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Behavior of the Zeta-Function of Open Surfaces at $s=1$

S. Lichtenbaum*

Dedicated to Professor Kenkichi Iwasawa

A major theme in the work of Iwasawa is the interplay between theorems and conjectures concerning zeta-functions in the number-field case with analogous theorems and conjectures in the function-field case. A particularly striking example of this was provided by Tate and M. Artin, who considered the function-field analogue of the conjecture of Birch and Swinnerton-Dyer, and largely showed that this conjecture was equivalent to a conjecture about the zeta-function of certain complete non-singular surfaces X over finite fields [T]. They also showed that this conjecture ($Z(X, 1)=\pm\chi(X, G_a)/\chi(X, G_m)$ in the notation of this paper) was true if and only if the Brauer group $H^2(X, G_m)$ of X was finite (which it may always be, as far as we know).

However, the number-theoretic case in someways resembles more closely the case where the surface, although still non-singular, is no longer complete. In this paper, we consider an open subset U of X obtained by removing a curve C, and show that the analogous conjecture ($Z(U, 1)=\pm\chi(X, G_a^U)/\chi(X, G_m^U)$) remains true, if the Brauer group of X is finite.

The reader should be cautioned that, because of a 2-torsion defect in [L2], all theorems are only valid up to 2-torsion groups or powers of 2, as the case may be.

§ 1. Definition and properties of Euler characteristics

We begin with some algebraic preliminaries. Let $\mathscr{P}$ be the abelian category whose objects are given by triples consisting of two finitely-generated abelian groups A, A' of the same rank and a non-degenerate bilinear map $\langle\ ,\ \rangle_A$: $A\times A'\to\boldsymbol{Q}$. A morphism from $(A, A', \langle\ ,\ \rangle_A)$ to $(B, B', \langle\ ,\ \rangle_B)$ is a pair of morphisms α: $A\to B$ and β: $B'\to A'$ such that $\langle\alpha(a), b'\rangle_B=\langle a, \beta(b')\rangle_A$ for all $a\in A, b'\in B'$.

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Definition 1.1. A *pairing* is an object of $\mathscr{P}$. Let $\mathscr{A}=\{A, A', \langle\ ,\ \rangle_A\}$ be a pairing. Let $a_1, \cdots, a_n$ be a basis of A modulo torsion, and $a'_1, \cdots, a'_n$ be a basis of A' modulo torsion.

Definition 1.2. The *regulator* $R(\mathscr{A})$ of $\mathscr{A}=|\det|\langle a_i, a'_j\rangle||$. Clearly $R(\mathscr{A})$ does not depend on the choice of bases for A and A'.

Definition 1.3. The *Euler characteristic* $\chi(\mathscr{A})$ of $\mathscr{A}=\dfrac{(\# A_{\text{tor}})(\# A'_{\text{tor}})}{R(\mathscr{A})}$.

Definition 1.4. A sequence of pairings $\mathscr{A}_1\to\cdots\to\mathscr{A}_n$ is *exact* if the two induced sequences $A_1\to\cdots\to A_n$ and $A'_n\to\cdots\to A'_1$ are exact sequences of abelian groups.

Lemma 1.5. *Let $0\to\mathscr{A}_1\to\cdots\to\mathscr{A}_n\to 0$ be an exact sequence of pairings. Then $\prod_{i=1}^{n}\chi(\mathscr{A}_i)^{(-1)^i}=1$.*

Proof. We may assume as usual that $n=3$. Let the exact sequence of pairings be $0\to\mathscr{A}\to\mathscr{B}\to\mathscr{C}\to 0$ and the two associated exact sequences of abelian groups be $0\to A\to B\to C\to 0$ and $0\to C'\to B'\to A'\to 0$. We first observe that if all six groups are torsion-free the lemma follows immediately from standard facts about determinants.

In the general case, if D is any abelian group, let D_0 be D/D_{tor}. Let A_1 be the kernel of the surjective map $B_0\to C_0$, and let C'_1 be the kernel of the surjective map $B'_0\to A'_0$.

We claim there are natural exact sequences

$$0\longrightarrow A_{\text{tor}}\longrightarrow B_{\text{tor}}\longrightarrow C_{\text{tor}}\longrightarrow A_1/A_0\longrightarrow 0$$

and

$$0\longrightarrow C'_{\text{tor}}\longrightarrow B'_{\text{tor}}\longrightarrow A'_{\text{tor}}\longrightarrow C'_1/C'_0\longrightarrow 0.$$

The snake lemma applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A\otimes Q & \longrightarrow & B\otimes Q & \longrightarrow & C\otimes Q & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & A_2 & \xrightarrow{\ \psi\ } & B_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

yields $0\to A_{\text{tor}}\to B_{\text{tor}}\to C_{\text{tor}}\to\operatorname{Ker}\psi\to 0$. (Here the exact sequence serves to define A_2 and B_2).

Now apply the snake lemma again to

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A\otimes \boldsymbol{Q} & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \psi & & \\
 & & B_0 & \longrightarrow & B\otimes \boldsymbol{Q} & \longrightarrow & B_2 & & \\
 & & & & \downarrow & & & & \\
 & & & & C\otimes \boldsymbol{Q} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

to obtain

$$0 \longrightarrow \operatorname{Ker} \psi \longrightarrow B_0/A_0 \longrightarrow C\otimes \boldsymbol{Q}.$$

But the image of B_0/A_0 in $C\otimes \boldsymbol{Q}$ is clearly $C_0 = B_0/A_1$. Hence $\operatorname{Ker}\psi$ is isomorphic to A_1/A_0, and the proof of the claim is completed.

Let $\mathscr{A}_0$ be the pairing $(A_0, A_0', \langle\ ,\ \rangle_A)$ and similarly for $\mathscr{B}_0$ and $\mathscr{C}_0$. Let $\mathscr{A}_1$ be the pairing $(A_1, A_0', \langle\ ,\ \rangle_A)$ and $\mathscr{C}_1$ be the pairing $(C_0, C_1', \langle\ ,\ \rangle_C)$. Now we have the following list of identities:

(1) $\chi(\mathscr{B}_0) = \chi(\mathscr{C}_1)\chi(\mathscr{A}_1)$ (by the torsion-free case),

(2) $\chi(\mathscr{B}_0) = R(\mathscr{B}_0)^{-1} = R(\mathscr{B})^{-1} = \chi(\mathscr{B})(\# B_{\mathrm{tor}})^{-1}(\# B'_{\mathrm{tor}})^{-1}$,

(3) $\chi(\mathscr{C}_1) = \chi(\mathscr{C}_0)(\#(C_1'/C_0')) = \chi(\mathscr{C})(\# C_{\mathrm{tor}})^{-1}(\# C'_{\mathrm{tor}})^{-1}(\#(C_1'/C_0'))$,

(4) $\chi(\mathscr{A}_1) = \chi(\mathscr{A})(\# A_{\mathrm{tor}})^{-1}(\# A'_{\mathrm{tor}})^{-1}(\#(A_1/A_0))$,

(5) $(\# B_{\mathrm{tor}})(\#(A_1/A_0)) = (\# A_{\mathrm{tor}})(\# C_{\mathrm{tor}})$,

(6) $(\# B'_{\mathrm{tor}})(\#(C_1'/C_0)) = (\# A'_{\mathrm{tor}})(\# C'_{\mathrm{tor}})$,

which clearly imply $\chi(\mathscr{B}) = \chi(\mathscr{A})\chi(\mathscr{C})$.

§ 2. Pairings, duality and zeta-functions of curves

Let C be a geometrically reduced curve proper over a field k. If x is a point of C, let j_x: $\operatorname{Spec} k(x) \rightarrow C$ be the natural map. Let $C^{(i)}$ denote the set of points of C of codimension i. Define the étale sheaf $\boldsymbol{Z}'$ on C to be $\oplus_{x\in C^{(0)}} (j_x)_* \boldsymbol{Z}$. Following Deninger [D] we define the complex of sheaves G_m' on C to be $\otimes_{x\in C^{(0)}} (j_x)_* G_m \rightarrow \oplus_{x\in C^{(1)}} (j_x)_* \boldsymbol{Z}$. (Note that if C is regular, $\boldsymbol{Z}'$ is isomorphic to $\boldsymbol{Z}$ and G_m' is quasi-isomorphic to G_m, but not in general).

There is a natural map $\boldsymbol{Z}' \oplus G_m \rightarrow G_m'$ given by $\{n_x\} \oplus u \mapsto \{u_x^{n_x}\}$, where the notation is clear. This induces bilinear maps $H^0(C, \boldsymbol{Z}') \times H^1(C, G_m) \rightarrow H^1(C, G_m')$ and $H^0(C, \boldsymbol{Z}') \times H^3(C, G_m) \rightarrow H^3(C, G_m')$, which we wish to study.

We note first that we have the exact sequence

$$\text{(1.c)} \quad \bigoplus_{x\in C^{(0)}} H^0(C, (j_x)_*G_m) \longrightarrow \bigoplus_{x\in C^{(1)}} H^0(C, (j_x)_*Z) \longrightarrow H^1(C, G'_m) \longrightarrow 0,$$

since $H^1(C, (j_x)_*G_m)=0$ by Hilbert Theorem 90. There is the usual degree map from $\oplus_{x\in C^{(1)}} H^0(C, (j_x)_*Z)\to Z$ which factors through $H^1(C, G'_m)$ since the divisor of a function has degree zero, even on singular curves. This degree map then induces a bilinear map $\langle\ ,\ \rangle_C$ from $H^0(C, Z')\times H^1(C, G_m)$ to Z.

Now let k be a finite field.

Proposition 2.1. a) $H^2(C, G_m)=0$.

b) *$H^3(C, G_m)$ is naturally isomorphic to $(Q/Z)^r$, where r is the number of irreducible components of C.*

c) *$H^0(C, Z')$ and $H^1(C, G_m)$ are finitely-generated groups of rank r and the bilinear map $\langle\ ,\ \rangle_C$ is non-degenerate if taken mod torsion, with regulator R_C equal to* 1.

d) *There is a natural trace map from $H^3(C, G'_m)$ to Q/Z, and the induced bilinear map from $H^0(C, Z')\times H^3(C, G_m)$ to Q/Z identifies $H^3(C, G_m)$ with the Q/Z-dual of $H^0(C, Z')$.*

Proof. Let $\bar{C}$ be the normalization of C in its total ring of quotients, and let π be the natural map from $\bar{C}$ to C. Then we have the exact sequence of étale sheaves on C

$$0\longrightarrow G_m \longrightarrow \pi_* G_{m,\bar{C}} \longrightarrow Q_m \longrightarrow 0,$$

where this sequence serves to define Q_m. Since π is an isomorphism outside of the singular set S_C of C, Q_m is a punctual sheaf. Since $H^1(G_m)$ is locally trivial for the Zariski topology, the map from Zariski stalks of $\pi_*G_{m,\bar{C}}$ to Zariski stalks of Q_m is surjective, which lets us identify $H^0(C, Q_m)$ as $\oplus_{P\in S_C} \bar{O}_P^*/\bar{O}_P^*$. (Recall that since π is finite, π_* is exact, so $H^1(C, \pi_*G_{m,\bar{C}})$ may be identified with $H^i(\bar{C}, G_m)$. Also, $\bar{O}_P$ is the integral closure of O_P in its total rings of quotients.)

Next we claim that $H^1(C, Q_m)=0$ for $i\geq 1$. It is shown in [Se] when C is irreducible and in [O] in general that Q_m is represented by a connected commutative algebraic group over the finite field k, and hence $H^1(k, Q_m)=0$ by Lang's theorem and $H^i(k, Q_m)=0$ for $i>1$ because $cd(k)=1$.

Since $H^2(\bar{C}, G_m)$ is well-known to be zero ($\bar{C}$ is the disjoint union of complete non-singular connected curves), it now follows that $H^2(C, G_m)=0$. Since $H^3(C', G_m)=Q/Z$ for complete non-singular C' (loc. cit.), we also get *b*). Since $\bar{O}_P^*/O_P^*$ is finite, we see that $H^1(C, G_m)$ has rank r. It is immediate that $H^0(C, Z')$ also has rank r. Up to torsion, the pairing may be computed on $\bar{C}$, where it is clearly non-degenerate. Since we may easily construct a divisor on $\bar{C}$ with support concentrated on one component, of degree one there, and disjoint from $\pi^{-1}(S_C)$, the regulator is 1, which proves c).

Since $H^0(C, Z')$ may be identified with $H^0(\bar{C}, Z')=H^0(\bar{C}, Z)$ and $H^3(C, G_m)$ may be identified with $H^3(\bar{C}, G_m)$, d) follows from the standard duality theory on $\bar{C}$. (See [M1]).

Definition 2.2. The *Euler characteristic* $\chi(C, G_m)$ is equal to

$$\# H^0(C, G_m)(\# H^1(C, G_m)_{\mathrm{tor}})^{-1} \# H^2(C, G_m)(\# H^3(C, G_m)_{\mathrm{cot}})^{-1} \cdot R_C$$
$$= \text{(in view of Proposition 2.1) } \# H^0(C, G_m)/\# H^1(C, G_m)_{\mathrm{tor}}.$$

Definition 2.3.

$$\chi(C, G_a)=\# H^0(C, G_a)/\# H^1(C, G_a)=\# H^0_{\mathrm{zar}}(C, O_C)/\# H^1_{\mathrm{zar}}(C, O_C).$$

Definition 2.4. If P is a point,

$$\chi^*(P)=\# H^0(P, G_a)/\# H^0(P, G_m).$$

Theorem 2.5.

$$\frac{\chi(\bar{C}, G_a)}{\chi(\bar{C}, G_m)} \cdot \frac{\chi(C, G_m)}{\chi(C, G_a)} = \prod_{P \in S_C} \Big(\Big(\prod_{Q \to P} \chi^*(Q)\Big)/\chi^*(P)\Big).$$

Proof. We also have the exact sequence of étale sheaves on C

$$0 \longrightarrow G_a \longrightarrow \pi_* G_{a,\bar{C}} \longrightarrow Q_a \longrightarrow 0,$$

where this sequence defines Q_a, which is punctual as in the case of Q_m. Also as before we may identify $H^0(C, Q_a)$ as $\oplus_{P \in S_C} \bar{O}_P/O_P$. Since $H^i(C, G_a) = H^i_{\mathrm{zar}}(C, G_a)=0$ for $i \geq 2$, and $H^i(\bar{C}, G_a)=0$ we have $H^i(C, Q_a)=0$ for $i \geq 2$, and $H^1(C, G_a) \xrightarrow{\phi} H^1(\bar{C}, G_a) \to H^1(C, Q_a) \to 0$. Looking at this sequence in the Zariski topology shows that ϕ is surjective, hence $H^1(C, Q_a) = 0$ as well.

Now the theorem immediately follows from

$$\# \bar{O}_P/O_P = \# \bar{O}_P^*/O_P^* \cdot \prod_{Q \to P} \chi^*(Q)/\chi^*(P)$$

which we will now proceed to prove.

Let $\mathcal{O}=O_P$. Let $I=\{x \in \bar{\mathcal{O}}: x\bar{\mathcal{O}} \subseteq \mathcal{O}\}$ = conductor of $\mathcal{O}$. Let m be the maximal ideal of $\mathcal{O}$ and $\bar{m}=\ker(\bar{\mathcal{O}} \to \oplus_{Q \to P} k(Q))$. We claim first that,

$$\#((1+m)/(1+I))=\#(m/I)$$

and

$$\#((1+\bar{m})/(1+I))=\#(\bar{m}/I).$$

Proof of claim. Since $\bar{\mathcal{O}}$ is a finitely-generated $\mathcal{O}$-module contained in the total quotient ring of $\bar{\mathcal{O}}$, I contains a non-zero-divisor in $\mathcal{O}$. Hence $\mathcal{O}/I$ is zero-dimensional, hence finite, so $\exists n$: $m^n \subseteq I$.

Next, we have $\#(m/m^n)=\#((1+m)/(1+m^n))$. It suffices to show $\#(m^k/m^{k+1})=\#((1+m^k)/(1+m^{k+1}))$, for all $k \geq 1$. But the map of $1+m^k$ to m^k/m^{k+1} given by $(1+x) \mapsto$ class (x) is clearly a surjective homomorphism with kernel $1+m^{k+1}$.

Now, $\#((I \cap m^k)/(I \cap m^{k+1}))=\#((1+I \cap m^k)/(1+I \cap m^{k+1}))$ by the same argument, hence as before,

$$\#((I \cap m)/(I \cap m^n))=\#((1+I \cap m)/(1+I \cap m^n)),$$

or

$$\#(I/m^n)=\#((1+I)/(1+m^n)).$$

But

$$\#(m/I)\#(I/m^n)=\#(m/m^n),$$

and

$$\#((1+m)/(1+I))\#((1+I)/(1+m^n))=\#((1+m)/(1+m^n)),$$

which imply:

$$\#(m/I)=\#((1+m)/(1+I)).$$

Similarly,

$$\#(\bar{m}/I)=\#((1+\bar{m})/(1+I)),$$

hence $\#(\bar{m}/m)=\#((1+\bar{m})/(1+m))$, and the proof of the claim is completed. Finally, $\#(\bar{O}_P/O_P)\cdot\#(\bar{O}_P^*/O_P^*)^{-1}=\#(\bar{O}_P/\bar{m}_P)\cdot\#(O_P/m_P)^{-1}\cdot\#(\bar{O}_P^*/(1+\bar{m}_P))^{-1}\cdot\#(O_P^*/(1+m_P))\cdot\#(\bar{m}_P/m_P)\cdot\#((1+\bar{m}_P)/(1+m_P))^{-1}$. But $\bar{O}_P/\bar{m}_P=\oplus_{Q\to P}k(Q)$, $\bar{O}_P^*/(1+\bar{m}_P)=\oplus_{Q\to P}k(Q)^*$, $O_P/m_P=k(P)$ and $O_P^*/m_P^*=k(P)^*$. So

$$\#(\bar{O}_P/O_P)\cdot\#(\bar{O}_P^*/O_P^*)^{-1}=\Big(\prod_{Q\to P}\chi^*(Q)\Big)/\chi^*(P),$$

which was what we claimed, and we have finished the proof of Theorem 2.5.

Let q be the order of k and W a scheme of finite type over k. Recall that the zeta-function $\zeta(W,s)$ is a rational function $Z(W,t)$ of $t=q^{-s}$.

Assume $Z(W, t)$ has a pole of order r_W at $t=q^{-1}$.

Definition 2.6. $Z(W, 1)=\lim_{t\to q^{-1}}(1-qt)^r Z(W, t)$.
Recall that if $W=C$ as above, r_W is the number of irreducible components of C.

Corollary 2.7. $Z(C, 1)=\pm\chi(C, G_a)/\chi(C, G_m)$.

Proof. It is immediate that $Z(\bar{C}, 1)/Z(C, 1)=\prod_{P\in S_C}((\prod_{Q\to P}Z(Q, 1))/Z(P, 1)^{-1})$ and that $Z(Q, 1)=\chi^*(Q)$, $Z(P, 1)=\chi^*(P)$. Theorem 2.5 then reduces Corollary 2.7 to the case where $C=\bar{C}$, where it is classical.

Now let $\mathscr{H}_1^C$ be the pairing $\{H^0(C, Z'), H^1(C, G_m), \langle\ ,\ \rangle_C\}$ and let $\mathscr{H}_0^C$ be the pairing $\{0, H^0(C, G_m), 0\}$.

Proposition 2.8. $\chi(C, G_m)=\chi(\mathscr{H}_0^C)\chi(\mathscr{H}_1^C)^{-1}$.

Proof. This follows immediately from the definitions and Proposition 2.1.

§ 3. Construction of various regulators

We begin by reviewing the regulator pairings for complete non-singular surfaces.

In [L2], we have defined a sequence of complexes of étale sheaves $\Gamma(X, i)$ for $i=0, 1, 2$ on any regular noetherian scheme X, such that $\Gamma(X, 0)=\boldsymbol{Z}$, $\Gamma(X, 1)=G_m[-1]$, and $\Gamma(X, 2)$ is given by a two-term complex of sheaves in degrees 1 and 2. For the basic properties of these complexes, we refer the reader to [L2] and [L3].

Let X be a complete non-singular surface over a field k (X/k is proper smooth and geometrically connected). We have shown in [L2] and [L3] that there is a natural map ν in the derived category of étale sheaves on X from $\Gamma(X, 1)\otimes^L\Gamma(X, 1)$ to $\Gamma(X, 2)$. Taking hypercohomology, this map induces a pairing $H^2(X, \Gamma(1))\otimes H^2(X, \Gamma(1))\to H^4(X, \Gamma(2))$.

But $H^2(X, \Gamma(1))$ is by definition $\mathrm{Pic}(X)=H^1(X, G_m)$, and it is shown in [L3] that $H^4(X, \Gamma(2))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is $CH^2(X)\otimes\boldsymbol{Q}$, where $CH^2(X)$ is the group of cycles of codimension two on X modulo rational equivalence. It is also shown in [L3] that, as one might guess, the bilinear map from $\mathrm{Pic}(X)\otimes\mathrm{Pic}(X)$ to $CH^2(X)\otimes\boldsymbol{Q}$ is the intersection pairing.

We wish to compare this bilinear map with the one defined in § 2 for curves. To do this, we recall the Gersten complex for motivic cohomology from [L3].

Theorem 3.1. *Let X be a regular noetherian scheme. Then there exists an object T in the derived category of étale sheaves on X and distinguished* *triangles* (*up to* 2-*torsion*)

$$\Gamma(X,2) \longrightarrow \bigoplus_{x\in X^{(0)}} t_{\le 3}R(j_x)_*\Gamma(k(x),2) \longrightarrow T \longrightarrow \Gamma(X,2)[1]$$

and

$$T \longrightarrow \bigoplus_{x\in X^{(1)}} t_{\le 2}R(j_x)_*\Gamma(k(x),1))[-1]$$
$$\longrightarrow \bigoplus_{x\in X^{(2)}} t_{\le 1}R(j_x)_*\Gamma(k(x),0)[-2] \longrightarrow T[1].$$

We can more concretely describe T by considering the exact sequence of sheaves:

$$0 \longrightarrow V \longrightarrow \bigoplus_{x\in X^{(1)}} (j_x)_*G_m \longrightarrow \bigoplus_{x\in X^{(2)}} (j_x)_*Z \longrightarrow 0,$$

and observing that $T = V[-2]$ in the derived category.

We must also recall the description of the map from T to $\Gamma(X,2)$ [1], or, equivalently, from V to $\Gamma(X,2)$ [3].

Lemma 3.2. *Let $B_{\bullet}$ and $D_{\bullet}$ be complexes of objects in an abelian category $\mathscr{A}$ which are acyclic outside of* [1, 2]. *Let $\varphi: B_{\bullet} \to D_{\bullet}$ be a map of complexes such that φ induces an isomorphism on H^1 and such that there exists an exact sequence*

$$0 \longrightarrow H^2(B_{\bullet}) \longrightarrow H^2(D_{\bullet}) \longrightarrow W \longrightarrow 0.$$

Then there exists a natural map ρ from W to $B_{\bullet}[3]$ in the derived category of $\mathscr{A}$ such that if $B'_{\bullet}$, $D'_{\bullet}$, φ', W' have the analogous properties and we have maps $\lambda_B: B \to B'$, $\lambda_D: D \to D'$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\lambda_B} & B' \\ \varphi\downarrow & & \downarrow\varphi' \\ D & \xrightarrow{\lambda_D} & D' \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} W & \xrightarrow{\rho} & B_{\bullet}[3] \\ \downarrow & & \downarrow \\ W' & \xrightarrow{\rho'} & B'_{\bullet}[3] \end{array}$$

commutes, where the vertical maps are clear.

Proof. This is a routine derived category exercise.

In order to use Lemma 3.2 to define our map from V to $\Gamma(X, 2)[3]$, we choose $B_{\bullet}$ to be $\Gamma(X, 2)$ and $D_{\bullet}$ to be $t_{\leq 2}Rj_*\Gamma(x, 2)$, where x is the generic point of X and j: $x \to X$. We refer to the proof of Theorem 4.4 of [L3] for a proof that $B_{\bullet}$ and $D_{\bullet}$ satisfy the hypotheses of the lemma. Of course W is now V.

We next note that if C lies on X, the pairing previously defined from $Z' \otimes G_m$ to G'_m on C can naturally be viewed as having its image in V. In view of our first triangle, this induces a map μ: $i_*Z' \otimes^L i_*\Gamma(C, 1) \to \Gamma(X, 2)[2]$, where i: $C \to X$.

There is of course a natural map ρ from $G_{m,X}$ to $i_*G_{m,C}$, so from $\Gamma(X, 1)$ to $i_*\Gamma(C, 1)$. In addition, the Gersten complex for G_m on X:

$$0 \longrightarrow G_m \longrightarrow \bigoplus_{x \in X^{(0)}} (j_x)_* G_m \longrightarrow \bigoplus_{x \in X^{(1)}} (j_x)_* Z \longrightarrow 0$$

induces a map θ from i_*Z' to $G_m[1]$ or $\Gamma(X, 1)[2]$. We now claim

Theorem 3.3. *The maps μ and ν are compatible with ρ and θ, i.e. the diagram*

$$\begin{array}{ccccc} & & i_*Z' \overset{L}{\otimes} i_*\Gamma(C, 1) & & \\ & \overset{1\otimes\rho}{\nearrow} & & \overset{\mu}{\searrow} & \\ i_*Z' \overset{L}{\otimes} \Gamma(X, 1) & & & & \Gamma(X, 2)[2] \\ & \underset{\theta\otimes 1}{\searrow} & & \underset{\nu[2]}{\nearrow} & \\ & & \Gamma(X, 1)[2] \overset{L}{\otimes} \Gamma(X, 1) & & \end{array}$$

commutes.

Proof. Pick U étale over X and fix $a \in G_m(U)$. Let $Z'_U = \bigoplus_{u \in U^{(1)}} (j_\omega)_* Z$. It then suffices to show the diagram

$$\begin{array}{ccccc} i_*Z'_C & \longrightarrow & Z'_U & \longrightarrow & \Gamma(U, 1)[2] \\ & \underset{\otimes\bar{a}}{\searrow} & \downarrow{\scriptstyle \otimes a} & & \downarrow{\scriptstyle \otimes a} \\ & & V_U & \longrightarrow & \Gamma(U, 2)[3] \end{array}$$
$$\theta : i_*Z'_C \to \Gamma(U, 1)[2]$$

commutes, where C is now a curve on U, $V_U = V_X \mid U$, and $\bar{a}$ is the class of $a \in G_m(C)$.

It is evident that the triangle commutes, and the square commutes by Lemma 3.2. Here, as before, $B'_{\bullet}$ is $\Gamma(U, 2)$, $D'_{\bullet}$ is $t_{\leq 2}Rj_*\Gamma(u, 2)$. B is $\Gamma(U, 1)[-1]$ and D is $j_*\Gamma(u, 1)[-1]$, where j maps the generic point u of

U into U. W' is V_U, and W is Z'_U. We here take for $\Gamma(U, 1)$ and $\Gamma(u, 1)$ the torsion-free complexes defined in [L2], § 2, so that $\otimes a$: $\Gamma(U, 1)$ to $\Gamma(U, 2)[1]$ and $\Gamma(U, 1)$ is acyclic outside of degrees 0 and 1. We remind the reader of the commutative diagram of sheaves on U:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & G_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & j_* G_m & & \\
 & & & & & & \downarrow & & \\
 & & & & & & Z'_U & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

3.4. We now come to the case of open surfaces. Let $U = X - C$ and let G_m^U be the kernel of the map from $G_{m,X}$ to $i_* G_{m,C}$.

The long exact sequence of cohomology coming from the short exact sequence $0 \to G_m^U \to G_{m,X} \to i_* G_{m,C} \to 0$ yields

$$
\begin{aligned}
0 &\longrightarrow H^0(X, G_m^U) \longrightarrow H^0(X, G_m) \longrightarrow H^0(C, G_m) \longrightarrow H^1(X, G_m^U) \\
&\longrightarrow H^1(X, G_m) \longrightarrow H^1(C, G_m) \longrightarrow H^2(X, G_m^U) \longrightarrow H^2(X, G_m).
\end{aligned}
$$

Since $H^2(X, G_m) = Br(X)$ is torsion, after tensoring with $\boldsymbol{Q}$ we obtain

$$
\begin{aligned}
H^1(X, G_m^U) \otimes \boldsymbol{Q} &\longrightarrow H^1(X, G_m) \otimes \boldsymbol{Q} \longrightarrow H^1(C, G_m) \otimes \boldsymbol{Q} \\
&\longrightarrow H^2(X, G_m^U) \otimes \boldsymbol{Q} \longrightarrow 0.
\end{aligned}
$$

On the other hand, we have the standard sequence

$$
\begin{aligned}
0 &\longrightarrow H^0(X, G_m) \longrightarrow H^0(U, G_m) \longrightarrow H^0(C, Z') \\
&\longrightarrow H^1(X, G_m) \longrightarrow H^1(U, G_m) \longrightarrow 0.
\end{aligned}
$$

which, after tensoring with $\boldsymbol{Q}$ becomes

$$
\begin{aligned}
H^0(U, G_m) \otimes \boldsymbol{Q} &\longrightarrow H^0(C, Z') \otimes \boldsymbol{Q} \longrightarrow H^1(X, G_m) \otimes \boldsymbol{Q} \\
&\longrightarrow H^1(U, G_m) \otimes \boldsymbol{Q} \longrightarrow 0.
\end{aligned}
$$

It is also easily checked that the map from $H^0(C, Z')$ to $H^1(X, G_m)$ is the same as that induced by the map θ above from $i_* Z'$ to $G_m[1]$.

Proposition 3.5. *θ and ρ define a commutative diagram*

$$\begin{array}{ccc} H^0(C, \mathbf{Z}') \times H^1(C, G_m) & \longrightarrow & \mathbf{Z} \\ \downarrow \theta \qquad \quad \uparrow \rho & & \\ H^1(X, G_m) \times H^1(X, G_m) & \longrightarrow & \mathbf{Z}. \end{array}$$

Proof. This is immediate from Theorem 3.3 and the fact that the degree map from $H^1(C, G'_m)$ to $\mathbf{Z}$ is compatible with the degree map from $H^4(X, \Gamma(2))$ to $\mathbf{Z}$, since both are induced by the usual degree map on points.

Proposition 3.6. *The commutative diagram of Proposition* 3.5 *induces bilinear maps*

$$\langle\ ,\ \rangle_1 \colon H^1(U, G_m) \times H^1(X, G_m^U) \longrightarrow \mathbf{Q}$$
$$\langle\ ,\ \rangle_2 \colon H^0(U, G_m) \times H^2(X, G_m^U) \longrightarrow \mathbf{Q}.$$

Proof. Evident.

Now assume that k is finite. Theorem 3.3. implies that we have a compatible system of bilinear maps:

$$\begin{array}{ccc} H^0(X, i_*\mathbf{Z}') \otimes H^3(X, i_*G_{m,c}) & \searrow & \\ \downarrow \qquad\qquad \uparrow & & H^6(X, \Gamma(2)) \xrightarrow[\sim]{\mathrm{Tr}} \mathbf{Q}\ \mathbf{Z} \\ H^1(X, G_m) \otimes H^3(X, G_m) & \nearrow & \end{array}$$

It is known that the duality on X is induced from the pairing $\Gamma(1) \otimes^L \Gamma(1) \rightarrow \Gamma(2)$ [Sa]. We also wish to show that the map from $H^0(X, i_*\mathbf{Z}') \otimes H^3(X, i_*G_{m,c}) \rightarrow \mathbf{Q}/\mathbf{Z}$ agrees with the duality $H^0(C, i_*\mathbf{Z}') \otimes H^3(C, G_m) \rightarrow \mathbf{Q}/\mathbf{Z}$ described in § 2. In view of the definitions of these two maps, it suffices to prove that the two trace maps from $H^6(X, \Gamma(2))$ and from $H^3(C, G'_m)$ into $\mathbf{Q}/\mathbf{Z}$ are compatible. Since the trace map on curves is compatible with the trace map on points ([D]), it suffices to show the following:

Proposition 3.7. *There exists a trace isomorphism $H^6(X, \Gamma(2)) \xrightarrow{\mathrm{Tr}} \mathbf{Q}/\mathbf{Z}$, which is compatible with the cycle class map in the sense that, if P is any closed point of X, the map $(i_P)_*\mathbf{Z} \rightarrow \Gamma(2)[4]$ defined in* [L3] *induces a commutative diagram*

$$\begin{array}{ccc} H^2(X, i_*\mathbf{Z}) & \longrightarrow & H^6(X, \Gamma(2)) \\ \downarrow \mathrm{Tr}_P & & \downarrow \\ \mathbf{Q}/\mathbf{Z} & \xrightarrow{\ 1\ } & \mathbf{Q}/\mathbf{Z} \end{array}$$

where Tr_P is the usual identification of $H^2(P, \mathbf{Z})$ with $\mathbf{Q}/\mathbf{Z}$.

Proof. We begin with a lemma.

Lemma 3.8. *Let* $\bar{X}=X\times_k \bar{k}$, $\bar{P}=P\times_k \bar{k}$. *Then*

$$H^6(\bar{X}, \Gamma(2))=H^5(\bar{X}, \Gamma(2))=H^5_{\bar{P}}(\bar{X}, \Gamma(2))=0.$$

Proof. We first observe that it follows from Theorem 3.1 that $H^i_{\bar{P}}(\bar{X}, \Gamma(2))$ and $H^i(\bar{X}, \Gamma(2))$ are both torsion for $i\geq 5$. First, $H^i(\bar{X}, V)$ is clearly torsion for $i\geq 2$, so $H^i(\bar{X}, T)$ is torsion for $i\geq 4$. Next, $R^q(j_x)_*\Gamma(k(x), 2)$ is a torsion sheaf for $q\geq 3$, so up to torsion $t_{\leq 3}R(j_x)_*\Gamma(k(x), 2)$ is equal to $R(j_x)_*\Gamma(k(x), 2)$. But $H^i(\bar{X}, R(j_x)_*\Gamma(k(x), 2))=H^i(k(x), \Gamma(k(x), 2))$ which is torsion for $i\geq 3$. The cohomology sequence of the first distinguished triangle then shows that $H^i(\bar{X}, \Gamma(2))$ is torsion for $i\geq 5$. Let $\bar{U}=\bar{X}-\bar{P}$. Comparing the cohomology sequences of $H^i(\bar{X}, \Gamma(2))$ and $H^i(\bar{U}, \Gamma(2))$ and observing that the map from $\oplus_{x\in\bar{X}^{(2)}} H^0(\bar{X}, (j_x)_*Z)$ to $\oplus_{x\in(\bar{U})^{(2)}} H^0(\bar{U}, (j_x)_*Z)$ is surjective we see that $H^i_{\bar{P}}(\bar{X}, \Gamma(2))$ is also torsion for $i\geq 5$.

First let n be prime to p. In [L3] it was shown that there is a commutative diagram

$$\begin{array}{ccc} H^0_{\bar{P}}(\bar{X}, (i_{\bar{P}})_*Z)/n & \longrightarrow & H^4_{\bar{P}}(\bar{X}, \Gamma(2))/n \\ \downarrow \wr & & \downarrow \phi \\ H^0_{\bar{P}}(\bar{X}, (i_{\bar{P}})_*Z/nZ) & \xrightarrow{\psi} & H^4_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2}), \end{array}$$

where ψ is the Gysin map, hence in this situation an isomorphism. Hence ϕ is surjective, and the Kummer sequence for $\Gamma(2)$ ([L2]) implies that $H^5_{\bar{P}}(\bar{X}, \Gamma(2))_n=0$.

Similarly, there is a commutative diagram

$$\begin{array}{ccc} H^0_{\bar{P}}(\bar{X}, (i_{\bar{P}})_*Z'/p^m) & \longrightarrow & H^4_{\bar{P}}(\bar{X}, \Gamma(2))/p^m \\ \downarrow & & \downarrow \\ H^0_{\bar{P}}(\bar{X}, i_*\nu(0)) & \longrightarrow & H^2_{\bar{P}}(\bar{X}, \nu_2(m)), \end{array}$$

which implies that $H^5_{\bar{P}}(\bar{X}, \Gamma(2))_{p^n}=0$. (See [M4] for the analogue of the Kummer sequence deduced from the distinguished triangle $\Gamma(2)\xrightarrow{p^n}\Gamma(2)\rightarrow\nu_2(m)\rightarrow\Gamma(2)[1]$, and [M3] for the analogue of the Gysin homomorphism). It follows readily from Theorem 3.1 (see also [Sa]) that $H^5_{\bar{P}}(\bar{X}, \Gamma(2))$ is torsion, which then implies that $H^5_{\bar{P}}(\bar{X}, \Gamma(2))=0$

This proof works equally well if the supports are removed, since the Gysin map is still an isomorphism. The Gysin and Milne sequences similarly show that since $H^6(\bar{X}, \Gamma(2))$ is torsion, it is zero.

We resume the proof of Proposition 3.7.

In [L3], it was shown that the cycle class map defined there was compatible with the classical l-adic and p-adic cycle class maps, which implies the existence of a commutative diagram

$$\begin{array}{ccc} 0 & & H^3_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2})=0 \\ \downarrow & & \downarrow \\ H^0_{\bar{P}}(\bar{P}, Z) & \longrightarrow & H^4_{\bar{P}}(\bar{X}, \Gamma(2)) \\ \downarrow n & & \downarrow \\ H^0_{\bar{P}}(\bar{P}, Z) & \longrightarrow & H^4_{\bar{P}}(\bar{X}, \Gamma(2)) \\ \downarrow & & \downarrow \\ H^0_{\bar{P}}(\bar{P}, Z/nZ) & \longrightarrow & H^4_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2}) \\ \downarrow & & \downarrow \\ 0 & & H^5_{\bar{P}}(\bar{X}, \Gamma(2))=0 \end{array}$$

of G_k-modules. $H^3_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2})=0$ by purity ([M1]), and $H^5_{\bar{P}}(\bar{X}, \Gamma(2))=0$ by Lemma 3.8. Hence there is a chain of commuting squares

$$\begin{array}{ccccc} H^1(P, Z/nZ) & \xrightarrow{\sim} & H^1(G_k, H^0(\bar{P}, Z/nZ)) & \longrightarrow & H^1(G_k, H^4_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2})) \\ \downarrow & & \downarrow \delta & & \downarrow \delta \\ H^2(P, Z) & \xrightarrow{\sim} & H^2(G_k, H^0(\bar{P}, Z)) & \longrightarrow & H^2(G_k, H^4_{\bar{P}}(\bar{X}, \Gamma(2)) \end{array}$$

and

$$\begin{array}{ccccc} H^1(G_k, H^4_{\bar{P}}(\bar{X}, \mu_n^{\otimes 2})) & \longrightarrow & H^1(G_k, H^4(\bar{X}, \mu_n^{\otimes 2})) & \xrightarrow{\sim} & H^5(X, \mu_n^{\otimes 2}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(G_k, H^4_{\bar{P}}(\bar{X}, \Gamma(2))) & \longrightarrow & H^2((G_k, H^4(\bar{X}, \Gamma(2))) & \underset{\sim}{\xrightarrow{\phi}} & H^6(X, \Gamma(2)) \end{array}$$

where ϕ is an isomorphism again by Lemma 3.8.

We now recall the well-known commutative diagram

$$\begin{array}{ccc} H^0(\bar{P}, Z/nZ) & \xrightarrow{\text{Gysin}} & H^4(\bar{X}, \mu_n^{\otimes 2}) \\ \downarrow \wr & & \downarrow \mathrm{Tr} \\ Z/nZ & \xrightarrow{1} & Z/nZ, \end{array}$$

and apply the functor $H^1(G_k, \cdot)$ to obtain

$$\begin{array}{ccc} H^1(P, Z/nZ) & \longrightarrow & H^5(X, \mu_n^{\otimes 2}) \\ \downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} \\ Z/nZ & \xrightarrow{1} & Z/nZ, \end{array}$$

which still commutes. Now define the prime-to-p part of the trace map by using the Kummer sequence, and the prime-to-p part of the proposition follows at once. By referring to [M2] instead of [M1] for the corresponding purity and Gysin results, and using the Milne sequence $\Gamma(2)\xrightarrow{p^n}\Gamma(2)\to\nu_2(n)\to\Gamma(2)[1]$ ([M3]) instead of the Kummer sequence, we complete the proof by proving the p-power part in exactly the same way.

Corollary 3.9. a) *There is a natural duality pairing* $H^3(X, G_m^U)\times H^1(U, G_m)$ *into* $H^6(X, \Gamma(2))=\boldsymbol{Q}/\boldsymbol{Z}$ *and hence* $\# H^3(X, G_m^U)_{\text{cot}}=\# H^1(U, G_m)_{\text{tor}}$.

b) $\# H^4(X, G_m^U)_{\text{cot}}=\# H^0(U, G_m)_{\text{tor}}$.

Proof. We have the two long exact sequences

$$0\longrightarrow H^3(X, G_m^U)\longrightarrow H^3(X, G_m)\xrightarrow{\alpha} H^3(X, i_*G_m)$$
$$\longrightarrow H^4(X, G_m^U)\longrightarrow H^4(X, G_m)\longrightarrow 0$$

and

$$0\longleftarrow H^1(U, G_m)\longleftarrow H^1(X, G_m)\xleftarrow{\beta} H^0(C, Z')$$
$$\longleftarrow H^0(U, G_m)\longleftarrow H^0(X, G_m)\longleftarrow 0.$$

There are perfect pairings compatible with α and β:

$$H^3(X, G_m)\times H^1(X, G_m)\longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

and

$$H^3(X, i_*G_m)\times H^0(C, Z')\longrightarrow \boldsymbol{Q}/\boldsymbol{Z}.$$

Hence $H^3(X, G_m^U)$ is the dual of $H^1(U, G_m)$, which proves a).

Also coker α is dual to ker β, and we have the exact sequences

$$0\longrightarrow \text{Coker}\,\alpha \longrightarrow H^4(X, G_m^U)\longrightarrow H^4(X, G_m)\longrightarrow 0$$

and

$$0\longrightarrow H^0(X, G_m)\longrightarrow H^1(U, G_m)\longrightarrow \ker\beta \longrightarrow 0.$$

Since $H^4(X, G_m)$ and $H^0(X, G_m)$ are finite, these give rise to the exact sequences:

$$0\longrightarrow(\text{Coker}\,\alpha)_{\text{cot}}\longrightarrow H^4(X, G_m^U)_{\text{cot}}\longrightarrow H^4(X, G_m)\longrightarrow 0$$

and

$$0\longrightarrow H^0(X, G_m) \longrightarrow H^0(U, G_m)_{\text{tor}}\longrightarrow (\text{Ker}\,\beta)_{\text{tor}} \longrightarrow 0.$$

By duality, $\#(\text{Coker}\,\alpha)_{\text{cot}}=\#(\text{Ker}\,\beta)_{\text{tor}}$ and $\# H^0(X, G_m)=\# H^4(X, G_m)$ hence $\# H^4(X, G_m^U)_{\text{cot}}=\# H^0(U, G_m)_{\text{tor}}$.

Remark 3.10. Note that this stops just short of proving that $H^4(X, G_m^U)$ is dual to $H^0(U, G_m)$, which must certainly be true.

§ 4. Zeta-functions of surfaces

Let X, C, and U be as in § 3 but now assume that k is finite. Also assume that the Brauer group $H^2(X, G_m)$ is finite. (That this is true of all such X would follow from the Tate conjecture for divisors). Then ([T], [M2], [Sa]) $H^i(X, G_m)$ is finite for $i \neq 1, 3$, $H^0(X, G_m)$ and $H^4(X, G_m)$ are dual abelian groups, $H^1(X, G_m)$ is a finitely-generated abelian group of rank $r=$ the order of the pole of the zeta-function of X at $s=1$. The map from $G_m \otimes^L G_m \to \Gamma(2)[2]$ induces an isomorphism of $H^3(X, G_m)$ with the $\boldsymbol{Q}/\boldsymbol{Z}$-dual of $H^1(X, G_m)$ by means of the induced pairing of these two groups into $H^6(X, \Gamma(2))$ which is canonically isomorphic to $\boldsymbol{Q}/\boldsymbol{Z}$. It also induces a pairing $\mathscr{H}_1^X = \{H^1(X, G_m), H^1(X, G_m), \langle\, , \rangle_X\}$ where $\langle\, , \rangle_X$ is given by intersection of divisors.

Definition 4.1. Let

$$\chi(X, G_m) = \frac{\# H^0(X, G_m) \# H^2(X, G_m) \# H^4(X, G_m) R(\mathscr{H}_1^X)}{\# H^1(X, G_m)_{\mathrm{tor}} \# H^3(X, G_m)_{\mathrm{cot}}}.$$

Let $\quad \chi(X, G_a) = \# H^0(X, G_a) \# H^2(X, G_a) \# H^1(X, G_a)^{-1}.$

Let $\mathscr{H}_0^X$ be the pairing $\{0, H^0(X, G_m), 0\}$ and $\mathscr{H}_2^X$ be the pairing $\{H^0(X, G_m), H^2(X, G_m), 0\}$.

Proposition 4.2. $\chi(X, G_m) = \chi(\mathscr{H}_0^X)\chi(\mathscr{H}_1^X)^{-1}\chi(\mathscr{H}_2^X)^{-1}$.

Proof. We need only observe that the duality theorems for $H^i(X, G_m)$ imply that $\# H^0(X, G_m) = \# H^4(X, G_m)$ and $\# H^1(X, G_m)_{\mathrm{tor}} = \# H^3(X, G_m)_{\mathrm{cot}}$, and that $H^2(X, G_m)$ is finite.

Proposition-Definition 4.3. *Under the current hypothesis that k is finite, the triples $\{H^1(U, G_m), H^1(X, G_m^U), \langle\, , \rangle_1\}$ and $\{H^0(U, G_m), H^2(X, G_m^U), \langle\, , \rangle_2\}$ of Proposition 3.6 are pairings, and we call these pairings $\mathscr{H}_1^U$ and $\mathscr{H}_2^U$ respectively. We denote their regulators by R_1^U and R_2^U. Let $\mathscr{H}_0^U$ be the pairing $\{0, H^0(X, G_m), 0\}$.*

Proof. This follows immediately from Proposition 3.5, Proposition 2.1.c and the non-degeneracy of the intersection pairing on X.

Definition 4.4. Let

$$\chi(X, G_m^U) = \frac{\# H^0(X, G_m^U) \# H^2(X, G_m^U)_{\mathrm{tor}} \# H^4(X, G_m^U)_{\mathrm{cot}} R_1^U}{\# H^1(X, G_m^U)_{\mathrm{tor}} \# H^3(X, G_m^U)_{\mathrm{cot}} \cdot R_2^U}.$$

Proposition 4.5. $\chi(X, G_m^U) = \chi(\mathscr{H}_0^U)\chi(\mathscr{H}_2^U)\chi(\mathscr{H}_1^U)^{-1}$.

Proof. $\chi(\mathscr{H}_0^U) = \# H^0(X, G_m^U)$, $\chi(\mathscr{H}_1^U) = \dfrac{\# H^1(X, G_m^U)_{\mathrm{tor}} \# H^1(U, G_m)_{\mathrm{tor}}}{R_1^U}$, but by corollary 3.9, $\# H^1(U, G_m)_{\mathrm{tor}} = \# H^3(X, G_m^U)_{\mathrm{cot}}$.

$$\chi(\mathscr{H}_2^U) = \frac{\# H^2(X, G_m^U)_{\mathrm{tor}} \# H^0(X, G_m^U)_{\mathrm{tor}}}{R_2^U},$$

but again by Corollary 3.9 $\# H^0(X, G_m^U)_{\mathrm{tor}} = \# H^4(X, G_m^U)_{\mathrm{cot}}$. The proposition follows immediately.

Theorem 4.6. $\chi(X, G_m) = \chi(X, G_m^U)\chi(C, G_m)$.

Proof. This follows immediately from Propositions 4.5, 4.2, and 2.8, Lemma 1.5, and the exact sequence of pairings

$$0 \longrightarrow \mathscr{H}_2^X \longrightarrow \mathscr{H}_2^U \longrightarrow \mathscr{H}_1^C \longrightarrow \mathscr{H}_1^X \longrightarrow \mathscr{H}_1^U \longrightarrow \mathscr{H}_0^C \longrightarrow \mathscr{H}_0^X \longrightarrow \mathscr{H}_0^U \longrightarrow 0.$$

Now let G_a^U be the kernel of the map from $G_{a,X}$ to $(i_C)_* G_{a,C}$. Then $H^i(X, G_a^U)$ is zero for $i \geq 3$ and finite for all i. Let

$$\chi(X, G_a^U) = \frac{\# H^0(X, G_a^U) \# H^2(X, G_a^U)}{\# H^1(X, G_a^U)}.$$

Proposition 4.7. $\chi(X, G_a) = \chi(X, G_a^U)\chi(C, G_a)$.

Proof. This is evident from the exact sequence

$$0 \longrightarrow G_a^U \longrightarrow G_{a,X} \longrightarrow i_* G_{a,C} \longrightarrow 0.$$

Theorem 4.8. $Z(U, 1) = \pm \chi(X, G_a^U)\chi(X, G_m^U)^{-1}$.

Proof. Since $Z(X, t) = Z(U, t)Z(C, t)$, it follows that $Z(X, 1) = Z(U, 1)Z(C, 1)$. In [T] and [M2] it is shown that (see [L1] for a translation of language) $Z(X, 1) = \pm \chi(X, G_a)\chi(X, G_m)^{-1}$. The theorem now follows from Proposition 4.7, Theorem 4.6, and Corollary 2.7.

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Department of Mathematics
Cornell University
Ithaca, NY 14853
U.S.A.