Advanced Studies in Pure Mathematics 16, 1988
Conformal Field Theory and Solvable Lattice Models
pp. 1–15

Scaling Limit Formula for 2-Point Correlation Function of Random Matrices

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In this article we give some results about 1 and 2 point correlation functions of the Gibbs measure of random matrices

$$\Phi d\tau = \Phi dx_1 \wedge \cdots \wedge dx_N \tag{0.1}$$

with the weight function $\Phi = \exp(-1/2(x_1^2 + \cdots + x_N^2)) \prod_{1 \le j < k \le N} |x_j - x_k|^\lambda$ for a constant $\lambda > 0$. As in [A1] we use the notations $(j, k) = x_j - x_k$, $d\tau_{N,p} = dx_{p+1} \wedge \cdots \wedge dx_N$ (which means a differential $(N-p)$-form) for $0 \le p < N$. We put $n = N - p$. We consider more generally the density

$$\Phi_{N,p} = \exp\left(-\frac{1}{2}(x_1^2 + \cdots + x_N^2)\right) \sum_{1 \le \mu < \nu \le n} |x_{p+\mu} - x_{p+\nu}|^\lambda \cdot \prod_{j=1}^{p} \prod_{1 \le \mu \le n} |x_{p+\mu} - x_j|^{\lambda'_j} \tag{0.2}$$

on the Euclidean space $\boldsymbol{R}^{N-p}$ of the variables $x_{p+1}, \cdots, x_N$. Here $\lambda'_1, \cdots, \lambda'_p$ denote some positive constants. For $\varepsilon_j = \pm 1$ we denote by $\langle (i_1, j_1)^{\varepsilon_1} \cdots (i_l, j_l)^{\varepsilon_l} | \lambda'_1, \cdots, \lambda'_p \rangle$ the correlation functions

$$\int_{\boldsymbol{R}^n} (i_1, j_1)^{\varepsilon_1} \cdots (i_l, j_l)^{\varepsilon_l} \Phi_{N,p} d\tau_{N,p}. \tag{0.3}$$

We abbreviate it by $\langle (i_1, j_1)^{\varepsilon_1} \cdots (i_l, j_l)^{\varepsilon_l} \rangle$ if $\lambda'_1 = \cdots = \lambda'_p = 0$. This is a l-point correlation function for the density $\Phi d\tau$.

The reduced density of p points

$$F_{N,p} = \int_{\boldsymbol{R}^n} \Phi_{N,p} d\tau_{N,p} \tag{0.4}$$

is known to be analytic in $x_1, \cdots, x_p$ and $\lambda, \lambda'_1, \cdots, \lambda'_p$. However the following problem seems difficult and interesting:

Received December 18, 1986.

Problem. *p being fixed, is $F_{N,p}$, as a function of n, a restriction to the set of positive integers of an analytic function? If it is so, what kind of asymptotic nature has it for $n\to\infty$?*

Our main purpose is to give an answer for the 2 points correlation functions, in case where $\lambda_1'=\lambda_2'=0$, and to give a limit formula when n tends to the infinity by using Bessel functions (Theorem in Section 3). This result extends the well-known formula obtained by M. L. Mehta as early as in 1960 (see [M1]).

1. One point correlation function (case where $p=1$)

We fix n $(=N-1)$ positive integers $f_2, \cdots, f_N$. Consider the integral

$$\begin{aligned}(1.1)\qquad &\exp\left\{\frac{1}{2}x_1^2\right\}\langle(2,1)^{f_2}\cdots(p,1)^{f_p}\rangle\\ &=\exp\left\{\frac{1}{2}x_1^2\right\}\int_{R^n}\Phi_{N,1}(2,1)^{f_2}\cdots(p,1)^{f_p}d\tau_{N,1}\end{aligned}$$

which is a polynomial of x_1 of degree $f_2+\cdots+f_N$. We have shown in Part 1 (see [A1]) the following Lemma:

Lemma 1.1. *For $0\leq r\leq n$,*

$$(1.2)\qquad \varphi_r(x_1)=\langle(2,1)\cdots(r+1,1)\rangle\exp\left\{\frac{1}{2}x_1^2\right\}$$

is equal to $M_n(-\sqrt{\lambda}/2)^r H_r(x_1/\sqrt{\lambda})$, where M_n and $H_r(x)$ denote the constant $(2\pi)^{n/2}\Gamma(1+\lambda/2)^{-n}\prod_{j=1}^n\Gamma(1+\lambda j/2)$ and the r-th Hermite polynomial $r!\sum_{\nu=0}^{[r/2]}(-1)^\nu(2x)^{r-2\nu}/\nu!\,(r-2\nu)!$ respectively. We call the system of polynomial $\{\varphi_r(x)\}_{r=1,2,3\ldots}$ "basic polynomials".

We are interested in writing correlation functions in terms of the basic polynomials.

Proposition 1. *For an arbitrary integer $l\geq 0$,*

$$\begin{aligned}(1.3)\quad &\langle(2,1)^l\cdots(N,1)^l\rangle\\ &=\exp\left\{-\frac{1}{2}x_1^2\right\}M_n^{-l+1}\sum_{n\geq k_2,k_2+k_3,\cdots,k_{l-1}+k_l}\varphi_{n-k_2}(x_1)\varphi_{n-k_2-k_3}(x_1)\cdots\\ &\quad\varphi_{n-k_{l-1}-k_l}(x)\varphi_{n-k_l}(x_1)\cdot(\lambda/2)^{k_2+\cdots+k_l}(2/\lambda)_{k_2}\cdots\left(\frac{2(l-1)}{\lambda}\right)_{k_l}\\ &\quad\times\frac{(n-k_3)!\cdots(n-k_l)!\,n!}{(n-k_2-k_3)!\cdots(n-k_{l-1}-k_l)!\,(n-k_l)!\,k_2!\cdots k_l!}\end{aligned}$$

where $(a)_k$ denotes the product $a(a+1)\cdots(a+k-1)$.

Before proving the Proposition we need two Lemmas. For arbitrary $0\leq r\leq s\leq n$ we abbreviate by $\langle\!\langle r, s \mid \lambda_1'\rangle\!\rangle$ the correlation function $\langle(2, 1)^2\cdots(r+1, 2)^2(r+2, 1)\cdots(s+1, 1)|\lambda_1'\rangle$. First we prove the following recurrence equations:

Lemma 1.2.

$$(1.4)\quad \langle\!\langle r, s|\lambda_1'\rangle\!\rangle=-x_1\langle\!\langle r-1, s|\lambda_1'\rangle\!\rangle+\{1+\lambda_1'+\lambda(n-s)/2\}\langle\!\langle r-1, s-1|\lambda_1'\rangle\!\rangle -(r-1)\lambda/2\langle\!\langle r-2, s|\lambda_1'\rangle\!\rangle.$$

Proof. This Lemma can be proved by using Stokes formula and symmetry property, due to the fact that an integral over $\boldsymbol{R}^n$ vanishes if its integrand changes the sign by the transposition between i and j for $p+1\leq i, j\leq N$. Since

$$(1.5)\quad d\log\Phi_{N,p}=\sum_{\mu=1}^{n}\sum_{j=1}^{p}\lambda_j' d\log(x_{p+\mu}-x_j)+\sum_{1\leq\mu<\nu\leq n}\lambda d\log(x_{p+\mu}-x_{p+\nu}),$$

we have a formula of exterior differentiation:

$$\begin{aligned}(1.6)\quad & d\Big\{(2, 1)(3, 1)^2\cdots(r+1,1)^2(r+2, 1)\cdots(s+1, 1)\Phi d\tau_{N,2}\}\\ &=\Phi\{-x_2(2, 1)(3, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)\\ &\quad+(1+\lambda_1')(3, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)\\ &\quad+\lambda\sum_{j=3}^{r+1}\frac{(2, 1)(3, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)}{(2, j)}\\ &\quad+\lambda\sum_{j=r+2}^{s+1}\frac{(2, 1)(3, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)}{(2, j)}\\ &\quad+\lambda\sum_{j=s+2}^{N}\frac{(2, 1)(3, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)}{(2, j)}\Big\}d\tau_{N,1}.\end{aligned}$$

By Stokes formula the integral over $\boldsymbol{R}^n$ of the left hand side vanishes so does it for the right hand side.

i) The integration of the third term in the right hand side is transformed as follows:

$$\begin{aligned}(1.7)\quad & \Big\langle(2, 1)(j, 1)\Big\{-1+\frac{(2, 1)}{(2, j)}\Big\}(3, 1)^2\cdots(j-1, 1)^2(j+1,1)^2\cdots(r+1, 1)^2\\ &\times(r+2, 1)\cdots(s+1, 1)\,|\,\lambda_1'\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\langle (2,1)(j,1)(3,1)^2\cdots(j-1,1)^2(j+1,1)^2\cdots \\
&\quad \times (r+1,1)^2(r+2,1)\cdots(s+1,1) \mid \lambda_1' \rangle \\
&\quad + \left\langle \frac{(2,1)^2(j,1)}{(2,j)}(3,1)^2\cdots(j-1,1)^2(j+1,1)^2\cdots(r+1,1)^2 \right. \\
&\quad \left. \times (r+2,1)\cdots(s+1,1) \mid \lambda_1' \right\rangle
\end{aligned}
$$

By symmetry property the last term is equal to the minus of the left hand side. Hence the left hand side is equal to

$$
\begin{aligned}
(1.8)\qquad & -\frac{1}{2}\langle (2,1)(j,1)\,(3,1)^2\cdots(j-1,1)^2(j+1,1)^2\cdots \\
& \times (r+1,1)^2(r+2,1)\cdots(s+1,1) \mid \lambda_1' \rangle \\
& = -\frac{1}{2}\langle\!\langle r-2, s \mid \lambda_1' \rangle\!\rangle.
\end{aligned}
$$

ii) For the fourth term in the right hand side, the integral vanishes, because $(2,1)(j,1)/(2,j)$ changes the sign by the transposition between 2 and j.

iii) In the same manner, in case where $s+2 \leq j \leq n$, one has

$$
(1.9)\qquad \frac{(2,1)}{(2,j)} = 1 + \frac{(j,1)}{(2,j)}
$$

and the corresponding integral is equal to

$$
(1.10)\qquad \frac{1}{2}\langle\!\langle r-1, s-1 \mid \lambda_1' \rangle\!\rangle,
$$

whence (1.6) implies (1.4), because $x_2 = -(2,1) - x_1$.

Lemma 1.3.

$$
\begin{aligned}
(1.11)\qquad & \langle\!\langle r, s \mid \lambda_1' \rangle\!\rangle \\
& = \frac{1}{M_n}\sum_{k=0}^{r} \varphi_{r-k}(x_1)\,(\lambda/2)^k \left(\frac{2(\lambda_1'+1)}{\lambda} + n - s\right)_k \frac{r!}{k!(r-k)!}\langle\!\langle 0, s-k \mid \lambda_1' \rangle\!\rangle.
\end{aligned}
$$

Proof. We want to prove this by induction in r. When r is equal to 0, nothing is to be proved because $\varphi_0(x_1) = M_n$. So we assume (1.11) holds for $r < r'$ and prove it for $r = r'$. (1.4) shows

$$
(1.12)\qquad \langle\!\langle r', s \mid \lambda_1' \rangle\!\rangle
$$

$$= -x_1 \langle\!\langle r'-1, s \mid \lambda_1' \rangle\!\rangle - \frac{r'-1}{2} \lambda \langle\!\langle r'-2, s \mid \lambda_1' \rangle\!\rangle$$
$$+ \{\lambda_1' + 1 + \lambda(n-s)/2\} \langle\!\langle r'-1, s-1 \mid \lambda_1' \rangle\!\rangle .$$

By induction hypothesis the right hand side is expressed as

$$(1.13) \qquad -x_1 \sum_{k=0}^{r'-1} \varphi_{r'-k-1}(x_1)(\lambda/2)^k \left(\frac{2(\lambda_1'+1)}{\lambda} + n - s \right)_k \frac{(r'-1)!}{k!(r'-k-1)!} \cdot$$
$$\langle\!\langle 0, s-k \mid \lambda_1' \rangle\!\rangle$$
$$- \frac{r'-1}{2} \lambda \sum_{k=0}^{r'-2} \varphi_{r'-k-2}(x_1)(\lambda/2)^k \left\{ \frac{2(\lambda_1'+1)}{\lambda} + n - s \right)_k \cdot \frac{(r'-2)!}{k!(r'-k-2)!}$$
$$\langle\!\langle 0, s-k \mid \lambda_1' \rangle\!\rangle$$
$$+ \{\lambda_1' + 1 + \lambda(n-s)/2\} \sum_{k=0}^{r'-1} \varphi_{r'-k-1}(x_1) \left(\frac{\lambda}{2} \right)^k \left\{ \frac{2(\lambda_1'+1)}{\lambda} + n - s + 1 \right\}_k$$
$$\times \frac{(r'-1)!}{k!(r'-k-1)!} \langle\!\langle 0, s-k-1 \mid \lambda_1' \rangle\!\rangle$$
$$= \sum_{k=0}^{r'} \Bigg\{ -x_1 \varphi_{r'-k-1}(x_1)(\lambda/2)^k \left(\frac{2(\lambda_1'+1)}{\lambda} + n - s \right)_k \frac{(r'-1)!}{k!(r'-k)!}$$
$$- \frac{r'-1}{2} \lambda \varphi_{r'-k-2}(x_1)(\lambda/2)^k \left(\frac{2(\lambda_1'+1)}{\lambda} + n - s \right)_k \frac{(r'-2)!}{k!(r'-k-2)!}$$
$$+ \{\lambda_1' + 1 + \lambda(n-s)/2\} \varphi_{r'-k}(x_1) \left(\frac{\lambda}{2} \right)^{k-1} \left(\frac{2(\lambda_1'+1)}{\lambda} + n - s + 1 \right)_k$$
$$\times \frac{(r'-1)}{(k-1)!(r'-k)!} \langle\!\langle 0, s-k \mid \lambda_1' \rangle\!\rangle \Bigg\} .$$

Since $\varphi_j(x_1)$ satisfy the 3-term recurrence relation (see [A1]):

$$(1.14) \qquad \varphi_{j+1}(x_1) + \frac{\lambda j}{2} \varphi_{j-1}(x_1) + x_1 \varphi_j(x_1) = 0$$

we can eliminate the term $x_1 \varphi_{r'-k-1}(x_1)$ in the above and get the formula (1.11) for $r = r'$. Lemma 1.3 has now been proved.

In particular when we put $\lambda_1' = 0$, Lemma 1.3 is simplified into

Corollary.

$$(1.15) \quad \langle (2, 1)^2 \cdots (r+1, 1)^2 (r+2, 1) \cdots (s+1, 1) \rangle$$
$$= (1/M_n) \exp\left(-\frac{x_1^2}{2} \right) \sum_{k=0}^{r} \varphi_{r-k}(x_1) \varphi_{s-k}(x_1)(\lambda/2)^k \left(\frac{2}{\lambda} + n - s \right)_k \frac{r!}{k!(r-k)!} .$$

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*) This remark has been pointed out by Mr. T. Tomohisa.

In fact by definition, $\langle\!\langle 0, s-k\,|\,0\rangle\!\rangle$ is equal to $\varphi_{s-k}(x_1)\exp(-(x_1^2/2))$.

Remark.*) When λ_1' is equal to a non-negative integer, the recurrence relations (1.4) and (1.11) still hold, if $\langle\!\langle r, s\,|\,\lambda_1'\rangle\!\rangle$ is replaced by $\langle\!\langle r, s\,|\,\underline{\lambda_1'}\rangle\!\rangle = \langle (2, 1)^2\cdots(r+1, 1)^2(r+2, 1)\cdots(s+1, 1)\prod_{j=2}^{N}(j, 1)^{\lambda_1'}\rangle$.

Proof of Proposition 1. The formula (1.12) and the above Remark enable us to give a recurrence relation for $\langle (2, 1)^l\cdots(N, 1)^l\rangle = \langle\!\langle 0, n\,|\,\underline{l-1}\rangle\!\rangle$ as follows: For arbitrary r such the $0\le r\le n$, we apply (1.11) for $\langle\!\langle 0, r\,|\,\underline{l-1}\rangle\!\rangle$. Then

$$(1.16)\quad \langle\!\langle 0, r\,|\,\underline{l-1}\rangle\!\rangle = \langle\!\langle r, n\,|\,\underline{l-2}\rangle\!\rangle$$
$$= (1/M_n)\sum_{k_2=0}^{r}\varphi_{r-k_2}(x)(\lambda/2)^{k_2}(2(l-1)/\lambda)_{k_2}\cdot\frac{r!}{k_2!(r-k_2)!}$$
$$\times\langle\!\langle 0, n-k_2\,|\,\underline{l-2}\rangle\!\rangle = M_n^{-2}\sum_{r\ge k_2, n\ge k_2+k_3}(\lambda/2)^{k_2+k_3}\left(\frac{2(l-1)}{\lambda}\right)_{k_2}$$
$$\times\left(\frac{2(l-2)}{\lambda}\right)_{k_3}\frac{r!(n-k_2)!}{k_2!(r-k_2)!k_3!(n-k_2-k_3)!}$$
$$\times\varphi_{r-k_2}(x_1)\varphi_{n-k_2-k_3}(x_1)\langle\!\langle 0, n-k_3\,|\,\underline{l-3}\rangle\!\rangle.$$

We can again apply (1.11) for $\langle\!\langle 0, n-k_3\,|\,\underline{l-3}\rangle\!\rangle$ and so on, and arrive at the Proposition by putting $r=n$ (Remark that $\langle\!\langle 0, n-k_l\,|\,\underline{0}\rangle\!\rangle = \exp(-(x_1^2/2))\cdot\varphi_{n-k_l}(x_1)$.)

2. 2 point correlation functions (case where $p=2$)

It is more difficult to evaluate 2 point correlation functions in terms of basic polynomials. For $0\le r\le s\le n$, we write simply $\varphi_{r,s}(x_1, x_2)\exp\{-(x_1^2+x_2^2)/2\}$ in place of $\langle (3, 1)\cdots(r+2, 1)(3, 2)\cdots(s+2, 2)\rangle$ in the sequel. Then $\varphi_{r,s}(x_1, x_2)$ is a polynomial of degree $r+s$.

The only result that we can give is the following:

Proposition 2. *For* $0\le r\le s\le n$ *and* $\lambda_1'=\lambda_2'=0$,

$$(2.1)\quad \varphi_{r,s}(x_1, x_2) = \frac{1}{M_n}\sum_{k=0}^{r}\varphi_{r-k}(x_1)\varphi_{s-k}(x_2)(\lambda/2)^k\left(\frac{2}{\lambda}+n-s\right)_k\frac{r!}{k!(r-k)!}.$$

In particular when r and s coincide with n, we have

$$(2.2)\quad \varphi_{n,n}(x_1, x_2) = \frac{1}{M_n}\sum_{k=0}^{n}\varphi_{n-k}(x_1)\varphi_{n-k}(x_1)(\lambda/2)^k\left(\frac{2}{\lambda}\right)_k\cdot\frac{n!}{k!(n-k)!}.$$

Lemma 2.1.

$$\varphi_{r,s}(x_1, x_2) = -x_1\varphi_{r-1,s}(x_1, x_2) - (r-1)\frac{\lambda}{2}\varphi_{r-2,s}(x_1, x_2) + \left(1+\frac{\lambda(n-s)}{2}\right)\varphi_{r-1,s-1}(x_1, x_2). \tag{2.3}$$

Proof. The proof is similar to the one of Lemma 1.2. We use the vanishing of the integral of the following exterior differentiation:

$$\begin{aligned} &d\{\Phi(4,1)\cdots(r+2,1)(3,2)\cdots(s+2,2)d\tau_{N,3}\} \\ &=\Phi\Big\{-x_3(4,1)\cdots(r+2,1)(3,2)\cdots(s+2,2) \\ &+(4.1)\cdots(r+2,1)(4,2)\cdots(s+2,2) \\ &+\lambda\sum_{4\le j\le r+2}\frac{(4,1)\cdots(r+2,1)(3,2)\cdots(s+2,2)}{(3,j)} \\ &+\lambda\sum_{r+3\le j\le s+2}\frac{(4,1)\cdots(r+2,1)(3,2)\cdots(s+2,2)}{(3,j)} \\ &+\lambda\sum_{j=s+3}^{N}\frac{(4,1)\cdots(r+2,1)(3,2)\cdots(s+2,2)}{(3,j)}\Big\}d\tau_{N,2}. \end{aligned} \tag{2.4}$$

In the same way as in the proof of Lemma 1.2 one can prove that the integrals of the third term, the fourth term and the last term are equal to $-\frac{1}{2}\varphi_{r-1,s-1}(x_1, x_2)$, 0 and $\frac{1}{2}\varphi_{r-1,s-1}(x_1, x_2)$ respectively. As a consequence of it, we get

$$\begin{aligned} 0 = &-\varphi_{r,s}(x_1, x_2) - x_1\varphi_{r-1,s}(x_1, x_2) \\ &+\{1+\lambda(n-s)/2)\}\varphi_{r-1,s-1}(x_1, x_2) \\ &-(r-1)\frac{\lambda}{2}\varphi_{r-2,s}(x_1, x_2) \end{aligned} \tag{2.5}$$

which is the same thing as (2.3).

We see that (2.3) has the same expression as (1.4) for $\lambda_1'=0$. However one cannot expect that (2.3) can be generalized for arbitrary λ_1' and λ_2' for 2 point correlation functions.

Lemma 2.2.

$$\varphi_{r,s}(x_1, x_2) = \frac{1}{M_n}\sum_{k=0}^{r}\varphi_{r-k}(x_1)\varphi_{s-k}(x_2)\left(\frac{\lambda}{2}\right)^k\left(\frac{2}{\lambda}+n-s\right)_k\frac{r!}{k!(r-k)!}. \tag{2.6}$$

In particular

$$(2.7)\qquad \varphi_{n,n}(x_1, x_2)=\left\langle(3, 1)\cdots(N, 1)(3, 2)\cdots(N, 2)\rangle \exp\left(\frac{x_1^2+x_2^2}{2}\right)\right\rangle$$
$$=\frac{1}{M_n}\sum_{k=0}^{n}\varphi_{n-k}(x_1)\varphi_{n-k}(x_2)\left(\frac{\lambda}{2}\right)^k\left(\frac{2}{\lambda}\right)_k\frac{n!}{k!(n-k)!}.$$

Proof. We can prove it like Lemma 1.3 if we use Lemma 2.1 in place of Lemma 1.2.

Proof of the Proposition 2. We have only to put $r=s=n$ in Lemma 2.2.

Remark. When $\lambda=2$, the formula (2.2) coincides with the one for $K_N(x, y)$ in [M1] p. 76.

3. A new integral formula for $\langle(3, 1)\cdots(N, 1)(3, 2)\cdots(N, 2)\rangle$

We start from giving an integral representation for the basic polynomials:

Lemma 3.1.

$$(3.1)\qquad \varphi_n(x)=(M_n/\sqrt{\lambda\pi})\int_{-\infty}^{\infty}\exp[-\zeta^2/\lambda](i\zeta-x)^n d\zeta$$

Proof. In fact the right hand side is equal to $M_n(-(\sqrt{\lambda}/2))^n \cdot H_n(x/\sqrt{\lambda})$, which is nothing else than $\varphi_n(x)$.

Lemma 3.2. *Suppose* $0<R<2/\lambda$. *Then*

$$(3.2)\qquad \varphi_{n,n}(x_1, x_2)$$
$$=n!\frac{M_n}{\lambda\pi}\frac{1}{2\pi i}\int_{|\zeta|=R}\int_{R^2}\exp\{-(\zeta_1^2+\zeta_2^2)/\lambda+(i\zeta_1-x_1)(i\zeta_2-x_2)\zeta\}$$
$$\times(1-\lambda\zeta/2)^{-2/\lambda}\zeta^{-n-1}d\zeta d\zeta_1 d\zeta_2.$$

Proof. A Taylor expansion of the integrand in the right hand side is equal to:

$$(3.3)\qquad \frac{n!M_n}{\lambda\pi}\frac{1}{2\pi i}\int_{|\zeta|=R}\int_{R^2}\exp\{-(\zeta_1^2+\zeta_2^2)/\lambda)$$
$$\times\sum_{\substack{n\geq k'>-\infty\\ k\geq 0}}\frac{(i\zeta_1-x_1)^{n-k'}(i\zeta_2-x_2)^{n-k'}\zeta^{n-k'}}{(n-k')!\,k!}$$
$$\times(2/\lambda)_k(\lambda/2)^k\zeta^{k-n-1}d\zeta d\zeta_1 d\zeta_2$$

$$=\frac{n!M}{\lambda\pi}\int_{R^2}\exp\{-(\zeta_1^2+\zeta_2^2)/\lambda\}.$$

$$\times\sum_{k=0}^{n}\frac{(i\zeta_1-x_1)^{n-k}(i\zeta_2-x_2)^{n-k}}{(n-k)!k!}(2/\lambda)_k(\lambda/2)^k d\zeta_1 d\zeta_2$$

which is equal to $\varphi_{n,n}(x_1, x_2)$ owing to (2.7) and (3.1).

It is more convenient to express (3.2) as follows.

Lemma 3.3.

$$\varphi_{n,n}(x_1, x_2)=\frac{n!M_n}{2\pi i}(\lambda/2)^n\int_{|\zeta'|=R'}\exp\left\{-\frac{(x_1^2+x_2^2)\zeta'^2-2x_1x_2\zeta'}{\lambda(1-\zeta'^2)}\right\}$$
$$\times(1-\zeta'^2)^{-1/2}(1-\zeta')^{-2/\lambda}\zeta'^{-n-1}d\zeta' \tag{3.4}$$

for $0<R'<1$.

Proof. We first make the integration of the right hand side of (3.2) with respect to the variables ζ_1 and ζ_2 (This is easy because it is Gaussian) and next make the change of variable $\zeta'=\lambda\zeta/2$.

4. Asymptotic behaviour of the 2 point correlation function for $n\to\infty$

We put $\xi=\sqrt{2n}\cdot x_1$ and $\sqrt{2n}\cdot x_2$. We are interested in the asymptotic behaviour of $\varphi_{n,n}(x_1, x_2)$ for $n\to\infty$, *under the condition that* ξ_1 *and* ξ_2 *are fixed*, because otherwise it will have an oscillatory nature whose study is beyond the range of our approach.

Our purpose is to prove the following Theorem:

Theorem. *For* $n\to\infty$,

$$\varphi_{n,n}(x_1, x_2)=n!M_n(\lambda/2)^n n^{(2/\lambda)-(1/2)}2^{(2/\lambda)-1}J_{(2/\lambda)-(1/2)}\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)$$
$$\times\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)^{-(2/\lambda)+(1/2)}\left\{1+0\left(\frac{1}{\sqrt[3]{n}}\right)\right\}. \tag{4.1}$$

where $J_\lambda(x)$ *denotes the Bessel function of order* λ:

$$J_\lambda(x)=(x/2)^\lambda\sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,\Gamma(\lambda+l+1)}(x/2)^{2l}. \tag{4.2}$$
$$=\frac{x^\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left[\frac{1}{2}\left(t-\frac{x^2}{t}\right)\right]t^{-\lambda-1}dt$$

for a positive constant c.

When $\lambda=2$, $J_{(2/\lambda)-(1/2)}(x)$ turns out to be $\sqrt{2x/\pi}(\sin x/x)$. In this case the Theorem shows

$$\varphi_{n,n}(x_1, x_2) \sim n!\, M_n \sqrt{2n/\pi}\, \sin\left(\frac{\xi_1-\xi_2}{\sqrt{2}}\right) \Big/ \frac{\xi_1-\xi_2}{\sqrt{2}}. \tag{4.3}$$

This formula coincides with the one (A9.2) in [M1] p. 198.

In this note we shall denote by $C_1, C_2, C_3, \cdots$ suitable positive constants.

By the Cauchy integral formula in the integral (3.4) the circle: $|\zeta'| = R'$ can be replaced by two lines γ_1 and γ_2:

$$\begin{aligned} &\gamma_1:\ \operatorname{Re}\zeta'=c_1 \\ &\gamma_2:\ \operatorname{Re}\zeta'=c_2 \end{aligned} \tag{4.4}$$

for constants c_1 and c_2 such that $0<c_1<1$ and $-1<c_2<0$ respectively.

In γ_1 $\operatorname{Im}\zeta'$ runs from $-\infty$ to $+\infty$ while in γ_2 $\operatorname{Im}\zeta'$ runs from $+\infty$ to $-\infty$. We denote the parts of integration over γ_1 and γ_2 by $\varphi'_{n,n}$ and $\varphi''_{n,n}$ respectively:

$$\varphi_{n,n}=\varphi'_{n,n}+\varphi''_{n,n}. \tag{4.5}$$

The saddle point method for the function in the integrand (3.4)

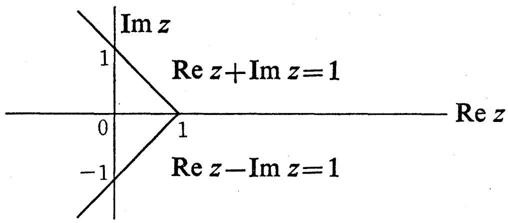
$$\operatorname{Re}\left\{-\left(\frac{2}{\lambda}+\frac{1}{2}\right)d\log(1-\zeta')-\frac{1}{2}d\log(1+\zeta')-(n+1)d\log\zeta'\right\} \tag{4.6}$$

suggests that it is convenient to make the change of variable $\zeta'=1-z/n$ in (3.4). Then

Lemma 4.1. *$\varphi'_{n,n}$ is expressed as*

$$\begin{aligned} \varphi'_{n,n} &= n!\, M_n n^{-(1/2)+(2/\lambda)}\left(\frac{\lambda}{2}\right)^n J, \\ J &= \frac{1}{2\pi i}\int_{\tilde{\gamma}_1}\left(2-\frac{z}{n}\right)^{-1/2} z^{-(1/2)-(2/\lambda)}\left(1-\frac{z}{n}\right)^{-n-1} \\ &\quad \times \exp\left[-\frac{(\xi_1^2+\xi_2^2)(1-z/n)^2-2\xi_1\xi_2(1-z/n)}{2\lambda z(2-z/n)}\right]dz \end{aligned} \tag{4.7}$$

where $\tilde{\gamma}_1$ can be chosen as follows:



We devide $\tilde{\gamma}_1$ into 3 parts; $\tilde{\gamma}_{1,1}+\tilde{\gamma}_{1,2}+\tilde{\gamma}_{1,3}$, where

$$\tilde{\gamma}_{1,1}:\ |z|\leq \sqrt[3]{n} \tag{4.8}$$

$$\tilde{\gamma}_{1,2}:\ \sqrt[3]{n}\leq|z|\leq\frac{n}{2}$$

$$\tilde{\gamma}_{1,3}:\ \frac{n}{2}\leq|z|.$$

and denote by J_1, J_2 and J_3 the corresponding integrals over $\tilde{\gamma}_{1,1}$, $\tilde{\gamma}_{1,2}$ and $\tilde{\gamma}_{1,3}$ respectively:

$$J=J_1+J_2+J_3. \tag{4.9}$$

Now we want to show the following:

Lemma 4.2.

i) $J_1=\frac{1}{2\pi i}\int_{\tilde{\gamma}_1}\frac{1}{\sqrt{2}}z^{-(1/2)-(2/\lambda)}e^z\exp\left\{-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right\}dz\{1+0(n^{-1/3})\}$

(4.10) ii) $J_2=0(n^{-1/3})$

iii) $J_3=0(n^{-1})$.

Proof i). First remark that the length of $\tilde{\gamma}_{1,1}$ is of growth order $0(\sqrt[3]{n})$. Since $|z|\leq\sqrt[3]{n}$ we have a bound

$$\left|\left(1-\frac{z}{n}\right)^{-n}e^{-z}\right|=\exp \mathrm{Re}\left\{\frac{z^2}{2n}+\frac{z^3}{3n^2}+\cdots\right\} \tag{4.11}$$

$$=\exp \mathrm{Re}\left\{\frac{z^2}{2n}\left(1+\frac{z}{3n}+\cdots\right)\right\}$$

$$\leq C_1|z|^2/n$$

namely

(4.12) $$\left|\left(1-\frac{z}{n}\right)^{-n}\right| \leq C_1|e^z||z^2|/n$$

on $\tilde{\gamma}_{1,1}$, Now

(4.13) $$\left|\frac{1}{2\pi i}\int_{\tilde{\gamma}_{1,1}}\left(1-\frac{z}{n}\right)^{-n-1}\left(2-\frac{z}{n}\right)^{-1/2} z^{-(1/2)-(2/\lambda)}\cdot\right.$$
$$\times \exp\left\{-\frac{(\xi_1^2+\xi_2^2)(1-z/n)^2-2\xi_1\xi_2(1-z/n)}{2\lambda z(2-z/n)}\right\}dz$$
$$\left.-\frac{1}{2\pi i}\int_{\tilde{\gamma}_{1,1}}\frac{e^z}{\sqrt{2}}z^{-(1/2)-(2/\lambda)}\exp\left\{-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right\}dz\right|$$
$$\leq A_1+A_2,$$

where A_1 and A_2 denote

(4.14) $$A_1=\left|\frac{1}{2\pi i}\int_{\tilde{\gamma}_{1,1}}\left[\left(1-\frac{z}{n}\right)^{-n}-e^z\right]\frac{z^{-(1/2)-(2/\lambda)}}{\sqrt{2}}\exp\left\{-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right\}dz\right|$$

(4.15) $$A_2=\left|\frac{1}{2\pi i}\int_{\tilde{\gamma}_{1,1}}\left(1-\frac{z}{n}\right)^{-n} z^{-(1/2)-(2/\lambda)}\left\{\left(1-\frac{z}{n}\right)^{-1}\left(2-\frac{z}{n}\right)^{-1/2}\right.\right.$$
$$\times\exp\left(-\frac{(\xi_1^2+\xi_2^2)(1-z/n)^2-2\xi_1\xi_2(1-z/n)}{2\lambda z(2-z/n)}\right.$$
$$\left.\left.-\frac{1}{\sqrt{2}}\exp\left(-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right)\right\}dz\right|.$$

Owing to the inequality (4.12),

(4.16) $$A_1 \leq \frac{C_1}{n\sqrt{2}}\int_{\tilde{\gamma}_{1,1}}|e^z||z^2||z|^{-(1/2)-(2/\lambda)}\left|\exp\left(-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right)\right||dz|$$
$$\leq C_2\max_{z\in\tilde{\gamma}_{1,1}}|z|^2/n\int_{\tilde{\gamma}_{1,1}}|z|^{-(1/2)-(2/\lambda)}\left|\exp\left(z-\frac{(\xi_1-\xi_2)^2}{4\lambda z}\right)\right||dz|$$
$$\leq C_3\max|z|^2/n \leq C_4/\sqrt[3]{n}$$

in view of $1/|z| \leq \sqrt{2}$, $|e^z| \leq e$. On may assume that z/n is near 0, and therefore

(4.17) $$|\text{The part } \{\quad\} \text{ in the integrand in (4.15)}| \leq C_5|z|/n \leq C_5 n^{-2/3}.$$

Moreover

(4.18) $$\left|\left(1-\frac{z}{n}\right)^{-n}\right| \leq |e^z|(1+C_1|z|^2/n) \leq e(1+C_1/\sqrt[3]{n})$$

whence

$$A_2 \leq C_6 n^{-1/3} \tag{4.19}$$

because the length of the path $\tilde{\gamma}_{1,1}$ is of growth order $0(n^{1/3})$. This proves the first part of Lemma 4.2.

To prove the second part we need the following inequality:

Lemma 4.3. *For* $\operatorname{Re} z \leq 0$ *and* $|z| \leq n/2$,

$$\left|\left(1-\frac{z}{n}\right)\right|^{-n} \leq e^{(2/3)\operatorname{Re} z}. \tag{4.20}$$

Proof. It can be seen that

$$-\log(1-x) \leq 2x/3 \tag{4.21}$$

for $-1/2 \leq x \leq 0$. Hence

$$-\log\left(1-\frac{r}{n}\cos\theta\right) \leq \frac{2r}{3n}\cos\theta \tag{4.22}$$

for $x = r\cos\theta$. Namely

$$\left(1-\frac{r}{n}\cos\theta\right)^{-n} \leq e^{(2r/3)\cos\theta}. \tag{4.23}$$

On the other hand we have for $z = re^{i\theta}$,

$$\left|\left(1-\frac{z}{n}\right)\right| \geq 1-\frac{r}{n}\cos\theta, \tag{4.24}$$

which implies Lemma 4.3.

Proof ii) of *Lemma* 4.2. On $\tilde{\gamma}_{1,2}$ we have

$$2 \leq \left|2-\frac{z}{n}\right| \tag{4.25}$$

$$n^{1/3} \leq |z| \tag{4.26}$$

$$\left|1-\frac{z}{n}\right|^{-n} \leq e^{(2/3)\operatorname{Re} z} \qquad \text{(Lemma 4.3)} \tag{4.27}$$

$$\left|-\frac{(\xi_1^2+\xi_2^2)(1-z/n)^2-2\xi_1\xi_2(1-z/n)}{4\lambda z(2-z/n)}\right| \leq C_7. \tag{4.28}$$

Hence the absolute value of the integrand in J is dominated by $C_8 e^{(2/3)\operatorname{Re} z}$. ii) is now proved, for

$$|J_2| \leq C_7 \int_{\gamma_{1,2}} e^{(2/3)\operatorname{Re} z} |dz| = 0(n^{-1/3}). \tag{4.29}$$

iii) If $n \geq 4$, then

$$|1 - z/n| \geq \frac{3}{4} + \frac{1}{(2\sqrt{2})} \tag{4.30}$$

for $|z| \geq n/2$. Hence

$$|1 - z/n|^{-n-1} \leq 1 \bigg/ \left(\frac{3}{4} + \frac{1}{2\sqrt{2}}\right)^{n+1} \tag{4.31}$$

for $n/2 \leq |z| \leq n$. Since for $kn \leq |z| \leq (k+1)n$, $k = 1, 2, 3, \cdots$

$$|1 - z/n| \geq k + 1 \tag{4.32}$$

we have

$$|J_3| \leq C_9 \left\{ \sum_{k=1}^{\infty} n(1+k)^{-n-1} + n\left(\frac{3}{4} + \frac{1}{2\sqrt{2}}\right)^{-n-1} \right\} = 0\left(\frac{1}{n}\right) \tag{4.33}$$

which proves iii).

Lemma 4.2 shows immediately the formula

$$J = \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^z}{\sqrt{2}} z^{-(1/2)-(2/\lambda)} \exp\left\{ -\frac{(\xi_1 - \xi_2)^2}{4\lambda z} \right\} dz \{1 + 0(n^{-1/3})\}, \tag{4.34}$$

or equivalently

$$\varphi'_{n,n} = n!\, M_n \left(\frac{\lambda}{2}\right)^n n^{-(1/2)+(2/\lambda)} \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^z}{\sqrt{2}} z^{-(1/2)-(2/\lambda)} \times \exp\left\{ -\frac{(\xi_1 - \xi_2)^2}{4\lambda z} \right\} dz \{1 + 0(n^{-1/3})\}. \tag{4.35}$$

As to $\varphi''_{n,n}$ we make the change of variable $\zeta' = -1 + z/n$ as before (refer to this with Lemma 4.1) and get a similar integral representation

$$n!\, M_n \left(\frac{\lambda}{2}\right)^n \cdot \frac{(-1)^{n+1}}{2\pi i} \int_{-\gamma_1} (2 - z/n)^{-(2/\lambda)-(1/2)} (z/n)^{-1/2} (1 - z/n)^{-n-1} \cdot \times \exp\left(-\frac{(\xi_1^2 + \xi_2^2)(1 - z/n)^2 + 2\xi_1 \xi_2 (1 - z/n)}{2\lambda z (2 - z/n)} \right) dz \tag{4.36}$$

A similar argument to the proof of Lemma 4.2 shows that for $n\to\infty$ $\varphi''_{n,n}$ is approximately equal to

$$(4.37)\qquad n!M_n\left(\frac{\lambda}{2}\right)^n\frac{(-1)^{n+1}}{2\pi i}n^{-1/2}2^{-(2/\lambda)-(1/2)}\cdot\int_{-\gamma_1}z^{-1/2}e^z \times\exp\left(-\frac{(\xi_1+\xi_2)^2}{4\lambda z}\right)\{1+0(n^{-1/3})\}.$$

Summing up (4.36) and (4.37), we have

(4.38)

$$\varphi_{n,n}=M_n n!\left(\frac{\lambda}{2}\right)^n\left\{n^{(2/\lambda)-(1/2)}\cdot 2^{(2/\lambda)-1}J_{(2/\lambda)-(1/2)}\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)^{-(2/\lambda)+(1/2)}\right.$$
$$\times[1+0(n^{-1/3})]+(-1)^n n^{-1/2}2^{-(2/\lambda)-1}$$
$$\left.\times J_{-1/2}\left(\frac{\xi_1+\xi_2}{\sqrt{\lambda}}\right)\left(\frac{\xi_1+\xi_2}{\sqrt{\lambda}}\right)^{1/2}[1+0(n^{-1/3})]\right\}$$
$$=n!M_n\left(\frac{\lambda}{2}\right)^n n^{(2/\lambda)-(1/2)}2^{(2/\lambda)-1}J_{(2/\lambda)-(1/2)}\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)\left(\frac{\xi_1-\xi_2}{\sqrt{\lambda}}\right)^{-(2/\lambda)+(1/2)}$$
$$\times\{1+0(n^{-1/3})\}$$

which implies the Theorem.

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