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Poincaré Bundle and Chern Classes

Mitsuhiro Itoh

§1. Theorems

Let (X, g) be a compact Kähler surface and P an $SU(2)$ bundle over X of index $k = c_2(P \times_\rho \mathbb{C}^2)$. We denote by $M_k = M_{k,X}$ the set of all gauge equivalence classes of anti-self-dual connections on P.

It is known that (i) the moduli space M_k is a Kähler manifold (possibly with singularities) and the dimension of the non-singular part of M_k is from the Atiyah-Singer index theorem $4k - 3(1 - q(X) + p_g(X))$ ([12],[14]) and (ii) in particular when (X, g) is Ricci flat Kähler, i.e., hyperkähler, the non-singular part $\widehat{M_k}$ is also hyperkähler ([13]).

So, we have

Theorem 1. *The first Chern class $c_1(\widehat{M_k})$ vanishes provided X is hyperkähler.*

This fact was shown also by S. Kobayashi by using complex symplectic geometry ([17]). See also [19].

This theorem shows that the moduli space of anti-self-dual connections inherits the Ricci flatness from the base manifold.

With respect to this, one can raise the following problem: does the moduli space M_k inherit the positivity (or negativity) of the first Chern class from the base manifold ?

For this problem we can say the following. On M_k a Kähler metric is defined naturally by means of the L_2 inner product over X. The curvature and hence the Ricci tensor of this metric are expressed in terms of integration over X by using the Green operator for an elliptic operator ([14]).

However, the Ricci form and hence $c_1(\widehat{M_k})$ can not in general be computed in a straightforward way.

In this paper we will discuss the positivity (resp., negativity) of the first Chern class $c_1(\widehat{M_k})$ by observing that it is the first Chern class of

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the holomorphic tangent bundle $T\widehat{M}_k$ and this bundle can be regarded as an index bundle arising from the Dolbeault operator coupling to all anti-self-dual connections on P.

Each anti-self-dual connection A on the bundle P gives rise to an elliptic complex

$$0 \to \Omega^0(\mathrm{ad}P) \to \Omega^1(\mathrm{ad}P) \to \Omega_+(\mathrm{ad}P) \to 0,$$

($\mathrm{ad}P$ is the adjoint bundle of P, Ω^k denotes the space of k-forms and Ω_+ is the space of self-dual 2-forms).

We call anti-self-dual connection A generic if the associated cohomology groups H^0 and H^2 vanish.

Then the moduli space $\widehat{M}_k$ of generic anti-self-dual connections is non-singular.

The fibre of $T\widehat{M}_k$ at $[A] \in \widehat{M}_k$ can be identified with $\mathrm{Ker}D_A$ of the Dolbeault operator coupled to the connection A ([12]);

$$D_A = (\overline{\partial}_A^*, \overline{\partial}_A) : \Omega^{0,1}(\mathrm{ad}P^{\mathbb{C}}) \to \Omega^{0,0}(\mathrm{ad}P^{\mathbb{C}}) \oplus \Omega^{0,2}(\mathrm{ad}P^{\mathbb{C}}).$$

Since D_A is gauge equivariant, the set of all formal differences $[\mathrm{Ker}D_A] - [\mathrm{Coker}D_A]$ for all connections A on P defines over the space of gauge equivalence classes of connections $B_{k,X}$ a virtual vector bundle, which we call index bundle $\mathrm{Ind}D$.

We notice that the index bundle becomes a proper vector bundle when restricted to the moduli space $\widehat{M}_k$.

The virtual rank of $\mathrm{Ind}D$, $\dim\mathrm{Ker}D_A - \dim\mathrm{Coker}D_A$, is given by $\int_X \mathcal{T}_X \wedge \mathrm{ch}(\mathrm{ad}P)$, where $\mathcal{T}_X$ is the total Todd class and $\mathrm{ch}(\mathrm{ad}P)$ denotes the Chern character of $\mathrm{ad}P$.

The index theorem for elliptic operators is generalized by Atiyah and Singer ([2]) as that for a family of elliptic operators $D_t, t \in T$. The formal differences $[\mathrm{Ker}D_t] - [\mathrm{Coker}D_t]$, $t \in T$ define over the parameter space T a virtual vector bundle.

Atiyah-Singer index theorem for a family gives an expression of its Chern character ($\in H^*(T)$).

For our index bundles over $B_{k,X}$ the following Chern character formula is basic ([3]):

$$\mathrm{ch}(\mathrm{Ind}D) = \int_X \mathcal{T}_X \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P}).$$

Here $\mathbb{P}$ is the $PU(2)$ principal bundle over $X \times B_{k,X}$, which we call the Poincaré bundle and $\mathrm{ch}(\mathrm{ad}\mathbb{P})$ implies a total differential form ($\in \Omega^*(X \times B_{k,X})$) representing the Chern character.

The Poincaré bundle $\mathbb{P}$ admits a natural connection $\mathbb{A}$ with curvature $\mathbb{F}$ so that the total differential form defined by $\mathrm{Tr}(\exp(\frac{i}{2\pi}\mathrm{ad}\mathbb{F}))$ represents the Chern character $\mathrm{ch}(\mathrm{ad}\mathbb{P})$ ([3],[15]).

In what follows, we identify Chern classes and forms representing them. We denote by $\mathrm{ch}(\mathrm{ad}\mathbb{P})_k$ the $2k$-component of $\mathrm{ch}(\mathrm{ad}\mathbb{P})$. So, $\mathrm{ch}(\mathrm{ad}\mathbb{P})_0 = 3$, $\mathrm{ch}(\mathrm{ad}\mathbb{P})_1 = 0$ and $\mathrm{ch}(\mathrm{ad}\mathbb{P})_2 = (-1/8\pi^2)\ \mathrm{Tr}\ \mathrm{ad}\mathbb{F}\ \mathrm{ad}\mathbb{F}$, for example.

Then, $c_1(T\widehat{M_k})$ is written by

$$c_1(T\widehat{M_k}) = \frac{1}{2}\int_X c_1(X) \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P})_2 + \int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3.$$

We have the following theorem for compact Kähler surfaces of definite first Chern class.

Theorem 2. *Let X be a compact Kähler surface. Assume that $c_1(X) > 0$ or < 0 (we choose a Kähler metric g with Kähler form ω_X so that $c_1(X) = [\epsilon_X \omega_X]$, $\epsilon_X = \pm 1$ corresponding to the sign of $c_1(X)$). Let P be an $SU(2)$ bundle over X of $c_2 = k$. Then, the first Chern class of the moduli space $\widehat{M_k}$ of generic anti-self-dual connections on P is given by*

$$c_1(\widehat{M_k}) = \frac{1}{4\pi^2}\epsilon_X\ \omega_M + \int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3,$$

where ω_M denotes the naturally defined Kähler metric on $\widehat{M_k}$.

It is not easy to calculate the term $\int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3$ over the entire space $\widehat{M_k}$. However, its estimation can be made at an end $M_0 \times S^k(X)$, the moduli space of ideal anti-self-dual connections, by using Donaldson's compactification ([7]).

In fact, if we let $[A]$ tend to the end, then $\int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3 \to 0$. Therefore, we get the following inheritance theorem with respect to the positivity (or the negativity) of the first Chern class.

Theorem 3. *Let X be a compact Kähler surface with $c_1(X) = [\epsilon_X \omega_X]$, where $\epsilon_X = \pm 1$. Then, the first Chern class of the moduli space $\widehat{M_k}$, $c_1(\widehat{M_k})$ tends to $1/4\pi^2\ \epsilon_X\ \omega_M$ if $[A]$ goes to $M_0 \times S^k(X)$ ($S^k(X)$ denotes the k-fold symmetric product of X).*

Remarks. (1) The 2-form $\int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3$ is closed and of type (1,1). It is moreover an exact form in the case of hyperkähler surfaces X.

(2) Estimation of $c_1(\widehat{M_k})$ at other ends, for example, at $M_{k-1} \times X$ is also available.

(3) We can identify over a nonsingular algebraic surface X the moduli space of anti-self-dual connections and the moduli of stable holomorphic vector bundles which are topologically isomorphic with $P \times_\rho \mathbb{C}^2$. Over $X = P^2(\mathbb{C})$ the moduli is asserted to be rational ([4]). Since $c_1 > 0$, Theorem 3 can be regarded as a differential-geometrical approach to this assertion.

Finally we should make additional remarks. One is on the moduli space of Einstein-Hermitian bundles over a Riemann surface. Another is on the moduli space $M_{k,X}$ over a Hodge surface.

Let E be a holomorphic vector bundle over a Riemann surface Σ. We denote by M_{EH} the moduli space of Einstein-Hermitian fibre metrics on E. Since each Einstein-Hermitian fibre metric exactly induces an Einstein-Hermitian connection on the associated $U(n)$ bundle, M_{EH} parametrizes gauge equivalence classes of Einstein-Hermitian connections on the $U(n)$ bundle (see [13], [16] for the definition of Einstein-Hermitian connection).

If we denote by $M_{EH,0}$ the subspace $\{[A] \in M_{EH}; \mathrm{Tr}F(A) = \omega\}$ for a fixed harmonic (1,1)-form ω, then it gives the fibre of the natural projection : $M_{EH} \to \mathrm{Jac}(\Sigma)$, the Jacobian variety of Σ.

We can here apply the Chern character formula for index bundles and then get the following positivity theorem.

Theorem 4. *The first Chern class $c_1(M_{EH,0})$ is positive.*

We already know that the moduli space $M_{EH,0}$ of connections of vanishing traceless curvature carries a Kähler metric of non-negative scalar curvature ([14]). So, the above theorem is consistent with this result. As we will see at Theorem 5.1 in §5, the Chern class $c_1(M_{EH,0})$ is in fact represented by $c\,\omega_M$ for a constant $c > 0$ and the Kähler form ω_M.

For the moduli space over a Hodge surface we obtain also the following positivity theorem

Theorem 5. *Let X be a Hodge surface and $\widehat{M}_{k,X}$ the moduli space of anti-self-dual connections of $c_2 = k$. Then, there exists a holomorphic line bundle $\mathbb{L}$ over $\widehat{M}_{k,X}$ whose first Chern class is $1/4\pi^2\ \omega_M$.*

The line bundle $\mathbb{L}$ in this theorem is associated with the determinant bundle of a suitable index bundle over $\widehat{M}_k$ induced by the holomorphic line bundle L on X defining the Kähler class $[\omega_X]$.

This theorem gives a complete answer to the observation on the positivity of the moduli space over a Hodge surface, conjectured by the author ([15]). An approach to this positivity from algebro-geometrical argument is given by Donaldson ([9]).

In the subsequent sections we will prove briefly the above theorems, which seem to be related essentially with physical anomalies.

§2. The Poincaré bundle

Let P be an $SU(2)$-bundle over a compact Kähler surface X. Associated with P the Poincaré bundle $\mathbb{P}$ is defined over the product space $X \times B_{k,X}$. The bundle $\mathbb{P}$ parametrizes gauge equivalently P with connections A where A runs over the set $\mathcal{A}(P)$ of all irreducible connections on P. In fact, the group $\mathcal{G}(P)$ of gauge transformations of P acts freely on the product $P \times \mathcal{A}(P)$ as

$$(u, A) \longrightarrow (gu, g(A)),$$

where

$$g(A) = gAg^{-1} + dgg^{-1}.$$

This action commutes with the right translation of P. So, it is easily seen that the quotient $P \times \mathcal{A}(P)/\mathcal{G}(P)$ has a fibration over $X \times B_{k,X} = SU(2)\backslash P \times \mathcal{A}(P)/\mathcal{G}(P)$ with fibre $PU(2)$. This fibration is the Poincaré bundle associated with the bundle P.

Note that local trivializing neighborhoods U of P and slices S of $B_{k,X}$ at any connection A give local trivializing neighborhoods of the bundle $\mathbb{P}$.

The Poincaré bundle admits a connection $\mathbb{A}$ in a natural way ([3], [15]).

The connection $\mathbb{A}$ is defined as A when restricted to $X \times [A]$ and as $\mathrm{ev}_x(\omega)$ over $\{x\} \times B_{k,X}$, here ev_x is the evaluation map at x; $\Omega^0(\mathrm{ad}P) \to (\mathrm{ad}P)_x$ and ω is the $\Omega^0(\mathrm{ad}P)$-valued 1-form over each slice S given by $\omega(\alpha) = G_A(d_A^*\alpha)$, $\alpha \in T_AS$, and $(\mathrm{ad}P)_x$ is identified with $\mathfrak{su}(2)$ through a trivialization over U around x.

Proposition 2.1 ([15]). *The curvature $\mathbb{F} = \mathbb{F}(\mathbb{A})$ of $\mathbb{A}$ is written with respect to the product space decomposition in the following form;*

$$\mathbb{F} = \mathbb{F}^{2,0} + \mathbb{F}^{1,1} + \mathbb{F}^{0,2},$$

where $\mathbb{F}^{2,0} = F(A)$, the curvature of A, $\mathbb{F}^{1,1}(u, \alpha) = -\alpha(u)$ for $(u, \alpha) \in T_{(x,[A])}(X \times B_{k,X})$ and $\mathbb{F}^{0,2}(.,.) = -2\mathrm{ev}_x(G_A\{.,.\})$.

Then, the adjoint bundle $\mathrm{ad}\mathbb{P}$ of the Poincaré bundle has the induced connection $\nabla^{\mathbb{A}}$ whose curvature is $\mathrm{ad}\mathbb{F}$.

Remark. The connection $\mathbb{A}$ on $\mathbb{P}$ is natural in the sense that its curvature is type (1,1) when $\mathbb{P}$ is restricted to the product Kähler manifold $X \times M_{k,X}$. Then, any complex vector bundle associated with $\mathbb{P}$ carries a holomorphic structure induced by the connection (see Proposition 5.2, [15]).

§3. The Chern character formula

The index formula for the family of Dolbeault operator $D = (\overline{\partial}^*, \overline{\partial})$ coupled to anti-self-dual connections A on P;

$$D_A = (\overline{\partial}_A^*, \overline{\partial}_A) : \Omega^{0,1}(\mathrm{ad}P^{\mathbb{C}}) \to \Omega^{0,0}(\mathrm{ad}P^{\mathbb{C}}) \oplus \Omega^{0,2}(\mathrm{ad}P^{\mathbb{C}})$$

is stated in the form of the Chern character for the index bundle as

$$\mathrm{ch}(\mathrm{Ind}D) = \int_X \mathcal{T}_X \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P}).$$

So, since $\mathcal{T}_X = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1^2 + c_2)(X)$, $c_1(\mathrm{Ind}D) = \mathrm{ch}_1(\mathrm{Ind}D)$ is given by

$$c_1(\mathrm{Ind}D) = \frac{1}{2}\int_X c_1(X) \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P})_2 + \int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3.$$

Since $\mathrm{ch}(\mathrm{ad}\mathbb{P})_2 = -1/8\pi^2$ Tr $\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F}$ and $\mathbb{F}$ is decomposed as in Proposition 2.1, only $\mathrm{Tr}(\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F})^{2,2}$ together with $c_1(X)$ is valid for the first integration term, where $(\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F})^{2,2}$ is the (2,2)-component of $\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F}$.

We have Tr $\mathrm{ad}Y\mathrm{ad}Z = 4\mathrm{Tr}\ YZ, Y, Z \in \mathfrak{su}(2)$. So,

$$\mathrm{Tr}(\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F})^{2,2} = 4(2\mathrm{Tr}\ \mathbb{F}^{2,0} \wedge \mathbb{F}^{0,2} + \mathrm{Tr}\ \mathbb{F}^{1,1} \wedge \mathbb{F}^{1,1}).$$

At $(x, [A]) \in X \times M_k$, Tr $\mathbb{F}^{2,0} \wedge \mathbb{F}^{0,2} = \mathrm{Tr}F(A) \wedge \mathbb{F}^{0,2}$ is anti-self-dual 2-form in the X-direction.

Assume that $c_1(X)$ of X is positive (or negative), and hence X has a Kähler form ω_X with $[\epsilon_X \omega_X] = c_1(X)$ (ϵ_X is the sign of $c_1(X)$). Then $c_1(X) \wedge \mathrm{Tr}\ \mathbb{F}^{2,0} \wedge \mathbb{F}^{0,2}$ vanishes at every point of X so that $\int_X c_1(X) \wedge \mathrm{Tr}\ \mathrm{ad}\mathbb{F}^{2,0} \wedge \mathrm{ad}\mathbb{F}^{0,2} = 0$.

For the second term we have the following

Lemma.

$$\int_X \omega_X \wedge \mathrm{Tr}\ \mathbb{F}^{1,1} \wedge \mathbb{F}^{1,1} = -\frac{1}{2}\omega_M.$$

This is derived from the expression of $\mathbb{F}^{1,1}$ and straightforward computation of the integrand. Therefore we have as Theorem 2 in §1 over the moduli space M_k

$$c_1(\widehat{M_k}) = \frac{1}{4\pi^2}\epsilon_X \omega_M + \int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3.$$

§4. The compactification of the moduli space

In order to estimate the remainder term $\int_X \mathrm{ch}(\mathrm{ad}\mathbb{P})_3$ we must study ends of the moduli space M_k giving rise to its compactification.

A tuple $(A; x_1, ..., x_\ell)$ is called an ideal anti-self-dual connection of index k when A is an anti-self-dual connection of index $k-\ell$ and $x_1, ..., x_\ell$ are points of X, not necessarily distinct.

The curvature density of $(A; x_1, ..., x_\ell)$ is defined by

$$|F(A)|^2 + 8\pi^2 \sum_j \delta(x_j)$$

so that its action integral is $8\pi^2(k-\ell) + 8\pi^2\ell = 8\pi^2 k$.

Denote by M_k^{id} the set of gauge equivalence classes of ideal anti-self-dual connections of index k. Then,

$$M_k^{\mathrm{id}} = M_k \sqcup (M_{k-1} \times X) \sqcup ...$$

$$\sqcup (M_1 \times S^{k-1}(X)) \sqcup (M_0 \times S^k(X)),$$

$S^j(X)$ denoting the j-fold symmetric product of X.

Relative to the naturally defined topology the closure of M_k in M_k^{id} is compact. Indeed, the compactness theorem due to Uhlenbeck states that if a sequence $\{[A_i]\}$ in M_k is not convergent, then

$$|F(A_i)|^2 \to |F(A_0)|^2 + 8\pi^2 \sum_{j=1}^{\ell} \delta(x_j)(i \to \infty)$$

for an anti-self-dual connection A_0 of index $k-\ell$ and points $x_1, ..., x_\ell$ in X so that $[A_i]$ goes to a point $([A_0]; x_1, ..., x_\ell)$ of M_k^{id} ([10]).

Now we let $[A] \in \widehat{M_k}$ be close to $M_0 \times S^k(X)$. Then the curvature density $|F(A)|^2 = -\mathrm{Tr}\ F \wedge F$ concentrates like delta functions at some points $x_1, ..., x_k$ of X. Here there are certain number of constraints on local scales, that is, degrees of curvature concentrations $\lambda_1, ..., \lambda_k$ ([7]).

We may assume $x_i \neq x_j$, $i \neq j$ without loss of generality.

The terms Tr $\mathbb{F}^{2,0} \wedge \mathbb{F}^{2,0} \wedge \mathbb{F}^{0,2}$ and Tr $\mathbb{F}^{2,0} \wedge \mathbb{F}^{1,1} \wedge \mathbb{F}^{1,1}$ appear as the $(4,2)$-component of the character $\mathrm{ch}(\mathrm{ad}\mathbb{P})_3$.

Each basic anti-instanton $I = I_{(0,\lambda)}$ with center $0 \in \mathbb{R}^4$ and scale λ satisfies the curvature identity

$$F_I \wedge F_I = \frac{24\lambda^2}{(\lambda^2 + |x|^2)^4} \,\mathrm{dvol} \otimes \mathrm{Id}_{su(2)},$$

(dvol denotes the standard volume element of the 4-space $\mathbb{R}^4$).

The Green operator G_A appeared in the curvature term $\mathbb{F}^{0,2}$ has an expression of integrated form with respect to a certain Green kernel ([18]).

For the tangent space $T_{[A]}\widehat{M}_k$ at $[A]$ which is close to $M_0 \times S^k(X)$ there exists a subspace $T^a_{[A]}$ in $\Omega^1(\mathrm{ad}P)$ approximating $T_{[A]}\widehat{M}_k$ in a suitable way ([11]).

Using these facts and applying also a convergence theorem on Schwartz hyperfunctions (see Theorem 13, Chapter II, [20]), we see

$$\int_X \mathrm{Tr}\ \mathbb{F}^{2,0} \wedge \mathbb{F}^{2,0} \wedge \mathbb{F}^{0,2} \to 0,$$

as $[A]$ goes to $M_0 \times S^k(X)$.

Similarly,

$$\int_X \mathrm{Tr}\ \mathbb{F}^{2,0} \wedge \mathbb{F}^{1,1} \wedge \mathbb{F}^{1,1} \to 0.$$

So, the proof of Theorem 3 is completed.

§5. Einstein-Hermitian bundles and Riemann surfaces

Let P be a $U(n)$ bundle over a compact Riemann surface Σ. A connection A on P is called Einstein-Hermitian if its curvature $F(A)$ equals $\lambda\, \mathrm{Id}_E \otimes \omega_\Sigma$ for a constant λ, where Id_E is the identity endomorphism of the associated vector bundle $E = P \times_\rho \mathbb{C}^n$ and ω_Σ is the volume form of Σ.

The moduli space $\widehat{M}_{EH}$ of irreducible Einstein-Hermitian connections on P is identified with the moduli $M(n,k)$ of stable holomorphic vector bundles of degree $k = c_1(E)$ which are topologically isomorphic to the bundle E.

Assigning the trace component to each A induces naturally a fibration of $\widehat{M}_{EH}$ over the Jacobian variety $\mathrm{Jac}(\Sigma)$ of Σ whose fibre is $\widehat{M}_{EH,0}$, the subset $\{[A] \in \widehat{M}_{EH}; \mathrm{Tr}\ F(A) = n\ \lambda\ \omega_\Sigma\}$. This corresponds exactly to the map : $M(n,k) \to J_k$ by taking determinants, where J_k is the Jacobian variety of Σ, which parametrizes holomorphic line bundles of degree k (see Section 9, [1]).

From the Chern character formula we get similarly as in the anti-self-dual moduli space case

$$\mathrm{ch}(\widehat{M}_{EH,0}) = \int_\Sigma \mathcal{T}_\Sigma \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P}),$$

where $\mathbb{P}$ is the Poincaré bundle defined over $\Sigma \times \widehat{M}_{EH,0}$ with structure group $PU(n)$.

The first Chern class $c_1(\widehat{M}_{EH,0})$ is then $\int_\Sigma \mathrm{ch}(\mathrm{ad}\mathbb{P})_2$ and $\mathrm{ch}(\mathrm{ad}\mathbb{P})_2$ is represented by $-1/8\pi^2$ Tr $\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F}$. The $(2,2)$-component of Tr $\mathrm{ad}\mathbb{F} \wedge \mathrm{ad}\mathbb{F}$ is $\mathrm{Tr}(2\ \mathrm{ad}\mathbb{F}^{2,0} \wedge \mathrm{ad}\mathbb{F}^{0,2} + \mathrm{ad}\mathbb{F}^{1,1} \wedge \mathrm{ad}\mathbb{F}^{1,1})$.

Since $\mathbb{F}^{2,0}$, the traceless part of $F(A)$, vanishes at all $[A]$ in $\widehat{M}_{EH,0}$, the first Chern class of the moduli space reduces to $-c(n)/8\pi^2 \int_\Sigma \mathrm{Tr}\ \mathbb{F}^{1,1} \wedge \mathbb{F}^{1,1}$. Here the constant $c(n) > 0$ is given by $\mathrm{Tr}\ \mathrm{ad}Y \mathrm{ad}Z = c(n)\mathrm{Tr}\ YZ, Y, Z \in \mathfrak{su}(n)$.

Therefore we have by using the formula for $\mathbb{F}^{1,1}$

Theorem 5.1. *The first Chern class of $\widehat{M}_{EH,0}$ is positive. In fact, $c_1(\widehat{M}_{EH,0}) = c(n)/16\pi^2\ \omega_M$.*

Remark. If $(n,k) = 1$, then $\widehat{M}_{EH,0}$ turns out to be compact and the second Betti number is one ([1]). So, from the fact that the scalar curvature is non-negative but not identically zero, the first Chern class must be positive. However, the above theorem asserts the positivity of the Chern class even in cases of $(n,k) \neq 1$.

§6. Hodge structure and the determinant bundle

Finally we assume that (X,g) is a Hodge surface. So, the Kähler form ω_X represents the first Chern class of a holomorphic line bundle L and also defines on L a connection a whose curvature form coincides with ω_X.

Consider the following twisted Dolbeault operators coupling to not only connections on a bundle P but also the connection a;

$$D_{a,A}\colon \Omega^{0,1}(L \otimes \mathrm{ad}P^{\mathbb{C}}) \to \Omega^{0,0}(L \otimes \mathrm{ad}P^{\mathbb{C}}) \oplus \Omega^{0,2}(L \otimes \mathrm{ad}P^{\mathbb{C}}).$$

While the index bundle $\mathrm{Ind}D_a = \{[\mathrm{Ker}D_{a,A}] - [\mathrm{Coker}D_{a,A}]\}$ for operators $D_{a,A}$ is virtual, its determinant $\mathrm{detInd}D_a = \{(\bigwedge^{\max} \mathrm{Ker}D_{a,A}) \otimes$

$(\bigwedge^{\max} \mathrm{Coker} D_{a,A})^*\}$ defines a proper complex line bundle over the moduli space $\widehat{M}_k$ of generic anti-self-dual connections on P ([5], [8]).

To this determinant bundle we use the Chern character formula. Then we have

$$\mathrm{ch}(\mathrm{Ind} D_a) = \int_X \mathcal{T}_X \wedge \mathrm{ch}(L) \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P}),$$

from which

$$c_1(\mathrm{detInd} D_a) = c_1(\widehat{M}_k) + \int_X c_1(L) \wedge \mathrm{ch}(\mathrm{ad}\mathbb{P}).$$

Because $c_1(L)$ is represented by ω_X, the second term can be reduced in the same way as in §2 to $1/4\pi^2\ \omega_M$. So, the complex line bundle $\mathbb{L} = (\mathrm{detInd} D_a) \otimes (\bigwedge^{\max} T\widehat{M}_k)^*$ has positive first Chern class given by the Kähler form on $\widehat{M}_k$. Hence, $\mathbb{L}$ becomes a holomorphic line bundle. Thus we obtain Theorem 5.

Remark. The determinant bundle $\mathrm{detInd} D_a$ carries a holomorphic structure because of $\mathrm{detInd} D_a = \mathbb{L} \otimes (\bigwedge^{\max} T\widehat{M}_k)$. See [6] for an argument on holomorphic structure of determinant bundles.

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Institute of Mathematics
University of Tsukuba
Tsukuba, 305
Japan