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Compactification of the Moduli Space of Einstein-Kähler Orbifolds

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Dedicated to Professor Ichiro Satake on his sixtieth birthday

Introduction

In the study of degenerations of various geometric objects such as Einstein metrics or complex structures, the construction of natural compactifications of the corresponding moduli spaces is of crucial importance. For instance, the Satake compactification [17] of the moduli spaces of principally polarized abelian varieties plays a beautiful role in the study of modular forms, while the recent study of Donaldson [8] on the ends of moduli spaces of anti-self-dual connections provides us with new aspects of differentiable 4-manifolds.

The purpose of this paper is to give a natural compactification of the moduli space of polarized Einstein-Kähler orbifolds with a given Hilbert polynomial. We shall then show that, for compact Riemann surfaces, our compactification coincides with those of Mumford [15] and Bers [5] (see 2.6 and 2.7). A couple of other examples of our compactification will be given in Section 4 where we discuss the moduli spaces of polarized Abelian varieties and also of a special type of del Pezzo surfaces (see [12]). As to Abelian varieties, for instance, our approach has some relation to Igusa's compactification [11] in view of both heavy dependence on theta functions. We would also say that Anderson [1], Bando, Kasue and Nakajima [3] recently succeeded in applying Gromov's theory to compactifying the moduli spaces of Einstein manifolds of $\dim_{\mathbb{R}} \leq 4$ with a fixed volume and bounded diameters, though our compactification needs no boundedness of diameters because of the algebro-geometric nature of our construction.

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§1. Construction of the compactification

By a *polarized orbifold*, we shall mean a compact complex connected V-manifold X (cf. Satake [18]) endowed with an ample holomorphic line bundle L. Further, K_X will denote the canonical bundle of X. In this paper, we fix once for all a Hilbert polynomial $P = P(m)$, and consider the associated moduli space $\mathcal{M}_P$ of polarized orbifolds (X, L) by setting

$$\mathcal{M}_P := \{\, (X, L)\, ;\, \chi(X, \mathcal{O}(L^{\otimes m})) = P \,\} / \equiv,$$

where we write $(X_1, L_1) \equiv (X_2, L_2)$ if there exists an isomorphism $\varphi\colon X_1 \cong X_2$ of complex V-manifolds such that $L_1 = \varphi^* L_2$. Moreover, define subsets $\mathcal{M}_P^+$, $\mathcal{M}_P^-$, $\mathcal{M}_P^0$ of $\mathcal{M}_P$ by

$$\mathcal{M}_P^+ := \{\, [X, L] \in \mathcal{M}_P\, ;\, c_1(X)_{\mathbb{R}} > 0,\ L = K_X^{-1} \},$$
$$\mathcal{M}_P^- := \{\, [X, L] \in \mathcal{M}_P\, ;\, c_1(X)_{\mathbb{R}} < 0,\ L = K_X \ \},$$
$$\mathcal{M}_P^0 := \{\, [X, L] \in \mathcal{M}_P\, ;\, c_1(X)_{\mathbb{R}} = 0 \,\},$$

where $[X, L] \in \mathcal{M}_P$ denotes the equivalence class of (X, L) in the equivalence relation $\equiv$ above. Hence $\mathcal{M}_P^+$ (resp. $\mathcal{M}_P^-$) is the set of the isomorphism classes of compact complex connected V-manifolds X such that $\chi(X, \mathcal{O}(K_X^{\otimes m})) = P(-m)$ (resp. $P(m)$) for all integers m. Now, a triple (X, L, ω) is called an *Einstein-Kähler* polarized orbifold, if ω is an Einstein-Kähler V-metric on X and in addition if (X, L) is a polarized orbifold such that the Kähler class $2\pi c_1(L)_{\mathbb{R}}$ is represented by ω. Recall that a Kähler V-metric ω on X is called an Einstein-Kähler V-metric if

$$\mathrm{Ric}(\omega) := \sqrt{-1}\bar{\partial}\partial \log(\omega^n)$$

is a constant multiple of ω, where $n := \dim_{\mathbb{C}} X$. Now for an Einstein-Kähler polarized orbifold (X, L, ω), replacing ω by its suitable constant multiple if necessary, we have either $\mathrm{Ric}(\omega) = 0$ or $\mathrm{Ric}(\omega) = -\omega$ or $\mathrm{Ric}(\omega) = \omega$. Note that $\mathrm{Ric}(\omega)$ represents $2\pi c_1(X)_{\mathbb{R}}$. Therefore, for (X, L, ω) as above, we always assume one of the following:

(1) $c_1(X)_{\mathbb{R}} = 0$ and ω is a Ricci-flat Kähler V-metric;
(2) $c_1(X)_{\mathbb{R}} < 0$, $L = K_X$ and $\mathrm{Ric}(\omega) = -\omega$;
(3) $c_1(X)_{\mathbb{R}} > 0$, $L = K_X^{-1}$ and $\mathrm{Ric}(\omega) = \omega$.

Choose moreover a Hermitian metric $h(\omega)$ for L, unique up to constant multiple, such that the Chern form $c_1(L, h(\omega))$ coincides with $\omega/2\pi$. For instance, for (2) (resp. (3)) above, we choose $(\omega^n)^*$ (resp. ω^n) as $h(\omega)$. Let us now define the moduli space $\mathcal{E}_P$ of Einstein-Kähler polarized orbifolds (X, L, ω) with Hilbert polynomial P by

$$\mathcal{E}_P := \{\, (X, L, \omega)\, ;\, [X, L] \in \mathcal{M}_P \} / \sim,$$

where each (X, L, ω) is required to satisfy one of the conditions (1), (2), (3) above, and we write $(X_1, L_1, \omega_1) \sim (X_2, L_2, \omega_2)$ if there exists an isomorphism $\varphi \colon X_1 \cong X_2$ of complex V-manifolds such that $L_1 = \varphi^* L_2$ and $\omega_1 = \varphi^* \omega_2$. Set further

$$\begin{aligned}
\mathcal{E}_P^+ &:= \{\, [X, L, \omega] \in \mathcal{E}_P \,;\, c_1(X)_{\mathbb{R}} > 0 \,\},\\
\mathcal{E}_P^- &:= \{\, [X, L, \omega] \in \mathcal{E}_P \,;\, c_1(X)_{\mathbb{R}} < 0 \,\},\\
\mathcal{E}_P^0 &:= \{\, [X, L, \omega] \in \mathcal{E}_P \,;\, c_1(X)_{\mathbb{R}} = 0 \,\},
\end{aligned}$$

where $[X, L, \omega] \in \mathcal{E}_P$ denotes the equivalence class of (X, L, ω) in terms of the equivalence relation $\sim$ above. By sending $[X, L, \omega]$ to $[X, L]$, we have a natural projection

$$\mathrm{pr} \colon \mathcal{E}_P = \mathcal{E}_P^+ \cup \mathcal{E}_P^- \cup \mathcal{E}_P^0 \to \mathcal{M}_P, \qquad \mathrm{pr}([X, L, \omega]) := [X, L],$$

together with its restrictions

$$\mathrm{pr}^+ \colon \mathcal{E}_P^+ \to \mathcal{M}_P^+, \quad \mathrm{pr}^- \colon \mathcal{E}_P^- \to \mathcal{M}_P^-, \quad \mathrm{pr}^0 \colon \mathcal{E}_P^0 \to \mathcal{M}_P^0$$

to $\mathcal{E}_P^+$, $\mathcal{E}_P^-$, $\mathcal{E}_P^0$, respectively. In view of the results of Aubin-Calabi-Yau (see for instance [7]) and Bando-Mabuchi [4], one obtains

Lemma 1.1. *The map* $\mathrm{pr} \colon \mathcal{E}_P \to \mathcal{M}_P$ *is injective and so is* pr^+. *Moreover, both* $\mathrm{pr}^- \colon \mathcal{E}_P^- \to \mathcal{M}_P^-$ *and* $\mathrm{pr}^0 \colon \mathcal{E}_P^0 \to \mathcal{M}_P^0$ *are bijective.*

Proof. Note that, by standard methods, arguments in [4] and [7] are valid also for orbifolds. Hence, in view of such orbifold versions, the maps pr^-, pr^0, pr^+, pr are injective by the uniqueness of Einstein-Kähler metrics (cf., e.g., [4], [7]) and the surjectiveness of pr^- and pr^0 follows from the existence results of Aubin and Yau (cf., e.g., [7]). Q.E.D.

We now assume $\mathcal{E}_P \neq \phi$. For each positive integer μ, let $\mathcal{E}_{P;\mu}$ be the set of all $[X, L, \omega] \in \mathcal{E}_P$ satisfying

(1) $h^i(X, \mathcal{O}(L^{\otimes m})) = 0, \qquad i > 0;$
(2) $L^{\otimes m}$ is very ample on X,

for all m with $m \geq \mu$. Then $\mathcal{E}_{P;\mu}$, $\mu = 1,2, \ldots$, form an increasing sequense of subsets of $\mathcal{E}_P$ such that

$$\cup_{\mu=1}^{\infty} \mathcal{E}_{P;\mu} \;=\; \mathcal{E}_P.$$

Let us now take a subset $\mathcal{F}$ of $\mathcal{E}_P$ such that $\mathcal{F} \subset \mathcal{E}_{P;\mu}$ for some μ. Let $\mathcal{F}^\mu$ be the image of the mapping

$$i_\mu \colon \mathcal{F} \to \mathcal{E}_{P_\mu}, \qquad [X, L, \omega] \mapsto i_\mu([X, L, \omega]) := [X, L^{\otimes \mu}, \mu\omega],$$

where we put $P_\mu(m) := P(\mu m)$. Then the mapping $i_\mu : \mathcal{F} \to \mathcal{F}^\mu$ is bijective, for instance, if either $\mathcal{F} \subset \mathcal{E}_P^+$ or $\mathcal{F} \subset \mathcal{E}_P^-$. However, it is in general finite-to-one. Now, our main purpose in this section is to construct a natural compactification of $\mathcal{F}^\mu$. Let us first give two typical examples of $\mathcal{F}$.

Example 1.2. (1) Let $\mathcal{E}_P^{n.s.}$ be the set of all $[X, L, \omega] \in \mathcal{E}_P$ such that X is nonsingular. Then by Matsusaka's big theorem (cf. [14]), there exists a positive integer $\mu_0 = \mu_0(P)$, depending only on P, such that $\mathcal{E}_P^{n.s.} \subset \mathcal{E}_{P;\mu}$ for all $\mu \geq \mu_0$.

(2) Let c_1^2, $c_2 \in \mathbb{Z}$ be such that $c_1^2 > 0$. Put $P(m) := \frac{1}{2}(m^2 - m)c_1^2 + \frac{1}{12}(c_1^2 + c_2)$. We then consider the subset $\mathcal{S}_{c_1^2, c_2}$ of all $[X, K_X, \omega]$ in $\mathcal{E}_P$ such that X is a canonical model of a minimal algebraic surface of general type. Note that $\mathcal{S}_{c_1^2, c_2} \subset \mathcal{E}_{P;\mu}$ whenever $\mu \geq 5$ (cf. Bombieri [6]). Moreover, by Lemma 1.1, $\mathcal{S}_{c_1^2, c_2}$ coincides with the set of isomorphism classes of all minimal algebraic surfaces S of general type with $c_1^2(S) = c_1^2$ and $c_2(S) = c_2$.

We shall now explain how to compactify $\mathcal{F}^\mu$. Let $[X, L, \omega] \in \mathcal{F}$. Then by setting $N := h^0(X, \mathcal{O}(L^{\otimes\mu})) - 1 = P(\mu) - 1$, we have a projective embedding

$$\Phi_\Sigma : X \hookrightarrow \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto \Phi_\Sigma(x) := (\sigma_0(x) : \sigma_1(x) : \cdots : \sigma_N(x)),$$

where $\Sigma := \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ is a unitary $\mathbb{C}$-basis for $H^0(X, \mathcal{O}(L^{\otimes\mu}))$, so that the natural L^2-pairing on $H^0(X, \mathcal{O}(L^{\otimes\mu}))$ induced by ω satisfies

$$< \sigma_i, \sigma_j >_{L^2, \omega} = \delta_{ij}, \qquad i, j \in \{0, 1, \ldots, N\},$$

where $< \sigma_i, \sigma_j >_{L^2, \omega}$ denotes the quantity $\int_X (\sigma_i, \sigma_j)_{h(\omega)}\, \omega^n / n!$. Since $(\Phi_\Sigma)^* \mathcal{O}_{\mathbb{P}^N}(1)$ is $\mathcal{O}(L^{\otimes\mu})$, the image $X_\Sigma := \Phi_\Sigma(X)$ satisfies

$$\chi(X_\Sigma, \mathcal{O}_{X_\Sigma}(m)) = P_\mu(m), \qquad m \in \mathbb{Z}.$$

Therefore, X_Σ is regarded as a point of the Hilbert scheme $H^\mu := (\mathrm{Hilb}_{\mathbb{P}^N})^{P_\mu}$ which parametrizes the subschemes of $\mathbb{P}^N(\mathbb{C})$ with Hilbert polynomial P_μ. For another unitary $\mathbb{C}$-basis $\Sigma' := \{\sigma'_0, \sigma'_1, \ldots, \sigma'_N\}$ for $H^0(X, \mathcal{O}(L^{\otimes\mu}))$, there exists a unitary matrix $u = (u_{ij}) \in \mathrm{U}(N+1)$ such that

$$\sigma'_j = \Sigma_{i=0}^N \sigma_i u_{ij}, \qquad j = 0, 1, \ldots, N.$$

Let us now consider $\mathbb{C}^{N+1}$ as a set of row vectors on which $\mathrm{U}(N+1)$ acts naturally from the right. This then induces a right $\mathrm{U}(N+1)$-action on $\mathbb{P}^N(\mathbb{C})$ and also on the Hilbert scheme H^μ. Since

$$\Phi_{\Sigma'}(x) = \Phi_\Sigma(x) \cdot u \qquad \text{for all } x \in X,$$

we can relate $X_{\Sigma'} := \Phi_{\Sigma'}(X)$ to X_Σ by $X_{\Sigma'} = X_\Sigma \cdot u$ in H^μ, so that by sending $[X, L^{\otimes\mu}, \mu\omega] \in \mathcal{F}$ to

$$\Phi([X, L^{\otimes\mu}, \mu\omega]) := X_\Sigma \cdot \mathrm{U}(N+1) \in H^\mu / \mathrm{U}(N+1),$$

we have a well-defined map $\Phi\colon \mathcal{F}^\mu \to H^\mu / \mathrm{U}(N+1)$. Note that, if $[X, L^{\otimes\mu}, \mu\omega] = [\hat{X}, \hat{L}^{\otimes\mu}, \mu\hat{\omega}]$ in $\mathcal{F}^\mu$, i.e., if there exists $\varphi\colon \hat{X} \cong X$ such that $\varphi^* L^{\otimes\mu} = \hat{L}^{\otimes\mu}$ and $\varphi^*\omega = \hat{\omega}$, then the pullback $\hat{\Sigma} := \varphi^*\Sigma$ is again a unitary $\mathbb{C}$-basis for $H^0(\hat{X}, \mathcal{O}(\hat{L}^{\otimes\mu}))$, and therefore $\Phi_\Sigma(X) = \Phi_{\hat{\Sigma}}(\hat{X})$ by $\Phi_{\hat{\Sigma}} = \varphi^*\Phi_\Sigma$. We shall now show that Φ yields an inclusion $\mathcal{F}^\mu \hookrightarrow H^\mu / \mathrm{U}(N+1)$.

Lemma 1.3. *Φ is injective.*

Proof. Let $[X_1, L_1, \omega_1]$, $[X_2, L_2, \omega_2]$ be elements in $\mathcal{F}$ such that $\Phi([X_1, L_1^{\otimes\mu}, \mu\omega_1]) = \Phi([X_2, L_2^{\otimes\mu}, \mu\omega_2])$. Then by choosing unitary $\mathbb{C}$-bases Σ_1, Σ_2 for $H^0(X_1, \mathcal{O}(L_1^{\otimes\mu}))$, $H^0(X_2, \mathcal{O}(L_2^{\otimes\mu}))$, respectively, we have the identity $X_1' = X_2' \cdot u$ for some $u \in \mathrm{U}(N+1)$, where $X_1' := \Phi_{\Sigma_1}(X_1)$ and $X_2' := \Phi_{\Sigma_2}(X_2)$. Hence $(X_1, L_1^{\otimes\mu}) \equiv (X_1', \mathcal{O}_{X_1'}(1)) \equiv (X_2', \mathcal{O}_{X_2'}(1)) \equiv (X_2, L_2^{\otimes\mu})$. Then, as in the proof of Lemma 1.1, the orbifold version of the uniqueness of Einstein-Kähler metrics asserts that $[X_1, L_1^{\otimes\mu}, \mu\omega_1] = [X_2, L_2^{\otimes\mu}, \mu\omega_2]$, as required. Q.E.D.

Recall that H^μ is projective algebraic. Then H^μ, endowed with an ordinary topology induced by the manifold topology of the projective space, is a compact Hausdorff space, and so is $H^\mu / \mathrm{U}(N+1)$ with quotient topology. Now by Lemma 1.3, $\mathcal{F}^\mu$ is regarded as a subset of $H^\mu / \mathrm{U}(N+1)$. Let $\overline{\mathcal{F}}^\mu$ be the closure of $\mathcal{F}^\mu$ in $H^\mu / \mathrm{U}(N+1)$. Then $\overline{\mathcal{F}}^\mu$ is a compact Hausdorff space containing $\mathcal{F}^\mu$ as a dense subset. The compactification of $\mathcal{F}^\mu$, we are looking for, is just $\overline{\mathcal{F}}^\mu$ for some suitably chosen μ, and will be studied in detail in subsequent sections. Here, a crucial point is that we have no difficulty in taking the quotient $H^\mu / \mathrm{U}(N+1)$, since the group $\mathrm{U}(N+1)$ is compact. We note, in taking group quotient, that the Einstein-Kähler form ω on the orbifold X (where $[X, L, \omega] \in \mathcal{E}_P$) allows us the reduction of the group $\mathrm{GL}(N+1, \mathbb{C})$ to $\mathrm{U}(N+1)$. This reveals the fact that the existence of Einstein-Kähler forms can somehow be a differential-geometric substitute for the concept of semistability in Mumford's geometric invariant theory.

§2. The moduli space of compact Riemann surfaces

Throughout this section, we fix integers $g \geq 2$ and $\mu \geq 4$, and put $P(m) = (2m-1)(g-1)$, $m \in \mathbb{Z}$. Moreover, all curves are assumed to be projective algebraic and defined over $\mathbb{C}$. Let $\mathcal{E}_g$ be the set of all $[C, K_C, \omega_C] \in \mathcal{E}_P^-$ such that C is a nonsingular irreducible curve of genus g, i.e., a compact Riemann surface of genus g. We then consider the universal covering

$$\rho\colon \Delta = \{\, z \in \mathbb{C};\ |z| < 1\,\} \to \Delta/\Gamma = C,$$

where $\Gamma := \pi_1(C)$ acts on Δ as the covering transformation group. Since $\mathrm{Ric}(\omega_C) = -\omega_C$, we can characterize ω_C by

$$\rho^*(\omega_C) = \frac{2\sqrt{-1}}{(1-|z|^2)^2}\, dz \wedge d\bar{z}.$$

Here, the Kähler metric on C associated with ω_C is called the Poincaré metric of C and is denoted often by ω_C by abuse of terminology. Let $\mathcal{M}_g$ be the isomorphism classes of all nonsingular irreducible curves of genus g. Then by Lemma 1.1, we have the set-theoretical identification:

$$\mathcal{E}_g \simeq \mathcal{M}_g, \qquad [C, K_C, \omega_C] \leftrightarrow C.$$

Moreover, the compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ by Mumford [15] (cf. Bers [5] for a Teichmüller-theoretic approach) is defined to be the set of all stable curves of genus g (see below for the definition of stable curves).

Definition 2.1 (cf. [15]). An irreducible reduced curve C is called a *stable curve* of genus g, if it satisfies the following conditions:

(a) $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$.
(b) Singular points of C are, if any, ordinary double points.
(c) No smooth rational components of C meet the remainder of the curve in fewer than three points.

Let C be a stable curve of genus g, and write it as a union $\cup_{i=1}^r C_i$ of irreducible components. Taking its desingularization

$$\nu\colon \tilde{C} \to C, \tag{2.1.1}$$

we can express $\tilde{C}$ as a disjoint union $\cup_{i=1}^r \tilde{C}_i$ of nonsingular irreducible curves $\tilde{C}_i$ of genus g_i such that $\nu(\tilde{C}_i) = C_i$. Let m (resp. n_i) be the number of the points in C_{sing} (resp. $\nu^{-1}(C_{\mathrm{sing}}) \cap \tilde{C}_i$), where C_{sing} (resp. C_{reg})

denotes the set of singular (resp. nonsingular) points in C. We obviously have $2m = n_1 + n_2 + \cdots + n_r$. Then the condition (a) above says that

$$(2.1.2) \qquad g - 1 = m + \Sigma_{i=1}^{r}(g_i - 1).$$

Note that, by the condition (b), each point p of C_{sing} has an open neighbourhood U_p in C written in the form

$$(2.1.3) \qquad U_p = \{ (z, w) \in \mathbb{C}^2 \,;\, zw = 0,\ |z| < 1/2,\ |w| < 1/2 \},$$

for some holomorphic functions z, w on U_p, and if $q \in C_{\text{reg}}$, we have a coordinate neighbourhood $U_q = \{ v \in \mathbb{C};\, |v| < 1 \}$ with holomorphic local coordinate v centered at q. Moreover, in view of (c) above,

$$(2.1.4) \qquad n_i \geq 3 - 2g_i.$$

Let K_C be the line bundle on C associated to the dualizing sheaf of C. Then $K_C^{\otimes\mu}$ has a holomorphic local base

$$\frac{dz^{\otimes\mu}}{z^{\mu}} + (-1)^{\mu}\frac{dw^{\otimes\mu}}{w^{\mu}} \quad (\text{resp. } \ dv^{\otimes\mu})$$

over the open neighbourhood U_p (resp. U_q) as above. By (2.1.4) above, one can easily check that $K_C^{\otimes\mu}$ is very ample, and moreover

$$(2.1.5) \qquad h^1(C, \mathcal{O}(K_C^{\otimes\mu})) = 0.$$

For the rest of this section, we set $\mathcal{F} = \mathcal{E}_g$, and apply the argument in Section 1. Then by $\mathcal{F} \subset \mathcal{E}_P^-$, the mapping $i_\mu \colon \mathcal{F} \to \mathcal{F}^\mu$ is bijective. This together with Lemma 1.3 implies

$$(2.2.1) \qquad \mathcal{M}_g \simeq \mathcal{E}_g = \mathcal{F} \simeq \mathcal{F}^\mu \subset H^\mu/\,\mathrm{U}(N+1),$$

where $N = P(\mu) - 1$. Therefore, the closure $\overline{\mathcal{F}}^\mu$ of $\mathcal{F}^\mu$ in $H^\mu/\,\mathrm{U}(N+1)$ is regarded as a compactification of $\mathcal{M}_g$. The purpose of this section is to give an explicit construction of a natural homeomorphism between two compactifications $\overline{\mathcal{F}}^\mu$ and $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$. Let C be a stable curve of genus g as above. In view of (2.1.4), we first observe that each $C_i^0 := C_{\text{reg}} \cap C_i$ is covered by a disc $\Delta = \{ z \in \mathbb{C};\, |z| < 1 \}$. Hence $C^0 := C_{\text{reg}}$ carries a unique Einstein-Kähler form ω_C such that its restriction to each C_i^0 is characterized by

$$\rho_i^*(\omega_{C|C_i^0}) = \frac{2\sqrt{-1}}{(1 - |z|^2)^2}\, dz \wedge d\bar{z},$$

where $\rho_i\colon \Delta \to C_i^0$ is the universal covering. By abuse of terminology, this ω_C is often called the Poincaré metric of the stable curve C. We now write C_{sing} as $\{p_0, p_1, \dots, p_{m-1}\}$. Since $K_C^{\otimes\mu}$ is very ample, there exists a $\mathbb{C}$-basis $\{\theta_0, \theta_1, \dots, \theta_N\}$ for $H^0(C, \mathcal{O}(K_C^{\otimes\mu}))$ such that

(1) $\theta_\alpha(p_\alpha) \neq 0, \qquad 0 \leq \alpha < m;$
(2) $\theta_\beta(p_\alpha) = 0, \qquad 0 \leq \alpha < m,\ 0 \leq \beta \leq N,\ \alpha \neq \beta.$

For θ', $\theta'' \in H^0(C, \mathcal{O}(K_C^{\otimes\mu}))$ with $\{\theta' = 0\} \cap \{\theta'' = 0\} \cap C_{\text{sing}} = \phi$, we can define a positive definite Hermitian pairing $< \theta', \theta'' > \in \mathbb{C}$ by

$$< \theta', \theta'' > = \int_{C^0} (\sqrt{-1}\,\theta' \wedge \bar{\theta}'')/\omega_C^{\otimes\mu-1}.$$

Therefore, we may assume that θ_m, θ_{m+1}, $\dots$, θ_N form a unitary $\mathbb{C}$-basis for $\Lambda^\mu := \{\,\theta \in H^0(C, \mathcal{O}(K_C^{\otimes\mu}))\,;\, \theta_{|C_{\text{sing}}} = 0\,\}$, i.e.,

$$< \theta_\alpha, \theta_\beta > = \delta_{\alpha\beta}, \qquad m \leq \alpha \leq N,\ m \leq \beta \leq N. \tag{2.2.2}$$

Moreover, replacing each θ_α $(0 \leq \alpha < m)$ by an element of $\theta_\alpha + \Lambda^\mu$, we may further assume

$$< \theta_\alpha, \theta_\beta > = 0, \qquad 0 \leq \alpha < m,\ m \leq \beta \leq N, \tag{2.2.3}$$

where for each $0 \leq \alpha < m$, such a θ_α is unique up to a constant multiple. Note also that $< \theta_\alpha, \theta_\beta > < +\infty$ for $0 \leq \alpha < m$, $0 \leq \beta < m$ with $\alpha \neq \beta$. We now put $\Lambda_i^\mu = \{\,\theta \in \Lambda^\mu\,;\, \theta_{|C_j} = 0$ for all $j \neq i\,\}$ for each i. Then Λ^μ is written as a direct sum of vector subspaces:

$$\Lambda^\mu = \oplus_{i=1}^r \Lambda_i^\mu.$$

Put $m_0 := m$ and $m_i := \dim_{\mathbb{C}} \Lambda_i^\mu$. Moreover, put $N_i := (\Sigma_{j=0}^i m_j) - 1$, so that $N_0 = m - 1$ and $N_r = N$. Since $\Lambda_i^\mu \perp \Lambda_j^\mu$ for $i \neq j$ with respect to the Hermitian pairing $<\,,\,>$ above, we may further assume that

$$\{\,\theta_\alpha\,;\, N_{i-1} < \alpha \leq N_i\,\}$$

is a unitary $\mathbb{C}$-basis for Λ_i^μ $(1 \leq i \leq r)$. Now, for $0 \leq \alpha < m$, let b_α be the point of $\mathbb{P}^N(\mathbb{C}) = \{\,(\zeta_0 : \zeta_1 : \cdots : \zeta_N)\,\}$ defined by

$$\zeta_\beta(b_\alpha) = \delta_{\alpha\beta}, \qquad 0 \leq \beta \leq N. \tag{2.2.4}$$

We then put $B := \{\,b_0, b_1, \dots, b_{m-1}\}$. For $1 \leq i \leq r$, let Π_i be the linear subspace of $\mathbb{P}^N(\mathbb{C})$ defined by

$$\Pi_i := \{\,\zeta_\alpha = 0 \text{ if either } \alpha \leq N_{i-1} \text{ or } N_i < \alpha\,\} \cong \mathbb{P}^{m_i-1}(\mathbb{C}),$$

where the isomorphism between Π_i and $\mathbb{P}^{m_i-1}(\mathbb{C})$ is obtained by the projection

$$\Pi_i \ni (\zeta_0 : \zeta_1 : \cdots : \zeta_N) \mapsto (\zeta_{N_{i-1}+1} : \zeta_{N_{i-1}+2} : \cdots : \zeta_{N_i}).$$

Identifying Π_i with $\mathbb{P}^{m_i-1}(\mathbb{C})$, we can now define meromorphic maps $\psi_i \colon \tilde{C}_i \to \Pi_i$, $1 \le i \le r$, by

$$\psi_i(q) := (\theta_{N_{i-1}+1}(\nu(q)) : \theta_{N_{i-1}+2}(\nu(q)) : \cdots : \theta_{N_i}(\nu(q))), \quad q \in \tilde{C}_i.$$

Lemma 2.3. *Each ψ_i defines an embedding, so that we have an isomorphism $\psi_i \colon \tilde{C}_i \cong \psi_i(\tilde{C}_i)$.*

Proof. Let D_i be the reduced divisor $\nu^{-1}(C_{\text{sing}}) \cap \tilde{C}_i$ on $\tilde{C}_i$. Then by (2.1.4), deg $D_i = n_i \ge 3 - 2g_i$. Note also that $\Lambda_i^\mu \cong H^0(\tilde{C}_i, \mathcal{K}_i^\mu(D_i'))$, where $\mathcal{K}_i^\mu := \mathcal{O}(K_{\tilde{C}_i}^{\otimes\mu})$ and $D_i' := (\mu - 1)D_i$. Hence, ψ_i is associated with the complete linear system $|\mathcal{K}_i^\mu(D_i')|$. Since $\mu \ge 4$, we have

$$\deg\, \mathcal{K}_i^\mu(D_i') = \mu(2g_i - 2) + \deg\, D_i' \ge 2g_i + \mu - 3 \ge 2g_i + 1.$$

Thus, $\mathcal{K}_i^\mu(D_i')$ is very ample, and therefore ψ_i defines a projective embedding, as required. Q.E.D.

Now, for simplicity, we put $\tilde{C}_i' := \psi_i(\tilde{C}_i')$. Moreover, let $\psi \colon \tilde{C} \to \mathbb{P}^N(\mathbb{C})$ be the mapping defined by

$$\psi_{|\tilde{C}_i} = \psi_i, \qquad 1 \le i \le r.$$

For each α with $0 \le \alpha < m$, the set $\psi(\nu^{-1}(p_\alpha))$ consists of just two points, so that we put

$$\psi(\nu^{-1}(p_\alpha)) = \{ p_\alpha', p_\alpha'' \}. \tag{2.4.1}$$

Note that the sets B, $\tilde{C}_1'$, $\tilde{C}_2'$, $\ldots$, $\tilde{C}_r'$ are mutually disjoint. We then consider the complex projective line ℓ_α' (resp. ℓ_α'') which passes through p_α' (resp. p_α'') and b_α. Let C^* be the reduced subvariety

$$\{ \cup_{\alpha=0}^{m-1} (\ell_\alpha' \cup \ell_\alpha'') \} \cup (\cup_{i=1}^{r} \tilde{C}_i'). \tag{2.4.2}$$

in $\mathbb{P}^N(\mathbb{C})$. Let H^μ be as in Section 1, where $P_\mu(m) := (2m\mu - 1)(g-1)$.

Lemma 2.5. *$C^* \in H^\mu$.*

Proof. In view of the proof of Lemma 2.3, we have

$$\deg\, \tilde{C}_i' = \deg\, \mathcal{K}_i^\mu(D_i') = (\mu - 1)\deg\, D_i + \mu(2g_i - 2).$$

Together with $\Sigma_{i=1}^{r} \deg D_i = 2m$, it now follows that $\Sigma_{i=1}^{r} \deg \tilde{C}_i' = 2\{(\mu-1)m + \mu\Sigma_{i=1}^{r}(g_i - 1)\}$. Hence, by (2.1.2), we obtain

$$\deg C^* = 2m + 2\{(\mu-1)m + \mu\Sigma_{i=1}^{r}(g_i-1)\} = 2\mu(g-1).$$

On the other hand, $\dim_{\mathbb{C}} H^0(C^*, \mathcal{O}_{C^*})$ is just 1, since C^* is connected. Moreover, $\dim_{\mathbb{C}} H^1(C^*, \mathcal{O}_{C^*}) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$. Thus, by the Riemann-Roch theorem,

$$\chi(C^*, \mathcal{O}_{C^*}(m)) = 1 - g + m \deg C^* = 1 - g + 2\mu m(g-1) = P_\mu(m),$$

as required. Q.E.D.

For simplicity, we put $h(C) := C^* \cdot \mathrm{U}(N+1) \in H^\mu/\,\mathrm{U}(N+1)$. Then, from our construction, one can easily check that $h(C)$ is uniquely determined by C. Note that, if C is nonsingular, then $h(C)$ is the corresponding element of $\mathcal{F}^\mu$ in (2.2.1), i.e., the restriction of h to $\mathcal{M}_g$ defines nothing but the bijection $\mathcal{M}_g \simeq \mathcal{F}^\mu$ in (2.2.1). We are now able to define a homeomorphism between $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{F}}^\mu$ as follows:

Theorem 2.6. *$h\colon \overline{\mathcal{M}}_g \to H^\mu/\,\mathrm{U}(N+1)$ is continuous.*

Corollary 2.7. *The image $h(\overline{\mathcal{M}}_g)$ coincides with $\overline{\mathcal{F}}^\mu$, and the mapping $h\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{F}}^\mu$ is a homeomorphism.*

The proof of Theorem 2.6 will be postponed until the next section, and we shall here show how Corollary 2.7 is obtained from Theorem 2.6.

Proof of 2.7 (*with* 2.6 *taken for granted*). Since $h(\mathcal{M}_g) = \mathcal{F}^\mu$, the continuity of h implies that $h(\overline{\mathcal{M}}_g) = \overline{\mathcal{F}}^\mu$. Each $C \in \overline{\mathcal{M}}_g$ is obtained from C^* by collapsing the projective lines $\cup_{\alpha=0}^{m-1}(\ell_\alpha' \cup \ell_\alpha'')$ to a point, and therefore the mapping $h\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{F}}^\mu$ is injective. Thus $h\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{F}}^\mu$ is a continuous bijection. Since both $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{F}}^\mu$ are compact Hausdorff spaces, $h\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{F}}^\mu$ is a homeomorphism, as required. Q.E.D.

Remark 2.8. Since the integer $\mu \geq 4$ in 2.6 and 2.7 can be chosen arbitrarily, Corollary 2.7 above in particular says that $\overline{\mathcal{F}}^\mu$ is topologically independent of the choice of the integer μ, as long as $\mu \geq 4$. On the other hand, even if $\mu = 1$, the compactification $\overline{\mathcal{F}}^\mu$ of $\mathcal{F}^\mu$ says something, since the natural Hermitian metric on $H^0(C, \mathcal{O}(K_C^{\otimes\mu})) \simeq H^{1,0}(C)$, $C \in \mathcal{M}_g$, is defined purely homologically, appearing often in the theory of arithmetic surfaces by Arakelov and Quillen. In this special case $\mu = 1$, however, the mapping $\Phi\colon \mathcal{F}^\mu \to H^\mu/\,\mathrm{U}(N+1)$ is no longer injective, losing for instance most information in parametrization of hyperelliptic curves.

§3. Proof of Theorem 2.6

In this section, using the same notation as in the previous section, we give a complete proof of Theorem 2.6, where our proof is divided into five steps. Before getting into the proof, we here introduce some notation. Let $C \in \overline{\mathcal{M}}_g$, i.e., let C be a stable curve of genus g. Numbering the singular points of C as in Section 2, we put

$$C_{\text{sing}} = \{p_0, p_1, \ldots, p_{m-1}\}.$$

Then for an open neibourhood V of the point C in $\overline{\mathcal{M}}_g$, we have a system of local uniformizing holomorphic coordinates $s = (s_0, s_1, \ldots, s_{m-1}, s_m, \ldots, s_{3g-4})$ on a uniformizing open set

$$\tilde{V} = \{\, s \in \mathbb{C}^{3g-3}\,;\ \|s\|^2 = \Sigma_{\beta=0}^{3g-4} |s_\beta|^2 < 1/4 \,\}$$

satisfying the following properties (cf. Earle and Marden [9]; see also Masur [13], Wolpert [23]):

(a) $V = \tilde{V}/F$ for some finite subgroup F of $\mathrm{U}(3g-3)$ acting on $\tilde{V}$;

(b) the natural projection $\lambda\colon \tilde{V} \to V$ maps the origin 0 of $\tilde{V}$ to the point C of V;

(c) there exists a proper morphism $\eta\colon X \to \tilde{V}$ of complex manifolds such that the scheme-theoretic fibre X_s over each $s \in \tilde{V}$ is just the stable curve $\lambda(s) \in \overline{\mathcal{M}}_g$;

(d) if $p \in C_{\text{sing}}$, i.e., $p = p_\alpha$ for some α, then by regarding p as a point of X (via the natural identification of C with X_0), we have an open neighbourhood U_α of p in X and holomorphic functions z_α, w_α on U_α centered at p such that the mapping

$$U_\alpha \ni q \mapsto (z_\alpha(q), w_\alpha(q), (\eta^* s)(q)) \in \mathbb{C}^{3g-1}$$

defines an isomorphism of U_α onto

$$\{(z_\alpha, w_\alpha, s) \in \mathbb{C}^{3g-1}; z_\alpha w_\alpha = s_\alpha, |z_\alpha| < 1/2, |w_\alpha| < 1/2, s \in \tilde{V}\}.$$

Note here that the restriction of (z_α, w_α) in (d) to $s = 0$ gives the expression (2.1.3). If there is no fear of confusion, $\eta^* s$ is simply denoted by s, and $s_0, \ldots, s_{3g-4}$ are regarded as holomorphic coordinates on U_α.

3.1. Let $K_{X/\tilde{V}}$ be the relative canonical bundle $K_X \otimes \eta^* K_{\tilde{V}}^{-1}$. Then the restriction of $K_{X/\tilde{V}}$ to each fibre X_s is naturally identified with K_{X_s}. Moreover, by (2.1.5),

$$h^0(X_s, \mathcal{O}(K_{X_s}^{\otimes \mu})) = P(\mu) = N + 1$$

for all $s \in \tilde{V}$. Hence the direct image sheaf

$$\mathcal{E} := \eta_*(\mathcal{O}(K_{X/\tilde{V}}^{\otimes\mu}))$$

over $\tilde{V}$ is locally free, and by shrinking $\tilde{V}$ if necessary, we may choose a holomorphic local basis $\{\tau_0, \tau_1, \dots, \tau_N\}$ for $\mathcal{E}$ such that the restriction of $\tau_\beta = \tau_\beta(s)$ to $s = 0$ satisfies

$$\tau_\beta(0) = \theta_\beta, \qquad 0 \le \beta \le N, \tag{3.1.1}$$

where each θ_β is just as in Section 2, satisfying (2.2.2) and (2.2.3), and $\tau_\beta(0)$ is naturally regarded as an element of $H^0(C, \mathcal{O}(K_C^{\otimes\mu}))$. By the same argument as in Masur [13], each $\tau_\beta = \tau_\beta(s)$ is written in the form

$$\tau_\beta = a_{\alpha\beta}\frac{dz_\alpha^{\otimes\mu}}{z_\alpha^\mu} = (-1)^\mu\, a_{\alpha\beta}\frac{dw_\alpha^{\otimes\mu}}{w_\alpha^\mu}, \qquad 0 \le \beta \le N, \tag{3.1.2}$$

on U_α for some holomorphic function $a_{\alpha\beta} = a_{\alpha\beta}(z_\alpha, w_\alpha, s)$ in z_α, w_α, s. We multiply each τ_β, if necessary, by a holomorphic function on $\tilde{V}$ whose value at the origin is 1. Then we may assume

$$a_{\alpha\beta} = \{e_{\alpha\beta} + z_\alpha f_{\alpha\beta}(z_\alpha, w_\alpha, s) + w_\alpha g_{\alpha\beta}(z_\alpha, w_\alpha, s)\} \tag{3.1.3}$$

for some $e_{\alpha\beta} \in \mathbb{C}$ $(0 \le \alpha < m,\ 0 \le \beta \le N)$ and holomorphic functions $f_{\alpha\beta} = f_{\alpha\beta}(z_\alpha, w_\alpha, s)$, $g_{\alpha\beta} = g_{\alpha\beta}(z_\alpha, w_\alpha, s)$ in z_α, w_α, s. Note here that, by (3.1.1), we have

$$e_{\alpha\beta} = 0 \quad \text{if and only if} \quad \alpha \ne \beta. \tag{3.1.4}$$

For each $s \in \tilde{V}$, let ω_s be the Poincaré metric of the stable curve X_s. There exists a constant $1 < k \in \mathbb{R}$ independent also of α and s such that

$$\frac{k^{-1}|dz_\alpha|^2}{\eta_\alpha|z_\alpha|^2} = \frac{k^{-1}|dw_\alpha|^2}{\eta_\alpha|w_\alpha|^2} \le \omega_s \le \frac{k|dz_\alpha|^2}{\eta_\alpha|z_\alpha|^2} = \frac{k|dw_\alpha|^2}{\eta_\alpha|w_\alpha|^2} \tag{3.1.5}$$

on U_α for all α, where we put $\eta_\alpha := \mathrm{Min}\{(\log|z_\alpha|)^2, (\log|w_\alpha|)^2\}$, $|dz_\alpha|^2 := \sqrt{-1}\, dz_\alpha \wedge d\bar{z}_\alpha$ and $|dw_\alpha|^2 := \sqrt{-1}\, dw_\alpha \wedge d\bar{w}_\alpha$ (see Masur [13]). For each $s \in \tilde{V}$, regarding $\tau_\beta(s)$, $\tau_\gamma(s)$ as elements of $H^0(X_s, \mathcal{O}(K_{X_s}^{\otimes\mu}))$, we define a Hermitian pairing by

$$< \tau_\beta(s), \tau_\gamma(s) > := \int_{X_s} \sqrt{-1}\, \tau_\beta(s) \wedge \bar{\tau}_\gamma(s)/\omega_s^{\otimes\mu-1}.$$

Take a sufficiently small $0 < \varepsilon \in \mathbb{R}$. Compare (3.1.2), (3.1.3) and (3.1.5) above. Then, when restricted to $\{|s_\alpha|^{1/2} \le |z_\alpha| \le |s_\alpha|^{1/2} + \varepsilon\}$ on U_α,

$$(3.1.6) \qquad \tau_\beta(s) \wedge \bar{\tau}_\gamma(s)/\omega_s^{\otimes \mu-1} = \frac{\{\,\varphi\, e_{\alpha\beta}\, e_{\alpha\gamma} + O(|z_\alpha|)\,\}|dz_\alpha|^2}{|z_\alpha|^2\,(\log\,|z_\alpha|)^{2(1-\mu)}},$$

for some function φ with $k^{1-\mu} \le |\varphi| \le k^{\mu-1}$, where $O(|z_\alpha|)$ denotes a function whose absolute value is bounded by some constant multiple of $|z_\alpha|$, the constant being independent also of s and ε. Similarly, on the subset $\{|s_\alpha|^{1/2} \le |w_\alpha| \le |s_\alpha|^{1/2} + \varepsilon\}$ of U_α,

$$(3.1.7) \qquad \tau_\beta(s) \wedge \bar{\tau}_\gamma(s)/\omega_s^{\otimes \mu-1} = \frac{\{\,\varphi\, e_{\alpha\beta}\, e_{\alpha\gamma} + O(|w_\alpha|)\,\}|dw_\alpha|^2}{|w_\alpha|^2\,(\log\,|w_\alpha|)^{2(1-\mu)}},$$

for some φ with $k^{1-\mu} \le |\varphi| \le k^{\mu-1}$. Let $U_\alpha^{s,\varepsilon}$ be the subset of $U_\alpha \cap X_s$ defined to be

$$\{\,(z_\alpha, w_\alpha, s) \in U_\alpha \cap X_s\,;\, |z_\alpha| \le |s_\alpha|^{1/2} + \varepsilon,\ |w_\alpha| \le |s_\alpha|^{1/2} + \varepsilon\,\}.$$

Moreover, put $U^{s,\varepsilon} := \cup_{\alpha=0}^{m-1} U_\alpha^{s,\varepsilon}$ and $W^\varepsilon := X - \cup_{s\in\tilde{V}} U^{s,\varepsilon}$. By (3.1.4) together with (3.1.6) and (3.1.7), we have

$$\int_{U_\alpha^{s,\varepsilon}} \tau_\beta(s) \wedge \bar{\tau}_\gamma(s)/\omega_s^{\otimes \mu-1} = \begin{cases} O((-\log|s_\alpha|)^{2\mu-1}) & \text{if } 0 \le \beta = \gamma < m, \\ O(\sqrt{\varepsilon}\,) & \text{otherwise,} \end{cases}$$

where $\beta,\ \gamma \in \{0, 1, \ldots, N\}$ and $\|s\| \ll 1$. If $0 \le \beta < m$, there exists a constant $k_\varepsilon > 1$ (which possibly depends on ε but is independent of s) such that we actually have the following stronger inequality:

$$(3.1.8) \qquad k_\varepsilon^{-1} \le \frac{|\int_{U_\alpha^{s,\varepsilon}} \tau_\beta(s) \wedge \bar{\tau}_\beta(s)/\omega_s^{\otimes \mu-1}|}{(-\log|s_\beta|)^{2\mu-1}} \le k_\varepsilon,$$

with $s_\beta \ne 0$ and $\|s\| \ll 1$. Moreover, $\int_{X_s - U^{s,\varepsilon}} \tau_\beta(s) \wedge \bar{\tau}_\gamma(s)/\omega_s^{\otimes \mu-1}$ depends continuously on $s \in \tilde{V}$ for any β and γ in $\{0, 1, \ldots, N\}$ (cf. Wolpert [23]). Let ε tend to 0. Shrinking $\tilde{V}$ if necessary, we have

$$< \tau_\beta(s), \tau_\gamma(s) > \to < \tau_\beta(0), \tau_\gamma(0) > = < \theta_\beta, \theta_\gamma > = \delta_{\beta\gamma} \quad (\text{as } s \to 0),$$

when $\beta \ge m$ or $\gamma \ge m$ or $\beta \ne \gamma$.

3.2. Let $\tilde{V}^*$ be the open subset of $\tilde{V}$ consisting of all points s such that $s_\alpha \ne 0$ for all $0 \le \alpha < m$. From now on, until the end of 3.2, the notation s is assumed to denote an element of $\tilde{V}^*$. In particular, X_s is always nonsingular. We now put

$$(3.2.1) \qquad \tilde{\tau}_\beta(s) = \begin{cases} \|\tau_\beta(s)\|^{-1}\tau_\beta(s) & 0 \le \beta < m \\ \tau_\beta(s) & m \le \beta \le N, \end{cases}$$

where $\|\tau_\beta\|^2 := < \tau_\beta(s), \tau_\beta(s) >$. Since $\|\tau_\alpha(s)\| \to +\infty$ as $s_\alpha \to 0$ for $0 \le \alpha < m$ (cf. (3.1.8)), it then follows that

$$(\,T_{\beta\gamma}(s)\,) := < \tilde{\tau}_\beta(s), \tilde{\tau}_\gamma(s) > \to \delta_{\beta\gamma}, \qquad \text{as } s \to 0,$$

for all β, $\gamma \in \{0, 1, \dots, N\}$. Since the square matrix $T(s) = (T_{\beta\gamma}(s))$ of degree $N+1$ is positive definite and Hermitian, $A(s) := T(s)^{-1/2}$ tends to the identity matrix I as s tends to 0. In terms of the matrix $A(s) = (A_{\beta\gamma}(s))$, we put

$$(3.2.2) \qquad \sigma_\beta(s) = \Sigma_{\gamma=0}^N A_{\beta\gamma}(s)\,\tilde{\tau}_\gamma(s), \qquad \beta = 0, 1, \dots, N.$$

Then $\Sigma_s := \{\,\sigma_0(s), \sigma_1(s), \dots, \sigma_N(s)\,\}$ is a unitary $\mathbb{C}$-basis for the vector space $H^0(X_s, \mathcal{O}(K_{X_s}^{\otimes\mu}))$. Let $\Phi_s\colon X_s \to \mathbb{P}^N(\mathbb{C})$ and $\Phi'_s\colon X_s \to \mathbb{P}^N(\mathbb{C})$ be the projective embeddings defined by

$$\Phi_s\colon X_s \hookrightarrow \mathbb{P}^N(\mathbb{C}), \quad x \mapsto \Phi_s(x) := ((\sigma_0(s))(x) : \cdots : (\sigma_N(s))(x)),$$
$$\Phi'_s\colon X_s \hookrightarrow \mathbb{P}^N(\mathbb{C}), \quad x \mapsto \Phi'_s(x) := ((\tilde{\tau}_0\,(s))(x) : \cdots : (\tilde{\tau}_N\,(s))(x)).$$

Then their images $\Phi_s(X_s)$ and $\Phi'_s(X_s)$ are regarded as elements of the Hilbert scheme H^μ (cf. Lemma 2.5). Let $\{\,t_j = (t_{j,0}, t_{j,1}, \dots, t_{j,3g-4})\,;\ j = 1, 2, \dots\,\}$ be an arbitrary sequence of points t_j in $\tilde{V}^*$ such that $t_j \to 0$ as $j \to \infty$. Since $A(t_j) \to I$ as $j \to \infty$,

$$(3.2.3) \qquad \lim_{j\to\infty} \Phi_{t_j}(X_{t_j}) = \lim_{j\to\infty} \Phi'_{t_j}(X_{t_j}) \quad \text{in } H^\mu$$

if the right-hand side exists. Let us now fix a sufficiently small $\varepsilon > 0$. Recall that, in view of (3.1.1), the line bundle $K_C^{\otimes\mu}$ is generated by the sections θ_m, θ_{m+1}, $\dots$, θ_N over the subset $C - C_{\text{sing}}$ of $C = X_0$. Hence, by shrinking $\tilde{V}$ if necessary, we have

$$W^\varepsilon = \cup_{\alpha=m}^N W_\alpha^\varepsilon = \cup_{\alpha=m}^N \hat{W}_\alpha^\varepsilon,$$

where we put $W_\alpha^\varepsilon := W^\varepsilon \cap \{\tau_\alpha \neq 0\}$ for $m \le \alpha \le N$, and $\hat{W}_\alpha^\varepsilon$ is a suitably chosen relatively compact open subset of W_α^ε. Let $\{x_j\}$ be a convergent sequence of points in $\hat{W}_\alpha^\varepsilon$, such that $x_j \in X_{t_j}$, with a limit

$$x_\infty := \lim_{j\to\infty} x_j \in \hat{W}_\alpha^\varepsilon.$$

Then for $\beta \in \{m, m+1, \dots, N\}$, we have

$$(\tilde{\tau}_\beta(t_j)/\tilde{\tau}_\alpha(t_j))(x_j) = (\tau_\beta(t_j)/\tau_\alpha(t_j))(x_j) \to (\theta_\beta/\theta_\alpha)(x_\infty),$$

as $j \to \infty$. On the other hand, if $0 \le \beta < m$, then by (3.1.8) and (3.2.1),

$$\begin{aligned} 0 \le |(\tilde{\tau}_\beta(t_j)/\tilde{\tau}_\alpha(t_j))(x_j)| &\le k_\varepsilon(-\log|t_{j,\beta}|)^{\frac{1}{2}-\mu}|(\tau_\beta(t_j)/\tau_\alpha(t_j))(x_j)| \\ &\le k'k_\varepsilon(-\log|t_{j,\beta}|)^{\frac{1}{2}-\mu} \to 0, \quad \text{as } j \to \infty, \end{aligned}$$

where $k' > 0$ is a constant independent of j. Write C as a union $\cup_{i=1}^r C_i$ of irreducible components, and identify X_0 with C. Then by the notation in (2.1.1) and 2.3, the limit set of $\Phi'_{t_j}(X_{t_j} - U^{t_j,\varepsilon})$, as $j \to \infty$, is

$$\tilde{C}'_\varepsilon := \cup_{i=1}^r \psi_i \circ \nu^{-1}(C_i - C_i \cap U^{0,\varepsilon}),$$

i.e., the set $\tilde{C}'_\varepsilon$ coincides with

$$\{\lim_{j\to\infty} y_j\,;\, \{y_j\} \in \mathcal{Y}_\varepsilon\},$$

where $\mathcal{Y}_\varepsilon$ is the set of all convergent sequences of points $\{y_j\}$ in $\mathbb{P}^N(\mathbb{C})$ such that $y_j \in \Phi'_{t_j}(X_{t_j} - U^{t_j,\varepsilon})$ for all j. Then by setting $\tilde{C}' := \cup_{i=1}^r \tilde{C}'_i$ in (2.4.2), we have $\tilde{C}'_\varepsilon \subset \tilde{C}'$ for all sufficiently small $\varepsilon > 0$. Moreover, the limit set of $\tilde{C}'_\varepsilon$ as $\varepsilon \to 0$ is $\tilde{C}'$.

3.3. Let $\{t_j\}$ be again an arbitrary sequence of points in $\tilde{V}^*$ such that $t_j \to 0$ as $j \to \infty$. We further fix an arbitrary singular point p_α $(0 \le \alpha < m)$ of C, and shrink $\tilde{V}$ if necessary. By setting

$$d_j := \|\tau_\alpha(t_j)\| \cdot (-\log|t_{j,\alpha}|)^{\frac{1}{2}-\mu} > 0,$$

we see, from (3.1.8), that $\{d_j\}$ is a bounded sequence of positive real numbers uniformly away from 0, so that some subsequence of $\{d_j\}$ converges to a positive real number d_∞. Replacing $\{t_j\}$ by its suitable subsequence, we have $d_j \to d_\infty$ as $j \to \infty$. By the notation in the proof of Lemma 2.3, $\mathcal{K}_i^\mu(D'_i)$ is very ample for each i, and hence

$$f_{\alpha\hat{\beta}}(0,0,0) \ne 0 \text{ and } g_{\alpha\hat{\gamma}}(0,0,0) \ne 0$$

in (3.1.3) for some $\hat{\beta}$, $\hat{\gamma} \in \{m, m+1, \dots, N\}$. For $u \in \mathbb{C}$, let q'_u, q''_u be the points in $\mathbb{P}^N(\mathbb{C}) = \{ (\zeta_0 : \zeta_1 : \cdots : \zeta_N) \}$ defined by

$$\zeta_\beta(q'_u) := \begin{cases} u \cdot e_{\alpha\beta}/d_\infty & \text{if } 0 \le \beta < m, \\ f_{\alpha\beta}(0,0,0) & \text{if } m \le \beta \le N, \end{cases}$$
$$\zeta_\beta(q''_u) := \begin{cases} u \cdot e_{\alpha\beta}/d_\infty & \text{if } 0 \le \beta < m, \\ g_{\alpha\beta}(0,0,0) & \text{if } m \le \beta \le N. \end{cases}$$

Then in view of (2.4.2), the set $\{ q'_u, q''_u \}$ coincides with the set $\{ p'_\alpha, p''_\alpha \}$ when $u = 0$. Hence, we may assume $q'_0 = p'_\alpha$ and $q''_0 = p'_\alpha$. In particular,

$$\ell'_\alpha - \{b_\alpha, p'_\alpha\} = \{ q'_u \,;\, u \in \mathbb{C}^* \} \quad \text{and} \quad \ell''_\alpha - \{b_\alpha, p''_\alpha\} = \{ q''_u \,;\, u \in \mathbb{C}^* \}.$$

For $u \in \mathbb{C}^*$, let $p'_{j,u}$, $p''_{j,u}$ (where $j \gg 1$) be the points in $U_\alpha^{t_j,\varepsilon}$ defined by

$$(z_\alpha(p'_{j,u}), w_\alpha(p'_{j,u})) = (u^{-1}(-\log|t_{j,\alpha}|)^{\frac{1}{2}-\mu},\, u \cdot t_{j,\alpha} \cdot (-\log|t_{j,\alpha}|)^{\mu-\frac{1}{2}}),$$
$$(z_\alpha(p''_{j,u}), w_\alpha(p''_{j,u})) = (u \cdot t_{j,\alpha} \cdot (-\log|t_{j,\alpha}|)^{\mu-\frac{1}{2}},\, u^{-1}(-\log|t_{j,\alpha}|)^{\frac{1}{2}-\mu}).$$

Since $w_\alpha(p'_{j,u})/z_\alpha(p'_{j,u}) \to 0$ as $j \to \infty$, we have

$$(\tilde{\tau}_\beta(t_j)/\tilde{\tau}_{\hat{\beta}}(t_j))(p'_{j,u}) = (\tau_\beta(t_j)/\tau_{\hat{\beta}}(t_j))(p'_{j,u}) = (a_{\alpha\beta}(t_j)/a_{\alpha\hat{\beta}}(t_j))(p'_{j,u})$$
$$\to f_{\alpha\beta}(0,0,0)/f_{\alpha\hat{\beta}}(0,0,0), \quad m \le \beta \le N, \qquad \text{as } j \to \infty.$$

Moreover, by (3.1.4), if $\{0, 1, \dots, m-1\} \ni \beta \neq \alpha$, then

$$(\tilde{\tau}_\beta(t_j)/\tilde{\tau}_{\hat{\beta}}(t_j))(p'_{j,u}) = \|\tau_\beta(t_j)\|^{-1}(\tau_\beta(t_j)/\tau_{\hat{\beta}}(t_j))(p'_{j,u})$$
$$\to (\lim_{j\to\infty} \|\tau_\beta(t_j)\|^{-1})\, f_{\alpha\beta}(0,0,0)/f_{\alpha\hat{\beta}}(0,0,0) = 0$$

as $j \to \infty$. Finally, considering the case $\beta = \alpha$, we have

$$(\tilde{\tau}_\alpha(t_j)/\tilde{\tau}_{\hat{\beta}}(t_j))(p'_{j,u}) = \|\tau_\alpha(t_j)\|^{-1}(\tau_\alpha(t_j)/\tau_{\hat{\beta}}(t_j))(p'_{j,u})$$
$$\to u \cdot e_{\alpha\alpha}\, d_\infty^{-1}/f_{\alpha\hat{\beta}}(0,0,0), \qquad \text{as } j \to \infty.$$

Therefore, it follows that

$$\lim_{j\to\infty} \Phi'_{t_j}(p'_{j,u}) = q'_u, \qquad u \in \mathbb{C}^*.$$

Similary, by computing $(\tilde{\tau}_\gamma(t_j)/\tilde{\tau}_{\hat{\gamma}}(t_j))(p''_{j,u})$, $0 \le \gamma \le N$, we obtain

$$\lim_{j\to\infty} \Phi'_{t_j}(p''_{j,u}) = q''_u, \qquad u \in \mathbb{C}^*.$$

Thus, the limit set of $\Phi'_{t_j}(U^{t_j,\varepsilon})$ (as $j \to \infty$) always contains ℓ'_α, ℓ''_α, $\alpha = 0, 1, \ldots, m-1$.

3.4. For the Hilbert scheme H^μ (cf. Section 2), we have the corresponding universal family $\pi^\mu \colon Z^\mu \to H^\mu$ with a natural embedding $Z^\mu \hookrightarrow H^\mu \times \mathbb{P}^N(\mathbb{C})$ such that π^μ is the proper flat morphism induced by the projection $H^\mu \times \mathbb{P}^N(\mathbb{C}) \to H^\mu$. Moreover, let $\iota^\mu \colon Z^\mu \to \mathbb{P}^N(\mathbb{C})$ be the restriction to Z^μ of the projection $H^\mu \times \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$. For each $\xi \in H^\mu$, let Z^μ_ξ denote the scheme-theoretic fibre of π^μ over ξ. Then

$$[\xi] := \iota^\mu(Z^\mu_\xi)\,(\cong Z^\mu_\xi)$$

is the closed subscheme of $\mathbb{P}^N(\mathbb{C})$ associated with ξ. We can then assign, to each $s \in \tilde{V}^*$, an element $\xi(s)$ of H^μ such that $\Phi'_s(X_s)$ is just $[\xi(s)]$. Let $\{t_j\} \subset \tilde{V}^*$ be an arbitrary sequence of points converging to 0, and the associated element $\xi(t_j)$ in H^μ will be denoted simply by ξ_j. First, we assume that ξ_j converges to a point ξ_∞ in H^μ as $j \to \infty$. Then by 3.2 and 3.3, there exists a subsequence $\{\xi_{j_\nu}\}$ of $\{\xi_j\}$ such that

$$C^* \subset \operatorname{limset}\{\, \Phi'_{t_{j_\nu}}(X_{t_{j_\nu}}) \,\},$$

where C^* is as in (2.4.2), and on the right-hand side, limset$\{\ldots\}$ means the limit set of $\{\ldots\}$ as $\nu \to \infty$. On the other hand, since π^μ is flat,

$$\operatorname{limset}\{\, \Phi'_{t_{j_\nu}}(X_{t_{j_\nu}}) \,\} = \operatorname{limset}\{\, [\xi_{j_\nu}] \,\} = [\xi_\infty].$$

Hence, $[\xi_\infty] \supset C^*$ set-theoretically. Recall that, by C^*, we also denote the corresponding element in H^μ (cf. Lemma 2.5). Since both ξ_∞ and C^* belong to H^μ,

$$\deg\,[\xi_\infty] = \deg\, C^*.$$

Hence, $[\xi_\infty]$ coincides with the reduced scheme C^*, except $[\xi_\infty]$ can have embedded primes at singular points of $[\xi_\infty]_{\text{red}} = C^*$. Again by using the fact that both $[\xi_\infty]$ and C^* belong to H^μ, we have

$$\chi(\mathcal{O}_{[\xi_\infty]}) = \chi(\mathcal{O}_{C^*}),$$

and in particular, $[\xi_\infty]$ is reduced. Therefore, regarding C^* as an element of H^μ, we obtain the convergence $\xi_j \to C^* \in H^\mu$ as $j \to \infty$. We must next consider the more general case that the sequence $\{\xi_j\}$ does not necessarily converge in H^μ. However, $\{\xi_j\}$ always converges. Because, otherwise, the compactness of H^μ would yield two convergent subsequences $\{\xi'_k\}$, $\{\xi''_l\}$ of $\{\xi_j\}$ such that

$$\lim_{k\to\infty} \xi'_k \neq \lim_{l\to\infty} \xi''_l, \qquad \text{in } H^\mu,$$

contradicting the fact that these limits must coincide with C^* by the above consequence for convergent cases. Thus $\xi_j \cdot \mathrm{U}(N+1)$ converges to $h(C) = C^* \cdot \mathrm{U}(N+1)$ in $H^\mu/\mathrm{U}(N+1)$, as $j \to \infty$, in all cases. Here, our stable curve C can be chosen arbitrarily. Hence if $D_j \in \mathcal{M}_g$, $j = 1, 2, \dots$, are such that $D_j \to D_\infty$ in $\overline{\mathcal{M}}_g$ for some $D_\infty \in \overline{\mathcal{M}}_g$, then

$$h(D_j) \to h(D_\infty) \text{ in } H^\mu/\mathrm{U}(N+1), \quad \text{as } j \to \infty.$$

3.5. Let $\{D_j\}$ be a convergent sequence of points in $\overline{\mathcal{M}}_g$, and D_∞ its limit. Then the proof of Theorem 2.6 is reduced to showing

$$h(D_j) \to h(D_\infty) \text{ in } H^\mu/\mathrm{U}(N+1), \quad \text{as } j \to \infty.$$

For contradiction, assume that the sequence $\{h(D_j)\}$ does not converge to $h(D_\infty)$ in $H^\mu/\mathrm{U}(N+1)$. Then replacing $\{D_j\}$ by its subsequence, we may assume

$$h(D_j) \to F \text{ in } H^\mu/\mathrm{U}(N+1), \quad \text{as } j \to \infty, \tag{3.5.1}$$

for some $F \in H^\mu/\mathrm{U}(N+1)$ with $F \neq h(D_\infty)$. By Urysohn's lemma, endow both $H^\mu/\mathrm{U}(N+1)$ and $\overline{\mathcal{M}}_g$ with a metric structure. Now, put

$$\delta := \operatorname{dist}(F, h(D_\infty)) > 0, \tag{3.5.2}$$

by the distance function. For each j, take a sequence $\{D_{j,\nu}\}_{\nu=1}^\infty \subset \mathcal{M}_g$ such that $D_{j,\nu} \to D_j$ $(\nu \to \infty)$. By 3.4, $h(D_{j,\nu}) \to h(D_j)$ $(\nu \to \infty)$. Hence, for each j, there exists ν_j satisfying the following:

$$\operatorname{dist}(D_j, D_{j,\nu_j}) < 1/j; \tag{3.5.3}$$

$$\operatorname{dist}(h(D_j), h(D_{j,\nu_j})) < 1/j. \tag{3.5.4}$$

Since $D_j \to D_\infty$ in $\overline{\mathcal{M}}_g$, the inequality (3.5.3) yields the convergence $D_{j,\nu_j} \to D_\infty$ in $\overline{\mathcal{M}}_g$ as $j \to \infty$. Hence, by 3.4,

$$h(D_{j,\nu_j}) \to h(D_\infty) \text{ in } H^\mu/\mathrm{U}(N+1), \quad \text{as } j \to \infty.$$

In view of this and (3.5.1), the identity (3.5.2) implies

$$\operatorname{dist}(h(D_j), h(D_{j,\nu_j})) \to \delta, \qquad \text{as } j \to \infty,$$

in contradiction to (3.5.4). Thus, $h(D_j) \to h(D_\infty)$ in $H^\mu/\mathrm{U}(N+1)$ as $j \to \infty$. The proof of Theorem 2.6 is now complete.

§4. Concluding remarks

The preceding section shows that our compactification of the moduli space coincides with the classical one for compact Riemann surfaces. In this section, we shall discuss other examples of our compactification by posing a couple of related conjectures.

4.1. Let d be an integer with $1 \leq d \leq 6$, and put

$$P(m) := \frac{1}{2}(m^2 + m)\,d + 1, \qquad m \in \mathbb{Z}.$$

Let $\mathcal{M}_{2,d}$ be the set of isomorphism classes of all nonsingular del Pezzo surfaces of degree d. Now, using freely the notation in Section 1, we have a natural inclusion

$$\mathcal{M}_{2,d} \subset \mathcal{M}_P^+.$$

Let $\mathcal{E}_{2,d}$ be the preimage of $\mathcal{M}_{2,d}$ under the mapping $\mathrm{pr}^+ \colon \mathcal{E}_P^+ \to \mathcal{M}_P^+$. By the works of Siu [19], Tian [20, 21], Tian and Yau [22], every nonsingular del Pezzo surface X carries an Einstein-Kähler metric ω_X. Hence, in view of Lemma 1.1, the injection pr^+ restricts to a bijection

$$\mathcal{E}_{2,d} \simeq \mathcal{M}_{2,d}, \qquad [X, K_X^{-1}, \omega_X] \leftrightarrow X,$$

where ω_X is chosen in such a way that $\mathrm{Ric}(\omega_X) = \omega_X$. By setting $\mathcal{F} = \mathcal{E}_{2,d}$, we now apply the argument in Section 1. Let μ be the integer 1, 2 or 3 according as $d > 2$, $d = 2$, or $d = 1$. Then if $m \geq \mu$, the line bundle $L^{\otimes m} = K_X^{-m}$ for $X \in \mathcal{M}_{2,d}$ is very ample, and moreover

$$h^i(X, \mathcal{O}(L^{\otimes m})) = 0, \qquad i > 0,$$

by the vanishing theorem. In view of $\mathcal{F} \subset \mathcal{E}_P^+$, the mapping $i_\mu \colon \mathcal{F} \to \mathcal{F}^\mu$ is bijective, so that by Lemma 1.3,

$$\mathcal{M}_{2,d} \simeq \mathcal{E}_{2,d} = \mathcal{F} \simeq \mathcal{F}^\mu \subset H^\mu / \mathrm{U}(N+1),$$

with $N = P(\mu) - 1$. Therefore, the closure $\overline{\mathcal{F}}^\mu$ of $\mathcal{F}^\mu$ in $H^\mu / \mathrm{U}(N+1)$ can be regarded as a compactification of $\mathcal{M}_{2,d}$. On the other hand, Anderson [1], Bando, Kasue and Nakajima [3] (see also Bando [2]) succeeded in constructing a natural compactification $\overline{\mathcal{E}}_{2,d}$ of $\mathcal{E}_{2,d}$ $(= \mathcal{M}_{2,d})$ by using Gromov's theory, where the topological structure of $\overline{\mathcal{E}}_{2,d}$ is given by the Hausdorff convergence. We now pose the following:

Conjecture A. $\overline{\mathcal{F}}^\mu = \overline{\mathcal{E}}_{2,d}$ *for all* d *with* $1 \leq d \leq 6$.

This is obviously true for $d = 5, 6$, since both $\mathcal{E}_{2,5}$ and $\mathcal{E}_{2,6}$ consists of a single point. Moreover, one can prove this for $d = 4$, which will be given elsewhere (cf. [12]).

4.2. We next consider the moduli space of polarized Abelian varieties. Take a positive integers e, n and put $P(m) := e \cdot m^n$. Using the notation in Section 1, let $\mathcal{A}_{n,e}$ denote the set of all $[X, L, \omega] \in \mathcal{E}_P$ such that (X, L) is a polarized Abelian variety with $c_1(L)^n[X]/n! = e$. Then for instance, $\mathcal{A}_{n,1}$ coincides set-theoretically with the moduli space of principally polarized Abelian varieties. Let μ be an integer satisfying $\mu \geq 3$. Now, it is well-known that $\mathcal{A}_{n,e} \subset \mathcal{E}_{P,\mu}$. Hence, by setting $\mathcal{F} = \mathcal{A}_{n,e}$, we see that the mapping $i_\mu \colon \mathcal{F} \to \mathcal{F}^\mu$ (cf. Section 1) is bijective in view of the $\mathrm{Aut}^0(X)$-action on $\mathrm{Pic}(X)$ for any Abelian variety X. It now follows from Lemma 1.3 that

$$\mathcal{A}_{n,e} = \mathcal{F} \simeq \mathcal{F}^\mu \subset H^\mu / \mathrm{U}(N+1),$$

where $N = P(\mu) - 1$. Then the closure $\overline{\mathcal{F}}^\mu$ of $\mathcal{F}^\mu$ in $H^\mu / \mathrm{U}(N+1)$ is regarded as a compactification of $\mathcal{A}_{n,e}$. In order to see this compactification more clearly, we assume $e = 1$ for simplicity. Note, in the construction of our compactification, the main difficulty arises from the fact that there are no explicit methods, in general, to find out a unitary $\mathbb{C}$-basis for $H^0(X, \mathcal{O}(L^{\otimes \mu}))$. However, as you see below, there is a method so far as principally polarized Abelian varieties are concerned. Let $\mathfrak{S}_n$ be the n-th Siegel's upper half-plane consisting of all $n \times n$ complex symmetric matrices $\Omega = \mathrm{Re}\,\Omega + \sqrt{-1}\,\mathrm{Im}\,\Omega$ such that $\mathrm{Im}\,\Omega$ is positive definite. Moreover, let $\mathrm{Sp}(n, \mathbb{Z})$ denote the group of all $2n \times 2n$ integral matrices h satisfying

$${}^t h\, J\, h = J, \qquad \text{where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $\mathrm{Sp}(n, \mathbb{Z})$ acts on $\mathfrak{S}_n$ in such a way that, for each $\Omega \in \mathfrak{S}_n$, the action is expressible as

$$h \cdot \Omega := (A\Omega + B)(C\Omega + D)^{-1}, \qquad h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}).$$

Recall that $\mathcal{A}_{n,1}$ is identified with $\mathfrak{S}_n / \mathrm{Sp}(n, \mathbb{Z})$ as follows. For Ω in $\mathfrak{S}_n$, we put $\Gamma_\Omega := \mathbb{Z}^n + \Omega\mathbb{Z}^n$ and consider the Abelian variety $V_\Omega := \mathbb{C}^n / \Gamma_\Omega$. Following the notation in Mumford [16], define the theta function

$$\vartheta(\mathbf{z}, \Omega) := \sum_{\mathbf{m} \in \mathbb{Z}^n} \exp(\pi\sqrt{-1}\,{}^t\mathbf{m}\,\Omega\,\mathbf{m} + 2\pi\sqrt{-1}\,{}^t\mathbf{m} \cdot \mathbf{z}), \qquad \mathbf{z} \in \mathbb{C}^n,$$

and let Θ_Ω be the associated theta diviosor on V_Ω defined as the zeroes of $\vartheta = \vartheta(\mathbf{z}, \Omega)$. Then the line bundle $L_\Omega := \mathcal{O}(\Theta_\Omega)$ on V_Ω carries a

Hermitian metric, denoted by $h(\omega_\Omega)$, such that (cf. Faltings [10])

$$\| 1 \|_{h(\omega_\Omega)}(\mathbf{z}) := (\det \Omega)^{1/4} \cdot \exp(-\pi\, {}^t\mathbf{y}\, (\operatorname{Im}\Omega)^{-1}\mathbf{y}) \cdot |\vartheta(\mathbf{z}, \Omega)|,$$

where $\mathbf{z} = \mathbf{x} + \sqrt{-1}\,\mathbf{y} \in \mathbb{C}^n$, and functions on V_Ω are identified with Γ_Ω-invariant functions on $\mathbb{C}^n$. The corresponding Chern form is (see Section 1 for how ω_Ω and $h(\omega_\Omega)$ are related)

$$c_1(L_\Omega, h(\omega_\Omega)) = \frac{\omega_\Omega}{2\pi} = \frac{\sqrt{-1}}{2}\, {}^t d\mathbf{z}\, (\operatorname{Im}\Omega)^{-1}\, d\bar{\mathbf{z}},$$

which obviously defines a flat Kähler metric on V_Ω. Now, we have the identification

$$\mathfrak{S}_n / \operatorname{Sp}(n, \mathbb{Z}) \simeq \mathcal{A}_{n,1}, \qquad \operatorname{Sp}(n, \mathbb{Z}) \cdot \Omega \leftrightarrow [V_\Omega, L_\Omega, \omega_\Omega].$$

Next, for an integer $\mu \geq 3$, let R_μ^Ω denote the space of all holomorphic functions $\theta = \theta(\mathbf{z})$ on $\mathbb{C}^n$ such that θ is quasi-periodic of weight μ, i.e.,

$$\begin{cases} \theta(\mathbf{z} + \mathbf{m}) = \theta(\mathbf{z}), \\ \theta(\mathbf{z} + \Omega\,\mathbf{m}) = \exp(-\pi\mu\sqrt{-1}\, {}^t\mathbf{m}\,\Omega\,\mathbf{m} - 2\pi\mu\sqrt{-1}\, {}^t\mathbf{z} \cdot \mathbf{m}) \cdot \theta(\mathbf{z}) \end{cases}$$

for all $\mathbf{m} \in \mathbb{Z}^n$ and $\mathbf{z} \in \mathbb{C}^n$. To find a basis for R_μ^Ω, recall now that theta functions of rational characteristic are defined as translates of ϑ multiplied by an exponential factor. Namely, for $\mathbf{a}$, $\mathbf{b} \in \mathbb{Q}^n$, put

$$\vartheta \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (\mathbf{z}, \Omega) = \exp(\pi\sqrt{-1}\, {}^t\mathbf{a}\,\Omega\,\mathbf{a} + 2\pi\sqrt{-1}\, {}^t\mathbf{a} \cdot (\mathbf{z} + \mathbf{b})) \cdot \vartheta(\mathbf{z} + \Omega\,\mathbf{a} + \mathbf{b}, \Omega).$$

Let $\mathcal{E}_\mu$ be the set of all $\mathbf{a} = {}^t(a_1, \ldots, a_n) \in \mathbb{Q}^n$ such that, for all i, the multiple $\mu\, a_i$ is an integer satisfying $0 \leq \mu\, a_i < \mu$. Then the functions

$$\theta_{\mathbf{a}}(\mathbf{z}) := \vartheta \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix} (\mu\,\mathbf{z}, \mu\,\Omega), \qquad \mathbf{a} \in \mathcal{E}_\mu, \tag{4.2.1}$$

form a $\mathbb{C}$-basis for R_μ^Ω (cf. [16; II-§1]). Moreover, we have

$$R_\mu^\Omega \cong H^0(V_\Omega, \mathcal{O}(L_\Omega^{\otimes\mu})), \qquad \theta \leftrightarrow \tilde{\theta} := \theta(z)/\vartheta(\mu z, \mu\Omega),$$

where Γ_Ω-invariant meromorphic function $\tilde{\theta}$ on $\mathbb{C}^n$ is naturally regarded as a global section of $L_\Omega^{\otimes\mu}$. Now, the Hermitian metric $h^\mu := h(\omega_\Omega)^{\otimes\mu}$ on the line bundle $L_\Omega^{\otimes\mu}$ is written in the form

$$\| \tilde{\theta} \|_{h^\mu}(\mathbf{z}) := \{ (\det\Omega)^{1/4} \cdot \exp(-\pi\, {}^t\mathbf{y}\, (\operatorname{Im}\Omega)^{-1}\mathbf{y}) \}^\mu \cdot |\theta(\mathbf{z})|, \tag{4.2.2}$$

for all $\theta \in R_\mu^\Omega$. Then by the quasi-periodicity of ϑ,

$$\theta_{\mathbf{a}}(\mathbf{z}+\mathbf{b}) = \exp(2\pi\mu\sqrt{-1}\,{}^t\mathbf{a}\cdot\mathbf{b})\cdot\theta_{\mathbf{a}}(\mathbf{z}), \qquad \mathbf{a},\,\mathbf{b}\in\mathcal{E}_\mu, \tag{4.2.3}$$

so that, for each $\mathbf{a}\in\mathcal{E}_\mu$, we have a $\mathbb{C}$-linear automorphism $T_{\mathbf{a}}$ of R_μ^Ω by setting

$$T_{\mathbf{a}}(\theta)(\mathbf{z}) = \theta(\mathbf{z}+\mathbf{a}), \qquad \theta\in R_\mu^\Omega.$$

In view of (4.2.2),

$$\begin{aligned}\|\widetilde{T_{\mathbf{a}}(\theta)}\|_{h^\mu}(\mathbf{z}) &= \{\,(\det\Omega)^{1/4}\cdot\exp(-\pi\,{}^t\mathbf{y}\,(\operatorname{Im}\Omega)^{-1}\mathbf{y})\,\}^\mu\cdot|T_{\mathbf{a}}(\mathbf{z})|\\ &= \{\,(\det\Omega)^{1/4}\cdot\exp(-\pi\,{}^t\mathbf{y}\,(\operatorname{Im}\Omega)^{-1}\mathbf{y})\,\}^\mu\cdot|\theta(\mathbf{z}+\mathbf{a})| = \|\tilde\theta\|_{h^\mu}(\mathbf{z}+\mathbf{a})\end{aligned}$$

for all $\theta\in R_\mu^\Omega$. Integrating this identity by the translation-invariant volume form $\omega_\Omega^n/n!$ over the space V_Ω, we have

$$\|\widetilde{T_{\mathbf{a}}(\theta)}\|^2_{L^2,\omega_\Omega} = \int_{V_\Omega}\|\widetilde{T_{\mathbf{a}}(\theta)}\|^2_{h^\mu}\,\omega_\Omega^n/n! = \int_{V_\Omega}\|\tilde\theta\|^2_{h^\mu}\,\omega_\Omega^n/n! = \|\tilde\theta\|^2_{L^2,\omega_\Omega}$$

for all $\theta\in R_\mu^\Omega$ and $\mathbf{a}\in\mathcal{E}_\mu$. We endow R_μ^Ω with the L^2-norm defined by

$$\|\theta\|_{L^2} := \|\tilde\theta\|_{L^2,\omega_\Omega}, \qquad \theta\in R_\mu^\Omega.$$

In terms of this L^2-norm, the automorphism $T_{\mathbf{a}}$ induces an isometry of R_μ^Ω for all $\mathbf{a}\in\mathcal{E}_\mu$. Together with (4.2.3), we have

$$\theta_{\mathbf{a}}\perp\theta_{\mathbf{b}}, \qquad \mathbf{a}\neq\mathbf{b}. \tag{4.2.4}$$

Note, in (4.2.1), our $\theta_{\mathbf{o}}(\mathbf{z})$ is nothing but $\vartheta(\mu\,\mathbf{z},\mu\,\Omega)$. Then the definition of theta functions of rational characteristics yields

$$|\theta_{\mathbf{a}}(\mathbf{z})| = \{\,\exp(-\pi\,{}^t\mathbf{a}\,(\operatorname{Im}\Omega)\,\mathbf{a} - 2\pi\,{}^t\mathbf{a}\cdot\mathbf{y})\,\}^\mu\cdot|\theta_{\mathbf{o}}(\mathbf{z}+\Omega\,\mathbf{a})|$$

for $\mathbf{a}\in\mathcal{E}_\mu$. Therefore, in the identity (4.2.2), substituting $\theta_{\mathbf{o}}$ for θ and also $\mathbf{z}+\Omega\,\mathbf{a}$ for $\mathbf{z}$, we obtain

$$\begin{aligned}\|\tilde\theta_{\mathbf{o}}\|_{h^\mu}(\mathbf{z}+\Omega\,\mathbf{a}) &= \{(\det\Omega)^{1/4}\cdot\exp(-\pi\,{}^t\mathbf{y}\,(\operatorname{Im}\Omega)^{-1}\mathbf{y})\}^\mu\cdot|\theta_{\mathbf{a}}(\mathbf{z})|\\ &= \|\tilde\theta_{\mathbf{a}}\|_{h^\mu}(\mathbf{z}).\end{aligned}$$

Integrate this by the volume form $\omega_\Omega^n/n!$ over the space V_Ω. Then,

$$\|\theta_{\mathbf{o}}\|_{L^2} = \|\theta_{\mathbf{a}}\|_{L^2}, \qquad \mathbf{a}\in\mathcal{E}_\mu.$$

Let $c > 0$ be the left-hand side of this identity. In view of (4.2.4), we now see that $\{ c^{-1}\theta_{\mathbf{a}} ; \mathbf{a} \in \mathcal{E}_\mu \}$ is a unitary $\mathbb{C}$-basis for R^{Ω}_{μ}, i.e.,

$$\Sigma_\Omega := \{ c^{-1}\tilde{\theta}_{\mathbf{a}} ; \mathbf{a} \in \mathcal{E}_\mu \}$$

is a unitary $\mathbb{C}$-basis for $H^0(V_\Omega, \mathcal{O}(L_\Omega^{\otimes\mu}))$. Then the corresponding projective embedding is given explicitly by

$$\Phi_{\Sigma_\Omega} : V_\Omega \hookrightarrow \mathbb{P}^N(\mathbb{C}), \quad \mathbf{z} + \Gamma_\Omega \mapsto (\theta_{\mathbf{a}_0}(\mathbf{z}) : \theta_{\mathbf{a}_1}(\mathbf{z}) : \ \dots \ : \theta_{\mathbf{a}_N}(\mathbf{z})),$$

with $\mathbf{z} \in \mathbb{C}^n$, where elements of $\mathcal{E}_\mu$ are so numbered that

$$\mathcal{E}_\mu = \{ \mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_N \}.$$

Then the image $X_{\Sigma_\Omega} := \Phi_{\Sigma_\Omega}(V_\Omega)$ is a point in the Hilbert scheme H^μ (cf. Section 1), and the closure of

$$\{ X_{\Sigma_\Omega} \cdot \mathrm{U}(N+1) ; \Omega \in \mathfrak{S}_n \}$$

in $H^\mu / \mathrm{U}(N+1)$ is nothing but our compactification $\overline{\mathcal{F}}^\mu$ of $\mathcal{A}_{n,1}$. This construction suggests that our compactification have some relation to Igusa's compactification [11], at least, in view of both heavy dependence on theta functions.

Let us now assume $n = 1$, and describe explicitly our compactification $\overline{\mathcal{F}}^\mu$ of $\mathcal{A}_{1,1}$ for $\mu = 4$. Since $\mathrm{Sp}(1, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$, it follows that

$$\mathcal{A}_{1,1} \simeq \mathfrak{S}_1 / \mathrm{SL}(2, \mathbb{Z}),$$

where $\mathfrak{S}_1 = \{ z \in \mathbb{C}; \mathrm{Im}\, z > 0 \}$. In view of (4.2.1), we put

$$\theta_{j/4}(z) := \vartheta \begin{bmatrix} j/4 \\ 0 \end{bmatrix} (4z, 4\tau), \qquad j = 0, 1, 2, 3,$$

$$\vartheta_{a,b}(z) := \vartheta \begin{bmatrix} a/2 \\ b/2 \end{bmatrix} (2z, \tau), \qquad a,\, b \in \{0, 1\},$$

for $z \in \mathbb{C}$ and $\tau \in \mathfrak{S}_1$. Then (cf. [16]),

$$\begin{cases} \vartheta_{0,0} = \theta_0 + \theta_{1/2} \\ \vartheta_{0,1} = \theta_0 - \theta_{1/2} \\ \vartheta_{1,0} = \theta_{1/4} + \theta_{3/4} \\ \vartheta_{1,1} = \sqrt{-1}\,(\theta_{1/4} - \theta_{3/4}), \end{cases}$$

so that, for the elliptic curve $V_\tau := \mathbb{C}^2/\mathbb{Z} + \mathbb{Z}\tau$, the theta functions

$$c^{-1}\vartheta_{a,b} \qquad a,\, b \in \{0,1\}$$

form a unitary $\mathbb{C}$-basis for $R_4^\tau \cong H^0(V_\tau, \mathcal{O}(L_\tau^{\otimes 4}))$, with $c := \|\vartheta_{0,0}\|_{L^2}$. Therefore the corresponding projective embedding is given by

$$\Phi_{\Sigma_\tau} : V_\tau \hookrightarrow \mathbb{P}^3(\mathbb{C}), \quad z + \Gamma_\tau \mapsto (\vartheta_{0,0}(z) : \vartheta_{0,1}(z) : \vartheta_{1,0}(z) : \vartheta_{1,1}(z)),$$

where $z \in \mathbb{C}$ and $\Gamma_\tau := \mathbb{Z} + \mathbb{Z}\tau$. Recall that the image $X_{\Sigma_\tau} := \Phi_{\Sigma_\tau}(V_\tau)$ in $\mathbb{P}^3(\mathbb{C}) := \{\zeta = (\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3)\}$ is just the scheme-theoretic complete intersection of quadrics (cf. [16; p.23])

$$\vartheta_{0,0}(0)^2\,\zeta_0^2 = \vartheta_{0,1}(0)^2\,\zeta_1^2 + \vartheta_{1,0}(0)^2\,\zeta_2^2,$$
$$\vartheta_{0,0}(0)^2\,\zeta_3^2 = \vartheta_{1,0}(0)^2\,\zeta_1^2 - \vartheta_{0,1}(0)^2\,\zeta_2^2.$$

Moreover, the fundamental region of $\mathrm{SL}(2,\mathbb{Z})$ in $\mathfrak{S}_1$ is

$$\{\,\tau = x + \sqrt{-1}\,y\,;\; x^2 + y^2 \geq 1,\; |x| \leq 1/2,\; y > 0\,\},$$

where $\{x = -1/2\}$ and $\{x = 1/2\}$ are identified by the map $\tau \mapsto \tau + 1$, and $\{x^2 + y^2 = 1\}$ is identified with itself by the map $\tau \mapsto -1/\tau$. Hence

$$\mathcal{A}_{1,1} \simeq \mathfrak{S}_1/\,\mathrm{SL}(2,\mathbb{Z}) \simeq \mathbb{R}^2.$$

As $y = \mathrm{Im}\,\tau \to +\infty$, we have (cf. [16;p.40])

$$\begin{cases} \vartheta_{0,0}(0) & = 1 + \mathrm{O}(\exp(-\pi y)), \\ \vartheta_{0,1}(0) & = 1 + \mathrm{O}(\exp(-\pi y)), \\ \vartheta_{1,0}(0) & = \mathrm{O}(\exp(-\pi y/4)), \end{cases}$$

so that the curve X_{Σ_τ} in $\mathbb{P}^3(\mathbb{C})$ converges uniformly to the union of four lines

$$X_\infty := \{\,\zeta \in \mathbb{P}^3(\mathbb{C})\,;\; \zeta_0^2 = \zeta_1^2,\; \zeta_3^2 = -\zeta_2^2\,\},$$

which is regarded as a point of the Hilbert scheme H^μ (where $\mu = 4$). We now put $p_\infty := X_\infty \cdot \mathrm{U}(N+1) \in H^\mu/\,\mathrm{U}(N+1)$. Then our $\overline{\mathcal{F}}^\mu$ is nothing but a one-point compactification of $\mathcal{A}_{1,1}$ as follows:

$$\overline{\mathcal{F}}^\mu = \mathcal{A}_{1,1} \cup \{p_\infty\} \simeq S^2.$$

4.3. Let c_1^2, c_2, $P(m)$, $\mathcal{S}_{c_1^2,c_2}$ be the same as in (2) of Example 1.2. Then $\mathcal{F} = \mathcal{S}_{c_1^2,c_2}$ coincides with the set of isomorphism classes of all minimal algebraic surfaces of general type of given Chern numbers c_1^2, c_2. Moreover,

$$\mathcal{S}_{c_1^2, c_2} = \mathcal{F} \simeq \mathcal{F}^{\mu} \subset \mathcal{E}_{P_\mu}, \qquad \mu \geq 5.$$

Now, look at our compactification $\overline{\mathcal{F}}^{\mu}$ of $\mathcal{S}_{c_1^2, c_2}$. As in the curve case (cf. 2.7), we can expect the following:

Conjecture B. *There exists an integer $\mu_0 \geq 5$ such that for every $\mu \geq \mu_0$, one has a natural homeomorphism between $\overline{\mathcal{F}}^{\mu}$ and $\overline{\mathcal{F}}^{\mu_0}$.*

We also hope that our compactification says something on the moduli spaces of polarized $K3$-surfaces. Finally, in view of the existence of Einstein-Hermitian metrics for stable bundles, an idea as in Section 1 is also applicable to compactifying the moduli space of stable vector bundles over a nonsingular projective algebraic variety.

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