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On Nearly Ordinary Hecke Algebras for $GL(2)$ over Totally Real Fields

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Dedicated to Kenkichi Iwasawa for his 70th birthday

Since this work is a continuation of our previous paper [8], we shall suppose the familiarity on the reader's part with the result and the notation in [8]. Especially we fix a rational prime p and a totally real field F of degree d. Let $\mathcal{O}$ be a valuation ring finite flat over $\boldsymbol{Z}_p$ containing all the conjugates of the integer ring r of F, and let Z (resp. $\bar{Z}$) be the Galois group of the maximal abelian extension of F unramified outside p and ∞ (resp. outside p). We denote by W the torsion-free part of Z. Then, for each weight $v>0$, we have constructed in [8] the ordinary Hecke algebra finite and without torsion over the continuous group algebra $\mathcal{O}[\![W]\!]$. The significance of this algebra lies in the fact that for every non-negative weight n parallel to $-2v$ (i.e. the sum $n+2v$ is a parallel weight), the Hecke algebra $\mathfrak{h}^{\mathrm{ord}}_{k,w}(p^\alpha; \mathcal{O})$ $(k=n+2t,\ w=v+k-t)$ over $\mathcal{O}$, for the space of holomorphic Hilbert cusp forms for F of level p^α and of weight (k, w), can be uniquely obtained as a residue algebra of $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$. Strictly speaking, this algebra $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ cannot be called ordinary when $v>0$ since the Hecke operator $T(p)$ in $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ is in fact divisible exactly by p^v. This algebra $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ cannot be either said universal because of the restriction to the weight n which requires n to be parallel to the fixed weight $-2v$. Thus, there exists infinitely many Hecke algebras $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ parametrized by the weights v modulo parallel ones. In this paper, we shall unify these infinitely many Hecke algebras $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ and construct a unique universal one $\boldsymbol{h}^{\mathrm{ord}}(1; \mathcal{O})$ from which each $\boldsymbol{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ can be obtained as a residue algebra for all non-negative v.

Although we shall postpone the exact formulation of our result to § 2 of this paper, let us make it a little more precise. Let $\bar{Z}_0$ be the subgroup of Z generated by the inertia groups at all the prime ideals of r over p, and put $G=\bar{Z}_0\times r_p^\times$, where $r_p=r\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$ is the p-adic completion of the integer ring r of F. Let I be the set of all embeddings of F into $\bar{\boldsymbol{Q}}$, and put $t=\sum_\sigma \sigma \in \boldsymbol{Z}[I]$. We shall say that a weight $k \in \boldsymbol{Z}[I]$ is parallel if k

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is an integer multiple of t; in this case, we write $k=[k]t$ for $[k]\in \boldsymbol{Z}$. Each weight $v=\sum_\sigma v_\sigma\sigma \in \boldsymbol{Z}[I]$ can be considered as a character of $r_p^\times$ by

$$r_p^\times \ni x \longmapsto x^v=\prod_{\sigma\in I} x^{\sigma v_\sigma}.$$

Let $\chi\colon Z\to\mathcal{O}^\times$ denote the cyclotomic character, and define, for each pair (n, v) of weights with parallel $n+2v\mu t$, a character $P_{n,v}\colon \boldsymbol{G}\to\mathcal{O}^\times$ by

$$P_{n,v}(z, a)=\chi^\mu(z)a^v \qquad \text{for } (z, a)\in \bar{Z}_0\times r_p^\times=\boldsymbol{G}.$$

This character $P_{n,v}$ naturally induces an $\mathcal{O}$-algebra homomorphism $P_{n,v}\colon \mathcal{O}[\![\boldsymbol{G}]\!]\to\mathcal{O}$. With a little abuse of symbols, we use the same symbol $P_{n,v}$ for the prime ideal of $\mathcal{O}[\![\boldsymbol{G}]\!]$ which is the kernel of the homomorphism $P_{n,v}$. Let $\boldsymbol{W}$ be the torsion-free part of $\boldsymbol{G}$ and K denote the quotient field $\mathcal{O}$. Then we have

Theorem. *There exists an $\mathcal{O}[\![\boldsymbol{G}]\!]$-algebra $\boldsymbol{h}^{n,\mathrm{ord}}$ and specified elements $T(\mathfrak{l})\in h$ for each ideal $\mathfrak{l}$ of r satisfying the following conditions*:

(i) *$\boldsymbol{h}^{n,\mathrm{ord}}$ is a torsion-free $\mathcal{O}[\![\boldsymbol{W}]\!]$-module of finite type*;

(ii) *for each non-negative (n, v) with parallel $n+2v$, there is an isomorphism*:

$$\boldsymbol{h}^{n,\mathrm{ord}_P}/P\boldsymbol{h}^{n,\mathrm{ord}_P}\cong \mathfrak{h}^{\mathrm{ord}}_{k,w}(U_0(p); K) \qquad (P=P_{n,v})$$

which takes $T(\mathfrak{l})$ to the Hecke operator $T(\mathfrak{l})$ in $\mathfrak{h}^{\mathrm{ord}}_{k,w}(U_0(p); K)$. Here $\boldsymbol{h}^{n,\mathrm{ord}_P}$ denotes the localization of $\boldsymbol{h}^{n,\mathrm{ord}}$ at P and $U_0(p)$ is the Γ_0-type open compact subgroup of level p of the adelized group $GL_2(F_A)$ (for details of these definitions, see § 1).

Here are several remarks about the theorem. If one admits the Leopoldt conjecture for p and F, the conditions (i) and (ii) characterize $\boldsymbol{h}^{n,\mathrm{ord}}$. In § 2, we shall give a formulation (Theorem 2.4) of this fact, which not only includes the level structure prime to p but also describes the behaviour of $\boldsymbol{h}^{n,\mathrm{ord}}$ under the localization at primes of $\mathcal{O}[\![\boldsymbol{G}]\!]$ corresponding to finite order characters of $\bar{Z}_0$. Then $\boldsymbol{h}^{n,\mathrm{ord}}$ is determined uniquely without supposing the Leopoldt conjecture.

This theorem implies that each normalized Hilbert eigenform has at least $d+1$ dimensional p-adic deformation inside the space of p-adic modular forms (Corollary 2.5). In appearance, the base group $\boldsymbol{G}$ looks a little artificial, but $\boldsymbol{G}$ is isomorphic to the quotient of $r_p^\times\times r_p^\times$ by the closure of the diagonal image of $r^\times$ and hence comes from the $\boldsymbol{Z}_p$-points of the maximal split torus of GL_2. Thus, in GL_1-case, the base group $\bar{Z}$ is essentially given by $GL_1(r_p)/\bar{r}^\times$ and in GL_2-case, the base group $\boldsymbol{G}$ is given by $(GL_1(r_p)\times GL_1(r_p))/\bar{r}^{\times}$. From this observation, one might imagine which group should be offered as the base group of the nearly ordinary Hecke algebra for each linear algebraic group.

Without restricting ourselves to the nearly ordinary part, we actually prove in § 5 the universality for the whole Hecke algebra (Theorem 2.3). Some application of our result to large Galois representations into $GL_2(\mathcal{O}[\![X_1, \cdots, X_s]\!])$ for the $\boldsymbol{Z}_p$rank s of W $(s>d)$ and to p-adic L-functions of Hilbert cusp forms will be discussed in our subsequent papers.

At the symposium dedicated to K. Iwasawa held at the Mathematical Sciences Research Institute in Berkeley, the author presented the result already appeared in [8] and the result described in this paper was actually stated on that occasion as conjectures.

Throughout the paper, we shall use the notation introduced in [8] without any mentioning except those explained below. We take the algebraic closure $\bar{\boldsymbol{Q}}$ of $\boldsymbol{Q}$ inside $\boldsymbol{C}$. We fix an algebraic closure $\bar{\boldsymbol{Q}}_p$ of $\boldsymbol{Q}_p$ and an embedding $\iota: \bar{\boldsymbol{Q}}\rightarrow\boldsymbol{Q}_p$ once and for all. Therefore, each algebraic number in $\bar{\boldsymbol{Q}}$, can be considered to be a complex number as well as a p-adic number uniquely.

Notation. We keep the terminology of [8]. We explain here some of the notation which will be used often in this paper. For each quaternion algebra B over F, we denote by $G=G^B$, the linear algebraic group defined over $\boldsymbol{Q}$ such that $G^B(\boldsymbol{Q})=B^{\times}$. The reduced norm map: $B\rightarrow F$ is denoted by ν. We denote by $G^B(A)$ the adelization of G. Let I_B be the set of all infinite places of F at which the completion of B is isomorphic to $M_2(\boldsymbol{R})$. Then the connected component $G_{\infty+}$ of the infinite part G_∞ of $G(A)$ acts naturally on $\mathscr{Z}_B=\mathscr{H}^{I_B}$ via the linear fractional transformation, where $\mathscr{H}$ denotes the upper half complex plane. This action can be extended to a real analytic action of G_∞ (see [8, § 2]). We denote by $C_{\infty+}$ (resp. C_∞) the stabilizer in $G_{\infty+}$ (resp. G_∞) of $z_0=(i, \cdots, i)$ for $i=\sqrt{-1}$. Let $F_\infty^\times$ be the infinite part of the idele group $F_A^\times$ and $F_{\infty+}^\times$ be the connected component of $F_\infty^\times$ with the identity. For any element or subset x of an adelized object (for example, $G(A)$, $F_A^\times$), we denote by x_f, x_∞, x_σ the finite part of x, the infinite part of x and the component of x at each place σ of F; for example, $F_f^\times$ denote the finite part of $F_A^\times$. We put $F_{A+}^\times=F_f^\times F_{\infty+}^\times$, $G(A)_+=G_fG_{\infty+}$. For each ideal N of r, we write G_N (resp. x_N) for $\prod_\sigma G_\sigma$ (resp. $\prod_\sigma x_\sigma$), where σ runs over all finite places inside N. We put $\hat{\boldsymbol{Z}}=\prod_l \boldsymbol{Z}_l$, which is the maximal compact subring of $\boldsymbol{Q}_f=A_f$. For any group Γ and for each right Γ-module M, we denote by ${}^tM$ the left Γ-module whose underlying module is M and whose Γ-action is given by

$$\gamma m=m\cdot\gamma^{-1} \qquad \text{for } \gamma\in\Gamma.$$

§ 1. Hecke operators on spaces of cusp forms

In this section, we give the definition of Hecke algebras over a subring A of C and prove some basic facts of Hecke algebras. For each quaternion algebra B over F, let $G=G^B$ denote the linear algebraic group defined over F with $G^B(Q)=B^\times$. Let U be an open compact subgroup of G_f. We fix a non-negative weight $v \in Z[I]$ and take a non negative $n \in Z[I]$ such that $n+2v\equiv 0 \bmod Zt$ for $t=\sum_{\sigma\in I}\sigma$. Put $k=n+2t$, $w=v+k-t$ and fix a subset J of I_B, where I_B is the subset of I consisting of all places at which B is split. We consider the space of cusp forms $S_{k,w,J}(U; B; C)$ defined in [8, (2.4)], where we also defined the Hecke operator, for each $x \in G_f C_\infty$ ([8, below (2.6c)]),

$$[UxU]\colon S_{k,w,J}(U; B; C) \longrightarrow S_{k,w,J}(U; B; C).$$

Let Φ be the composite field of all the images of F under the elements of I and r_Φ denote the integer ring of Φ. We fix an r_Φ-algebra A inside C and suppose the following condition:

(1.1) For every $x \in F_f^\times$ and every $\sigma \in I$, the A-ideal $x^\sigma A$ is generated by a single element of A.

For each ideal $\mathfrak{a}$ of r, we consider an A-ideal $\mathfrak{a}^v A=\prod_\sigma \mathfrak{a}^{v_\sigma \sigma} A$. By (1.1), $\mathfrak{a}^\sigma A$ is principal. Thus we choose once and for all, for each prime ideal $\mathfrak{l}$ of r, an element $\{\mathfrak{l}^\sigma\} \in A$ which generates the ideal $\mathfrak{l}^\sigma A$. For a general fractional ideal $\mathfrak{a}$ of F, decomposing $\mathfrak{a}=\prod \mathfrak{l}^{e(\mathfrak{l})}$ as a product of prime ideals $\mathfrak{l}$, we define $\{\mathfrak{l}^v\}=\prod_\sigma \{\mathfrak{l}^{\sigma v_\sigma}\}^{e(\mathfrak{l})}$, which of course generates the ideal $\mathfrak{a}^V A$ in the quotient field of A. The map: $\mathfrak{a}\mapsto\{\mathfrak{a}^v\}$ gives a quasi-character of the ideal group of F into the quotient field of A but may not be a Hecke character. For each $x \in F_A^\times$, we write $\{x^v\}$ for $\{(xr)^v\}$. Then, we shall modify the Hecke operator $[UxU]$ and define $(UxU)\colon S_{k,w,J}(U; B; C) \to S_{k\ w,Jx}(U; B; C)$ by

$$f|(UxU)(g)=\{\nu(x)^v\}^{-1}(f|[UxU](g)),$$

where $\nu\colon G(A)\to F_A^\times$ is the reduced norm map. We fix a maximal order R of B. We suppose hereafter in this paper

$$\hat{R}=R\otimes_r \hat{r}\cong M_2(\hat{r}) \qquad (\hat{r}=r\otimes_Z \hat{Z}). \tag{1.2}$$

This identification will be fixed throughout the paper. Such a quaternion algebra always exists over F. We shall define the following subsemigroups of $M_2(\hat{r})$ for each r-ideal N:

$$\Delta = M_2(\hat{r}) \cap GL_2(F_f), \qquad \Delta(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \,\middle|\, a_N \in r_N^\times,\ c \in N\hat{r}\right\},$$

$$\Delta_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta(N) \,\middle|\, a-1 \in N\hat{r}\right\}, \qquad U_0(N) = GL_2(\hat{r}) \cap \Delta(N),$$

$$U_1(N) = GL_2(\hat{r}) \cap \Delta_1(N), \qquad U(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(N) \,\middle|\, d-1 \in N\hat{r}\right\}.$$

We fix an ideal N and an open compact subgroup S of $GL_2(\hat{r})$ such that $U_0(N) \supset S \supset U(N)$. We also fix a rational prime p outside N and suppose

$$(1.3) \qquad S = GL_2(r_p) \times S^p \qquad \text{for } S^p = \{x \in S \mid x_p = 1\}.$$

Then we put, for each positive integer α,

$$S(p^\alpha) = S \cap U(p^\alpha), \qquad S_0(p^\alpha) = S \cap U_0(p^\alpha) \quad \text{and} \quad S_1(p^\alpha) = S \cap U_1(p^\alpha).$$

Then the correspondence: $S(p^\alpha)xS(p^\alpha) \mapsto (S(p^\alpha)xS(p^\alpha))$ gives a representation of the abstract Hecke ring $R(S(p^\alpha), \Delta(Np^\alpha))$ (cf. [9, 3.1]) into $\mathrm{End}_C(S_{k,w,J}(S(p^\alpha); B; C))$. Let $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ be the A-subalgebra of $\mathrm{End}_C(S_{k,w,J}(S(p^\alpha); B; C))$ generated over A by all operators $(S(p^\alpha)xS(p^\alpha))$ for $x \in \Delta(Np^\alpha)$. Similarly, we can consider, for each finite order character $\varepsilon\colon S_0(p^\alpha)/S_1(p^\alpha) \to A^\times$,

$$S_{k,w,J}(S_0(p^\alpha), \varepsilon; C) = \{f \in S_{k,w,J}(S(p^\alpha); C) \mid f|_{k,w}s = \varepsilon(s)f \text{ for } s \in S_0(p^\alpha)\}$$

and the A-subalgebra $\mathfrak{h}_{k,w}(S_0(p^\alpha), \varepsilon; C)$ of $\mathrm{End}_C(S_{k,w,J}(S_0(p^\alpha), \varepsilon; C))$ generated over A by $(S_0(p^\alpha)xS_0(p^\alpha))$ for all $x \in \Delta(Np^\alpha)$. The change of the map: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\}$ affects to the operator $(S(p^\alpha)xS(p^\alpha))$ only by the multiplication of A-units. Thus the Hecke algebra $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ itself is determined independently of the choice of the map: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\}$. By [8, Theorems 2.1 and 2.2], the algebra $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ is also independent of the choice of B satisfying (1.2) and the subset J of I_B.

Note that $S_0(p^\alpha)/S_1(p^\alpha) \cong (r/p^\alpha r)^\times \times (r/p^\alpha r)^\times$. Put $G^\alpha = S_0(p^\alpha)r^\times/S(p^\alpha)r^\times$. Then G^α naturally acts on $S_{k,w,J}(S(p^\alpha); B; C)$ via the operator:

$$\omega(a_p^{n+2v})^{-1}[S(p^\alpha)xS(p^\alpha)] \qquad \text{for } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(p^\alpha),$$

where $\omega\colon Z_p^\times \to \bar{Q}^\times$ is the Teichmuller character when p is odd, and where ω is the Legendre symbol $\left(\frac{-4}{\ }\right)$ when $p=2$. In fact, if $g \in$

$S_{k,w,J}(S(p^\alpha); B; C)$, then by definition $g(ax)=g(x)$ for $a \in F^\times$ and $g|_{k,w,J}u=g$ for $u \in S(p^\alpha)C_{\infty+}$. Thus we see that for $\varepsilon \in r^\times$, ε_∞ belongs to $C_{\infty+}$ and

$$g|_{k,w}\varepsilon_f = g|_{k,w}\varepsilon = \varepsilon^{2w-k}g = \varepsilon^{n+2v}g.$$

Since $\omega(\varepsilon)=\varepsilon^{n+2v}$ if $n+2v\equiv 0 \bmod \boldsymbol{Z}t$, we know that $\varepsilon \in r^\times$ acts trivially on $S_{k,w,J}(S(p^\alpha); B; C)$. It is obvious that $S_0(p^\alpha)$ normalizes $S(p^\alpha)$ and hence acts on $S_{k,w,J}(S(p^\alpha); B; C)$. Therefore we have the action of $\boldsymbol{G}^\alpha$. For each prime ideal $\mathfrak{l}$ of r, we choose once and for all an element $\lambda \in \hat{r}$ such that $\lambda r=\mathfrak{l}$ and $\lambda_\sigma=1$ for all places σ outside $\mathfrak{l}$. Then we put, as an element in $\mathfrak{h}_{k,w}(S(p^\alpha); A)$,

$$T_0(\mathfrak{l})=(S(p^\alpha)\gamma S(p^\alpha)) \qquad \text{for } \gamma=\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix},$$

$$T_0(\mathfrak{l},\mathfrak{l})=(S(p^\alpha)\lambda S(p^\alpha)) \qquad \text{if } \mathfrak{l} \text{ is prime to } Np^\alpha.$$

If $\mathfrak{l}$ is prime to Np^α, these operators are determined independently of the choice of λ, but when $\mathfrak{l}$ divides Np^α, $T_0(\mathfrak{l})$, may depend on the element λ.

Proposition 1.1. (i) *$\mathfrak{h}_{k,w}(S(p^\alpha); A)$ is commutative if $U_0(N)\supset S\supset U(N)$.*

(ii) *If $U_0(N)\supset S\supset U_1(N)$, then $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ is generated by the operators induced from the action of $\boldsymbol{G}^\alpha$ and $T_0(\mathfrak{l})$ for all prime ideals $\mathfrak{l}$.*

Proof. We first prove the assertion for $U(N)$. In this case, we have $S(p^\alpha)=U(Np^\alpha)$. Taking an element $w \in GL_2(\hat{r})$ such that $w_{Np}=\begin{pmatrix}0 & -1\\ Np^\alpha & 0\end{pmatrix}$ and $w_\sigma=1$ for σ outside Np, we see that

$$w\begin{pmatrix}a & b\\ Np^\alpha c & d\end{pmatrix}=\begin{pmatrix}d & -c\\ -Np^\alpha b & a\end{pmatrix}w.$$

Thus if we define an involution: $x\mapsto x^*$ by $x^*=wx^\iota w^{-1}$ for the main involution ι satisfying, $x^\iota x=\det(x)$, then $\Delta(Np^\alpha)$ and $U(Np^\alpha)$ are stable under this involution "*", and one can verify (see (1.4) and (1.6) below) that

$$(U(Np^\alpha)xU(Np^\alpha))^*=U(Np^\alpha)xU(Np^\alpha) \qquad \text{for all } x \in \Delta(Np^\alpha).$$

Then, by [9, Proposition 3.8], $\mathfrak{h}_{k,w}(U(Np^\alpha); A)$ is commutative. Writing $X_\sigma=\{x_\sigma \mid x \in X\}$ for any subset X of $GL_2(F_f)$ for each place σ of F, we can decompose

$$U(Np^\alpha)=\prod_\sigma U(Np^\alpha)_\sigma \quad \text{and} \quad \Delta(Np^\alpha)=\prod_\sigma \Delta(Np^\alpha)_\sigma.$$

This shows that

$$R(U(Np^\alpha), \Delta(Np^\alpha))=\bigotimes_\sigma R(U(Np^\alpha)_\sigma, \Delta(Np^\alpha)_\xi).$$

For each prime ideal $\mathfrak{l}$ outside Np, it is well known (e.g. [9, III] or [2, Theorem 2.5]) that

$$R(GL_2(r_\mathfrak{l}), M_2(r_\mathfrak{l})\cap GL_2(F_\mathfrak{l}))\cong \boldsymbol{Z}[T(\mathfrak{l}), T(\mathfrak{l}, \mathfrak{l})], \tag{1.4}$$

where $T(\mathfrak{l})=GL_2(r_\mathfrak{l})\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}GL_2(r_\mathfrak{l})$ and $T(\mathfrak{l}, \mathfrak{l})=GL_2(r_\mathfrak{l})\lambda GL(r_\mathfrak{l})$ ($\lambda \in r_\mathfrak{l}$ is such that $\lambda r_\mathfrak{l}=\mathfrak{l}r_\mathfrak{l}$). Now we assume that $\mathfrak{l}$ is a prime factor of Np^α. Write e for the exponent of $\mathfrak{l}$ in Np^α. Define

$$\begin{aligned} B&=\left\{\begin{pmatrix}a & b\\ d & c\end{pmatrix} \in GL_2(r_\mathfrak{l}) \,\middle|\, c \in \mathfrak{l}^e r_\mathfrak{l}\right\},\\ U&=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in B \,\middle|\, a\equiv d\equiv 1 \bmod \mathfrak{l}^e r_\mathfrak{l}\right\},\\ \Delta'&=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in M_2(r_\mathfrak{l})\cap GL_2(F_\mathfrak{l}) \,\middle|\, c \in \mathfrak{l}^e r_\mathfrak{l},\ a \in r_\mathfrak{l}^\times\right\}. \end{aligned} \tag{1.5}$$

Then for each $x \in \Delta'$, we can find an integer $m\geq 0$ so that

$$BxB=B\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}B.$$

Since U is normal subgroup of B, we know that

$$B\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}B=\bigcup\nolimits_u Uu\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}U=\bigcup\nolimits_u U\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}U \quad \text{(disjoint union)},$$

where u runs over a complete representative set for B/U. Note that for $x=u\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}$ as above

$$B\cap x^{-1}Bx=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in B \,\middle|\, b \in \lambda^m r_\mathfrak{l}\right\},$$

$$B\cap x^{-1}Ux=U\cap x^{-1}Bx=U\cap x^{-1}Ux=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U \,\middle|\, b \in \lambda^m r_\mathfrak{l}\right\}.$$

Then we know from [9, 3.1]

$$BxB=\bigcup_v B\begin{pmatrix}1 & v\\ 0 & \lambda^m\end{pmatrix}, \qquad BxU=\bigcup_v B\begin{pmatrix}1 & v\\ 0 & \lambda^m\end{pmatrix}, \tag{1.6}$$

$$UxU=\bigcup_v Uu\begin{pmatrix}1 & v\\ 0 & \lambda^m\end{pmatrix}=\bigcup_v uU\begin{pmatrix}1 & v\\ 0 & \lambda^m\end{pmatrix} \quad \text{(disjoint union)},$$

where v runs over a complete representative set for $r_{\mathfrak{l}}/\lambda^m r_{\mathfrak{l}}$. This shows

(1.7a)
$$U\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}U=\left(U\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}U\right)^m,$$
$$Uu\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}U=UuU\cdot U\begin{pmatrix}1 & 0\\ 0 & \lambda^m\end{pmatrix}U,$$

(1.7b) The correspondence: $UxU\mapsto VxV\mapsto BxB$ induces surjective homomorphisms: $R(U, \Delta')\to R(V, \Delta')\to R(B, \Delta')$ for any subgroup V sitting between U and B.

The commutativity of $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ follows from (1.7b) for $S\supset U(N)$, and as for the generators, we may suppose that $S=U_1(N)$. For primes $\mathfrak{l}$ dividing N, $R(U_1(N)_{\mathfrak{l}}, \Delta')$ is genreated by $T_0(\mathfrak{l})$ and the double cosets $U_1(N)_{\mathfrak{l}}uU_1(N)_{\mathfrak{l}}$ for scalar matrices u insides $U_0(N)_{\mathfrak{l}}$. As for the element of type $U_1(N)_{\mathfrak{l}}uU_1(N)_{\mathfrak{l}}$ for scalar u, one can show in the same manner as the argument above Theorem 3.2 in [8, § 3] that $(S(p^\alpha)uS(p^\alpha))$ for scalar u is contained in the subring of $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ generated by $T_0(\mathfrak{q}, \mathfrak{q})$ for primes $\mathfrak{q}$ outside Np. For the primes $\mathfrak{l}$ dividing p, (1.7a) shows that the cosets UuU for $u\in B$ and $T_0(\mathfrak{l})$ generate $R(U, \Delta')$. Inside $\mathfrak{h}_{k,w}(S(p^\alpha); A)$, the action of G^α gives the operator corresponding to UuU for $u\in B$. $\mathfrak{h}_{k,w}$ Hence $\mathfrak{h}_{k,w}(S(p^\alpha); A)$ is generated by $T_0(\mathfrak{l})$, $T_0(\mathfrak{l}, \mathfrak{l})$ and the action of G^α.

Here we note byproducts of the proof of Proposition 1.1:

Corollary 1.2. *With the notation of* (1.5) *and* (1.7a, b), *we assume that* $\mathfrak{l}$ *divides* p. *Then the ring* $R(V, \Delta')$ *is isomorphic to the polynomial ring generated by the variable* $T(\mathfrak{l})$ *over the group algebra* $\boldsymbol{Z}[B/V]$ *for each subgroup* V *of* B *containing* U.

From (1.6), we know

Corollary 1.3. *We have the following commutative diagram for* $\beta>\alpha>0$:

$$\begin{array}{ccc} S_{k,w,J}(S(p^\alpha); \boldsymbol{C}) & \longrightarrow & S_{k,w,J}(S(p^\beta); \boldsymbol{C}) \\ \big\downarrow{\scriptstyle (S(p^\alpha)xS(p^\alpha)} & & \big\downarrow{\scriptstyle (S(p^\beta)xS(p^\beta))} \\ S_{k,w,J}(S(p^\alpha); \boldsymbol{C}) & \longrightarrow & S_{k,w,J}(S(p^\beta); \boldsymbol{C}) \end{array}$$

for all $x\in\Delta(Np^\beta)$. *Hence the correspondence*:

$$(S(p^\beta)xS(p^\beta))\longmapsto(S(p^\alpha)xS(p^\alpha))$$

for $x \in \Delta(Np^\alpha)$ induces surjective ring homomorphisms:

$$R(S(p^\beta), \Delta(Np^\beta)) \longrightarrow R(S(p^\alpha), \Delta(Np^\alpha))$$
$$\text{and} \quad \mathfrak{h}_{k,w}(S(p^\beta); A) \longrightarrow \mathfrak{h}_{k,w}(S(p^\alpha); A).$$

§ 2. Results on the Hecke algebras

Let Φ be the composite inside $\bar{Q}$ of the images of F under all the embeddings of I. We denote by $\mathcal{O}_\Phi$ the valuation ring of Φ corresponding to the fixed embedding of $\bar{Q}$ into $\bar{Q}_p$. Let $\hat{\Phi}$ be the p-adic closure of Φ in $\bar{Q}_p$ and let K be a finite extension of $\hat{\Phi}$ in $\bar{Q}_p$. We denote by $\mathcal{O}$ the p-adic integer ling of K. Then, $\mathcal{O}$ is naturally an algebra over $\mathcal{O}_\Phi$. We fix a rational prime p and an r-ideal N prime to p. We also fix an open compact subgroup S between $U_0(N)$ and $U_1(N)$ such that $S = GL_2(r_p) \times S^p$. Put $S(p^\alpha) = S \cap U_0(p^\alpha)$, and define groups as in § 1 by

$$G^\alpha = S_0(p^\alpha) r^\times / S(p^\alpha) r^\times, \qquad G = \varprojlim_\alpha G^\alpha.$$

Lemma 2.1. *Put $S_F = S \cap F_f^\times$, $S_F(p^\alpha) = S(p^\alpha) \cap F_f^\times$,*

$$\bar{Z}_0^\alpha = S_F r^\times / S_F(p^\alpha) r^\times \quad \text{and} \quad \bar{Z}_0 = \varprojlim_\alpha \bar{Z}_0^\alpha.$$

Then the map: $S \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a_p^{-1} d_p, a)$ induces isomorphisms:

$$G^\alpha \cong (r/p^\alpha r)^\times \times \bar{Z}_0^\alpha \quad \text{and} \quad G \cong r_p^\times \times \bar{Z}_0.$$

Proof. The maps: $d \mapsto \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in GL_2(r_p)$ and $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto a_p^{-1} d_p$ induce homomorphisms ι: $(r/p^\alpha r)^\times \to G^\alpha$ and ρ: $G^\alpha \to (r/p^\alpha r)^\times$ such that $\rho \circ \iota$ is the identity map. Thus we can decompose $G^\alpha = (r/p^\alpha r)^\times \times X$ with a subgroup X. Since $S \supset U_1(N)$ and $U_1(N) \hat{r}^\times = U_0(N)$, $\iota((r/p^\alpha r)^\times)$ and the image of $S_F r^\times$ generate G^α. This shows that $X \cong \bar{Z}_0^\alpha$. By taking the projective limit with respect to α, we obtain the isomorphism: $G \cong r_p^\times \times \bar{Z}_0$.

We define, for each $\mathcal{O}$-algebra A and for each character ε: $S_0(p^\alpha)/S_1(p^\alpha) \to \mathcal{O}^\times$,

$$\text{(2.1)} \qquad \mathfrak{h}_{k,w}(S(p^\alpha); A) = \mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}_\Phi) \otimes A,$$
$$\mathfrak{h}_{k,w}(S_0(p^\alpha), \varepsilon; A) = \mathfrak{h}_{k,w}(S_0(p^\alpha), \varepsilon; \mathcal{O}_\Phi(\varepsilon)) \otimes_{\mathcal{O}_{\Phi(\varepsilon)}} A,$$

where $\mathcal{O}_\Phi(\varepsilon)$ is the valuation ring, corresponding to the embedding: $\bar{Q} \to \bar{Q}_p$, of the extension of Φ generated by the values of ε.

Lemma 2.2. *If $n \geq 0$ and $v \geq 0$ ($k = n + 2t \geq 2t$ and $w = v + k - t$),* *then* $\mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O})$ *is free of finite rank over* $\mathcal{O}$.

Proof. This fact is well known; so, we shall give only a sketch of a proof. We take $M_2(F)$ as B and consider the locally constant shelf $\mathcal{L}(n, v; A)$ (for any $\mathcal{O}_\Phi$-algebra A) on $X(U)=GL_2(F)\backslash GL_2(F_A)/UC_{\infty+}$ defined in [8, § 6] for each open compact subgroup U of $S(p^\alpha)$. In order to have locally constant sheaves, we need

$$\bar{\Gamma}^i(U) \text{ is torsion-free for all } i, \tag{2.2}$$

where $\bar{\Gamma}^i(U)$ is the arithmetic subgroup of $G_Q=GL_2(F)$ defined in [8, § 2]; i.e., for a fixed representative set $\{t_i\}_i$ for $G_Q\backslash G_A/UG_{\infty+}$, $\Gamma^i(U)=G_Q\cap t_iUG_{\infty+}t_i^{-1}$ and $\bar{\Gamma}^i(U)=\Gamma^i(U)/\Gamma^i(U)\cap F^\times$. We can always find a normal open subgroup V of U satisfying (2.2). Let $H_p^i(X(V), \mathcal{L}(n, v; A))$ be the natural image of the cohomology group on $X(V)$ of compact support in the usual cohomology group. Then, by abusing language a little, for any field extension L of Φ, we write $H_p^i(X(U), \mathcal{L}(n, v; L))$ for $H^0(U/V, H_p^i(X(V), \mathcal{L}(n, v; L)))$. This space is determined independently of the choice of V. Then, by a work of Harder [3] and [4], writing d for $[F: \boldsymbol{Q}]$, we have

$$\begin{aligned} H_p^d(X(U), \mathcal{L}((n, v; \boldsymbol{C}))&=\mathrm{Inv}(U)\oplus(\oplus_J S_{k,w,J}(U; M_2(F); \boldsymbol{C})) \\ &\qquad\qquad \text{if } d \text{ is even and } n=0, \\ H_p^d(X(U), \mathcal{L}((n, v; \boldsymbol{C}))&=\oplus_J S_{k,w,I}(U; M_2(F); \boldsymbol{C}) \\ &\qquad\qquad \text{if either } d \text{ is odd or } n>0, \end{aligned} \tag{2.3}$$

where J runs over all subsets of I and $\mathrm{Inv}(U)$ is the space of $G_{\infty+}$-invariant harmonic forms on $X(U)$ defined in [8, § 6]. This is an analogue in the case where $B=M_2(F)$ of a theorem of Matsushima-Shimura which is given when B is division algebra (e.g. [8, Theorem 6.2]).

On the other hand, by the universal coefficient theorem [1, II, Theorem 18.3], if U satisfies (2.2),

$$H_p^i(X(U), \mathcal{L}(n, v; \mathcal{O}_\Phi))\otimes_{\mathcal{O}_\Phi} A\cong H_p^i(X(U), \mathcal{L}(n, v; A)) \tag{2.4}$$

for any algebra A flat over $\mathcal{O}_\Phi$. Suppose that A is an integral domain of characteristic 0, and denote by L the quotient field of A. For sufficiently large $\beta\geq\alpha$, the group $S(p^\beta)$ satisfies (2.2) ([8, Lemma 7.1]). We take such a β and write U for $S(p^\beta)$. When either $n>0$ or d is odd, we write $H(A)$ for the image of $H_p^d(X(U), \mathcal{L}(n, v; A))$ in $H_p^d(X(S(p^\alpha)), \mathcal{L}(n, v; L))$. When $n=0$ and d is even, it is known (e.g. [5, 4.6], [8, § 7]) that the decomposition (2.3) induces another decomposition:

$$H^d_p(X(S(p^\alpha)), \mathscr{L}(n, v; L)) = \mathrm{Inv}(S(p^\alpha))(L) \oplus S(L)$$

such that

$$\mathrm{Inv}(S(p^\alpha))(\Phi) \otimes_\Phi C \cong \mathrm{Inv}(S(p^\alpha)),\ S(\Phi) \otimes_\Phi C \cong \oplus_J S_{k,w,J}(S(p^\alpha); M_2(F); C)$$
$$\text{and} \quad \mathrm{Inv}(S(p^\alpha))(\Phi) \otimes_\Phi L \cong \mathrm{Inv}(S(p^\alpha))(L),\ S(\Phi) \otimes_\Phi L \cong S(L).$$

Then we define $H(A)$ to be the image of $H^d_p(X(U), \mathscr{L}(n, v; A))$ in $S(L)$. Then we know

(2.5a) $\qquad H(A)$ is stable under the operator $(S(p^\alpha)xS(p^\alpha))$ for $x \in \Delta(Np^\alpha)$,

(2.5b) $\qquad H(A) = H(\mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} A.$

By (2.5a, b), we have an injection:

$$\mathfrak{h}_{k,w}(S(p^\alpha); A) \longrightarrow \mathrm{End}_A(H(A)) \cong (\mathrm{End}_{\mathcal{O}_\Phi}(H(\mathcal{O}_\Phi))) \otimes_{\mathcal{O}_\Phi} A,$$

since $H(\mathcal{O}_\Phi)$ is $\mathcal{O}_\Phi$-free. Thus $\mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}_\Phi)$ is $\mathcal{O}_\Phi$-free and

(2.5c) $\qquad \mathfrak{h}_{k,w}(S(p^\alpha); A) = \mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} A.$

This finishes the proof.

Put $T_0(p) = (S(p^\alpha)x(S(p^\alpha))$ for $x \in M_2(\hat{r})$ such that $x_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $x_\sigma = 1$ for σ outside p. By Lemma 2.2, if A is an $\mathcal{O}$-algebra, we can decompose, as an A-algebra direct sum,

$$\mathfrak{h}_{k,w}(S(p^\alpha); A) = \mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\alpha); A) \oplus \mathfrak{h}^{ss}_{k,w}(S(p^\alpha); A),$$
$$\mathfrak{h}_{k,w}(S_0(p^\alpha), \varepsilon; A) = \mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S_0(p^\alpha), \varepsilon; A) \oplus \mathfrak{h}^{ss}_{k,w}(S_0(p^\alpha), \varepsilon; A),$$

so that the image of $T_0(p)$ in the first factor (i e. the nearly ordinary part) is a unit and the image in the other factor is topologically nilpotent. In our previous paper [8], we wrote $\mathfrak{h}^{\mathrm{ord}}$ instead of $\mathfrak{h}^{n,\mathrm{ord}}$. By the reason explained in the introduction of this paper, we hereafter use the symbol $\mathfrak{h}^{n,\mathrm{ord}}$ when $v \neq 0$ (however, when $v = 0$, we keep the symbol $\mathfrak{h}^{\mathrm{ord}}$ for $\mathfrak{h}^{n,\mathrm{ord}}$, since in this case, $T(p)$ itself is a unit in the ordinary part). By Corollary 1.3, we have natural surjective $\mathcal{O}$-algebra homomorphisms for $\alpha > \beta > 0$:

$$\rho^\alpha_\beta\colon \mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}) \longrightarrow \mathfrak{h}_{k,w}(S(p^\beta); \mathcal{O}),$$
$$\rho^\alpha_\beta\colon \mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\alpha); \mathcal{O}) \longrightarrow \mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\beta); \mathcal{O}),$$

which take $T_0(\mathfrak{l})$ to $T_0(\mathfrak{l})$ and $T_0(\mathfrak{l}, \mathfrak{l})$ to $T_0(\mathfrak{l}, \mathfrak{l})$ for all prime ideals $\mathfrak{l}$ outside p. Define the p-adic Hecke algebras by

$$(2.6)\qquad \mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})=\varprojlim_\alpha \mathfrak{h}_{k,w}(S(p^\alpha);\mathcal{O}),$$
$$\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})=\varprojlim_\alpha \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\alpha);\mathcal{O}).$$

Then $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}((S(p^\infty);\mathcal{O})$ is naturally an algebra factor of $\mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})$.

Theorem 2.3. *There exists an algebra isomorphism for each non-negative n and v with $n+2v\equiv 0 \bmod \boldsymbol{Z}t$:*

$$\mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})\cong\mathfrak{h}_{2t,t}(S(p^\infty);\mathcal{O})\qquad (k=n+2t,\ w=v+k-t),$$

which induces an isomorphism:

$$\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})\cong\mathfrak{h}_{2t,t}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})$$

and which takes $T(\mathfrak{l})$ (resp. $T(\mathfrak{l},\mathfrak{l})$ to $T(\mathfrak{l})$ (resp. $T(\mathfrak{l},\mathfrak{l})$) for all prime ideals $\mathfrak{l}$ prime to p.

This result will be proven in § 4 for $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})$ and in § 5 for larger $\mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})$. Since $\mathfrak{h}_{k,w}(S(p^\alpha);\mathcal{O})$ is an algebra over $\mathcal{O}[\boldsymbol{G}^\alpha]$ and since this $\mathcal{O}[\boldsymbol{G}^\alpha]$-algebra structure is compatible with the projective limit (2.6), $\mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})$ and $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})$ have natural algebra structure over $\mathcal{O}[\![\boldsymbol{G}]\!]$ for $\boldsymbol{G}=r_p^\times\times\bar{Z}_0$. We shall identify $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty),\mathcal{O})$ (resp. $\mathfrak{h}_{k,w}(S(p^\infty);\mathcal{O})$) with $\mathfrak{h}_{2t,t}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})$ (resp. $\mathfrak{h}_{2t,t}(S(p^\infty);\mathcal{O})$) and write the identified algebra as $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$ and $\boldsymbol{h}(S;\mathcal{O})$. We regard $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$ and $\boldsymbol{h}(S;\mathcal{O})$ as $\mathcal{O}[\![\boldsymbol{G}]\!]$-algebras through the action of $\boldsymbol{G}$ of weight $(2t,t)$.

Let $E=\{u\in r^\times\mid u^\sigma>0 \text{ for all } \sigma\in I\}$, and put

$$Z_0^\alpha=ES_F/ES_F(p^\alpha)\quad\text{and}\quad Z_0=\varprojlim_\alpha Z_0^\alpha,$$

where as in Lemma 2.1, $S_F(p^\alpha)=S(p^\alpha)\cap F_f^\times$ and $S_F=S\cap F_f^\times$, By definition, we have a natural surjective homomorphism of Z_0 onto $\bar{Z}_0$ whose kernel is annihilated by 2. Let $\varepsilon\colon Z_0\to\mathcal{O}^\times$ be a finite order character factoring through Z_0^α, and let $\chi\colon Z_0\to\boldsymbol{Z}_p^\times$ be the cyclotomic character; so, far $x\in r_p^\times$, considering it as an element of $\hat{r}^\times$,

$$\chi(x)=x^t=\mathcal{N}_{F/Q}(x)\in\boldsymbol{Z}_p^\times.$$

For each $\lambda\in\boldsymbol{Z}t$, writing $\lambda=[\lambda]t$ for $[\lambda]\in\boldsymbol{Z}$, we define $\chi_\lambda\colon Z_0\to\boldsymbol{Z}_p^\times$ by $\chi_\lambda(x)=\chi(x)^{[\lambda]}$. Let n and v be non-negative elements in $\boldsymbol{Z}[I]$ such that $n+2v\equiv 0 \bmod \boldsymbol{Z}t$ and $\varepsilon\chi_{n+2v}\colon Z_0\to\mathcal{O}^\times$ factors through $\bar{Z}_0$. Then we have a continuous group character:

$$\boldsymbol{G}=r_p^\times\times\bar{Z}_0\ni(a,z)\longmapsto\varepsilon\chi_{n+2v}(z)a^v\in\mathcal{O}^\times,$$

which induces an $\mathcal{O}$-algebra homomorphism $P_{n,v,\varepsilon}: \mathcal{O}[[G]] \to \mathcal{O}$ (by the universality of the continuous group algebra $\mathcal{O}[[G]]$). Thus $\mathrm{Ker}(P_{n,v,\varepsilon})$ is a prime ideal of $\mathcal{O}[[G]]$. Identifying the space $\mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{O}[[G]], \mathcal{O})$ with the subset of $\mathcal{O}$-valued points of $\mathrm{Spec}(\mathcal{O}[[G]])_{/\mathcal{O}}$, we shall use the same symbol $P_{n,v,\varepsilon}$ to indicate the prime ideal which is the kernel of the homomorphism $P_{n,v,\varepsilon}$. Let $\mathcal{O}[[G]]_P$ (resp. $\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O})$) denote the localization of $\mathcal{O}[[G]]$ (resp. $\boldsymbol{h}^{n,\mathrm{ord}}((S; \mathcal{O}))$ at each prime ideal P of $\mathcal{O}[[G]]$. By the definition (2.6), we have a surjective homomorphism:

$$\boldsymbol{h}^{n,\mathrm{ord}}(S; \mathcal{O}) \longrightarrow \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; \mathcal{O})$$

for $k=n+2t$ and $w=v+k-t$. Almost by definition (see (3.3) below), this morphism induces a surjection:

$$\boldsymbol{h}^{n,\mathrm{ord}}(S; \mathcal{O})/P\boldsymbol{h}^{n,\mathrm{ord}}(S; \mathcal{O}) \longrightarrow \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; \mathcal{O}) \qquad \text{for } P=P_{n,v,\varepsilon}.$$

Thus we have a natural surjection:

$$\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O})/P\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O}) \longrightarrow \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; K) \qquad \text{for } P=P_{n,v,\varepsilon},$$

which takes $T(\mathfrak{l})$ (resp. $T(\mathfrak{l}, \mathfrak{l})$) to $T(\mathfrak{l})$ (resp. $T(\mathfrak{l}, \mathfrak{l})$).

We shall now consider another decomposition $\boldsymbol{G}=\boldsymbol{W}\times\boldsymbol{G}_{\mathrm{tor}}$, where $\boldsymbol{G}_{\mathrm{tor}}$ is a finite subgroup of $\boldsymbol{G}$ and $\boldsymbol{W}$ is a torsion-free group. The choice of $\boldsymbol{G}_{\mathrm{tor}}$ is unique but $\boldsymbol{W}$ may not be canonically determined. We fix once and for all a torsion-free subgroup $\boldsymbol{W}$ as above. The subgroup $\boldsymbol{W}$ is independent of S in the sense that if we write $\boldsymbol{G}_{\mathrm{tor}}(S)$ (resp. $\boldsymbol{G}(S)$) for the group $\boldsymbol{G}_{\mathrm{tor}}$ (resp. $\boldsymbol{G}$) for S, the natural homomorphism: $\boldsymbol{G}(S) \to \boldsymbol{G}(GL_2(\hat{r}))$ induces an isomorphism of $\boldsymbol{G}(S)/\boldsymbol{G}_{\mathrm{tor}}(S)$ onto $\boldsymbol{G}(GL_2(\hat{r}))/\boldsymbol{G}_{\mathrm{tor}}(GL_2(\hat{r}))$. Moreover, the group $\boldsymbol{W}$ is isomorphic to the additive group $\boldsymbol{Z}_p^s$ for an integer s with $d<s<2d$ for $d=[F:\boldsymbol{Q}]$. If the Leopoldt conjecture holds for p and F, then $s=d+1$. Thus, if one fix an isomorphism $\boldsymbol{W}\cong\boldsymbol{Z}_p^s$, we have an isomorphism of $\mathcal{O}$-algebras: $\mathcal{O}[[\boldsymbol{W}]] \cong \mathcal{O}[[X_1, \cdots, X_s]]$, where the righthand side is the formal power series ring of s-variables over $\mathcal{O}$.

Theorem 2.4. *Let the notation be as above. Then the nearly ordinary Hecke algebra $\boldsymbol{h}^{n,\mathrm{ord}}(S; \mathcal{O})$ is a torsion-free $\mathcal{O}[[\boldsymbol{W}]]$-module of finite type. Moreover, the natural morphism: $\boldsymbol{h}^{n,\mathrm{ord}}(S; \mathcal{O}) \to \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; \mathcal{O})$ for (n, ν, ε) as above induces an isomorphism:*

$$\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O})/P\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O}) \cong \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; K) \qquad \textit{for } P=P_{n,\nu,\varepsilon},$$

and $\boldsymbol{h}_P^{n,\mathrm{ord}}(S; \mathcal{O})$ is free of finite rank over $\mathcal{O}[[G]]_P$ for $P=P_{n,\nu,\varepsilon}$.

We shall prove this theorem in § 4.

Let $\mathscr{L}$ be the quotient field of $\mathcal{O}[\![W]\!]$. We fix an algebraic closure $\bar{\mathscr{L}}$ of $\mathscr{L}$ and consider $\bar{\boldsymbol{Q}}_p$ as a subfield of $\bar{\mathscr{L}}$ by fixing (once and for all) an embedding of $\bar{\boldsymbol{Q}}_p$ into $\bar{\mathscr{L}}$. Let λ: $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})\to\bar{\mathscr{L}}$ be an $\mathcal{O}[\![W]\!]$-algebra homomorphism. By Theorem 2.4, the image of λ is contained in a finite extension $\mathscr{K}$ of $\mathscr{L}$ in $\bar{\mathscr{L}}$, or more precisely, it is contained in the integral closure I of $\mathcal{O}[\![W]\!]$ inside $\mathscr{K}$. Let

$$\mathscr{X}=\mathscr{X}(I)=\mathrm{Spec}(I)(\bar{\boldsymbol{Q}}_p)=\mathrm{Hom}_{\mathcal{O}\text{-alg}}(I,\bar{\boldsymbol{Q}}_p)$$

and $\mathscr{A}=\mathscr{A}(I)$ be the subset of $\mathscr{X}(I)$ consising of $\mathcal{O}$-algebra homomorphisms P: $I\to\bar{\boldsymbol{Q}}_p$ whose restriction to $\mathcal{O}[\![W]\!]$ is given by $P_{n,v,\varepsilon}|_{\mathcal{O}[[W]]}$ for a finite character ε: $Z_0\to\bar{\boldsymbol{Q}}^\times$ and non-negative pair n, $v\in\boldsymbol{Z}[I]$ such that $n+2v\equiv 0 \bmod \boldsymbol{Z}t$ and $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$. For each $P\in\mathscr{A}(I)$, we write $n(P)$ for n, $v(P)$ for v, $m(P)$ for $n(P)+2v(P)$ andε_P for the restriction of ε to W if $P|_{\mathcal{O}[[G]]}=P_{n,v,\varepsilon}|_{\mathcal{O}[[G]]}$. We write $C(P)$ for the conductor of ε_P in the torsion free part of Z_0; thus, the ideal $C(P)$ divides some power p^α and divisible by every prime factor of p. Let $\bar{Z}_{\mathrm{tor}}$ be the maximal torsion subgroup of $\bar{Z}_0$. Since $\bar{Z}_{\mathrm{tor}}$ is naturally a factor of $\boldsymbol{G}_{\mathrm{tor}}$ by Lemma 2.1, we have a natural character: $\bar{Z}_{\mathrm{tor}}\to\boldsymbol{h}^{n,\mathrm{ord}}(S,\mathcal{O})^\times$. Let ψ be the composite of this character with λ. This character ψ will be called the character of λ. Let μ be the maximal torsion subgroup of $r_p^\times$. Then, we have $\boldsymbol{G}_{\mathrm{tor}}=\bar{Z}_{\mathrm{tor}}\times\mu$ by Lemma 2.1.

Corollary 2.5. *Let λ: $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})\to I$ be an $\mathcal{O}[\![W]\!]$-algebra homomorphism as above, and for each $P\in\mathscr{X}(I)$, put $\lambda_P=P\circ\lambda$: $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})\to\bar{\boldsymbol{Q}}_p$. If $P\in\mathscr{A}(I)$ and if $\lambda|_\mu$ coincides with $\mu\ni\zeta\mapsto\zeta^{\nu(P)}$, then $\lambda_p(T(\mathfrak{n}))$ is algebraic, i.e., $\lambda_P(T(\mathfrak{n}))\in\bar{\boldsymbol{Q}}$ for all ideals $\mathfrak{n}$, and by regarding $\lambda_P(T(\mathfrak{n}))$ as a complex number via the embedding: $\bar{\boldsymbol{Q}}\to\boldsymbol{C}$, we can find a non-trivial common eigenform f_P of all Hecke operators $T(\mathfrak{n})$ in $S_{k,w,J}(U_0(NC(P)),\varepsilon_P\psi\omega^{-[m(P)]};\boldsymbol{C})$ so that $f_P|T(\mathfrak{n})=\lambda_P(T(\mathfrak{n}))f_P$ for all $\mathfrak{n}$, where ω: $Cl(p)\to\bar{\boldsymbol{Q}}^\times$ is the Teichmuller character and we write $k=n(P)+2t$, $w=n(P)+\nu(P)+t$ and $m(P)=[m(P)]t$ for $[m(P)]\in\boldsymbol{Z}$. The cusp form f_P is determined by P up to constant factor. Conversely, if it is given a common eigenform $f\in S_{k,w,J}(U_1(Np^\alpha))$ of all Hecke operators for $\alpha>0$ such that $f|T_0(p)=af$ with $|a|_p=1$, then there exists at least one homomorphism of $\mathcal{O}[\![W]\!]$-algebra λ; $\boldsymbol{h}^{n,\mathrm{ord}}(U_1(N);\mathcal{O})\to\bar{\mathscr{L}}$ and $P\in\mathscr{A}(I)$ such that f is a constant multiple of f_P.*

This fact can be deduced from Theorem 2.4 in exactly the same manner as the proof of [8, Corollary 3.5]; so, we omit the proof.

Let $f\in S_{k,w,J}(U_1(Np^\alpha))$ be a common eigenform of all Hecke operator. We say that f has s-dimensional p-adic deformation over $\mathcal{O}$ if the following conditions are satisfied:

(i) *there exists an algebra I oner $\mathcal{O}[\![X_1, \cdots, X_s]\!]$ which is finite and without torsion over $\mathcal{O}[\![X_1, \cdots, X_s]\!]$ and which is an integral domain.*

(ii) *there exists a Zariski dense subset $\mathcal{A}$ in* $\mathrm{Spec}(I)(\bar{Q}_p)$ *and an element $\Phi_{\mathfrak{n}}$ of I for each ideal $\mathfrak{n}$ prime to p such that there exists a non-zero common eigenform f_P for every $P \in \mathcal{A}$ in $S_{k',w',I}(U_1(Np^\beta))$ (for some β, k' and w') satisfying $f_P|T(\mathfrak{n}) = \Phi_{\mathfrak{n}}(P)f_P$ (this especially means that $\Phi_{\mathfrak{n}}(P) \in \bar{Q}$ for all $P \in \mathcal{A}$) and f is a constant multiple of f_P for one $P \in \mathcal{A}$.*

(iii) *the elements $\Phi_{\mathfrak{n}}$ topologically generate a subring of I with the same quotient field as I.*

From the above corollary, we know that any nearly ordinary common eigenform has s-dimensional deformation for $s = \mathrm{rank}_{Z_p}(W)$. Note here that $s \geq d+1$ for $d = [F: Q]$ and if the Leopoldt conjecture holds for F and p, then $s = d+1$.

§ 3. Result on cohomology group

In this section, we present several result on the Hecke module structure of cohomology groups on Shimura varieties. These results have their own merits but they are also vital to the proof of the main results in § 2.

Let S be an open compact subgroup of $GL_2(\hat{r})$ as in § 2. Once and for all, for each S, we shall fix a decomposition:

$$G^B(A) = \bigcup_{i=1}^h G^B(Q)t_i SG_{\infty+} \quad \text{with } t_i \in G^B(A) \quad \text{(disjoint union).}$$

We may (and will) assume that $(t_i)_{pN} = (t_1)_\infty = 1$ for all i. If B is indefinite, we can choose t_i independently of α so that

$$G^B(A) = \bigcup_{i=1}^h G^B(Q)t_i S_0(p^\alpha)G_{\infty+} \quad \text{(disjoint union)} \quad \text{for all } \alpha > 0.$$

We suppose for the moment that B is indefinite. Note that

$$\begin{aligned}(3.1\text{a}) \qquad & G^B(Q)\backslash G^B(Q)t_i S_0(p^\alpha)G_{\infty+}/S(p^\alpha)G_{\infty+} \\ & \cong G^B(Q)^{t_i} \cap S_0(p^\alpha)G_{\infty+}\backslash S_0(p^\alpha)G_{\infty+}/S(p^\alpha)G_{\infty+},\end{aligned}$$

where $G^B(Q)^{t_i} = t_i^{-1}G^B(Q)t_i$ and the isomorphism is given by $t_i s \mapsto s$. By the strong approximation theorem, if we put

$$(3.1\text{b}) \qquad X = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in S_0(p^\alpha) \,\middle|\, ad \equiv u \bmod p^\alpha \hat{r} \text{ for some } u \in E\right\}$$

(where $E = \{u \in r^\times \mid u^\sigma > 0 \text{ for all } \sigma \in I\}$), then the set (3.1a) is isomorphic to $X\backslash S_0(p^\alpha)/S(p^\alpha)$ and is thus independent of i. Now we shall choose a complete representative set $\mathscr{S}=\mathscr{S}_\alpha$ for

$$G(\boldsymbol{Q})\backslash S_0(p^\alpha)/S(p^\alpha)\cong X\backslash S_0(p^\alpha)/S(p^\alpha).$$

Then we have a decomposition:

$$G^B(A)=\bigcup_{i=1}^h \bigcup_{s\in\mathscr{S}_\alpha} G^B(\boldsymbol{Q})t_isS(p^\alpha)G_{\infty+}.$$

Since each element $s\in\mathscr{S}_\alpha$ normalizes $S(p^\alpha)$,

(3.2) $\qquad t_isS(p^\alpha)\, s^{-1}t_i^{-1}=t_iS(p^\alpha)t_i^{-1}$ is independent of $s\in\mathscr{S}_\alpha$.

As in [8], we put

$$\Gamma^i(p^\alpha)=t_iS(p^\alpha)t_i^{-1}\cap G^B(\boldsymbol{Q}),\qquad \Gamma_0^i(p^\alpha)=t_iS_0(p^\alpha)t_i^{-1}\cap G^B(\boldsymbol{Q}),$$
$$\bar{\Gamma}^i(p^\alpha)=\Gamma^i(P^\alpha)/\Gamma^i(p^\alpha)\cap F^\times,\qquad \bar{\Gamma}_0^i(p^\alpha)=\Gamma_0^i(p^\alpha)/\Gamma_0^i(p^\alpha)\cap F^\times.$$

Then, for X as in (3.1b), we know that

$$\bar{\Gamma}_0^i(p^\alpha)/\bar{\Gamma}^i(p^\alpha)\cong XS(p^\alpha)r^\times/S(p^\alpha)r^\times\hookrightarrow S_0(p^\alpha)r^\times/S(p^\alpha)r^\times=\boldsymbol{G}^\alpha.$$

We write this subgroup of $\boldsymbol{G}^\alpha$ as $\boldsymbol{E}^\alpha$, which is the image of $\bar{\Gamma}_0^i(p^\alpha)/\bar{\Gamma}^i(p^\alpha)$. Then we know that $\boldsymbol{G}^\alpha/\boldsymbol{E}^\alpha\cong\mathscr{S}^\alpha$ as $\boldsymbol{G}^\alpha$-sets.

Let K be a finite extension of $\hat{\Phi}$ inside $\bar{\boldsymbol{Q}}_p$, and let $\mathcal{O}$ be the p-adic integer ring of K. We suppose that

(3.3) $$R\otimes_{\boldsymbol{Z}}\mathcal{O}\cong M_2(\mathcal{O})^I.$$

For each character $\varepsilon\colon Z_0^\alpha\to\mathcal{O}^\times$, we consider the $S_0(p^\alpha)$-modules $L(n, v, \varepsilon; A)$ and $L^*(n, v, \varepsilon; A)$ for $A=K/\mathcal{O}$, $\mathcal{O}/p^r\mathcal{O}$ and $p^{-r}\mathcal{O}/\mathcal{O}$, which are defined in [8, § 7]. Let us recall the definition of these modules. We get $L(n, v, \varepsilon; A)$ and $L^*(n, v, \varepsilon; A)$ by modifying the action of $S_0(p^\alpha)$ on the underlying $S_0(p^\alpha)$-module $L(n, v; A)$ defined in [8, § 1]. We write the action of $u\in S_0(p^\alpha)$ on $m\in L(n, v; A)$ as $m\mapsto m\circ u$ $(=m\circ u_p)$. Then for $m\in L(n, v, \varepsilon; A)$ and $m^*\in L^*(n, v, \varepsilon; A)$, we define a new action by

(*) $$m\cdot u=\varepsilon(a)(m\circ u),\quad m^*\cdot u^\iota=\varepsilon(a)(m^*\circ u^\iota)\quad \text{for } u=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in S_0(p^\alpha).$$

Since we have

$$S_1(p^\alpha)=S\cap U_1(p^\alpha)=S\cap \varDelta_1(p^\alpha)\quad\text{and}\quad S_0(p^\alpha)=S\cap\varDelta(p^\alpha),$$

we know that

$$\varDelta(p^\alpha)/\varDelta_1(p^\alpha)E\cong S_0(p^\alpha)E/S_1(p^\alpha)E\cong Z_0^\alpha.$$

Thus we may consider the $S_0(p^\alpha)$-module $L(n, v, \varepsilon; A)$ (resp. $L^*(n, v, \varepsilon; A)$) as modules over the semi-group $\Delta(p^\alpha)$ (resp. $\Delta(p^\alpha)^\iota$), naturally. To have a $\bar{\Gamma}^i_0(p^\alpha)$-module out of $L(n, v, \varepsilon; A)$, strictly speaking, we have to suppose that $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$. For each $n \in \boldsymbol{Z}[I]$, let $\chi_n\colon r_p^\times \to \mathcal{O}^\times$ denote the character of $r_p^\times$ given by $\chi_n(x) = x^n$. This notation is consistent with the one already used for $n \in \boldsymbol{Z}t$, since if $n \in \boldsymbol{Z}t$, as above induces the character $\chi^{[n]}$ on Z_0. If $A = \mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0 \le \gamma \le \alpha$, we can let $\Delta(p^\alpha)$ act on $L(0, v; A)$ formally by replacing ε in $(*)$ by $\varepsilon\chi_n$. We write this module, with a little abuse of notation, as $L(n, v, \varepsilon\chi_n; A)$. If $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$, $r^\times$ acts on $L(n, v, \varepsilon\chi_n; A)$ through $\varepsilon\chi_{n+2v} \bmod p^\gamma$ and hence acts on it trivially. Thus $L(n, v, \varepsilon\chi_n; A)$ is a well defined module over $\Delta(p)$ as well as $\bar{\Gamma}^i_0(p^\alpha)$ if $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$ and if $A = \mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0 \le \gamma \le \alpha$. Similarly, we can define the $\Delta(p^\alpha)^\iota$-module $L^*(n, v, \varepsilon\chi_n; A)$. Hereafter, we shall always suppose that $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$ when we consider $L(n, v, \varepsilon; A)$ and $L^*(n, v, \varepsilon; A)$ and that $A = \mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0 \le \gamma \le \alpha$ when we handle the modules $L(n, v, \varepsilon\chi_n; A)$ and $L^*(n, v, \varepsilon\chi_n; A)$. Then, naturally, the cohomology group $H^q(\bar{\Gamma}^i_0(p^\alpha), {}^tL(n, v, \varepsilon; A))$ and $H^q(\bar{\Gamma}^i_0(p^\alpha), {}^tL(n, v, \varepsilon\chi_n; A))$ become $\boldsymbol{E}^\alpha$-modules. When $\bar{\Gamma}^i(p^\alpha)$ has no elements of finite order different from the identity for all i, we can consider the sheaf cohomology groups $H^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A))$ and $H^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon\chi_n; A))$ defined in [8, § 7], which are canonically isomorphic to

$$\bigoplus_{i=1}^h H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon; A))^{\mathscr{S}_\alpha}$$
$$\text{and} \quad \bigoplus_{i=1}^h H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon\chi_n; A))^{\mathscr{S}_\alpha},$$

respectively. Here for each module X, $X^{\mathscr{S}_\alpha}$ is the product of copies of X parameterized by the elements of $\mathscr{S}_\alpha$. Even when $\bar{\Gamma}^i(p^\alpha)$ has some non-trivial elements of finite order, by abusing the notation a little, we simply write $H^q((X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A))$ (resp. $H^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon\chi_n; A))$) for $\bigoplus_{i=1}^h H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon; A))^{\mathscr{S}_\alpha}$ (resp. $\bigoplus_{i=1}^h H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon\chi_n; A))^{\mathscr{S}_\alpha}$). These cohomology groups are naturally a right module over the group $\boldsymbol{G}^\alpha$. This shows

Proposition 3.1. *Suppose that B is indefinite, and let A denote one of the modules $K/\mathcal{O}$, $\mathcal{O}/p^\gamma\mathcal{O}$ and $p^{-\gamma}\mathcal{O}/\mathcal{O}$. Then, we have an isomorphisms of $\boldsymbol{G}^\alpha$-module:*

$$H^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A)) \cong \bigoplus_{i=1}^h \mathrm{Ind}_{\boldsymbol{E}^\alpha}^{\boldsymbol{G}^\alpha} H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon; A))$$
$$H^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon\chi_n; A)) \cong \bigoplus_{i=1}^h \mathrm{Ind}_{\boldsymbol{E}^\alpha}^{\boldsymbol{G}^\alpha} H^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon\chi_n; A)),$$

where we assume that $A = \mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0 \le \gamma \le \alpha$ in the latter isomorphism.

Here, for a finite group G and its subgroup H and for a given right H-module M, $\mathrm{Ind}_H^G(M)$ is the space of all functions $f\colon G \to M$ such that $f(gh)=f(g)h$ for $h \in H$. We let G act on $f \in \mathrm{Ind}_H^G(M)$ by $(f|g)(x)=f(gx)$. We also note that an assertion similar to Proposition 3.1 also holds for parabolic cohomology groups

$$H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A)) \quad \text{and} \quad H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon\chi_n; A)).$$

Put

$$\mathscr{Y}_q(n, v, \varepsilon)=\varinjlim_\alpha H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; K/\mathcal{O}))),$$

$$\mathscr{Y}_q^i(n, v, \varepsilon)=\varinjlim_\alpha H_P^q(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v, \varepsilon; K/\mathcal{O})),$$

$$\mathscr{Y}_q(0, v, \varepsilon\chi_n)=\varinjlim_\alpha H_P^q(X(S(p^\alpha)), \mathscr{L}(0, v, \varepsilon\chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})),$$

$$\mathscr{Y}_q^i(0, v, \varepsilon\chi_n)=\varinjlim_\alpha H_P^q(\bar{\Gamma}^i(p^\alpha), {}^tL(0, v, \varepsilon\chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})).$$

Here (and hereafter), we understand that $H_P^q=H^q$ if $X(S(p^\alpha))$ is compact (i.e, B is a division algebra). Then we have

Corollary 3.2. *Suppose that B is indefinite. Then, as $\boldsymbol{E}$-modules, we have*

$$\mathscr{Y}_q(n, v, \varepsilon)\cong \bigoplus_{i=1}^h \mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{G}}(\mathscr{Y}_q^i(n, v, \varepsilon)),$$

$$\mathscr{Y}_q(0, v, \varepsilon\chi_n)\cong \bigoplus_{i=1}^h \mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{G}}(\mathscr{Y}_q^i(0, v, \varepsilon\chi_n)).$$

Here for any right $\boldsymbol{G}$-module M, $\mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{G}}(M)$ denotes the space of all locally constant functions $f\colon \boldsymbol{G} \to M$ satisfying $f(gh)=f(g)h$ for all $h \in \boldsymbol{E}$. We let $\boldsymbol{G}$ act on $\mathrm{Ind}_{\boldsymbol{E}}^{\boldsymbol{G}}(M)$ by $(f|g)(x)=f(gx)$ for $g \in \boldsymbol{G}$.

Hereafter, we do not suppose that B is indefinite and treat general quaternion algebras. We define the modules $\mathscr{Y}_q(n, v, \varepsilon)$ and $\mathscr{Y}_q(0, v, \varepsilon\chi_n)$ by the formulae above Corollary 3.2 even when B is definite. These modules as well as the cohomology group appearing in the definition as above of these modules are equipped with the action of the modified Hecke operators $T_0(\mathfrak{l})$ and $T_0(\mathfrak{l}, \mathfrak{l})$ as introduced in [8, § 7] and usual ones $T(\mathfrak{l})$ and $T(\mathfrak{l}, \mathfrak{l})$. We note that the operators $T_0(\mathfrak{l})$ and $T_0(\mathfrak{l}, \mathfrak{l})$ are different from the usual operators $T(\mathfrak{l})$ and $T(\mathfrak{l}, \mathfrak{l})$ by $\{\mathfrak{l}^v\}$ or $\{\mathfrak{l}^{2v}\}$. We say that two modules equipped with the action of these operators are isomorphic as Hecke modules if there is an isomorphism between them compatible with the action of $T(\mathfrak{l})$ and $T(\mathfrak{l}, \mathfrak{l})$ for all $\mathfrak{l}$. Let U be an open compact subgroup of $GL_2(\hat{r})$ with $S_0(p^\alpha)\supset U\supset S(p^\alpha)$. Let $T_0(p^\beta)=(UxU)$ for $x \in M_2(\hat{r})$ with

$$x_p=\begin{pmatrix}1 & 0\\ 0 & p^\beta\end{pmatrix} \text{ and } x_\sigma=1 \text{ for every place outside } p.$$

We can then define an idempotent e in $\mathrm{End}_A(H_P^q(X(U)), \mathscr{L}(n, v, \varepsilon; A))$ attached to $T_0(p^\beta)$ ($\beta>0$) as in [8, § 8 below (8.5)] for $A=\mathcal{O}$, $K/\mathcal{O}$, $\mathcal{O}/p^\gamma\mathcal{O}$ and $p^{-\gamma}\mathcal{O}/\mathcal{O}$. This idempotent is determined independently of $\beta>0$ and makes the following diagram commutative for $\gamma>\alpha>0$:

$$\begin{array}{ccc} H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A)) & \longrightarrow & H_P^q(X(S(p^\gamma)), \mathscr{L}(n, v, \varepsilon; A)) \\ \downarrow e & & \downarrow e \\ H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A)) & \longrightarrow & H_P^q(X(S(p^\gamma)), \mathscr{L}(n, v, \varepsilon; A)), \end{array}$$

where the horizontal arrows indicate natural restriction maps. Thus there exists a unique idempotent $e\in \mathrm{End}(\mathscr{Y}_q(n, v, \varepsilon))$ which induces the above idempotent on $H_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A))$ for each α. Similarly, we can define the idempotent e for $H_P^q(X(S(p^\alpha)), \mathscr{L}(0, v, \varepsilon\chi_n; A))$ and $\mathscr{Y}_q(0, v, \varepsilon\chi_n)$. We define the nearly ordinary parts of these modules by $eH_P^q(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; A))$, $eH_P^q(X(S(p^\alpha)), \mathscr{L}(0, v, \varepsilon\chi_n; A))$, $e\mathscr{Y}_q(n, v, \varepsilon)$ and $e\mathscr{Y}_q(0, v, \varepsilon\chi_n)$. Since $T_0(p^\beta)$ $(=T_0(p)^\beta)$ for sufficiently large β takes $H_P^q(\Gamma^i(p^\alpha), {}^tL(n, v, \varepsilon; K/\mathcal{O}))$ into itself if B is indefinite (such a β may depend on α), the idempotent e sends $\mathscr{Y}_q^i(n. v, \varepsilon)$ and $\mathscr{Y}_q^i(0, v, \varepsilon\chi_n)$ into themselves.

We shall make use of the following morphisms which are defined in [8, § 8]: For an open compact subgroup U with $S_0(p^\alpha)\supset U\supset S(p^\alpha)$ and for $A=\mathcal{O}$, $K/\mathcal{O}$, $\mathcal{O}/p^\gamma\mathcal{O}$ and $p^{-\gamma}\mathcal{O}/\mathcal{O}$,

$$i_*\colon H_P^q(X(U), \mathscr{L}(n, v, \varepsilon; A))\longrightarrow H_P^q(X(U), \mathscr{L}(0, v, \varepsilon\chi_n; A)),$$
$$j_*\colon H_P^q(X(U), \mathscr{L}^*(0, v, \varepsilon\chi_n; A))\longrightarrow H_P^q(X(U), \mathscr{L}^*(0, v, \varepsilon\chi_n; A)),$$
$$W\colon H_P^q(X(U), \mathscr{L}^*(0, v, \varepsilon\chi_n; A))\longrightarrow H_P^q(X(U), \mathscr{L}^*(0, v, \varepsilon\chi_n; A)),$$
$$[U^\omega\delta U]\colon H_P^q(X(U^\omega), \mathscr{L}^*(n, v, \varepsilon; A))\longrightarrow H_P^q(X(U), \mathscr{L}(n, v, \varepsilon; A)),$$

where for the first three morphisms, we have assumed that $A=\mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0\leq\gamma\leq\alpha$, ω is an element of $M_2(\hat{r})$ such that $\omega_p=\begin{pmatrix}0 & 1\\ -p^\alpha & 0\end{pmatrix}$ and $\omega_\sigma=1$ for σ outside p and $U^\omega=\omega^{-1}U\omega$. Then we have

Theorem 3.3. *Let $A=\mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0\leq\gamma\leq\alpha$, $\varepsilon\colon Z_0^\alpha\to\mathcal{O}^\times$ be a character such that $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$ and U be an open compact subgroup of $GL_2(\hat{r})$ with $S_0(p^\alpha)\supset U\supset S(p^\alpha)$ for $\alpha>0$. Define*

$$\pi=\pi_\alpha\colon H_P^q(X(U), \mathscr{L}(0, v. \varepsilon\chi_n; A))\longrightarrow H_P^q(X(U), \mathscr{L}(n, v, \varepsilon; A)),$$
$$\iota=\iota_\alpha\colon H_P^q(X(U), \mathscr{L}(n, v, \varepsilon; A))\longrightarrow H_P^q(X(U), \mathscr{L}(0, v, \varepsilon\chi_n; A))$$

by $\iota=i_*$ *and* $\pi=[U^{\omega}\delta U]\circ j_*\circ W$. *Then we have*

$$\pi\circ\iota=p^{-\alpha v}\{p^{\alpha v}\}T_0(p^\alpha)\quad on\quad H^q_P(X(U),\ \mathscr{L}(n,v,\varepsilon;A)),$$
$$\iota\circ\pi=p^{-\alpha v}\{p^{\alpha v}\}T_0(p^\alpha)\quad on\quad H^q_P(X(U),\ \mathscr{L}(0,v,\varepsilon\chi_n;A)),$$

where $T_0(p^\alpha)=(UxU)$ *for* $x\in M_2(\hat{r})$ *with*

$$x_p=\begin{pmatrix}1&0\\0&p^\beta\end{pmatrix}\ \text{and}\ x_\sigma=1\ \text{for every place outside}\ p.$$

Moreover, ι *is equivariant under the operators* (UyU) *for* $y\in\Delta(p^\alpha)$.

Although in [8, Theorem 8.1], this theorem is proven for some special class of subgroups U, the proof given in [8, § 8] works well for general U here: so, we omit the details. The following fact is immediate from Theorem 3.3:

Corollary 3.4. *Let A be $\mathcal{O}/p^\gamma\mathcal{O}$ or $p^{-\gamma}\mathcal{O}/\mathcal{O}$ for $0<\gamma\leq\alpha$. Then the morphism ι induces an isomorphism:*

$$eH^q_P(X(U),\ \mathscr{L}(n,v,\varepsilon;A))\cong eH^q_P(X(U),\ \mathscr{L}(0,v,\varepsilon\chi_n;A))$$

for each U with $S_0(p^\alpha)\supset U\supset S(p^\alpha)$.

Corollary 3.5. *The morphism ι induces an isomorphism of Hecke modules for all triple (n,v,ε) with $n\geq 0$, $v\geq 0$ and $n+2v\equiv 0$ mod $\boldsymbol{Z}t$:*

$$e\mathscr{Y}_q(n,v,\varepsilon)\cong e\mathscr{Y}_q(0,v,\varepsilon\chi_n),$$

where we denote by id *the identity character of Z_0. Especially, if r denotes the number of split infinite places for B (i.e. $r=|I_B|$) and if $0\leq r\leq 1$, then $e\mathscr{Y}_r(n,v,\varepsilon)$ is p-divisible.*

Proof. Note that $\mathscr{L}(0,v,\varepsilon\chi_n;p^{-\gamma}\mathcal{O}/\mathcal{O})$ (resp. $\mathscr{L}(n,v,\varepsilon;K/\mathcal{O})$) is naturally isomorphic to the constant sheaf $p^{-\gamma}\mathcal{O}/\mathcal{O}$ (resp. the sheaf $\mathscr{L}(n,v\colon K/\mathcal{O})$) on $X(S(p^\gamma))$ for $\gamma\geqq\alpha$. Thus we see that

$$\begin{aligned}e\mathscr{Y}_q(n,v,\varepsilon)&=\varinjlim_{\gamma\geq\alpha} eH^q_P(X(S(p^\gamma)),\ \mathscr{L}(n,v;K/\mathcal{O}))\\&=\varinjlim_{\gamma}\varinjlim_{\beta} eH^q_P(X(S(p^\gamma)),\ \mathscr{L}(n,v;p^{-\beta}\mathcal{O}/\mathcal{O}))\\&=\varinjlim_{\gamma} eH^q_P(X(S(p^\gamma)),\ \mathscr{L}(n,v;p^{-\gamma}\mathcal{O}/\mathcal{O})))\\&\overset{\iota}{\cong}\varinjlim_{\gamma} eH^q_P(X(S(p^\gamma)),\ p^{-\gamma}\mathcal{O}/\mathcal{O})=e\mathscr{Y}(0,0,\mathrm{id}),\end{aligned}$$

since ι_γ is compatible with the above injective limit by the construction done in [8, § 8]. For sufficiently large γ, $\bar{\Gamma}^i(p^\gamma)$ becomes torsion-free for all i (cf. [8, Lemma 7.1 (i)]). Thus, if $r=1$, $X(S(p^\gamma))$ is smooth for large γ and thus $eH^1_P(X(S(p^\gamma)), K/\mathcal{O})$ is p-divisible. If $r=0$, for sufficiently large γ, $\bar{\Gamma}^i(p^\gamma)$ turns out to be trivial and hence $eH^0_P(X(S(p^\gamma)), K/\mathcal{O})$ is p-divisible for large γ. Since injective limit as a functor is exact, it preserves the p-divisibility and hence $e\mathcal{Y}_q(0, 0, \mathrm{id})$ is p-divisible if $r=0$ or 1. Then the first assertion proves the rest.

Although we can identify $e\mathcal{Y}_q(n, v, \varepsilon)$ for all triples (n, v, ε) by the isomorphism of Corollary 3.5, the action of $\boldsymbol{G}$ on $e\mathcal{Y}_q(n, v, \varepsilon)$ depends on (n, v, ε). Thus, identifying $e\mathcal{Y}_q(n, v, \varepsilon)$ for all (n, v, ε) by Corollary 3.5 and writing the identified module as $\mathcal{Y}_q^{n,\mathrm{ord}}$, we denote by $w \mapsto w|\langle g\rangle_{n,v,\varepsilon}$ the action of $g \in \boldsymbol{G}$ coming from $e\mathcal{Y}_q(n, v, \varepsilon)$. We take as a standard $\boldsymbol{G}$-action on $\mathcal{Y}_q^{n,\mathrm{ord}}$ the action of weight $(0, 0, \mathrm{id})$ and write this action simply as $w \mapsto w|\langle g\rangle$. By identifying $\boldsymbol{G}$ with $r_p^\times \times \bar{Z}_0$ as in Lemma 2.1, we see from the definition of ι (see [8, the formula above (8.2a)])

$$(3.3) \qquad w|\langle a, z\rangle = \varepsilon\chi_{n+2v}(z)a^v(w|\langle a, z\rangle_{n,v,\varepsilon}) \qquad \text{for } a \in r_p^\times \text{ and } z \in \bar{Z}_0.$$

For each $\mathcal{O}$-algebra homomorphism $P\colon \mathcal{O}[\![\boldsymbol{G}]\!] \to \mathcal{O}$ (resp. $P\colon \mathcal{O}[\![\boldsymbol{E}]\!] \to \mathcal{O}$) and for each $\mathcal{O}[\![\boldsymbol{G}]\!]$-module (resp. $\mathcal{O}[\![\boldsymbol{E}]\!]$-module) M, we define

$$M[P] = \{m \in M \mid m|g = P(g)m \text{ for } g \in \mathcal{O}[\![\boldsymbol{G}]\!] \text{ (resp. } g \in \mathcal{O}[\![\boldsymbol{E}]\!])\}.$$

For each triple (n, v, ε) as above, we have a continuous character $P_{n,v,\varepsilon}\colon \boldsymbol{G} \to \bar{\boldsymbol{Q}}_p^\times$ given by $P_{n,v,\varepsilon}(a, z) = \varepsilon\chi_{n+2v}(z)a^v$. This character induces naturally an $\mathcal{O}$-algebra homomorphism: $\mathcal{O}[\![\boldsymbol{G}]\!] \to \bar{\boldsymbol{Q}}_p$, which we denote again by $P_{n,v,\varepsilon}$.

Lemma 3.6. *Let $n, v \in \boldsymbol{Z}[I]$ with $n \geq 0$, $v \geq 0$ and $n+2v \equiv 0 \bmod \boldsymbol{Z}t$ and let $\varepsilon\colon Z_0^\alpha \to \mathcal{O}^\times$ be a character such that $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$ for $\alpha > 0$. Then the restriction map induces isomorphisms:*

$$e\mathcal{Y}_1^i(0, 0, \mathrm{id})[P_{n,\nu,\varepsilon}] \cong eH^1_P(\bar{\Gamma}^i(p^\alpha), {}^tL(n, \nu, K|\mathcal{O})) \qquad \text{if } r=1,$$
$$\mathcal{Y}_0^{n,\mathrm{ord}}[P_{n,\nu,\varepsilon}] \cong eH^0(X(S(p_\nu)), \mathcal{L}(n, v, \varepsilon; K|\mathcal{O})) \qquad \text{if } r=0.$$

Proof. This result can be proven in exactly the same way as in the proof of [8, Theorem 9.4], [6, Theorem 3.1] and [7, Theorem 1.9]; so, we shall give a brief indication of the proof. We shall prove the result only in the case where $r=1$ and $F \neq \boldsymbol{Q}$, since the case: $r=0$ can be treated in much easier way and the case: $F=\boldsymbol{Q}$ has already been treated in [7, Theorem 1.9]. We write simply $e\mathcal{Y}_1^i$ for $e\mathcal{Y}_1^i$ (0, 0, id). By (3.3), we have

$$e\mathcal{Y}_1^i[P_{n\ v,\varepsilon}] = \{x \in e\mathcal{Y}_1^i \mid x|\langle y\rangle_{n,v,\varepsilon} = x \qquad \text{for all } y \in E\}.$$

By Lemma 3.7 below, we know from the argument which proves [8, Corollary 9.3] that if ε factors through Z_0^α, then for all $\beta \geq \alpha$,

$$eH^1(\bar{\Gamma}^i(p^\beta), {}^tL(n, v, \varepsilon; K/\mathcal{O}))^{\Gamma_0^i(p^\beta)} \cong eH^1(\bar{\Gamma}_0^i(p^\beta), {}^tL(n, v, \varepsilon; K/\mathcal{O})).$$

As seen in [8, Proposition 8.3], we have

$$eH^1(\bar{\Gamma}_0^i(p^\beta), {}^tL(n, v, \varepsilon; K/\mathcal{O}))^{\Gamma_0^i(p^\beta)} \cong eH^1(\bar{\Gamma}_0^i(p^\beta), {}^tL(n, v, \varepsilon; K/\mathcal{O})).$$

Thus we have isomorphisms for all β:

$$eH^1(\bar{\Gamma}^i(p^\beta), {}^tL(n, v, \varepsilon; K/\mathcal{O}))^{\Gamma_0^i(p^\beta)} \cong eH^1(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon; K/\mathcal{O})).$$

By taking the injective limit with respect to β, we obtain the result.

We can prove the following result in exactly the same manner as in [8, Lemma 9.2]:

Lemma 3.7. *Suppose that B is indefinite (i.e $r>0$), and let Γ be a subgroup of $\bar{\Gamma}_0^i(p^\alpha)$ containing $\bar{\Gamma}^i(p^\alpha)$. Then the idempotent e annihilates $H^0(\Gamma, M)$ and $H_0(\Gamma, M)$ for $M={}^tL(n, v, \varepsilon; A))$ if A is either $\mathcal{O}/p^\gamma\mathcal{O}$, $p^{-\gamma}\mathcal{O}/\mathcal{O}$, $\mathcal{O}$ or $K/\mathcal{O}$.*

Theorem 3.8. *Suppose that $r=0$ or $r=1$ and that $\bar{\Gamma}^i(S)$ is torsion-free for all i. Let $\mathcal{R}$ be a local ring of the continuous group ring $\mathcal{O}[\![\boldsymbol{G}]\!]$ and $1_{\mathcal{R}}$ denote the idempotent of $\mathcal{R}$ in $\mathcal{O}[\![\boldsymbol{G}]\!]$. Let $Y_r^{n,\mathrm{ord}}$ be the Pontryagin dual module of $\mathcal{Y}_r^{n,\mathrm{ord}}$ and put $Y_r^{n,\mathrm{ord}}(\mathcal{R})=1_{\mathcal{R}}(Y_r^{n,\mathrm{ord}})$. Then $Y_r^{n,\mathrm{ord}}(\mathcal{R})$ is $\mathcal{R}$-free of finite rank. Especially $Y_r^{n,\mathrm{ord}}$ is free of finite rank over $\mathcal{O}[\![\boldsymbol{W}]\!]$ for the torsionfree part $\boldsymbol{W}$ of $\boldsymbol{G}$.*

To prove this theorem, we prepare several lemmas.

Lemma 3.9. *The subset of* $\mathrm{Spec}(\mathcal{O}[\![\boldsymbol{G}]\!])(\bar{\boldsymbol{Q}}_p)$ *consisting of the points of the form $P_{n,\nu,\varepsilon}$ for all triples (n, ν, ε) is Zariski dense.*

Proof. Let Ω denote the p-adic completion of $\bar{\boldsymbol{Q}}_p$, and write $\mathcal{P}(r_p^\times)$ for the space of all functions Φ on $r_p^\times$ of the form

$$\Phi(a)=\sum_v c_v a^v \qquad \text{with } c_v \in \Omega$$

($c_v \neq 0$ only for finitely many v). For each topological space T, let $C(T; \Omega)$ denote the Banach spaces consisting of all continuous functions on T with values in Ω. We equip $C(T; \Omega)$ the uniform norm $\| \quad \|$. We denote by $LC(T, \Omega)$ the space of all locally constant functions in $C(T; \Omega)$. Since $\mathcal{P}(r_p^\times)$ (resp. $LC(\bar{Z}_0; \Omega)$) is dense in $C(r_p^\times; \Omega)$ (resp. $C(\bar{Z}_0; \Omega)$)

and since $C(r_p^\times; \Omega)\otimes_\Omega C(\bar{Z}_0; \Omega)$ is dense in $C(\boldsymbol{G}; \Omega)$, the space $V_0 = \mathscr{P}(r_p^\times)\otimes LC(\bar{Z}_0; \Omega)$ is dense in $C(\boldsymbol{G}; \Omega)$. For any $f \in V_0$, we can write $f(a, z)=\sum_v f_v(z)a^v$ with $f_v \in LC(\bar{Z}_0; \Omega)$. We take $\xi \in \boldsymbol{Z}t$ sufficiently large so that $\xi \geq 2v$ if $f_v \neq 0$. For any given real number $\delta > 0$, we can find $f_{v,\delta} \in LC(\bar{Z}_0; \Omega)$ so that $\|f_{v,\delta} - f_v\chi_\xi^{-1}\| < \delta$. Then we know that

$$\|\chi_\xi(z)^{-1}f(a, z) - \sum_v f_{v,\delta}(z)a^v\| < \delta \quad \text{and} \quad \|f(a, z) - \sum_v f_{v,\delta}(z)\chi_\xi(z)a^v\| < \delta.$$

Since $LC(\bar{Z}_0; \Omega)$ is spanned by finite order characters of $\bar{Z}_0$, by taking $\xi - 2v$ as n, the above argument shows that the subspace V_1 of $C(\boldsymbol{G}; \Omega)$ spanned by the functions: $\boldsymbol{G} \ni (a, z) \mapsto \varepsilon\chi_{n+2v}(z)a^v$ for $v \geq 0$, $n \geq 0$ with $n+2v \equiv 0 \bmod \boldsymbol{Z}t$ and for finite order characters $\varepsilon\colon Z_0 \to \Omega^\times$ is dense in $C(\boldsymbol{G}; \Omega)$. We shall show, by using the density of V_1 in $C(\boldsymbol{G}; \Omega)$,

$$(*) \qquad \bigcap\nolimits_{n,v,\varepsilon} \mathrm{Ker}(P_{n,v,\varepsilon}) = \{0\} \qquad \text{in } \mathcal{O}[\![\boldsymbol{G}]\!],$$

from which the lemma follows. Note that $\mathcal{O}[\![\boldsymbol{G}]\!]$ is canonically embedded

into the space of all bounded measures on $\boldsymbol{G}$ so that for $\mu \in \mathcal{O}[\![\boldsymbol{G}]\!]$, writing $d\mu$ for the measure corresponding to μ.

$$P_{n,v,\varepsilon}(\mu) = \int_{\boldsymbol{G}} \varepsilon\chi_{n+2v}(z)a^v d\mu(a, z).$$

If $\mu \in \bigcap_{n,v,\varepsilon} \mathrm{Ker}(P_{n,v,\varepsilon})$, then $\int_{\boldsymbol{G}} \varepsilon\chi_{n+2v}(z)a^v d\mu(a, z) = 0$ for all (n, v, ε). Then the density of V_1 in $C(\boldsymbol{G}; \Omega)$ shows the vanishing of $d\mu$, and hence we know $(*)$.

Lemma 3.10. *Let $\mathscr{R}$ be a local ring of $\mathcal{O}[\![\boldsymbol{G}]\!]$, and let M be an $\mathscr{R}$-module of finite type. Suppose that whenever $P_{n,v,\varepsilon}$ (for $n \geq 0$, $v \geq 0$ with $n+2v \equiv 0 \bmod \boldsymbol{Z}t$ and for a finite order character $\varepsilon\colon Z_0 \to \bar{\boldsymbol{Q}}_p^\times$ such that $\varepsilon\chi_{n+2v}$ factors through $\bar{Z}_0$) belongs to* $\mathrm{Spec}(\mathscr{R})(\bar{\boldsymbol{Q}}_p)$ $(=\mathrm{Hom}^{\mathcal{O}\text{-alg}}(\mathscr{R}, \bar{\boldsymbol{Q}}_p))$, *$M/\mathrm{Ker}(P_{n,v,\varepsilon})M$ is $\mathcal{O}$-free. Then M is free of finite rank over $\mathscr{R}$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{R}$ and write s for $\dim_{\mathscr{R}/\mathfrak{m}} M/\mathfrak{m}M$. Then we have a surjective morphism of $\mathscr{R}$-module $\varphi\colon \mathscr{R}^s \to M$. Since the image $\mathcal{O}_{n,v,\varepsilon}$ in $\bar{\boldsymbol{Q}}_p$ of $\mathscr{R}$ under $P_{n,v,\varepsilon}$ is an $\mathcal{O}$-algebra which is $\mathcal{O}$-free of finite rank, $M/\mathrm{Ker}(P_{n,v,\varepsilon})M$ is $\mathcal{O}_{n,v,\varepsilon}$-free of rank s. This shows that φ induces an isomorphism: $(\mathcal{O}_{n,v,\varepsilon}) \cong M/\mathrm{Ker}(P_{n,v,\varepsilon})M$. Thus $\mathrm{Ker}(\varphi)$ is contained in the intersection of $\mathrm{Ker}(P_{n,v,\varepsilon})^s$ for all $P_{n,v,\varepsilon}$ in $\mathrm{Spec}(\mathscr{R})$, which is reduced to $\{0\}$ by Lemma 3.9. Therefore, we now know that M is $\mathscr{R}$-free.

Proof of Theorem 3.8. We firstly suppose that $F \neq \boldsymbol{Q}$ and $[F: \boldsymbol{Q}]$ is odd. In this case, we have $r=1$ since B is unramified everywhere at finite places. For each finite character $\varepsilon_0: Z_0^\alpha \to \mathcal{O}^\times$ such that $\varepsilon_0 \chi_{n+2v}$ factors through $\bar{Z}_0$, we know from [9, Proposition 8.2] that

$$H^2(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; \mathcal{O})) \cong H_0(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; \mathcal{O})),$$

and thus, by Lemma 3.7,

$$eH^2(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v_0, \varepsilon_0; \mathcal{O})) = 0.$$

From the exact sequence:

$$\begin{aligned} eH^1(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; K)) &\longrightarrow eH^1(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; K/\mathcal{O})) \\ &\longrightarrow eH^2(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; \mathcal{O})), \end{aligned}$$

we know that $e\mathcal{Y}_1^i[P_{n,v,\varepsilon_0}] \cong eH^1(\bar{\Gamma}_0^i(p^\alpha), {}^tL(n, v, \varepsilon_0; K/\mathcal{O}))$ is p-divisible, where we have written $e\mathcal{Y}_1^i$ for $e\mathcal{Y}_1^i(0, 0, \mathrm{id})$. Then, by Proposition 3.1 and Corollary 3.2,

$$\mathcal{Y}_1^{n,\mathrm{ord}}[P_{n,v,\varepsilon_0}] = \{w \in \mathcal{Y}_1^{n,\mathrm{ord}} \mid w|\langle g \rangle = P_{n,v,\varepsilon_0}(g)w \text{ for } g \in \boldsymbol{G}\}$$

is p-divisible. By the Pontryagin duality, $Y_1^{n,\mathrm{ord}}(\mathcal{R})/\mathrm{Ker}(P_{n,v,\varepsilon_0})Y_1^{n,\mathrm{ord}}(\mathcal{R})$ is $\mathcal{O}$-free of finite rank. Write s for the $\mathcal{O}$-rank of this module. Then we have a surjective homomorphism of $\mathcal{R}$-modules $\varphi: \mathcal{R}^s \to Y_1^{n,\mathrm{ord}}(\mathcal{R})$ and $\mathrm{Ker}(\varphi)$ is contained in $\mathrm{Ker}(P_{n,v,\varepsilon_0})^s$ by the proof of Lemma 3.10.

Now we take another finite order character $\varepsilon: Z_0 \to \bar{\boldsymbol{Q}}_p^\times$ (whose values may not be contained in $\mathcal{O}$). Let $K(e)$ denote the extension of K inside $\bar{\boldsymbol{Q}}_p$ generated by the values of ε and $\mathcal{O}(\varepsilon)$ be the p-adic integer ring of $K(\varepsilon)$. Since $\mathcal{O}(\varepsilon)[\![\boldsymbol{G}]\!]$ is naturally isomorphic to $\mathcal{O}[\![\boldsymbol{G}]\!] \otimes_{\mathcal{O}} \mathcal{O}(\varepsilon)$, $\mathcal{R}_\varepsilon = \mathcal{R} \otimes_{\mathcal{O}} \mathcal{O}(\varepsilon)$ is a local ring of $\mathcal{O}(\varepsilon)[\![\boldsymbol{G}]\!]$. The module $Y_r^{n,\mathrm{ord}}(\mathcal{R})$ is obtained relative to the base ring $\mathcal{O}$. If it is necessary to indicate its dependence on the base ring $\mathcal{O}$, we write $Y_r^{n,\mathrm{ord}}(\mathcal{R})_{/\mathcal{O}}$ instead of $Y_r^{n,\mathrm{ord}}(\mathcal{R})$. Thus we can consider $Y_1^{n,\mathrm{ord}}(\mathcal{R}_\varepsilon)_{/\mathcal{O}(\varepsilon)}$ which is naturally isomorphic to $Y_1^{n,\mathrm{ord}}(\mathcal{R}) \otimes_{\mathcal{O}} \mathcal{O}(\varepsilon)$ (e.g. [8, Theorem 6.3]).

Note that $P_{n,v,\varepsilon}: \mathcal{R} \to \bar{\boldsymbol{Q}}_p$ actually has values in $\mathcal{O}(\varepsilon)$. We denote by $\mathcal{O}_{n,v,\varepsilon}$ the subring of $\mathcal{O}(\varepsilon)$ which is the image of $P_{n,v,\varepsilon}$. Since $\mathcal{O}_{n,v,\varepsilon}$ is $\mathcal{O}$-free, we have a natural inclusion:

$$\mathcal{O}(\varepsilon) \otimes_{\mathcal{O}} \mathcal{O}_{n,v,\varepsilon} \to K(\varepsilon) \otimes_{\mathcal{O}} \mathcal{O}_{n,v\ \varepsilon} = K(\varepsilon) \otimes_K K(\varepsilon) \cong K(\varepsilon)^{\mathrm{Gal}(K(\varepsilon)/K)}.$$

From this, we know that $\mathcal{O}(\varepsilon) \otimes_{\mathcal{O}} \mathcal{O}_{n,v,\varepsilon}$ is a reduced $\mathcal{O}$-algebra (i.e., it has no nilpotent radical). We write $\varepsilon^\sigma: \mathcal{O}(\varepsilon) \otimes_{\mathcal{O}} \mathcal{O}_{n,v,\varepsilon} \to K(\varepsilon)$ for the projection corresponding to $\sigma \in \mathrm{Gal}(K(\varepsilon)/K)$, write simply $\mathfrak{a}$ for $\mathrm{Ker}(P_{n,v,\varepsilon})$ in $\mathcal{R}$

and write $\mathfrak{a}_\sigma$ for the kernel of the morphism:

$$\mathcal{R}_s=\mathcal{R}\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)\longrightarrow\mathcal{O}(\varepsilon)\otimes_{\mathcal{O}}\mathcal{O}_{n,v,\varepsilon}\longrightarrow\mathcal{O}(\varepsilon),$$

where the first arrow is given by $P_{n,v,\varepsilon}\otimes\mathrm{id}$ and the second is induced by ε^σ. Since we have an exact sequence:

$$0\longrightarrow\mathfrak{a}\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)\longrightarrow\mathcal{R}_\varepsilon\longrightarrow\mathcal{O}(\varepsilon)\otimes_{\mathcal{O}}\mathcal{O}_{n,v,\varepsilon}\longrightarrow 0,$$

we know that $\mathfrak{a}\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)=\bigcap_{\sigma\in\mathrm{Gal}(K(\varepsilon)/K)}\mathfrak{a}_\sigma$. Since φ yields an exact sequence: $0\to\mathrm{Ker}(\varphi)\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)\to(\mathcal{R}_\varepsilon)^s\to Y_1^{n,\mathrm{ord}}(\mathcal{R})\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)\to 0$ and since $Y_1^{n,}(\mathcal{R}_\varepsilon)_{/\mathcal{O}(\varepsilon)}\cong Y_1^{n,\mathrm{ord}}(\mathcal{R})\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)$, we know, from the same argument as in the case of ε_0, that $\mathrm{Ker}(\varphi)\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)$ is contained in $(\mathfrak{a}_\sigma)^s$ for all $\sigma\in\mathrm{Gal}(K(\varepsilon)/K)$, and hence $\mathrm{Ker}(\varphi)\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)$ is contained in $\mathfrak{a}^s\otimes_{\mathcal{O}}\mathcal{O}(\varepsilon)$. Since $\mathcal{O}(\varepsilon)$ is faithfully flat over $\mathcal{O}$, we conclude that $\mathrm{Ker}(\varphi)$ is contained in $\mathfrak{a}^s$ $(=\mathrm{Ker}(P_{n,v,\varepsilon})^s)$ whenever $P_{n,v,\varepsilon}$ belongs to $\mathrm{Spec}(\mathcal{R})$. Thus $Y_1^{n,\mathrm{ord}}(\mathcal{R})/\mathrm{Ker}(P_{n,v,\varepsilon})Y_1^{n,\mathrm{ord}}(\mathcal{R})$ is $\mathcal{O}$-free, and thus, we know from Lemma 3.10 that $Y_1^{n,\mathrm{ord}}(\mathcal{R})$ is $\mathcal{R}$-free of rank s.

Next, we suppose that $F=\boldsymbol{Q}$. In this case, $\boldsymbol{E}=\boldsymbol{Z}_p^\times/\{\pm 1\}$ or $\boldsymbol{Z}_p^\times$ according as $-1\in S$ or not. Thus $\boldsymbol{E}$ is isomorphic to $\bar{Z}_0$. If $p>2$, $\mathcal{R}\cong\mathcal{O}[\![\Gamma\times\Gamma]\!]$ for $\Gamma=1+p\boldsymbol{Z}_p$ in $\boldsymbol{Z}_p^\times$, and if $p=2$ $\mathcal{R}\cong\mathcal{O}[\![\boldsymbol{G}]\!]$. In [6, Theorem 3.1], we prove that the Pontryagin dual module eY_1^i of $e\mathcal{Y}_1^i$ is $\mathcal{O}[\![\Gamma]\!]$-free of finite rank if $p\geq 5$ and $S=U_1(N)$. The same argument works well even for $p=2$ or 3 and for arbitrary $S\supset U_1(N)$ provided that $\bar{\Gamma}^i(S)$ is torsion-free. The details can be found in [8, Corollary 10.3]. Therefore eY_1^i is $\mathcal{O}[\![\Gamma]\!]$-free if $p>2$ and eY_1^i is $\mathcal{O}[\![\boldsymbol{E}]\!]$-free if $p=2$. Since $\mathcal{Y}_1^{n,\mathrm{ord}}\cong\mathrm{Ind}_E^G(e\mathcal{Y}_1^i)$ by Corollary 3.2, we know that $Y_1^{n,\mathrm{ord}}(\mathcal{R})$ is $\mathcal{R}$-free.

Finally, we suppose that $[F\colon\boldsymbol{Q}]$ is even. Then we know that $r=0$. In this case, under the assumption that $\bar{\Gamma}_0^i(S)$ is torsion-free for all i, we know that $\bar{\Gamma}_0^i(p^\alpha)=\{1\}$ for all α and i. Thus we have

$$H^0(\bar{\Gamma}^i(p^\alpha),{}^tL(n,v,\varepsilon;K/\mathcal{O}))\cong{}^tL(n,v,\varepsilon;K/\mathcal{O}).$$

Thus, by Lemma 3.6, if ε has values in $\mathcal{O}$, $\mathcal{Y}_0^{n,\mathrm{ord}}(\mathcal{R})[P_{n,v,\varepsilon}]$ is p-divisible. Then the same argument as in the case where $r=1$ proves the result.

§ 4. Proof of Theorems 2.3 and 2.4 for nearly ordinary Hecke algebras

In this and the following section, we take a quaternion algebra B/F with $r=0$ or $r=1$ and which is unramified at all finite places of F. We take a finite extension K of $\hat{\Phi}$ inside $\bar{\boldsymbol{Q}}_p$ and let $\mathcal{O}$ be the p-adic integer ring of K. Fix a maximal order R of B, and we identify $R\otimes_{\boldsymbol{Z}}\mathcal{O}$ with $M_2(\mathcal{O})^I$ as in (3.3). We note that $r\equiv[F\colon\boldsymbol{Q}]\bmod 2$. When $r=0$, $H^0(X(S(p^\alpha)),\mathcal{L}(n,v;K/\mathcal{O}))$ is naturally isomorphic to the space

$S_{k,w}(S(p^\alpha); K/\mathcal{O})$ consisting of functions f: $G(A)\to L(n, v, K/\mathcal{O})$ satisfying

$$(*)\qquad f(axs)=f(x)s_p \text{ for all } a\in G^B(\mathbf{Q}) \text{ and } s\in S(p^\alpha)G_\infty.$$

In either case: $r=0$ or 1, it is known (e.g. [8, Lemma 7.1]) that

$$(4.1)\qquad \bar{\Gamma}^i(p^\alpha) \text{ is torsion-free for all } i \text{ if } \alpha \text{ is sufficiently large.}$$

Thus, if α is large and if $r=0$, then $\bar{\Gamma}^i(p^\alpha)=\{1\}$ and $H^0(X(S(p)), \mathscr{L}(n, v; K/\mathcal{O}))$ is isomorphic to the space of all functions on the finite set $X(S(p_v))$ with values in $L(n, v; K/\mathcal{O})$ ([8, (2.6a)]) and hence is p-divisible. When $r=1$, if (4.1) is satisfied (and thus, if α is sufficiently large), then by [9, Proposition 8.2],

$$H^2(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v; K/\mathcal{O}))\cong H_0(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v; K/\mathcal{O})).$$

Thus, by Lemma 3.7, $eH^2(\bar{\Gamma}^i(p^\alpha), {}^tL(n, v; K/\mathcal{O}))=0$. Then, in the same manner as in the proof of Theorem 3.8, we know that

$$eH^1(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O})) \quad \text{is } p\text{-divisible.}$$

When $B=M_2(\mathbf{Q})$, it is known that $H^1_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O}))$ is p-divisible (cf. [7, § 3]; in [7] the p-divisibility is proven for $S=U_1(N)$, but the argument there works well for general S). Thus, in each case, we know that, if α is sufficiently large,

$$(4.2\text{a})\qquad eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O})) \quad \text{is } p\text{-divisible,}$$

$$\begin{aligned}(4.2\text{b})\qquad &\text{The sequence: } 0\to eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; \mathcal{O}))\\ &\longrightarrow eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K))\\ &\longrightarrow eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O}))\longrightarrow 0 \quad \text{is exact.}\end{aligned}$$

We now take a finite extension K_0 of Φ inside K such that $B\otimes_{\mathbf{Q}}K_0\cong M_2(K_0)^I$. Let $\mathcal{O}_0=K_0\cap\mathcal{O}$. Then by the universal coefficient theorem ([1, II, Theorem 18.3] and [8, Theorem 6.3]), if (4.1) is satisfied, then

$$(4.3\text{a})\qquad H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; \mathcal{O}))\cong H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; \mathcal{O}_0))\otimes_{\mathcal{O}_0}\mathcal{O}.$$

On the other hand, we have, for any $\alpha\geq 0$,

$$(4.3\text{b})\qquad H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; \mathcal{O}_0))\otimes_{\mathcal{O}_0}\mathbf{C}\cong H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; \mathbf{C})).$$

By a theorem of Matsushima and Shimura (e.g. [8, Theorem 6.2], [9, Theorem 8.4]), (4.3c) $H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; C)) \cong S_{k,w,\phi}(S(p^\alpha); C)$ if $r=0$ and $n \neq 0$,

$H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; C)) \cong S_{k,w,I_B}(S(p_\alpha); C) \oplus S_{k,w,\phi}(S(p^\alpha); C)$ if $r=1$,

$H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; C)) \cong \mathrm{Inv}(S(p^\alpha)) \oplus S_{k,w,\phi}(S(p^\alpha); C)$ if $r=n=0$,

where $\mathrm{Inv}(S(p^\alpha))$ is the space of functions f: $G(\boldsymbol{Q})\backslash G(\boldsymbol{A}) \to \boldsymbol{C}$ satisfying $f(xu)=f(x)\nu(u_\infty)^v$ for all $u \in S(p^\alpha)G_\infty$. Thus, if $n \neq 0$ or $r=1$,

(4.4) $\mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\alpha); \mathcal{O})$ can be naturally embedded into $\mathrm{End}_{\mathcal{O}}(eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O})))$ if α is sufficiently large.

Now we shall see that (4.4) holds even if $n=r=0$ with some modification. Define, for $A=\mathcal{O}$, $\mathcal{O}_0$ and $K/\mathcal{O}$, $\mathrm{Inv}(S(p^\alpha); A)$ to be a subspace of functions on $G(\boldsymbol{A})$ with values in $L(0, v; A)$ satisfying $(*)$ and of the form: $f=f_0 \circ \nu$ for a function f_0: $F^\times_{\boldsymbol{A}+} \to L(0, v; A)$. If $f \in \mathrm{Inv}(S(p^\alpha); K/\mathcal{O})$ with $f=f_0 \circ \nu$ for f_0: $F^\times_{\boldsymbol{A}+} \to K/\mathcal{O}$, then

(4.5) $f_0(axu)=f_0(x)\chi_v(u_p)$ for $a \in \nu(B^\times)$ and $u \in \nu(S(p^\alpha))F^\times_{\infty+}$,

where $F^\times_{\infty+}$ is the identity component of $F^\times_\infty$. As seen in [8, §11], $\nu(B^\times)$ is the set $F^\times_+$ of all totally positive elements of F. Thus the reduced norm map ν induces a surjection of $G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/S(p^\alpha)G_{\infty+}$ onto $F^\times_+\backslash F^\times_{\boldsymbol{A}+}/\nu(S(p^\alpha))F^\times_{\infty+}$, and therefore, if we write $\mathscr{T}_\alpha$ for the space of functions on $F^\times_{\boldsymbol{A}+}$ satisfying (4.5), then the map: $f_0 \mapsto f_0 \circ \nu$ induces an isomorphism ν^*: $\mathscr{T}_\alpha \cong \mathrm{Inv}(S(p^\alpha); K/\mathcal{O})$. If we define another function $\nu^* f_0 | x$ on $G(\boldsymbol{A})$ (for $x \in G(\boldsymbol{A})$) by $(\nu^* f_0|x)(y)=\nu^* f_0(yx^{-1})\chi_v(\nu(x_p))$, $\nu^* f_0|x$ depends only on the double coset $(S(p^\alpha)xS(p^\alpha))$ but not on each x because of the fact that $\nu \in \boldsymbol{Z}t$. Thus, by the definition of the Hecke operator given in [8, § 2], we know that

$$\nu^* f_0 | T_0(p) = p^m \nu^* f_0 | x \qquad \text{for } p^m = |S(p^\alpha)\backslash S(p^\alpha)xS(p^\alpha)|,$$

where $x \in M_2(\hat{r})$ such that $x_p=\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ and $x_\sigma=1$ for σ outside p. This shows that the idempotent e annihilates $\mathrm{Inv}(S(p^\alpha); K/\mathcal{O})$, and thus (4.4) still holds even when $n=r=0$.

Since the restriction of operators in $\mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\beta); \mathcal{O})$ to the submodule $eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O}))$ for each $\beta > \alpha$ gives the transition morphism of $\mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\beta); \mathcal{O})$ onto $\mathfrak{h}^{n,\mathrm{odr}}_{k,w}(S(p^\alpha); \mathcal{O})$, by taking the limit of the morphisms of $\mathfrak{h}^{n,\mathrm{ord}}_{k,w}(S(p^\alpha); \mathcal{O})$ into $\mathrm{End}_{\mathcal{O}}(eH^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O})))$, we obtain the natural algebra homomorphism

$$\iota\colon \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})\longrightarrow \mathrm{End}_{\mathcal{O}}(e\mathcal{Y}_r(n,v,\mathrm{id})).$$

In the same manner as in the proof of Lemma 3.6, we know that the natural restriction map: $eH_P^r(X(S(p^\alpha)),\mathcal{L}(n,v;K/\mathcal{O}))\to e\mathcal{Y}_r(n,v,\mathrm{id})$ is injective (cf. [8, Theorem 9.4]). Thus, if $\iota(h)=0$, then the image of h in $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\alpha);\mathcal{O})$ vanishes for all sufficiently large α and hence $h=0$ because $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})=\varprojlim_\alpha \mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\alpha);\mathcal{O})$; namely, ι is injective. Since $e\mathcal{Y}_r(n,v,\mathrm{id})$ is independent of (n,v,id) as Hecke modules and $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p_\nu);\mathcal{O})$ is topologically generated by operators $T_0(\mathfrak{l})$, $T_0(\mathfrak{l},\mathfrak{l})$ and the action of the diagonal elements of $U_0(N)$ (cf. Proposition 1.1), $\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S(p^\infty);\mathcal{O})$ is independent of k and w as a subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathcal{Y}_r^{n,\mathrm{ord}})$. This shows Theorem 2.3 for the nearly ordinary part.

Now we prove Theorem 2.4. If $\bar{\Gamma}^i(S)$ is torsion free for all i, then by Theorem 3.8, the Pontryagin dual module $Y_r^{n,\mathrm{ord}}$ of $\mathcal{Y}_r^{n,\mathrm{ord}}$ is $\mathcal{O}[\![W]\!]$-free of finite rank. Thus the subalgebra $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$ of $\mathrm{End}_{\mathcal{O}[[W]]}(Y_r^{n,\mathrm{ord}})$ is a torsion-free $\mathcal{O}[\![W]\!]$-module of finite type. Now we shall show that this assertion holds even if $\bar{\Gamma}^i(S)$ has non-trivial torsion. We take a sufficiently small ideal M with $M+pr=r$ and $N\supset M$ and also take a subgroup S' such that $S\supset S'\supset U_1(M)$ and $\bar{\Gamma}^i(S')$ is torsion-free for all i. To indicate the dependence on S, we sometimes write $e\mathcal{Y}_r(n,v;S)$ and $eY_r(n,v;S)$ for $e\mathcal{Y}_r(n,v,\mathrm{id})$ and $eY_r(n,v,\mathrm{id})$ with respect to S. We define the trace operator

$$\mathrm{Tr}_\alpha\colon eH_P^r(X(S'(p^\alpha)),\mathcal{L}(n,v;K/\mathcal{O})\longrightarrow eH_P^r(X(S(p^\alpha)),\mathcal{L}(n,v;K/\mathcal{O}))$$

by the action of the double coset $[S(p^\alpha)]=[S'(p^\alpha)1S(p^\alpha)]$. We also consider the restriction map

$$\mathrm{res}_\alpha\colon eH_P^r(X(S(p^\alpha)),\mathcal{L}(n,v;K/\mathcal{O}))\longrightarrow eH_P^r(X(S'(p_\nu)),\mathcal{L}(n,v;K/\mathcal{O})).$$

These maps are compatible with the injective systems defining $e\mathcal{Y}_r(n,v,S)$ and $e\mathcal{Y}_r(n,v;S')$ and induce

$$\mathrm{Tr}\colon e\mathcal{Y}_r(n,v;S')\to e\mathcal{Y}_r(n,v;S)\quad\text{and}\quad \mathrm{res}\colon e\mathcal{Y}_r(n,v;S)\to e\mathcal{Y}_r(n,v;S').$$

By definition the composite map $\mathrm{Tr}\circ\mathrm{res}$ is the multiplication by the index $(F^\times S\colon F^\times S')$. Since $e\mathcal{Y}_r(n,v;S)$ and $e\mathcal{Y}_r(n,v;S')$ are p-divisible, Tr is surjective. By the Pontryagin duality, we have an embedding of $\mathcal{O}[\![W]\!]$-module $\mathrm{Tr}^*\colon eY_r(n,v;S)\to eY_r(n,v;S')$. Since $eY_r(n,v;S)$ is free of finite rank over $\mathcal{O}[\![W]\!]$, $eY_r(n,v;S)$ and $\mathrm{End}_{\mathcal{O}[[W]]}(eY_r(n,v;S))$ are torsion-free $\mathcal{O}[[W]]$-modules of finite type. Thus $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$ is a torsion-free $\mathcal{O}[[W]]$-module of finite type because of the fact that $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$ is naturally embedded into $\mathrm{End}_{\mathcal{O}[[W]]}(eY_r(n, v; S))$.

Now we shall prove the natural morphism:

$$\boldsymbol{h}_P^{n,\mathrm{ord}}(S:\mathcal{O})/P\boldsymbol{h}_P^{n,\mathrm{ord}}(S:\mathcal{O})\longrightarrow\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; K)$$

is a surjective isomorphism for $P=P_{n,v,\varepsilon}$. We write simply $\mathscr{A}$ for $\mathcal{O}[[G]]$ and $\mathscr{A}_P$ the localization of $\mathscr{A}$ at $P=P_{n,v,\varepsilon}\in\mathrm{Spec}(\mathscr{A})(\mathcal{O})$. By Lemma 3.6 combined with Corollary 3.2,

$$Y_r^{n,\mathrm{ord}}\otimes_{\mathscr{A}}\mathscr{A}_P/P\mathscr{A}_P\cong eH_P^r(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; K))$$

as $\boldsymbol{h}^{n,\mathrm{ord}}(S;\mathcal{O})$-modules. We write simply $\boldsymbol{H}$ for $\boldsymbol{h}^{n\mathrm{ord}}(S;\mathcal{O})$ and denote by $\boldsymbol{H}_P$ the localization of $\boldsymbol{H}$ at P. When $r=1$, we can let complex conjugation ρ acts on $Y_r^{n,\mathrm{ord}}$ and $eH_P^r(X(S(p^\alpha)), \mathscr{L}(n, v, \varepsilon; K))$ as in the proof of [8, Theorem 7.2]. Then

$$Y_r^{n,\mathrm{ord}}\otimes_{\mathscr{A}}\mathscr{A}_P=V_+\oplus V_-,$$

where on $V^\pm$, ρ acts by ± 1. We write V for one of the spaces $V_\pm$ if $r=1$ and for $Y_r^{n,\mathrm{ord}}\otimes_{\mathscr{A}}\mathscr{A}_P$ if $r=0$. By [8, Theorem 7.2] and its proof (see also, [6, Lemma 6.4]), we know that

$$V/PV\cong\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; K)\quad\text{as Hecke modules.}$$

We then take the element $\bar{v}\in V/PV$ corresponding to the identity of the Hecke algebra via the above isomorphism and choose $v\in V$ so that $v \bmod PV=\bar{v}$. Then we define a morphism $\pi\colon \boldsymbol{H}_P\to V$ by $\pi(h)=hv$. Then π is surjective by Nakayama's lemma. Since $\boldsymbol{H}$ can be embedded into $\mathrm{End}_{\mathcal{O}[[W]]}(Y_r^{n,\mathrm{ord}})$, π must be injective. Thus $\boldsymbol{H}_P\cong V$ aod $\boldsymbol{H}_P/P\boldsymbol{H}_P\cong V/PV\cong\mathfrak{h}_{k,w}^{n,\mathrm{ord}}(S_0(p^\alpha), \varepsilon; K)$, which finishes the proof of Theorem 2.4.

§ 5. Proof of Theorem 2.3 for the universal Hecke algebra

We shall keep the notation in § 4 and give only a sketch of the proof since the proof is quite similar to the case when $v=0$ (see [8, § 11]). We consider the following morphism for each α:

$$\iota^\alpha=\varinjlim_\beta \iota_\beta\circ\mathrm{res}_\beta\colon H_P^r(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathcal{O}))\longrightarrow\mathscr{Y}_r(0, 0, \mathrm{id}),$$

where res_β is the restriction map of $H_P^r(X(S(p^\alpha)), \mathscr{L}(n, v; p^{-\beta}\mathcal{O}/\mathcal{O}))$ to $H_P^r(X(S(p^\beta)), \mathscr{L}(n, v; p^{-\beta}\mathcal{O}/\mathcal{O}))$ and ι_β is the morphism in Theorem 3.3. In the same manner as in the proof of [8, Theorem 8.7], we know

(5.1) $\quad \mathrm{Ker}(\iota^\alpha)$ has only finitely many elements.

Let $\mathscr{V}_\alpha = \mathscr{V}_\alpha^{n,v}$ be the natural image of $H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K))$ in $H^r_P(X(S(p^\alpha)), \mathscr{L}(n, v; K/\mathscr{O}))$. Then, we know that $\mathfrak{h}_{k,w}(S(p^\alpha); \mathscr{O})$ can be embedded naturally into $\mathrm{End}_{\mathscr{O}}(\mathscr{V}_\alpha^{n,v})$ if r is odd or $n \neq 0$ (cf. 4.3c)).

Suppose that $r=1$. Then, for sufficiently large α, $H^r_P(X(S(p^\alpha)), K/\mathscr{O})$ is p-divisible (cf. (4.1)) and hence, by a theorem of Matsushima and Shimura (4.3c), $\mathfrak{h}_{2t,t}(S(p^\alpha); \mathscr{O})$ can be considered as a subalgebra of $\mathrm{End}_{\mathscr{O}}(H^r_P(X(S(p^\alpha)), K/\mathscr{O}))$. Then we have a natural morphism of algebras:

$$i\colon \mathfrak{h}_{2t,t}(S(p^\alpha); \mathscr{O}) \longrightarrow \mathrm{End}_{\mathscr{A}}(\mathscr{Y}_1(0, 0, \mathrm{id})) \qquad \text{for } \mathscr{A} = \mathscr{O}[\![G]\!].$$

We claim that i is injective. In fact, by (5.1), the restriction map

$$\iota^\alpha\colon H^1_P(X(S(p^\alpha)), K/\mathscr{O}) \longrightarrow \mathscr{Y}_1(0, 0, \mathrm{id})$$

is of finite kernel. Therefore if $i(h)=0$ for $h \in \mathfrak{h}_{2t,t}(S(p^\infty); \mathscr{O})$, then the restriction of h to $\mathrm{Im}(\iota^\alpha)$, which coincides with the natural projection of h to $\mathfrak{h}_{2t,t}(S(p^\alpha); \mathscr{O})$, vanishes. This shows that $h=0$ since $\mathfrak{h}_{2t,t}(S(p^\infty); \mathscr{O}) = \varprojlim_\alpha \mathfrak{h}_{2t,t}(S(p^\alpha); \mathscr{O})$. Thus we may consider $\mathfrak{h}_{2t,t}(S(p^\infty); \mathscr{O})$ as a subalgebra of $\mathrm{End}_{\mathscr{A}}(\mathscr{Y}_1(0, 0, \mathrm{id}))$. By restricting the elements of $\mathfrak{h}_{2t,t}(S(p^\infty); \mathscr{O})$ to the image of $\mathscr{V}_\alpha^{n,v}$, we have a surjective homomorphism:

$$\rho_\alpha\colon \mathfrak{h}_{2t,t}(S(p^\infty); \mathscr{O}) \longrightarrow \mathfrak{h}_{k,w}(S(p^\alpha); \mathscr{O}),$$

which is compatible with the transition morphisms of the projective system $\{\mathfrak{h}_{k,w}(S(p^\alpha); \mathscr{O}\}_\alpha$. By taking the projective limit, we have a surjective $\mathscr{A}$-algebra homomorphism

$$\mathfrak{h}_{2t,t}(S(p_\infty); \mathscr{O}) \longrightarrow \mathfrak{h}_{k,w}(S(p^\infty); \mathscr{O}), \tag{5.2}$$

which takes $T(\mathfrak{l})$ and $T(\mathfrak{l}, \mathfrak{l})$ to the corresponding ones.

We now show the existence of the surjective morphism (5.2) in the case of $r=0$. We put $\mathscr{W}_\alpha = \mathscr{V}_\alpha^{0,0}/\mathrm{Inv}(S(p^\alpha); K/\mathscr{O})$. With the notation of (4.5), we have $\nu_*\colon \mathscr{T}_\alpha \cong \mathrm{Inv}(S(p^\alpha); K/\mathscr{O})$ which takes $f_0 \in \mathscr{T}_\alpha$ to $f_0 \circ \nu$ for the reduced norm map ν. Since ν^* is compatible with the natural projective systems $\{\mathscr{V}_\alpha^{0,0}\}_\alpha$ and $\{\mathscr{T}_\alpha\}_\alpha$, we know that the restriction map: $\mathscr{W}_\alpha \to \mathscr{W}_\infty = \varinjlim_\beta \mathscr{W}_\beta$ is injective. We can show in exactly the same manner as in the proof of Theorem 3.2 in [8, § 11] that the map induced by $\iota^\alpha\colon \mathscr{V}_\alpha^{n,v} \to \mathscr{W}_\infty$ is still of finite kernel. Then the same argument as in the case of $r=1$ shows the existence of the surjective morphism (5.2) even when $r=0$.

When $r=0$ and $n\neq 0$, plainly, the limit $\iota^{\infty}: \mathcal{V}_{\infty}^{n,v}=\varinjlim \mathcal{V}_{\infty}^{n,v}\to\mathcal{W}^{\infty}$ of ι^{α} is surjective since $\mathcal{V}_{\alpha}^{n,v}$ coincides with the whole $H^0(X(S(p^{\alpha})), \mathcal{L}(n, v; K/\mathcal{O}))$ $(=L(n, v; K/\mathcal{O})^{X(S(p^{\alpha}))})$ for sufficiently large α (see (4.1)). On the other hand, in either case of $r=0$ or $r=1$, the argument in [8, § 11] shows that the surjectivity of the map $\iota^{\infty}: \mathcal{V}_{\infty}^{n,v}\to\mathcal{W}_{\infty}$ implies the injectivity of (5.2). Thus the assertion is now proven in the case where $r=0$.

When $r=1$, the surjectivity of the map of $\mathcal{V}_{\infty}^{n,v}$ to $\mathcal{Y}_1(0, 0, \mathrm{id})$ can be proven in exactly the same manner as in [8, § 11] and thus (5.2) induces an $\mathcal{A}$-algebra isomorphism of $\mathfrak{h}_{2t,t}(S(p^{\infty}); \mathcal{O})$ onto $\mathfrak{h}_{k,w}(S(p^{\infty}); \mathcal{O})$, which finishes the proof.

References

[1] G. E. Bredon, *Sheaf Theory* McGraw-Hill, 1967.
[2] W. Casselman, An assortment of results on representations of $GL_2(k)$, in "*Modular functions of one variable II*", Lecture notes in Math. No. 349, pp. 1–54, Springer, 1973.
[3] G. Harder, On the cohomology of $SL(2, O)$, Proc. Summer School of group representations, Budapest 1971: "*Lie groups and their representations*", pp. 139–150.
[4] ——, On the cohomology of discrete arithmetically defined groups, Proc. Int. Coll. on Discrete subgroups of Lie groups and applications to moduli, Bombay 1973, pp. 129–160.
[5] H. Hida, On abelian varieties with complex multiplication as factors of the abelian variety attached to Hilbert modular forms, Japan. J. Math., **5** (new series, 1979), 159–208.
[6] ——, Galois representations into $GL_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. Math., **85** (1986), 545–613.
[7] ——, Modules of congruence of Hecke algebras and L-functions associated with cusp forms, Amer. J. Math., **110** (1988), 323–382.
[8] ——, On p-adic Hecke algebras for GL_2 over totally real fields, Ann. of Math., **128** (1988), 295–384.
[9] G. Shimura, *Introduction to the arithmetic theory of automorphic functions,* Iwanami Shoten and Princeton University Press, 1971.

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