} : M \to M_{\mathrm{can}}$ for $m \gg 0$ gives the canonical morphism onto M_{can} which in the case of $\dim M = 2$, it is nothing but the contraction of (-2)-curves. Moreover it is well-known that M_{can} is a normal projective variety with only canonical singularities and its canonical divisor $K_{M_{\mathrm{can}}}$ is ample.*

§2. Construction of degenerate Einstein-Kähler metrics

In this section, we shall construct degenerate Einstein-Kähler metrics on canonical varieties. Let M be an n dimensional compact projective algebraic manifold and $E = \sum_i E_i$ effective divisors on M. Assume that the following condition is satisfied.

Assumption 2.1. *There exist positive numbers a_i $(1 \leq i \leq N)$ so that $K_M - \sum_{i=1}^N a_i E_i$ is ample.*

Let $\mathbf{h}_i$ be a Hermitian fibre metric of the line bundle $[E_i]$. Then, by the assumption (2.1), a C^∞ d-closed real (1.1)-form ω_R in $2\pi c_1(K_M)$ exists on M such that

$$\omega_R - \sum_{i=1}^{N} a_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$$

is positive definite everywhere on M. From [Y], there exists a Kähler metric $\mathbf{G}$ on M such that $\sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} = \omega_R$ and therefore

$$\omega_a := \sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_{i=1}^N a_i\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}_i$$

is positive definite everywhere on M. Let s_i be the non-zero holomorphic section of $[E_i]$ whose the divisor is just E_i. Now we consider the equation:

$$\det[\mathbf{G}(a)_{i\bar{j}} + u_{:i\bar{j}}] = \prod_{i=1}^N \| s_i \|_{\mathbf{h}_i}^{2a_i} e^u \det\mathbf{G}$$

where

$$\omega_a = \sqrt{-1}\textstyle\sum_{i\bar{j}}\mathbf{G}(a)_{i\bar{j}}dz_i \wedge d\bar{z}_j,$$

and

$$u_{:i\bar{j}} := \partial_i\partial_{\bar{j}}.$$

Let ϕ be a C^∞ solution of (2.2) on $M\backslash\cup_i E_i$. Then it is easy to see that $\tilde{\omega}_a := \omega_a + \sqrt{-1}\partial\bar{\partial}\phi$ is an Einstein-Kähler metric on $M\backslash\cup_i E_i$. In order to solve the equation the equation above, we consider the following perturbed equation:

$$\det[\mathbf{G}(a)_{i\bar{j}} + u_{:i\bar{j}}] = \prod_{i=1}^N (\| s_i \|_{\mathbf{h}_i} + \delta)^{2a_i} e^u \det\mathbf{G} \tag{2.2}$$

It is known by Yau (for instance, see [Y, Theorem 4]) that the equation (2.2) always has the unique C^∞ solution ϕ_δ for any positive number δ. We shall show that a subsequence of $\{\phi_\delta\}_{\delta>0}$ converges to a C^∞ function on $M\backslash\cup_i E_i$ with respect to certain norm as δ goes to 0. Firstly, we shall obtain a lower bound for ϕ_δ. In what follows, C_i always denotes a positive number which depends only on $\{a_i\}$. Let fix a positive number δ and x_0 a point of M such that $\phi_\delta(x_0) = \inf_M \phi_\delta$. From the definition of ϕ_δ, we have

$$\begin{aligned}\det(\mathbf{G}(a)_{i\bar{j}})(x_0) &\leq \det(\mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}})(x_0)\\ &= \prod_{i=1}^N (\| s_i \|_{\mathbf{h}_i}^2 + \delta)^{-a_i}(x_0) e_\delta^\phi(x_0)\det\mathbf{G}(x_0).\end{aligned}$$

Therefore, we see

$$\begin{aligned}\phi_\delta(x_0) &\geq \log\{\prod_{i=1}^N (\| s_i \|_{\mathbf{h}_i}^2 + \delta)^{-a_i}(x_0)\det(\mathbf{G}(a))(x_0)/\det\mathbf{G}(x_0)\}\\ &\geq -C_1.\end{aligned}$$

Hence we have the following lemma.

Lemma 2.3. *For any* $0 < \delta < 1$, *we have* $\inf_M \phi_\delta \geq -C_1$.

Next we shall obtain an upper bound for ϕ_δ. We set

$$\psi_\delta := \phi_\delta + \sum_{i=1}^{N} a_i \log \| s_i \|^2_{\mathbf{h}_i} . \tag{2.4}$$

Then ψ_δ is a C^∞ function on $M \backslash \cup_i E_i$ which is bounded above, and ψ_δ is $-\infty$ on $\cup_i E_i$. Since, on $M \backslash \cup_i E_i$,

$$\mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det \mathbf{G} + \psi_{\delta:i\bar{j}},$$

we have

$$\det(\partial_i \partial_{\bar{j}} \log \det \mathbf{G} + \psi_{\delta:i\bar{j}}) = \prod_{i=1}^{N} (\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i} \| s_i \|^{-2a_i}_{\mathbf{h}_i} e^{\psi_\delta} \det \mathbf{G}.$$

Now fix $0 < \delta < 1$, and choose y_0 so that $\psi_\delta(y_0) = \sup_M \psi_\delta$. Since $\sqrt{-1}\partial\bar{\partial}\psi_\delta(y_0) \leq 0$, we get

$$\begin{aligned} &\det(\partial_i \partial_{\bar{j}} \log \det \mathbf{G})(y_0) \\ &\geq \det(\partial_i \partial_{\bar{j}} \log \det \mathbf{G} + \psi_{\delta:i\bar{j}})(y_0) \\ &= \prod_{i=1}^{N} (\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i} \| s_i \|^{-2a_i}_{\mathbf{h}_i} (y_0) e^{\psi_\delta(y_0)} \det \mathbf{G}(y_0). \end{aligned}$$

Hence $\sup_M \psi_\delta \leq C_2$ for a certain positive number C_2. Now, from (2.4), we have

Lemma 2.5. $\phi_\delta \leq C_2 - \sum_{i=1}^N a_i \log \| s_i \|^2_{\mathbf{h}_i}$ *for any* $0 < \delta < 1$.

Note that

$$\log \| s_i \|^2_{\mathbf{h}_i} \in L^1(M, \mathbf{G}(a)).$$

By (2.3) and (2.5), we have

$$\int_M |\phi_\delta| dV_{\mathbf{G}(a)} < C_3$$

for a certain positive number C_3. On the other hand, since

$$\omega_{\mathbf{G}(a)} + \sqrt{-1}\partial\bar{\partial}\phi_\delta$$

is a Kähler form on M, we have $\Delta\phi_\delta + n > 0$, where Δ is the normalized Laplacian of $\mathbf{G}(a)$. Let $\mathbf{G}(x,y)$ be the Green's kernel function of Δ under the Neumann's condition, and set

$$\widehat{\phi}_\delta := \phi_\delta - \frac{1}{Vol_{\mathbf{G}(a)}(M)} \int_M \phi_\delta dV_{\mathbf{G}(a)}.$$

Choose a positive number C_4 so that $\mathbf{G}(x,y) + C_4 > 0$ on $M \times M$. Then

$$\begin{aligned}\widehat{\phi}_\delta(x) &= -\int_M \mathbf{G}(x,y)\Delta\widehat{\phi}_\delta(y)dV_{\mathbf{G}(a)} \\ &= -\int_M \{\mathbf{G}(x,y) + C_4\}\Delta\widehat{\phi}_\delta(y)dV_{\mathbf{G}(a)} \\ &\leq n\int_M \{\mathbf{G}(x,y) + C_4\}dV_{\mathbf{G}(a)}.\end{aligned}$$

Hence we obtain the following lemma.

Lemma 2.6. *There exists a positive number C_5 such that*

$$\sup_M \phi_\delta \leq C_5 \quad \textit{for any} \quad 0 < \delta < 1.$$

Now, following Yau's argument, we obtain the second order estimate for ϕ_δ.

Lemma 2.7. *There exists a positive number C_6 depending only on $\{a_i\}$ such that*

$$\| \phi_\delta \|_{C^2(M)} \leq C_6$$

and that

$$C_6^{-1} \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^2 \mathbf{g} \leq \mathbf{g}'_\delta \leq C_6 \mathbf{g},$$

where

$$\mathbf{g}'_{\delta:i\bar{j}} := \mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}}$$

and where

$$\mathbf{g}_{i\bar{j}} := \mathbf{G}(a)_{i\bar{j}}.$$

Now use [G-T, Theorem 17.14] and regularity theorem for elliptic operators, we finally obtain the following theorem.

Theorem 2.8. *There exists a unique*

$$\phi \in C^\infty(M \setminus \cup_i E_i) \cap C^{2-\alpha}(M)$$

such that, on $M \setminus \cup_i E_i$, $\omega_a + \sqrt{-1}\partial\bar{\partial}\phi$ *is a Kähler form and that satisfies*

$$\det(\mathbf{G}(a)_{i\bar{j}} + \phi_{:i\bar{j}}) = \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^{2a_i} e^{\phi} \det \mathbf{G}.$$

Moreover $\{\phi_{:i\bar{j}}\}$ *are bounded.*

Corollary 2.9. *There exists a d-closed real positive current of type* (1.1),

$$\widehat{\gamma} \in c_1(K_M - \sum_{i=1}^{N} a_i E_i)$$

such that $\widehat{\gamma}|(M \setminus \cup_i E_i)$ *is a* C^∞ *Einstein-Kähler metric and that*

$$\lim_{x \to \cup_i E_i} \det \widehat{\gamma}(x) = 0.$$

Next, we shall give examples which satisfy the assumption (2.1).

Example 2.10. Let X be a canonical variety i.e. X is a minimal variety whose canonical divisor is ample. And let $\mu : Y \to X$ be a Hironaka's desingularization. By the definition, the canonical divisor K_Y of Y is expressed as $K_Y = \mu^* K_X + \sum_i a_i E_i$, where the E_i varies all exceptional divisors for μ, and a_i is a non-negative rational number. It is well-known that there exist positive rational numbers c_i such that $\mu^* K_X - \sum_i (a_i + \delta c_i) E_i$ is ample for any sufficiently small δ. Then, by (2.9), we get an Einstein-Kähler metric ω_δ for any δ on X_{reg}.

Example 2.11. Let X be a nonsingular minimal variety of general type. Then, by (1.7), there exists an effective divisor E_0 such that $K_X - aE_0$ is ample for any sufficiently small positive rational number a. Therefore, by (2.9) again, we obtain a C^∞ Einstein-Kähler metric on $X \setminus E_0$.

It is not so hard to see that, in the example (2.10), there exists Hermitian fibre metric $\mathbf{h}_i$ of E_i and that there exists a Kähler metric $\mathbf{g}$ on Y such that

$$\sqrt{-1}\partial\bar{\partial} \log \det \mathbf{g} - \textstyle\sum_i (a_i + \delta c_i) \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$$

is positive definite everywhere for any $0 < \delta < 1$. Now repeating the previous arguments (although we need some modifications), we obtain the following theorem.

Theorem 2.12 ([B-K], [Sg]). *In Example* 2.10, *the Einstein-Kähler metrics* $\{\omega_\delta\}$ *converge to a Einstein-Kähler metric* ω_0 *which is* C^∞ *on* X_{reg}. *Moreover* ω_0 *defines a d-closed positive* (1.1)*-current by the integral*

$$\langle \omega_0, \varphi \rangle := \int_Y \omega_0 \wedge \varphi$$

for a C^∞ $(n-1.n-1)$*-form* φ, *and* ω_0 *is a representative of* $c_1(\mu^* K_X)$ *as a current.*

§3. Stability for tangent sheaves of minimal varieties

In the present section, we shall overview Enoki's results which states the tangent sheaves of minimal varieties are stable in some sence. We begin with the definition of stability. Let X be a non-singular projective variety with dimension n and $\mathcal{E}$ be a torsion-free coherent sheaf on X. Let H be an ample divisor on X. $\mathcal{E}$ is said to be H-semistable if, for any coherent subsheaf $\mathcal{S}$ of $\mathcal{E}$, the inequality of the first Chern class

$$\frac{1}{\text{rk}(\mathcal{S})} c_1(\mathcal{S}) \cdot H^{n-1} \leq \frac{1}{\text{rk}(\mathcal{E})} c_1(\mathcal{E}) \cdot H^{n-1} \tag{3.1}$$

holds. Here rk denotes the rank of a sheaf. One can generalize the definition of stability for a nef divisor. Let D be a nef divisor. $\mathcal{E}$ is said to be D-semistable if an inequality

$$\frac{1}{\text{rk}(\mathcal{S})} c_1(\mathcal{S}) \cdot D^{n-1} \leq \frac{1}{\text{rk}(\mathcal{E})} c_1(\mathcal{E}) \cdot D^{n-1} \tag{3.2}$$

holds arbitrary coherent subsheaf $\mathcal{S}$ of $\mathcal{E}$. Now, Enoki's result is as follows.

Theorem 3.3 ([E, Theorem 1.1]). *Let* X *be an* n *dimensional minimal variety and* $\mu : Y \to X$ *a desingularization of* X. *Then the tangent sheaf* $\mathcal{T}_Y$ *of* Y *is* $\mu^* K_X$ *-semistable.*

Remark 3.4. In his paper, Enoki proved the theorem in much more general situation. But for the later purpose, it is only enough in this form.

We give an outline of the proof of Theorem 3.3. Let Φ be a Kähler form on Y. By the definition, we have $K_Y = \mu^* K_X + \sum_i a_i E_i$ where a_i is a nonnegative rational number and E_i is an exceptional divisor for μ. We write $L = \sum_i a_i E_i$ and $L = aL_0$ with L_0 is an effective divisor and

$a > 0$. Fix a hermitian metric $\| \cdot \|^2$ and a holomorphic section s of $[L_0]$ which defines the divisor L_0. For each $\epsilon > 0$, define

$$\gamma(L, \epsilon) := \sqrt{-1}\partial\bar{\partial}\log \| s \|^{2a} - \sqrt{-1}\partial\bar{\partial}\log(\| s \|^2 + \epsilon)^a.$$

For real (1.1)-forms φ and ψ we mean by $\varphi \leq \psi$ that $\varphi - \psi$ is negative semidefinite everywhere. We need two lemmas.

Lemma 3.5 ([E, Lemma 3.1]). *$\gamma(L, \epsilon) \leq \chi_\epsilon \Phi$, where $\chi_\epsilon > 0$ is uniformly bounded and $\chi_\epsilon \to 0$ in L^1-sence as $\epsilon \to 0$.*

Lemma 3.6 ([E, Lemma 3.2]). *For each ϵ and t, there is a Kähler form $\Psi(\epsilon, t)$ on Y with the following properties:*

(1) *$\Psi(\epsilon, t) - t\Phi$ represents $2\pi c_1(\mu^* K_M)$ and*

$$\sqrt{-1}\,\mathrm{Ric}(\Psi(\epsilon, t)) = -\Psi(\epsilon, t) - \gamma(L, \epsilon) + t\Phi$$

(2) *$\Psi(\epsilon, t)$ remains bounded on Y as $\epsilon \to 0$ whenever t is fixed.*

Let $\mathcal{S}$ be a coherent subsheaf of the tangent sheaf $\mathcal{T}_Y$ with $r := \mathrm{rank}(\mathcal{S}) > 0$. Then $\mathcal{S}$ defines a holomorphic subbundle $S \subset \mathcal{T}_Y$ outside the analytic subset $W(\mathcal{S})$ of Y. Using Lemma 3.5, Lemma 3.6, Gauss-Codazzi's equation and the arguments of [Ko-S, (6.14) in p.23], and of [Ko-S, p.172-p.182], we have

$$\begin{aligned}\frac{2\pi}{r}\int_Y c_1(\mathcal{S}) \wedge \Psi(\epsilon, t)^{n-1} \leq \frac{2\pi}{n}&\int_Y c_1(\mathcal{T}_Y) \wedge \Psi(\epsilon, t)^{n-1} \\ &+ t(\frac{1}{r} - \frac{1}{n})\int_Y \Phi \wedge \Psi(\epsilon, t)^{n-1} \\ &+ \frac{1}{n}\int_Y \gamma(L, \epsilon) \wedge \Psi(\epsilon, t)^{n-1} \\ &+ \frac{1}{r}\int_Y \chi_\epsilon \Phi \wedge \Psi(\epsilon, t)^{n-1}.\end{aligned}$$

By Lemma 3.5 and (2) of Lemma 3.6, the last term tends to 0 as ϵ goes to 0 ; the other terms are independent of ϵ (they depend only on cohomology classes of Φ and $\Psi(\epsilon, t)$). On the other hand, since

$$\int_Y \gamma(L, \epsilon) \wedge \Psi(\epsilon, t)^{n-1} = 2\pi(\sum_i a_i E_i) \cdot 2\pi(\mu^* K_X + t\Phi)^{n-1}$$

and since E_i's are exceptional, $\int_Y \gamma(L, \epsilon) \wedge \Psi(\epsilon, t)^{n-1}$ goes to 0 as $t \to 0$.

Now let $t \to 0$, and we complete the proof.

§4. An inequality between Chern numbers

In the present section, we shall give an outline of a proof of an inequality between Chern numbers of a minimal variety of general type. Our main theorem is as follows. For details, see [Sg].

Theorem 4.1. *Let M be an n dimensional compact minimal variety of general type, and $\mu : N \to M$ an arbitrary resolution. Then we have an inequality between Chern numbers:*

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\mu^* K_M)^{n-2} \leq 0.$$

Theorem 4.1 and [Mi-3, Theorem 6.6] imply the following result.

Corollary 4.2. *Let M be an n dimensional compact minimal variety which is smooth in codimension* 2. *Then, for an arbitrary desingularization $\mu : N \to M$, we have the same inequality between Chern numbers of N as in Theorem* 4.1.

Remark 4.3. When $\mu^* K_M$ is cohomologous to zero, one can obtain a stronger result. Namely let M be an n dimensional compact minimal variety and $\mu : N \to M$ a desingularization. Assume that $\mu^*(K_M)$ is cohomologous to zero. Then we get

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\mu^* H)^{n-2} \leq 0.$$

for any ample divisor H on M.

In order to prove Theorem 4.1, we consider a more general situation. Let M be an n dimensional compact $\mathbf{Q}$-Gorenstein projective variety and $\mu : N \to M$ a birational morphism from an n dimensional non-singular projective variety N to M. Let E be a holomorphic vector bundle of rank r on N, and H an ample divisor on M. Assume that E is $\mu^* H$-semistable. Then we obtain the following proposition.

Proposition 4.4. *We have an inequality between Chern numbers of E*

$$\{(r-1)c_1(E)^2 - 2rc_2(E)\} \cdot (\mu^* H)^{n-2} \leq 0.$$

Now Theorem 4.1 follows from Theorem 3.1 and Proposition 4.4.

Proof of Theorem 4.1. By Fact 1.8, for a sufficiently large integer m, there exist an n dimensional compact projective variety Z with only canonical singularities such that K_Z is ample, and a proper birational morphism $\Phi : M \to Z$ defined by the linear system $|mK_M|$, which satisfies $K_M = \Phi^*(K_Z)$. We consider the commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\text{identity}} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\Psi} \\ M & \xrightarrow{\Phi} & Z \end{array}$$

By definition of minimal projective variety, we have

$$K_N = \mu^* K_M + \sum a_i E_i$$

for non-negative rational numbers a_i, where E_i is exceptional for μ. Therefore

$$K_N = \Psi^* K_M + \sum a_i E_i,$$

and by Theorem 3.1, $\mathcal{T}_N$ is $\Psi^*(K_Z)$-semistable. Hence, by Proposition 4.4, we obtain

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\Psi^* K_Z)^{n-2} \leq 0.$$

Note that $\Psi^*(K_Z) = \mu^*(K_M)$, and we finish the proof. Q.E.D.

In the rest of this section, we shall give an outline of the proof of Proposotion 4.4. See [Sg] for details.

Lemma 4.5. *The torsion free sheaf $\mu_* E$ is H-semistable.*

Combining Lemma 4.5 and [Mi-3, Theorem 2.5 and Corollary 3.6. (see also Remark 2.6)], we have the following proposition.

Proposition 4.6 (see also [Mi-3, Lemma 4.1]). *For sufficently large integers $m_1, \dots, m_{n-1}$, let choose a general smooth complete intersection curve C of $|m_i H|$'s so that $C \subset M \setminus M_1$, where M_1 is an analytic subset of M with codimension 2 such that μ is biholomorphic on $N \setminus \mu^{-1}(M_1)$. We set*

$$\tilde{C} := \mu^{-1} C (\cong C).$$

Then for any divisor with $\deg D > 0$, we have

$$\begin{aligned} &H^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E) - D)) \\ &= H^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) - D)) = 0 \end{aligned}$$

for every positive integer t. Here Sym *denotes the symmetric tensorial power.*

For sufficiently large integers $m_2, \dots, m_{n-1}$, choose a general complete intersection surface X of $|m_i H|$'s so that $X \cap M_1$ is a set of finite points and that $S := \mu^{-1} X$ is a compact smooth surface.

Lemma 4.7. *For any non-zero effective divisor D, we have*

$$\begin{aligned} &H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E) - D)) \\ &= H^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) - D)) = 0 \end{aligned}$$

for every positive integer t.

Lemma 4.8. *Let things be as in Proposition* 4.7 *and L a fixed Cartier divisor. Then the dimension*

$$h^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E) + L))$$

and

$$h^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + L))$$

are bounded by a polynomial of degree $r - 1$ in t.

Proposition 4.9. *The dimensions*

$$h^0(S, \operatorname{Sym}^{rt} E(-tc_1(E))$$

and

$$h^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S))$$

are bounded by a polynomial of degree $r - 1$.

Proof. Consider the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E) - \tilde{C})) \\ &\longrightarrow H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E))) \\ &\longrightarrow H^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E))). \end{aligned}$$

Then, by Lemma 4.7 and Lemma 4.8, we obtain the first statement. On the other hand, since

$$K_S = \mu^* K_X + \sum_i a_i E_i$$

where E_i is the exceptional divisor of $\mu\ |_S\colon S \to X$, and since $\tilde{C} \cap (\cup_i E_i) = \phi$ for a sufficiently general member C of $|\ m_i H\ |$'s,

$$\begin{aligned}&\operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C})\ |_{\tilde{C}} \\ &= \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + \mu^*(K_X - m_1 H))\ |_{\tilde{C}}\ .\end{aligned}$$

Let choose a positive integer m_1 so that $m_1 H - K_X$ is ample. Then, by Proposition 4.6 and Lemma 4.7, we get

$$H^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C})) = 0.$$

Now consider the exact sequence

$$\begin{aligned}0 &\longrightarrow H^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C})) \\ &\longrightarrow H^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S)) \\ &\longrightarrow H^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S)).\end{aligned}$$

Then, by Lemma 4.8, we obtain the second statement Q.E.D.

Now Proposition 4.4 easily follows from the argument of [Mi-3, Theorem 4.3] which uses Riemann-Roch theorem.

§5. A criterion for stability

Let M be an n dimensional compact projective algebraic variety with only canonical singularities whose canonical divisor K_M is ample, and $\mu : N \to M$ a Hironaka's resolution of singularities. Then, by Enoki's results, the tangent sheaf $\mathcal{T}_N$ of N is $\mu^* K_M$-semistable and it is easy to see that $\mathcal{T}_N$ admits the unique filtration of coherent sheaves

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \mathcal{T}_N$$

such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ is a torsion free sheaf of positive rank, $\mu^* K_M$-stable and that

$$\frac{1}{\operatorname{rk}(\mathcal{S}_i)}\{c_1(\mathcal{S}_i) \cdot (\mu^* K_M)^{n-1}\} = \frac{1}{n}\{c_1(\mathcal{T}_N) \cdot (\mu^* K_M)^{n-1}\}$$

for any i. We call such a filtration as $\mu^* K_M$-*stable filtration* of $\mathcal{T}_N$. Using the same argument as in Section 3, we can prove the following theorem. Note that we sometimes consider the regular part M_{reg} of M as an open subset of N via μ. For proofs of the following results, see [Sg].

Theorem 5.1. *M_{reg} admits a C^∞ Einstein-Kähler metric $\tilde{\gamma}$ and there exists a holomorphic vector bundle S_i on M_{reg} such that*

$$(\mathcal{S}_i/\mathcal{S}_{i-1})|_{M_{\mathrm{reg}}} = \mathcal{O}(S_i)$$

and that $\mathcal{T}_N$ orthogonally decomposes

$$\mathcal{T}_N = S_1 \oplus \cdots \oplus S_t$$

on M_{reg} with respect to $\tilde{\gamma}$. Moreover, for any point x of M_{reg}, there exists an open neighborhood U of x such that $(U, \tilde{\gamma})$ is isometric to the direct product of Einstein-Kähler manifolds

$$(U, \tilde{\gamma}) = (U_1, \tilde{\gamma}_1) \times \cdots \times (U_t, \tilde{\gamma}_t).$$

Here U_i is a complex submanifold of U characterized by $\mathcal{T}_{U_i} = S_i|_{U_i}$ and $\tilde{\gamma}_i = \tilde{\gamma}|U_i$.

Next we shall give an example of a projective variety M with desingularization $\mu : N \to M$ whose tangent sheaf $\mathcal{T}_N$ is $\mu^* K_M$-stable.

Let (B, ω) be an n dimensional complete simply connected Kähler manifold with $\mathrm{Ric}\,\omega = -\omega$ and we assume that

$$\mathrm{Isom}(B, \omega) := \{\text{all biholomorphic maps which preserves} \quad \omega\}$$

acts B transitively. Let Γ be a discrete subgroup of $\mathrm{Isom}(B, \omega)$ and we assume that Γ satisfies the following condition.

Condition 5.2. *The quotient variety $M := B/\Gamma$ is compact. And let $\Gamma_x := \{\gamma \in \Gamma | \gamma(x) = x\}$. Then cardinality of Γ_x is finite.*

It is easy to see that (5.2) implies the following lemma.

Lemma 5.3. *Let $F_\Gamma := \{x \in \Gamma | \gamma(x) = x \quad \text{for certain} \quad \gamma \neq \mathrm{id} \in \Gamma\}$. Then, for any element x of F_Γ, there exists an open neighborhood U of x and a nowhere vanishing holomorphic n form η_U on U which is Γ_x-invariant.*

From (5.3), it follows $\mathrm{codim} F_\Gamma \geq 2$, and M has only canonical singularities. Moreover M is a V-manifold and K_M is ample. Let y be a point of the singularities of M and $x \in \pi^{-1}(y)$ where $\pi : B \to M$ be the natural projection. We choose an open neighborhood W of x so small that there exists a Γ_x-invariant holomorphic n form η_W on W and we set $V := \pi(W)$, $\eta_V := \pi_* \eta_W$. Let $\mu : N \to M$ be a Hironaka's resolution of singularities. Since $\mu^* \eta_V$ is square integrable, $\mu^* \eta_V$ can be extended to a holomorphic n form on $\mu^{-1}(V)$ and we denote it $\mu^*\eta_V$ again (for instance, see [L]). Since

$$K_N = \mu^* K_M + \sum_i a_i E_i$$

for certain non-negative integers a_i, $\mu^*\eta_V$ vanishies along the exceptional divisor E_i with order a_i. We denote $\sum_i a_i E_i = aE$ for certain non-negative rational number a and for a non-zero effective divisor E. Let s be the section of $[E]$ whose divisor is E. Then we have

$$\omega^n = f \parallel s \parallel_{\mathbf{h}}^{2a} \Phi^n$$

on M_{reg}. Here we denote $\pi_*\omega$ by ω again and f is a C^∞ function on M_{reg} satisfying

$$C^{-1} \leq f \leq C$$

for a positive number C. Let ω_0 be the Einstein-Kähler metric which we have constructed in Section 2. Then we have the following proposition.

Proposition 5.4. *$\omega_0 = \omega$ on M_{reg}.*

Combining (5.1) and (5.4), we obtain the following result.

Corollary 5.5. *If $\mathcal{T}_N$ is not $\mu^* K_M$-stable, (B, ω) is isometric to the direct product*

$$(B, \omega) = (B_1, \omega_1) \times \cdots \times (B_t, \omega_t)$$

where $t \leq 2$. Therefore, in particular, if (B, ω) is irreducible, $\mathcal{T}_N$ is $\mu^* K_M$-stable.

Remark 5.6. About the uniqueness, we can obtain more general statement as follows. Let (M, ω) be an n dimensional compact Kähler V-manifold with $\mathrm{Ric}\,\omega = -\omega$. Then $\omega_0 = \omega$.

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