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Remarks on the Theorems of Takagi and Furtwängler

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Dedicated to Professor Kenkichi Iwasawa on his 70th birthday

The present article contains a short exposition of the fact that the fundamental inequality in Takagi's class field theory is almost a corollary of Furtwängler's theorems as far as unrestricted use of idele-theoretic terminology is allowed.

Let K/F be a cyclic extension of a finite degree n over an algebraic number field of a finite degree. Denote by I_L, P_L and C_L, the idele group, the principal idele group, and the idele class group of an algebraic number field L, respectively. Then, Takagi's fundamental inequality is written as $(C_F: N_{K/F}C_K) \geqq n$. (See e.g. [2], p. 154).

Let, on the other hand, F be an algebraic number field containing the group $\mu_{(l)}$ of the l-th roots of unity with a prime number l. Then, it was shown by Furtwängler [1] that the following two theorems hold:

Theorem I). *Product formula* $\prod_{\mathfrak{p}} (\alpha, \beta/\mathfrak{p})_l = 1$ *of the norm residue symbol* $(\alpha, \beta/\mathfrak{p})_l$, *where* $\alpha, \beta \in F^{\times} = F - \{0\}$, *and* $\mathfrak{p}$ *runs through all places of* F.

Theorem II). *Principal genus theorem* $H^{-1}(C_K, \mathrm{Gal}(K/F)) = 1$ *for an arbitrary Kummer extension* K/F *of degree* l, *which says that* $N_{K/F}a = 1$ *for* $a \in C_K$ *entails* $a = b^{1-S}$ *with some* $b \in C_K$, *where* S *is a generator of the Galois group* $\mathrm{Gal}(K/F)$.

Furtwängler's theorems are fully idele-theoretic. The second theorem was originally stated by him in the form that an element of F which is everywhere a local norm from K is a global norm from K.

We shall show that these two theorems of Furtwängler easily yields Takagi's fundamental inequality.

Proposition 1. *Let* K/F *be a cyclic extension of a finite degree over an algebraic number field* F *of a finite degree, and let* K' *be an intermediate field of* K/F. *Assume that* $H^{-1}(C_K, \mathrm{Gal}(K/K')) = 1$ *and* $H^{-1}(C_{K'}, \mathrm{Gal}(K'/F)) = 1$. *Then,* $H^{-1}(C_K, \mathrm{Gal}(K/F)) = 1$.

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Proof. Assume $N_{K/F}a=1$ with $a \in C_K$, and put $a'=a^{1+S+\cdots+S^{m-1}}$, where S is a generator of Gal (K/F) and $m=(K':F)$. Then, since $N_{K/K'}a'=1$ by the assumption, there is an idele class $b \in C_K$ such that $a'=b^{1-S^m}$, so that $a'^{1-S}=a^{1-S^m}=b^{(1-S)(1-S^m)}$. Therefore, $c=ab^{-(1-S)} \in C_{K'}$, and $N_{K'/F}c=a'b^{-(1-S^m)}=1$. This yields $c=b'^{1-S}$ with some $b' \in C_{K'}$, and $a=b^{1-S}c=(bb')^{1-S}$. q.e.d.

Proposition 2. *Let $K \supset K' \supset F$ be as in Proposition* 1 *and assume that* $H^{-1}(C_K, \mathrm{Gal}\,(K/F))=1$. *Then,* $H^{-1}(C_{K'}, \mathrm{Gal}\,(K'/F))=1$.

Proof. Let $a \in C_{K'}$ satisfy $N_{K'/F}a=1$. Then, $N_{K/F}a=1$, so that $a=b^{1-S}$ with some $b \in C_K$ by the assumption, where S is a generator of Gal (K/F). Put $(K':F)=m$; then

$$b^{1-S^m}=b^{(1-S)(1+S+\cdots+S^{m-1})}=N_{K'/F}a=1.$$

Hence, $b \in C_{K'}$. q.e.d.

These two propositions are quite elementary ones concerning merely modules with group operation, but the next proposition requires Furtwängler's Theorem II.

Proposition 3. *Let F be an algebraic number field of a finite degree, and let K'/F be a cyclic extension of a prime degree l. Then,*

$$H^{-1}(C_{K'}, \mathrm{Gal}\,(K'/F))=1.$$

Proof. If F contains the group $\mu_{(l)}$ of the l-th roots of unity, then the proposition reduces to Furtwängler's Theorem II.

Assume now $\mu_{(l)}$ is not contained in F, and assume that the proposition is true for any prime number less than l. Let F_1 be the field obtained by adjoining $\mu_{(l)}$ to F and put $K=K'F_1$. Then, K/F is cyclic and repeated application of Proposition 1 shows that $H^{-1}(C_K, \mathrm{Gal}\,(K/F))=1$. Therefore, Proposition 2 shows that $H^{-1}(C_{K'}, \mathrm{Gal}\,(K'/F))=1$. Since the case of $l=2$ reduces to Furtwängler's Theorem II, the proposition is proved. q.e.d.

Proposition 4. *Let F be an algebraic number field of a finite degree, and let K/F be a cyclic extension of a finite degree. Then,*

$$H^{-1}(C_K, \mathrm{Gal}\,(K/F))=1.$$

Proof. This proposition follows immediately from Proposition 3 and Proposition 1. q.e.d.

Let F be an algebraic number field of a finite degree and K/F be an abelian extension of a finite degree. We denote then by $(\mathfrak{a}, K/F)$ the Artin symbol of $\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of F composed of prime ideals which are unramified in K. In the present article, we mean by Artin's law of reciprocity for K/F the assertion that there exists a modulus $\mathfrak{m}$ of F such that $(\mathfrak{a}, K/F)=1$ whenever $\mathfrak{a}\equiv 1 \pmod{\mathfrak{m}}$. If Artin's law of reciprocity is valid for K/F, then the map $\mathfrak{a}\to(\mathfrak{a}, K/F)$ induces a homomorphism $a\to(a, K/F)$ from C_F to $\mathrm{Gal}(K/F)$; this is uniquely determined by the condition that $(a, K/F)=(\mathfrak{p}, K/F)$, if a is represented by the idele whose $\mathfrak{p}$-component is a generator of $\mathfrak{p}$ in the $\mathfrak{p}$-adic field $F_\mathfrak{p}$, and other components are 1. We call $C_K \ni a\to(a, K/F) \in \mathrm{Gal}(K/F)$ the reciprocity map; it is surjective, and its kernel contains $N_{K/F}C_K$. In fact, if the former assertion is false, then there would exist an intermediate field $K'\neq F$ of K/F in which almost all prime ideals of F split completely; therefore the Dedekind zeta function of K' would have a pole of order $m=(K'\colon F)>1$. The second assertion is a basic property of the Frobenius automorphism.

Takagi's fundamental inequality can now be obtained as

Theorem. *Let F be an algebraic number field of a finite degree, and let K/F be a cyclic extension of a finite degree n. Then, $(C_F\colon N_{K/F}C_K)\geqq n$.*

Proof. i) We proceed by induction on n. Let l be a prime divisor of n, and let K' be the intermediate field of K/F with $(K\colon K')=l$. If K' contains the group $\mu_{(l)}$ of the l-th roots of unity, then Theorem I of Furtwängler assures that Artin's reciprocity law is vaild for K/K'. If K' does not contain $\mu_{(l)}$, Artin's reciprocity holds for $KL/K'L$, where L is obtained by adjoining $\mu_{(l)}$ to $\boldsymbol{Q}$. So, $(\mathfrak{a}, KL/K'L)=1$ for an ideal $\mathfrak{a}$ of $K'L$ satisfying $\mathfrak{a}\equiv 1 \pmod{\mathfrak{m}}$, where $\mathfrak{m}$ is a suitable modulus of $K'L$. Let, in particular, $\mathfrak{a}$ be an ideal of K'; then a basic property of the Frobenius automorphism implies $(\mathfrak{a}, KL/K'L)=(\mathfrak{a}, KL/K')^r$, where r is prime to l as a divisor of $l-1$, Hence, $(\mathfrak{a}, K/K')=1$. Namely, the reciprocity law of Artin holds for any cyclic extension of a prime degree l.

ii) Let now H be the kernel of the reciprocity map: $C_{K'}\to\mathrm{Gal}(K/K')$ which is certainly well-defined due to i). We have $(C_{K'}\colon H)=l$, and the assumption of the induction yields $(C_F\colon N_{K'/F}C_{K'})\geqq l^{-1}n$. Furthermore, $N_{K'/F}a=1$ for $a\in C_{K'}$ implies $a=b^{1-S}$ by Proposition 4, where $b\in C_{K'}$ and S is a generator of $\mathrm{Gal}(K/F)$; since again a basic property of the Frobenius automorphism says $(b, K/K')=(b^S, K/K')$, a satisfies $(a, K/K')=1$, i.e., $a\in H$. This means the kernel of the norm map of $C_{K'}$ into C_F is contained in H. Thus, we see

$$\begin{aligned}(C_F\colon N_{K'/F}H)&=(C_F\colon N_{K'/F}C_{K'})(N_{K'/F}C_{K'}\colon N_{K'/F}H)\\ &=(C_F\colon N_{K'/F}C_{K'})(C_{K'}\colon H)\geqq l^{-1}n\cdot l=n.\end{aligned}$$

It follows from this and from $H\supset N_{K/K'}C_K$ that $(C_F: N_{K/F}C_K)\geqq n$. q.e.d.

References

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[2] G. J. Janusz, Algebraic number fields, Academic Press, New York, 1973.

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