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On Functional Equations of Zeta Distributions

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Dedicated to Prof. Ichiro Satake on his sixtieth birthday

Introduction

Recent development in the theory of prehomogeneous vector spaces (in particular the works of Gyoja-Kawanaka [10] on prehomogeneous vector spaces defined over finite fields and of Igusa [17] on prehomogeneous vector spaces defined over $\mathfrak{p}$-adic number fields) has revealed a striking resemblance between the theories over finite fields, $\mathfrak{p}$-adic number fields, real and complex number fields and algebraic number fields, as is common in the theory of representations of algebraic groups.

Now we give a brief sketch of the fundamental theorem in the theory of prehomogeneous vector spaces. Let K be one of the fields mentioned above and $(\boldsymbol{G}, \rho, \boldsymbol{V})$ be a K-regular prehomogeneous vector space (satisfying some additional conditions, if necessary). Take K-irreducible polynomials $P_1, \cdots, P_n$ defining the K-irreducible hypersurfaces contained in the singular set $\boldsymbol{S}$. Let $\Omega(K^\times)$ be the set of quasi-characters of the multiplicative group $K^\times$ and $\mathscr{S}(V(K))$ the space of Schwartz-Bruhat functions on $V(K)$. For an $\omega \in \Omega(K^\times)^n$ we can define a tempered distribution (zeta distribution) $Z(\omega)$ on $V(K)$ by analytic continuation of the integral

$$Z(\omega)(\phi)=\int_{V(K)-S(K)} \prod_{i=1}^{n} \omega_i(P_i(x))\phi(x)d_V^\times(x) \qquad (\phi \in \mathscr{S}(V(K))),$$

where $d_V^\times(x)$ is a certain relatively $\boldsymbol{G}(K)$-invariant measure on $\boldsymbol{V}(K)-\boldsymbol{S}(K)$. Starting from the prehomogeneous vector space $(\boldsymbol{G}, \rho^*, \boldsymbol{V}^*)$ dual to $(\boldsymbol{G}, \rho, \boldsymbol{V})$, we can obtain a tempered distribution $Z^*(\omega)$ $(\omega \in \Omega(K^\times)^n)$ on $V^*(K)$.

Roughly speaking, the fundamental theorem states that the Fourier transform of the tempered distribution $Z(\omega)$ coincides with $Z^*(\omega^*)$ for certain ω^* up to a constant multiple $\gamma(\omega)$ depending meromorphically on ω: $\hat{Z}(\omega)=\gamma(\omega)Z^*(\omega^*)$.

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The aim of this paper, which is expository in part, is to explain in detail the analogy between the theories over various fields. However our attention will be focused mainly on the cases of $\mathfrak{p}$-adic number fields and the rational number field $\boldsymbol{Q}$. The case of algebraic number fields can be reduced to the case of $\boldsymbol{Q}$ by using Weil's functor $R_{K/\boldsymbol{Q}}$.

When the base field K is a $\mathfrak{p}$-adic number field $k_{\mathfrak{p}}$, the fundamental theorem has been proved by Igusa [17] under the assumptions that

1) $\boldsymbol{G}$ is reductive and self-adjoint over $k_{\mathfrak{p}}$,

2) $\boldsymbol{S}$ is absolutely irreducible,

3) $\boldsymbol{S}$ contains only a finite number of $\boldsymbol{G}^1$-orbits, where $\boldsymbol{G}^1$ is the kernel of rational characters corresponding to relative invariants.

After recalling some basic properties of regular prehomogeneous vector spaces and the fundamental theorem for $K=\boldsymbol{R}$ in Section 1, the fundamental theorem for $k_{\mathfrak{p}}$ is proved in Section 2 under certain weaker conditions so as to include the case of several independent relative invariants. Namely the assumptions 1) and 2) will be replaced by the assumption that $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is $k_{\mathfrak{p}}$-regular; however some modified form of the assumption 3) on the finiteness of singular orbit is inevitable at present.

Compared with the previous result for $\boldsymbol{R}$ ([37], [40], [35]), the result for $k_{\mathfrak{p}}$ is still unsatisfactory in some respects. Firstly the finiteness assumption on singular orbits should be removed in future. Secondly the nature of the so-called Γ-matrices $\gamma(\omega)$, in particular a possible relation of Γ-matrices with b-functions, remains to be clarified. Hence it seems profitable at the present stage to compare the Γ-matrix over a $\mathfrak{p}$-adic number field with the Γ-matrix over $\boldsymbol{R}$ for a prehomogeneous vector space defined over $\boldsymbol{Q}$. In Section 3 we make such a comparison of Γ-matrices for some concrete examples. Our calculation of Γ-matrices along with the recent results of Muller [25] and Igusa [18] suggests that Γ-matrices are under the control of b-functions also in the case of $\mathfrak{p}$-adic number fields (see § 3.5).

For the case $K=\boldsymbol{Q}$, by introducing an analogue $\mathscr{S}(\boldsymbol{V}(\boldsymbol{Q}))$ of the Schwartz-Bruhat space and a relatively $\boldsymbol{G}(\boldsymbol{Q})$-invariant "measure" on $\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$, we reformulate in Section 4 the functional equations of zeta functions associated with prehomogeneous vector spaces in order to make it easier to see the analogy to the theories over the other fields. Moreover, as a result of the introduction of $\mathscr{S}(\boldsymbol{V}(\boldsymbol{Q}))$, we are able to discuss L-functions and zeta functions of Hurwitz type associated with prehomogeneous vector spaces as well (§ 4.6). The necessity of this kind of generalization of zeta functions, which has been recognized independently by Hoffmann [16], results from the fact that certain special values of L-functions and zeta functions of Hurwitz type appear in the contribution of parabolic conjugacy classes to the Selberg trace formula. This kind of application of prehomogeneous vector spaces will be reviewed in Section 4.7. Another by-product of our formulation is that the arithmetic meaning of zeta functions is clarified by considering the "measure" on $V(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ (§ 4.4).

The fundamental theorem over finite fields due to Gyoja-Kawanaka is explained in Section 4 in connection with L-functions associated with prehomogeneous vector spaces.

In the present paper we do not formulate the fundamental theorems in their full generality, namely we do not discuss partial Fourier transforms with respect to regular subspaces, since it is easy to extend most of the results in this paper to the case of partial Fourier transforms as treated in [35].

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§ 1. Preliminaries

1.1. Let K be a field of characteristic 0. We denote by $\bar{K}$ the algebraic closure of K. Let $\boldsymbol{G}$ be a connected linear algebraic group defined over K, $\boldsymbol{V}$ a finite dimensional $\bar{K}$-vector space with K-structure and $\rho: \boldsymbol{G}\rightarrow \boldsymbol{GL}(\boldsymbol{V})$ a rational representation of $\boldsymbol{G}$ on $\boldsymbol{V}$ defined over K. We assume that the triple $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is a prehomogeneous vector space defined over K. Then by definition there exists a proper algebraic subset $\boldsymbol{S}$ of $\boldsymbol{V}$ such that $\boldsymbol{V}(\bar{K})-\boldsymbol{S}(\bar{K})$ is a single $\boldsymbol{G}(\bar{K})$-orbit. The set $\boldsymbol{S}$ is called the singular set of $(\boldsymbol{G}, \rho, \boldsymbol{V})$ and is also defined over K. Denote by $K[\boldsymbol{V}]$ (resp. $K(\boldsymbol{V})$) the ring of polynomial functions (resp. the field of rational functions) defined over K. Let $\boldsymbol{S}_1, \cdots, \boldsymbol{S}_n$ be the K-irreducible hypersurfaces in $\boldsymbol{V}$ contained in $\boldsymbol{S}$. Take a K-irreducible polynomial function $P_i \in K[\boldsymbol{V}]$ defining $\boldsymbol{S}_i$ for each $i=1, \cdots, n$. Then $P_1, \cdots, P_n$ are relative invariants of $(\boldsymbol{G}, \rho, \boldsymbol{V})$ and any relative invariant in $K(\boldsymbol{V})$ can be written uniquely as $cP_1^{\nu_1}\cdots P_n^{\nu_n}$ with $c \in K^{\times}, \nu_1, \cdots, \nu_n \in \boldsymbol{Z}$ ([35, Lemma 1.3]). Let $\chi_1 \cdots \chi_n$ be the rational characters of $\boldsymbol{G}$ corresponding to $P_1, \cdots, P_n$, respectively and $X_\rho(\boldsymbol{G})_K$ the group of K-rational characters of $\boldsymbol{G}$ corresponding to relative invariants. The group $X_\rho(\boldsymbol{G})_K$ is a free abelian group of rank n generated by $\chi_1, \cdots, \chi_n$ ([35, Lemma 1.4]).

1.2. Let $\boldsymbol{V}^*$ be the $\bar{K}$-vector space dual to $\boldsymbol{V}$. The vector space $\boldsymbol{V}^*$ has a natural K-structure determined by the K-structure of $\boldsymbol{V}$. Denote by $\rho^*: \boldsymbol{G}\rightarrow \boldsymbol{GL}(\boldsymbol{V}^*)$ the rational representation of $\boldsymbol{G}$ on $\boldsymbol{V}^*$ contragredient to ρ. Fix a K-basis of $\boldsymbol{V}(K)$ and identify $\boldsymbol{V}$ with $\bar{K}^N$ ($N=\dim_K \boldsymbol{V}$). Then

$K[V]$ can be identified with the ring $K[x_1, \cdots, x_N]$ of polynomials in N variables with coefficients in K. We also identify V^* with $\bar{K}^N$ via the K-basis of $V^*(K)$ dual to the fixed basis of $V(K)$.

For a relative invariant $P(x) \in K[V]=K[x_1, \cdots, x_N]$, we define a mapping ϕ_P: $V-S \to V^*$ by

$$\phi_P(x)=\left(\frac{1}{P(x)} \frac{\partial P}{\partial x_1}(x), \cdots, \frac{1}{P(x)} \frac{\partial P}{\partial x_N}(x)\right).$$

The mapping ϕ_P is independent of the choice of K-basis of $V(K)$ and defines a rational mapping defined over K. Moreover it is known that

$$\phi_P(\rho(g)x)=\rho^*(g)\phi_P(x) \qquad (x \in V-S,\ g \in G).$$

Definition. If there exists a relative invariant $P(x) \in K[V]$ such that the mapping ϕ_P: $V-S \to V^*$ is dominant, then (G, ρ, V) is called *K-regular* and such a P is called *nondegenerate.*

Lemma 1.1 ([37, Chap. 1 § 1], [39, § 4 Prop. 10, Remark 11], [35, Lemmas 2.4, 2.5]). *Let (G, ρ, V) be a K-regular prehomogeneous vector space and $P(\in K[V])$ be a nondegenerate relative invariant. Then*

(1) *(G, ρ^*, V^*) is also a K-regular prehomogeneous vector space.*

(2) *Let S^* be the singular set of (G, ρ^*, V^*). The mapping ϕ_P induces a G-equivariant biregular rational mapping defined over K of $V-S$ onto V^*-S^*.*

(3) *For an $x \in V-S$, put $x^*=\phi_P(x)$, Then $G_x=G_{x^*}$, where $G_x=\{g \in G;\ \rho(g)x=x\}$ and $G_{x^*}=\{g \in G;\ \rho^*(g)x^*=x^*\}$.*

(4) *$X_\rho(G)_K=X_{\rho^*}(G)_K$.*

(5) *The rational character of G defined by $g \mapsto \det \rho(g)^2$ is in $X_\rho(G)_K$.*

(6) *S is a hypersurface in V if and only if S^* is a hypersurface in V^*.*

Let $S_1^*, \cdots, S_n^*$ be the K-irreducible hypersurfaces contained in S^*. Note that by Lemma 1.1 (4) the number of K-irreducible hypersurfaces contained in S^* is equal to the number n of K-irreducible hypersurfaces contained in S. Take a K-irreducible relative invariant $P_i^* \in K[V^*]$ defining S_i^* for each $i=1, \cdots, n$ and denote by χ_i^* the rational character of G corresponding to P_i^*. Since $\chi_1, \cdots, \chi_n$ and $\chi_1^*, \cdots, \chi_n^*$ form two systems of generators of the free abelian group $X_\rho(G)_K=X_{\rho^*}(G)_K$, there exists a unimodular matrix $U=(u_{ij}) \in GL(n, Z)$ satisfying

$$\chi_i=\prod_{j=1}^n \chi_j^{*u_{ij}} \qquad (1 \leqq i \leqq n).$$

Let λ and λ^* be the elements of $(\frac{1}{2}Z)^n$ defined by

$$\det \rho(g)^2 = \prod_{i=1}^{n} \chi_i(g)^{2\lambda_i}, \qquad \det \rho^*(g)^2 = \prod_{i=1}^{n} \chi_i^*(g)^{2\lambda_i^*}.$$

Since $\det \rho(g) = \det \rho^*(g)^{-1}$, we have $\lambda^* = -\lambda U$.

Another important fact derived immediately from Lemma 1.1 is the one-to-one correspondence between $G(K)$-orbits in $V(K)-S(K)$ and $G(K)$-orbits in $V^*(K)-S^*(K)$. In particular, if the number of $G(K)$-orbits in $V(K)-S(K)$ is finite, then so is the number of $G(K)$-orbits in $V^*(K)-S^*(K)$ and they coincide with each other. Note that this is the case if K is a local field ([42, III 4.2 Example d), 4.4 Theorem 5]).

1.3. Now we recall the definition of b-functions (for details, see [37, Chap. 1] and [38]). In the following we assume that (G, ρ, V) is K-regular.

For $\chi \in X_\rho(G)_K = X_{\rho^*}(G)_K$, let $\delta(\chi)$ and $\delta^*(\chi)$ be the elements in Z^n such that

$$\chi = \prod_{i=1}^{n} \chi_i^{\delta(\chi)_i} = \prod_{i=1}^{n} \chi_i^{*\delta^*(\chi)_i}.$$

Then $\delta(\chi)U = \delta^*(\chi)$.

We put

$$P^\chi(x) = \prod_{i=1}^{n} P_i(x)^{\delta(\chi)_i}, \qquad P^{*\chi}(y) = \prod_{i=1}^{n} P_i^*(y)^{\delta^*(\chi)_i} \qquad (\chi \in X_\rho(G)_K).$$

Define a partial differential operator $P^\chi(\mathrm{grad}_y)$ (resp. $P^{*\chi}(\mathrm{grad}_x)$) with constant coefficients by

$$P^\chi(\mathrm{grad}_y) e^{\langle x, y\rangle} = P^\chi(x) e^{\langle \chi, y\rangle}$$
$$(\text{resp. } P^{*\chi}(\mathrm{grad}_x) e^{\langle x, y\rangle} = P^{*\chi}(y) e^{\langle x, y\rangle}).$$

If $\delta^*(\chi)_i \geq 0$ for all i, then there exists a polynomial $b_\chi(s)$ in $s_1, \cdots, s_n$ of degree $\deg P^{*\chi}(y)$ satisfying

$$P^{*\chi}(\mathrm{grad}_y) P^s(x) = b_\chi(s) P^{s+\delta(\chi)}(x),$$

where $P^s(x) = \prod_{i=1}^{n} P_i(x)^{s_i}$. By the cocycle property

$$b_{\chi\psi}(s) = b_\chi(s) b_\varphi(s + \delta(\chi)),$$

we can define $b_\chi(s)$ for all $\chi \in X_\rho(G)$. The polynomial $b_\chi(s)$ is called the *b-function* of (G, ρ, V).

In the case of $\bar{K} = C$ the b-function $b_\chi(s)$ has the following expression in terms of the gamma function:

$$b_\chi(s)=c(\chi)\gamma(s)/\gamma(s+\delta(\chi))$$

$$\gamma(s)=\prod_{i=1}^{m}\left\{\prod_{j=1}^{\alpha_i}\Gamma\left(\sum_{l=1}^{n}a_l^{(i)}s_l-p_{ij}\right)\right\}\Big/\left\{\prod_{j=1}^{\beta_i}\Gamma\left(\sum_{l=1}^{n}a_l^{(i)}s_l-q_{ij}\right)\right\},$$

where c is a homomorphism of $X_\rho(G)_K$ into $C^\times$ and $a_l^{(i)}\in Z$, $a_l^{(i)}\geqq 0$, p_{ij}, $q_{ij}\in C$.

1.4. In this paragraph we assume that K is a subfield of R and we describe briefly the main result in the theory over R.

Put $V=V(R)$, $V^*=V^*(R)$, $S=S(R)$ and $S^*=S^*(R)$.

Let $V-S=V_1\cup\cdots\cup V_\nu$ and $V^*-S^*=V_1^*\cup\cdots\cup V_\nu^*$ be the decompositions into connected components. Each connected component is an orbit of the identity component G^+ of $G(R)$.

We denote by $\mathscr{S}(V)$ and $\mathscr{S}(V^*)$ the spaces of rapidly decreasing functions on V and V^*, respectively.

Consider the integrals

$$Z_j(s)(f)=\int_{V_j}\prod_{i=1}^{n}|P_i(x)|^{s_i-\lambda_i}f(x)dx \qquad (f\in\mathscr{S}(V),\ 1\leqq j\leqq\nu)),$$

$$Z_j^*(s)(f^*)=\int_{V_j^*}\prod_{i=1}^{n}|P_i(y)|^{s_i-\lambda_i^*}f^*(y)dy \qquad (f^*\in\mathscr{S}(V^*),\ 1\leqq j\leqq\nu),$$

where dx and dy are Euclidean measures on V and V^*, respectively. The integrals $Z_j(s)(f)$ (resp. $Z_j^*(s)(f^*)$) are absolutely convergent at least for $\operatorname{Re}s_i\geqq\lambda_i$ (resp. $\operatorname{Re}s_i\geqq\lambda_i^*$). The analytic continuation of $Z_j(s)(f)$ (resp. $Z_j^*(s)(f^*)$) to a meromorphic function of s in C^n always exists and defines a tempered distribution

$$Z_j(s): f\longmapsto Z_j(s)(f)$$
$$(\text{resp. } Z_j^*(s): f^*\longmapsto Z_j^*(s)(f^*))$$

on V (resp. V^*) depending on s meromorphically, which we call the *zeta distribution* associated with (G, ρ, V) over R.

The Fourier transform $\hat{Z}_j(s)$ of the zeta distribution $Z_j(s)$, which is a tempered distribution on V^*, is defined by

$$\hat{Z}_j(s)(f^*)=Z_j(s)(\hat{f}^*),$$

$$\hat{f}^*(x)=\int_{V^*}f^*(y)e^{2\pi i\langle x, y\rangle}dy.$$

Now the fundamental theorem over R can be stated as follows:

Theorem R (cf. [37, Chap. 2 § 1], [40, § 1], [35, § 5]). *Assume that (G, ρ, V) is R-regular and S is a hypersurface. Then*

$$\hat{Z}_i(s)=\sum_{j=1}^{\nu} \Gamma_{ij}(s-\lambda)Z_j^*(\lambda^*+s^*) \qquad (1 \leqq i \leqq \nu)$$

with

$$\Gamma_{ij}(s)=c(-s)(-2\pi\sqrt{-1})^{\sum_{l=1}^n s_l \deg P_l^*}\gamma(s)\cdot\prod_{l=1}^n \varepsilon_l(i)^{-s_l}\cdot\prod_{l=1}^n \varepsilon_l^*(j)^{s_l^*}\cdot t_{ij}(s),$$

where $\varepsilon_l(i)=\operatorname{sgn} P_l(x)$ $(x \in V_i)$, $\varepsilon_l^*(j)=\operatorname{sgn} P_l^*(y)$ $(y \in V_j^*)$, $s^*=(s_1^*, \cdots, s_n^*)$ $=sU$ and $t_{ij}(s)$ are polynomials in $e^{\pm 2\pi\sqrt{-1}s_1}, \cdots, e^{\pm 2\pi\sqrt{-1}s_n}$.

We call the matrix $(\Gamma_{ij}(s))_{i,j=1}^{\nu}$ the *Γ-matrix* of (G, ρ, V).

Remark. If we take $K=C$ and assume that (G, ρ, V) is (C-) regular, then $(\tilde{G}, \tilde{\rho}, \tilde{V})=R_{C/R}(G, \rho, V)$ is R-regular and $\tilde{V}(R)-\tilde{S}(R)=V(C)-S(C)$. Hence Theorem R for $(\tilde{G}, \tilde{\rho}, \tilde{V})$ immediately yields "Theorem C" (the fundamental theorem for prehomogeneous vector spaces defined over C) for (G, ρ, V). Note that $\nu=1$ in this case and hence the Γ-matrix is a scalar. However more is known for $K=C$. Namely the Γ-matrix is explicitly calculated (up to sign) at least when G is reductive ([37, § 3]). If we further assume that S is absolutely irreducible, then the sign left to be determined was settled by Igusa ([18, Theorem 2]). A remarkable fact in the case of $K=C$ is that the Γ-matrix is completely determined by the b-function.

§ 2. Fourier transforms of p-adic complex powers

2.1. Let K be a $\mathfrak{p}$-adic number field, namely a finite extension of the p-adic number field Q_p. Denote by $\mathfrak{o}$ the ring of integers of K and by $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$. Let π be a (fixed) prime element of K and $\mathfrak{o}^\times$ the unit group of $\mathfrak{o}$. We normalize the absolute value of $\alpha \in K^\times$ by

$$|\alpha|=q^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)}, \qquad q=N(\mathfrak{p})=\#(\mathfrak{o}/\mathfrak{p}).$$

Let $\Omega(K^\times)$ be the set of quasi-characters of $K^\times$. For a complex number $s \in C$, we define an $\omega_s \in \Omega(K^\times)$ by

$$\omega_s(\alpha)=|\alpha|^s \qquad (\alpha \in K^\times).$$

We denote the dual group of $\mathfrak{o}^\times$ by $\widehat{\mathfrak{o}^\times}$. We identify an element $\phi \in \widehat{\mathfrak{o}^\times}$ with the character of $K^\times$ obtained by extending ϕ to $K^\times$ so that $\phi(\pi)=1$.

Every $\omega \in \Omega(K^{\times})$ can be expressed as $\omega=\omega_s\phi$ with

$$s \in \boldsymbol{C}\Big/\Big(\frac{2\pi i}{\log q}\Big)\boldsymbol{Z} \quad \text{and} \quad \phi \in \widehat{\mathfrak{o}^{\times}}.$$

Then we put $\mathrm{Re}(\omega)=\mathrm{Re}\, s$ $(s=\log \omega(\pi)/\log q)$.

Put

$$\Omega(K^{\times})_{\phi}=\Big\{\omega_s\phi;\ s \in \boldsymbol{C}\Big/\Big(\frac{2\pi i}{\log q}\Big)\boldsymbol{Z}\Big\} \qquad (\phi \in \widehat{\mathfrak{o}^{\times}}).$$

Since

$$\Omega(K^{\times})=\coprod_{\phi\in\widehat{\mathfrak{o}^{\times}}} \Omega(K^{\times})_{\phi} \quad \text{and} \quad \Omega(K^{\times})_{\phi}\simeq \boldsymbol{C}\Big/\Big(\frac{2\pi i}{\log q}\Big)\boldsymbol{Z}\simeq \boldsymbol{C}^{\times},$$

we may regard $\Omega(K^{\times})$ as a 1-dimensional complex Lie group. We call a function F: $\Omega(K^{\times})^n\to \boldsymbol{C}$ a *rational function on* $\Omega(K^{\times})^n$, if the function $F_{\phi}(s_1, \cdots, s_n)=F(\omega_{s_1}\phi_1, \cdots, \omega_{s_n}\phi_n)$ is a rational function of $q^{-s_1}, \cdots, q^{-s_n}$ for each $\phi=(\phi_1, \cdots, \phi_n) \in \widehat{(\mathfrak{o}^{\times})^n}$.

2.2. We now consider a prehomogeneous vector space $(\boldsymbol{G}, \rho, \boldsymbol{V})$ defined over a $\mathfrak{p}$-adic number field K and assume that

(A.1) $(\boldsymbol{G}, \rho, \boldsymbol{V})$ *is K-regular.*

Retain the notation in Section 1.1 and Section 1.2 and put $G=\boldsymbol{G}(K)$, $V=\boldsymbol{V}(K)$, $S=\boldsymbol{S}(K)$ and $S^*=\boldsymbol{S}^*(K)$. As remarked at the end of Section 1.2, the number of $\rho(G)$-orbits in $V-S$ is finite and is equal to the number of $\rho^*(G)$-orbits in V^*-S^*. Let

$$V-S=V_1\cup\cdots\cup V_{\nu} \quad \text{and} \quad V^*-S^*=V_1^*\cup\cdots\cup V_{\nu}^*$$

be the $\rho(G)$-orbit decomposition of $V-S$ and the $\rho^*(G)$-orbit decomposition of V^*-S^*, respectively.

Denote by $\mathscr{S}(V)$ and $\mathscr{S}(V^*)$ the spaces of Schwartz-Bruhat functions on V and V^*, respectively. We fix a nontrivial additive character ψ of K and define the Fourier transform of $f^* \in \mathscr{S}(V)$ by

$$\hat{f}^*(x)=\int_{V^*} f^*(y)\psi(\langle x, y\rangle)dy,$$

where dy is a (fixed) Haar measure on V^*. Denote by dx the Haar measure on V dual to dy.

For $\omega=(\omega_1, \cdots, \omega_n) \in \Omega(K^{\times})^n$, put

$$\omega(P(x))=\prod_{i=1}^{n}\omega_i(P_i(x)) \quad \text{and} \quad \omega(P^*(y))=\prod_{i=1}^{n}\omega_i(P_i^*(y)).$$

Further put

$$\omega_\lambda=(\omega_{\lambda_1},\cdots,\omega_{\lambda_n}) \quad \text{and} \quad \omega_{\lambda^*}=(\omega_{\lambda_1^*},\cdots,\omega_{\lambda_n^*})$$

(for the definitions of λ and λ^*, see § 1.2).

Consider the integrals

$$Z_i(\omega)(f)=\int_{V_i}(\omega/\omega_\lambda)(P(x))f(x)dx \qquad (f\in\mathcal{S}(V),\ 1\leqq i\leqq\nu)$$

and

$$Z_j^*(\omega)(f^*)=\int_{V_j^*}(\omega/\omega_{\lambda^*})(P^*(y))f^*(y)dy \qquad (f^*\in\mathcal{S}(V^*),\ 1\leqq j\leqq\nu).$$

It is clear that $Z_i(\omega)(f)$ (resp. $Z_j^*(\omega)$ (f^*)) are absolutely convergent for $\operatorname{Re}\omega_i>\lambda_i$ $(1\leqq i\leqq n)$ (resp. $\operatorname{Re}\omega_i>\lambda_i^*$ $(1\leqq i\leqq n)$) and represent holomorphic functions in $\{\omega\in\Omega(K^\times)^n;\ \operatorname{Re}\omega_i>\lambda_i\ (1\leqq i\leqq n)\}$ (resp. $\{\omega\in\Omega(K^\times)^n;\ \operatorname{Re}\omega_i>\lambda_i^*\ (1\leqq i\leqq n)\}$).

Since the theorem of Denef (see [8, Theorems 3.2 and 7.4] and [9, Theorem 3.1]) on the rationality of p-adic integrals, which Igusa used in his proof of [17, Lemma 2], can be easily generalized to the case of several complex variables (cf. [8, Proof of Theorem 5.1], [2, § 2 Theorem A]), we can prove the following lemma in the same manner as in the proof of [17, Lemma 2].

Lemma 2.1. *The functions $Z_i(\omega)(f)$ $(1\leqq i\leqq\nu, f\in\mathcal{S}(V))$ have analytic continuations to rational functions in $\Omega(K^\times)^n$ in the sense of Section 2.1. Moreover for each $\phi\in\widehat{(\mathfrak{o}^\times)}^n$, there exists a collection of integers $\{a_1^{(l)},\cdots,a_n^{(l)},b^{(l)},m_l;\ 1\leqq l\leqq h,\ 0\leqq m_l\leqq[K\colon \boldsymbol{Q}_p]\dim V\}$ independent of $f\in\mathcal{S}(V)$ such that*

$$\prod_{l=1}^{h}(1-q^{-\Sigma_{j=1}^{n}a_j^{(l)}s_j-b^{(l)}})^{m_l}\cdot Z_i(\omega_s\phi)(f)$$

are holomorphic functions and hence polynomials in $q^{\pm s_1},\cdots,q^{\pm s_n}$.

The same statement holds also for $Z_j^*(\omega)(f^*)$.

The tempered distributions $Z_i(\omega)$ and $Z_j^*(\omega)$ defined by

$$Z_i(\omega)\colon \mathcal{S}(V)\ni f\longmapsto Z_i(\omega)(f)\in\boldsymbol{C}$$

and

$$Z_j^*(\omega)\colon \mathcal{S}(V^*) \ni f^* \longrightarrow Z_j^*(\omega)(f^*) \in \boldsymbol{C}$$

are called the *zeta distributions associated with* $(\boldsymbol{G}, \rho, \boldsymbol{V})$ and $(\boldsymbol{G}, \rho^*, \boldsymbol{V}^*)$, respectively. They depend on ω meromorphically.

2.3. In the following we further assume that

(A.2) *$\boldsymbol{S}$ decomposes into a finite number of $\boldsymbol{G}$-orbits. Moreover for any $\boldsymbol{G}$-orbit $\boldsymbol{S}'$ in $\boldsymbol{S}$, there exists a $\chi \in X_\rho(\boldsymbol{G})_K$ ($\chi \neq 1$) such that $\boldsymbol{S}'$ is a $\boldsymbol{G}^{(\chi)}$-orbit, where $\boldsymbol{G}^{(\chi)}$ denotes the kernel of χ.*

The assumption (A.2) enables us to analyze relatively invariant distributions on $S=\boldsymbol{S}(K)$ and singularities of zeta distributions.

The condition is equivalent to the following:

(A.2)$'$ *$\boldsymbol{S}$ decomposes into a finite number of $\boldsymbol{G}$-orbits. Moreover for any $x \in \boldsymbol{S}$, there exists a $\chi \in X_\rho(\boldsymbol{G})_K$ ($\chi \neq 1$) such that the restriction of χ to the identity component of $\boldsymbol{G}_x=\{g \in \boldsymbol{G};\ \rho(g)x=x\}$ is nontrivial.*

Define a homomorphism $\nu\colon G \to \boldsymbol{Z}^n$ by

$$\nu(g)=(\mathrm{ord}_{\mathfrak{p}}\chi_1(g), \cdots, \mathrm{ord}_{\mathfrak{p}}\chi_n(g)) \qquad (g \in G).$$

Lemma 2.2. *For any $x \in S$, $\nu(G_x)$ is a non-zero submodule of $\boldsymbol{Z}^n$, where $G_x=\boldsymbol{G}_x \cap G$.*

Proof. Let $S^{(1)}$ be the $\rho(G)$-orbit containing x and $\boldsymbol{S}_1$ be the $\rho(\boldsymbol{G})$-orbit in $\boldsymbol{S}$ such that $\boldsymbol{S}_1(K) \supset S^{(1)}$. By (A.2) there exists a $\chi \in X_\rho(\boldsymbol{G})_K$ ($\chi \neq 1$) such that $\boldsymbol{S}_1$ is a $\boldsymbol{G}^{(\chi)}$-orbit. Put $G^{(\chi)}=\boldsymbol{G}^{(\chi)}(K)$. Then $S^{(1)}$ is $G^{(\chi)}$-stable and decomposes into a finite number of $G^{(\chi)}$-orbits. Since $G^{(\chi)}$ is a normal subgroup of G, every element in G induces a permutation of $G^{(\chi)}$-orbits in $S^{(1)}$. Let G' be the subgroup of G consisting of all elements in G which stabilize all $G^{(\chi)}$-orbits. It is clear that the index $[G\colon G']$ is finite. Put $G'_x=G' \cap G_x$. Then we have $G'=G'_x \cdot G^{(\chi)}$. Recall that χ can be written as $\chi=\prod_{i=1}^n \chi_i^{r_i}$ with $r_1, \cdots, r_n \in \boldsymbol{Z}$. For an $\boldsymbol{m}=(m_1, \cdots, m_n) \in \boldsymbol{Z}^n$, put $\langle \boldsymbol{r}, \boldsymbol{m}\rangle=\sum_{i=1}^n r_i m_i$. If $\nu(G_x)=\{0\}$, then $\langle \boldsymbol{r}, \nu(G')\rangle = \langle \boldsymbol{r}, \nu(G'_x \cdot G^{(\chi)})\rangle = \langle \boldsymbol{r}, \nu(G'_x)\rangle=\{0\}$. Since G' is of finite index in G, we have $\langle \boldsymbol{r}, \nu(G)\rangle=\{0\}$. On the other hand $\chi_1, \cdots, \chi_n$ are multiplicatively independent K-rational characters of $\boldsymbol{G}$ and hence $\boldsymbol{G}$ contains a K-split algebraic torus $\boldsymbol{T}$ of dimension n such that the restictions of $\chi_1, \cdots, \chi_n$ to $\boldsymbol{T}$ generate a subgroup of the character group of $\boldsymbol{T}$ of finite index. This implies that $\nu(G)$ is a subgroup of $\boldsymbol{Z}^n$ of finite index. This contradicts $\langle \boldsymbol{r}, \nu(G)\rangle=\{0\}$, since $\boldsymbol{r}\neq 0$. Therefore $\nu(G)\neq\{0\}$.

Let

$$S=S^{(1)}\cup\cdots\cup S^{(m)}$$

be the $\rho(G)$-orbit decomposition of S. The number m of $\rho(G)$-orbits is finite by the assumption (A.2) and [42, III 4.4 Theorem 5]. We fix a $\rho(G)$-orbit in S, say $S^{(1)}$. Let $\mathcal{S}(S^{(1)})$ be the $\boldsymbol{C}$-vector space of locally constant functions on $S^{(1)}$ with compact support and $\mathcal{S}(S^{(1)})^*$ the space of distributions on $S^{(1)}$. By definition we have $\mathcal{S}(S^{(1)})^*=\mathrm{Hom}_{\boldsymbol{C}}(\mathcal{S}(S^{(1)}),\boldsymbol{C})$. The group G acts on $\mathcal{S}(S^{(1)})$ and $\mathcal{S}(S^{(1)})^*$ as follows:

$$\begin{aligned}&f^g(x)=f(\rho(g)x) && (f\in\mathcal{S}(S^{(1)}),\ g\in G),\\ &(gT)(f)=T(f^g) && (T\in\mathcal{S}(S^{(1)})^*,\ f\in\mathcal{S}(S^{(1)}),\ g\in G).\end{aligned}$$

For an $\omega\in\Omega(K^\times)^n$, we define a continuous homomorphism $\omega\circ\chi\colon G\to\boldsymbol{C}^\times$ by

$$\omega\circ\chi(g)=\prod_{i=1}^{n}\omega_i(\chi_i(g))\qquad(g\in G).$$

Put

$$\mathcal{E}_{S^{(1)}}(\omega)=\{T\in\mathcal{S}(S^{(1)})^*;\ gT=\omega\circ\chi(g)^{-1}T,\ g\in G\}\qquad(\omega\in\Omega(K^\times)^n)$$

and

$$\Omega(S^{(1)})=\{\omega\in\Omega(K^\times)^n;\ \mathcal{E}_{S^{(1)}}(\omega)\neq\{0\}\}.$$

Also put

$$\Omega(S^{(1)})_\phi=\Omega(S^{(1)})\cap(\Omega(K^\times)_{\phi_1}\times\cdots\times\Omega(K^\times)_{\phi_n})$$

for $\phi=(\phi_1,\cdots,\phi_n)\in(\widehat{\mathfrak{o}^\times})^n$.

We say that a subset E of

$$\Omega(K^\times)^n_\phi=\Omega(K^\times)_{\phi_1}\times\cdots\times\Omega(K^\times)_{\phi_n}\simeq\boldsymbol{C}^n\Big/\Big\{\Big(\frac{2\pi i}{\log q}\Big)\boldsymbol{Z}\Big\}^n$$

a *linear subspace*, if there exists a finite number of inhomogeneous linear functions $L_1,\cdots,L_t$ on $\boldsymbol{C}^n$ such that the coefficients of $s_1,\cdots,s_n$ in $L_1,\cdots,L_t$ are integral and

$$E=\Big\{(\omega_{s_1}\phi_1,\cdots,\omega_{s_n}\phi_n);\ L_i(s_1,\cdots,s_n)\in\Big(\frac{2\pi i}{\log q}\Big)\boldsymbol{Z}\ (1\leqq i\leqq t)\Big\}.$$

Lemma 2.3. *If $\Omega(S^{(1)})_\phi$ is not empty, then $\Omega(S^{(1)})_\phi$ is a linear subspace of $\Omega(K^\times)^n_\phi$. In fact let $\boldsymbol{m}^{(i)}=(m_1^{(i)},\cdots,m_n^{(i)})$ $(1\leqq i\leqq r)$ be a set of* *generators of the free abelian group* $\nu(G_x)$ $(x \in S^{(1)})$. *Then for some* $\alpha=(\alpha_1, \cdots, \alpha_n) \in \boldsymbol{C}^n$, *we have*

$$\Omega(S^{(1)})_\phi=\left\{(\omega_{s_1+\alpha_1}\phi_1, \cdots, \omega_{s_n+\alpha_n}\phi_n);\ \sum_{i=1}^n m_i^{(j)}s_i \in \left(\frac{2\pi i}{\log q}\right)\boldsymbol{Z}\quad (1\leqq j\leqq r)\right\}.$$

Proof. Assume that $\Omega(S^{(1)})_\phi$ is not empty. For $\omega=\omega_s\phi \in \Omega(K^\times)^n_\phi$, ω belongs to $\Omega(S^{(1)})_\phi$ if and only if $(\omega\circ\chi)\cdot\Delta_G|_{G_x}=\Delta_{G_x}$, where $x \in S^{(1)}$ and Δ_G (resp. Δ_{G_x}) are the modules of G (resp. G_x). Fix an $\alpha \in \boldsymbol{C}^n$ such that $\omega_\alpha\phi \in \Omega(S^{(1)})_\phi$. Then $\omega_{s+\alpha}\phi$ belongs to $\Omega(S^{(1)})_\phi$ if and only if $\omega_s\circ\chi|_{G_x}=1$. This is equivalent to the condition that

$$\sum_{i=1}^n m_i s_i \in \left(\frac{2\pi i}{\log q}\right)\boldsymbol{Z}\qquad \text{for any}\quad (m_1, \cdots, m_n) \in \nu(G_x).$$

This proves the lemma.

We define $\mathscr{E}_S(\omega)$ $(\omega \in \Omega(K^\times)^n)$, $\Omega(S)$ and $\Omega(S)_\phi$ in the same way as in the definitions of $\mathscr{E}_{S^{(1)}}(\omega)$, $\Omega(S^{(1)})_\phi$ and $\Omega(S^{(1)})$. Then by (A.2) and [17, p. 1016] we have $\Omega(S)_\phi=\bigcup_{i=1}^m \Omega(S^{(i)})_\phi$ and $\Omega(S)=\bigcup_{i=1}^m \Omega(S^{(i)})$, where $S=S^{(1)}\cup\cdots\cup S^{(m)}$ is the $\rho(G)$-orbit decomposition of S. Therefore Lemma 2.3 immediately implies the following

Corollary. *$\Omega(S)_\phi$ is a finite union of linear subspaces of $\Omega(K^\times)^n_\phi$.*

Proposition 2.4. *The poles of $Z_i(\omega)(f)$ are contained in $\Omega(S)$.*

Proof. Fix a $\phi \in \widehat{(\mathfrak{o}^\times)}^n$. Then by Lemma 2.1 we have

$$Z_i(\omega_s\phi)(f)=\frac{P(f, \omega_s\phi)}{\prod_{l=1}^h (1-q^{-\sum_{j=1}^n a_j^{(l)}s_j-b^{(l)}})^{m_l}},$$

where $P(f, \omega_s\phi)$ is a polynomial in $q^{\pm s_1}, \cdots, q^{\pm s_n}$, $a_j^{(l)}$, $b^{(l)}$, m_l are integers independent of f and $1\leqq m_l\leqq [k\colon \boldsymbol{Q}_p]\dim V$. Assume that $s \in \boldsymbol{C}^n$ satisfies the linear equation

$$\sum_{j=1}^n a_j^{(l)}s_j+b^{(l)} \in \left(\frac{2\pi i}{\log q}\right)\boldsymbol{Z}$$

for some l. Then the distribution T on V defined by

$$T(f)=(1-q^{-\sum_{j=1}^n a_j^{(l)}s_j-b^{(l)}})^{m_l}Z_i(\omega_s\phi)(f)\qquad (f \in \mathscr{S}(V))$$

satisfies the identity $gT=(\omega_s\phi)\circ\chi(g)^{-1}T$. We may assume that the integer m_l is chosen so that $T(f)\neq 0$ for some $f\in\mathscr{S}(V)$. Since $Z_i(\omega_s)(f)$ is entire for every $f\in\mathscr{S}(V-S)$, we have $T(f)=0$ for all $f\in\mathscr{S}(V-S)$. In other words, T is in $\mathscr{E}_S(\omega_s\phi)$. Hence $\omega=\omega_s\phi\in\Omega(S)$.

2.4. For $\omega\in\Omega(K^\times)^n$, put

$$\omega^*=\left(\prod_{i=1}^n \omega_i^{u_{i1}},\ \cdots,\ \prod_{i=1}^n \omega_i^{u_{in}}\right)\in\Omega(K^\times)^n,$$

where $U=(u_{ij})$ is the unimodular matrix defined in Section 1.2.

The Fourier transform $\hat{Z}_i(\omega)$ of the zeta distribution $Z_i(\omega)$ is defined by $\hat{Z}_i(\omega)\,(f^*)=Z_i(\omega)(\hat{f}^*)$ $(f^*\in\mathscr{S}(V^*))$. Then we have the following

Theorem $k_{\mathfrak{p}}$. *If* (G,ρ,V) *and* (G,ρ^*,V^*) *satisfy the assumptions* (A.1) *and* (A.2), *then the following functional equations hold*:

$$\hat{Z}_i(\omega)=\sum_{j=1}^{\nu}\Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*})\qquad(1\leqq i\leqq\nu),$$

where $\Gamma_{ij}(\omega)$ $(1\leqq i,j\leqq\nu)$ *are rational functions in* $\Omega(K^\times)^n$.

Proof. For $f\in\mathscr{S}(V)$ and $f^*\in\mathscr{S}(V^*)$, we put

$$f^g(x)=f(\rho(g)x),\qquad f^{*g}(y)=f^*(\rho^*(g)y)\qquad(g\in G).$$

Then we have

$$(f^{*g})^\wedge(x)=\omega_\lambda\circ\chi(g)(\hat{f}^*)^g(x)\qquad(f^*\in\mathscr{S}(V^*)).$$

Moreover it is easy to check the following identities:

$$Z_i(\omega)((f^{*g})^\wedge)=(\omega_\lambda/\omega)\circ\chi(g)Z_i(\omega)(\hat{f}^*),$$
$$Z_j^*(\omega^*\omega_{\lambda^*})(f^{*g})=(\omega_\lambda/\omega)\circ\chi(g)Z_j^*(\omega^*\omega_{\lambda^*})(f^*).$$

Therefore by the uniqueness of relatively invariant distributions on homogeneous spaces (cf. [17, Lemma 1]), we obtain

$$Z_i(\omega)(\hat{f}^*)=\Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*})(f^*)\qquad(f^*\in\mathscr{S}(V_j^*))$$

and hence

$$(*)\qquad Z_i(\omega)(\hat{f}^*)=\sum_{j=1}^{\nu}\Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*})(f^*)\qquad(f^*\in\mathscr{S}(V^*-S^*)),$$

where $\Gamma_{ij}(\omega)$ is independent of f^*. It is obvious that $\Gamma_{ij}(\omega)$ is a rational function on $\Omega(K^\times)^n$. Consider the distribution T_ω on V^* defined by

$$T_\omega(f^*)=Z_i(\omega)(\hat{f}^*)-\sum_{j=1}^{\nu}\Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*})(f^*) \qquad (f^*\in\mathscr{S}(V^*)).$$

By the identity (*) and Proposition 2.4, T_ω defines a distribution with support in S^* for ω in $U=\{\omega\in\Omega(K^\times)^n;\ \omega\notin\Omega(S),\ \omega^*\omega_{\lambda^*}\notin\Omega(S^*)\}$. Since $gT_\omega=(\omega_\lambda/\omega)\circ\chi(g)T$, we have $T_\omega=0$ unless $\omega^*\omega_{\lambda^*}\in\Omega(S^*)$. Therefore $T_\omega=0$ for ω in U. It follows from the assumption (A.2) and Corollary to Lemma 2.3 that U is a dense open subset of $\Omega(K^\times)^n$. Since T_ω is a distribution depending on ω meromorphically, we have $T_\omega=0$; namely the identity (*) holds for all $f^*\in\mathscr{S}(V^*)$.

Remark 1. As remarked in the introduction, Theorem $k_\mathfrak{p}$ has been proved by Igusa ([17]) under the following three conditions:

1) $\boldsymbol{G}$ is a connected reductive algebraic group defined over a $\mathfrak{p}$-adic number field and there exists an involution of End$(\boldsymbol{V})$ defined over the base field under which $\rho(\boldsymbol{G})$ is stable.

2) $\boldsymbol{S}$ is an absolutely irreducible hypersurface.

By the condition 2) there exists a unique (up to a constant factor) irreducible relative invariant. Let $\boldsymbol{G}^1$ be the kernel of its corresponding rational character of $\boldsymbol{G}$.

3) $\boldsymbol{S}$ decomposes into a finite number of $\boldsymbol{G}^1$-orbits.

By [39, § 4 Remark 26] the conditions 1) and 2) imply our condition (A.1). It is obvious that (A.2) is a natural generalization of the condition 3) above.

Remark 2. One can find many examples of prehomogeneous vector spaces satisfying the conditions (A.1) and (A.2) among reductive prehomogeneous vector spaces which admit only a finite number of orbits classified completely by Kimura, Kasai and Yasukura ([23]). Another examples can be obtained in the following manner. Let $(\boldsymbol{G},\rho,\boldsymbol{V})$ be a prehomogeneous vector space of commutative parabolic type defined over a $\mathfrak{p}$-adic number field K. Then the restriction of ρ to a K-parabolic subgroup remains to be prehomogeneous ([26, Prop. 3.21]) and supplies us an example with the properties (A.1) and (A.2). In the following section we shall examine three examples of this kind.

Remark 3. It is very likely that the condition (A.2) for a regular $(\boldsymbol{G},\rho,\boldsymbol{V})$ implies the same condition for $(\boldsymbol{G},\rho^*,\boldsymbol{V}^*)$. Note that $(\boldsymbol{G},\rho^*,\boldsymbol{V}^*)$ inherits the finiteness of $\boldsymbol{G}$-orbits from $(\boldsymbol{G},\rho,\boldsymbol{V})$ (Pyasetskii [31]).

§ 3. Calculation of Γ-matrices

3.1. In this section K is a local field of characteristic 0. We denote by ψ the additive character of K defined by

$$\psi(\alpha)=\begin{cases}\exp(2\pi i\,\alpha), & \text{if } K=\boldsymbol{R},\\ \exp(4\pi i\,\mathrm{Re}\,\alpha), & \text{if } K=\boldsymbol{C},\\ \exp(2\pi i\,\lambda_0(\mathrm{tr}_{K/\boldsymbol{Q}_p}(\alpha))), & \text{if } K \text{ is a finite extension of } \boldsymbol{Q}_p.\end{cases}$$

Here λ_0 stands for the canonical mapping

$$\boldsymbol{Q}_p \longrightarrow \boldsymbol{Q}_p/\boldsymbol{Z}_p \longrightarrow \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow \boldsymbol{R}/\boldsymbol{Z}.$$

A Haar measure $|d\alpha|_K$ on K is always normalized such that $|d\alpha|_K$ is auto dual with respect to ψ.

For a quasi-character $\omega \in \Omega(K^\times)$, the Tate Γ-factor $\Gamma_K(\omega)$ is defined by the functional equation

$$(3.1)\qquad (\omega(\alpha)/|\alpha|_K)^\wedge=\Gamma_K(\omega)\omega(\alpha)^{-1},$$

where $^\wedge$ means the Fourier transformation. The identity (3.1) is nothing but the fundamental theorems $\boldsymbol{R}$, $\boldsymbol{C}$ and k_p for the simplest prehomogeneous vector space obtained from the standard-one dimensional representation of $\boldsymbol{GL}(1)$.

If K is an archimedean local field, then $\Gamma_K(\omega)$ is given explicitly by

$$\Gamma_{\boldsymbol{R}}(|\cdot|_{\boldsymbol{R}}^s\cdot\mathrm{sgn}(\cdot)^\delta)=i^\delta\pi^{(1-2s)/2}\Gamma\left(\frac{s+\delta}{2}\right)\Big/\Gamma\left(\frac{1-s+\delta}{2}\right)\qquad(\delta=0,1),$$

$$\Gamma_{\boldsymbol{C}}\left(|\cdot|_{\boldsymbol{C}}^s\cdot\left(\frac{\cdot}{|\cdot|_{\boldsymbol{C}}^{1/2}}\right)^p\right)=i^{|p|}(2\pi)^{1-2s}\Gamma\left(s+\frac{|p|}{2}\right)\Big/\Gamma\left(1-s+\frac{|p|}{2}\right)\qquad(p\in\boldsymbol{Z}).$$

If K is a finite extension of $\boldsymbol{Q}_p$, then

$$\Gamma_K(|\cdot|_K^s\cdot\phi(\cdot))=N(\mathfrak{d})^{s-1/2}\times\begin{cases}(1-q^{-(1-s)})/(1-q^{-s}) & (\phi=1),\\ g_\phi\cdot q^{fs} & (\phi\neq 1),\end{cases}$$

where $q=N(\mathfrak{p})$, $\phi\in\widehat{\mathfrak{o}^\times}$, f is the conductor of ϕ, $\mathfrak{d}$ is the absolute different of K and

$$g_\phi=q^{-f}\sum_{u\in\mathfrak{o}^\times/1+\mathfrak{p}^f}\phi(u)\psi(u/\pi^{f+\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d})}).$$

For $\varepsilon\in K^\times/(K^\times)^2$, we put

$$(3.2)\qquad \Gamma_K(\omega;\varepsilon)=[K^\times:(K^\times)^2]^{-1}\sum_\phi\phi(\varepsilon)\Gamma_K(\omega\phi)\qquad(\omega\in\Omega(K^\times)),$$

where ϕ runs through all unitary characters of $K^\times$ of order 2.

Next we recall Weil's result on Fourier transforms of quadratic characters.

Let Y be a nondegenerate m by m symmetric matrix with entries in K. The matrix defines a quadratic character $\psi({}^tuYu)$ $(u \in K^m)$ on K^m. Let $|du|_K=|du_1|_K\cdots|du_m|_K$ be the autodual Haar measure on K^m with respect to the pairing $(u, v)=\psi(\langle u, v\rangle)$ with $\langle u, v\rangle=u_1v_1+\cdots+u_mv_m$.

The following formula for the Fourier transforms of quadratic characters was obtained by Weil:

$$\text{(3.3)}\quad \int_{K^m}\psi({}^tuYu)\psi(\langle u, v\rangle)|du|_K=|2|_K^{-m/2}|\det Y|_K^{-1/2}\gamma(Y)\psi(-{}^tvY^{-1}v/4),$$

for some constant $\gamma(Y)$ with absolute value 1 ([50, n^o 14, Theorem 2]). It is known that

$$\gamma(Y_1\perp Y_2)=\gamma(Y_1)\gamma(Y_2),\quad \gamma({}^tgYg)=\gamma(Y)\quad (g\in \boldsymbol{GL}(m, K)).$$

By considering an element $\alpha\in K^\times$ as a nondegenerate symmetric matrix of size 1, we can define a constant $\gamma(\alpha)$. The constant $\gamma(\alpha)$ depends only on $\mathrm{sgn}(\alpha)$, the residue class of α in $K^\times/(K^\times)^2$. As for the explicit values of $\gamma(\alpha)$, we quote the following results from [50, n° 26] and [32]:

(1) if $K=\boldsymbol{C}$, then $\gamma(\alpha)=1$ for all $\alpha\in\boldsymbol{C}^\times$,

(2) if $K=\boldsymbol{R}$, then $\gamma(\alpha)=\exp(\mathrm{sgn}(\alpha)\cdot\pi i/4)$ for $\alpha\in\boldsymbol{R}^\times$,

(3) if $K=\boldsymbol{Q}_p$, then for $\alpha\in\boldsymbol{Z}_p^\times$

$$\gamma(\alpha)=\begin{cases}1 & (p>2)\\ \exp(\pi i/4) & (p=2,\ \alpha\equiv 1 \pmod{4\boldsymbol{Z}_2})\\ \exp(-\pi i/4) & (p=2,\ \alpha\equiv -1 \pmod{4\boldsymbol{Z}_2})\end{cases}$$

and

$$\gamma(p\alpha)=\begin{cases}\left(\dfrac{\alpha}{p}\right)\exp((1-p)\pi i/4) & (p>2)\\ \psi(\alpha/8) & (p=2).\end{cases}$$

3.2. **((Trig(n), Sym(n)))**: Let $\boldsymbol{G}=\boldsymbol{G}^{(n)}=\mathbf{Trig}(n)$ be the group of all nondegenerate lower triangular matrices of size n and $\boldsymbol{V}=\boldsymbol{V}^{(n)}=\mathbf{Sym}(n)$ the vector space of all symmetric matrices of size n. Let ρ be the rational representation of $\boldsymbol{G}$ on $\boldsymbol{V}$ defined by $\rho(g)x=gx\,{}^tg$ $(g\in\boldsymbol{G},\ x\in\boldsymbol{V})$. With respect to the natural K-structure, the triple $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is a prehomogeneous vector space defined over K with the singular set

$$\boldsymbol{S}=\bigcup_{i=1}^n\boldsymbol{S}_i,\qquad \boldsymbol{S}_i=\{x\in\boldsymbol{V};\ d_i(x)=0\},$$

where $d_i(x)$ is the determinant of the upper left i by i block of x. For

$i=n$, $d_n(x)=\det x$. The character χ_i corresponding to $d_i(x)$ is given by

$$\chi_i\left(\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}\right)=(a_1\cdots a_i)^2 \qquad (1\leqq i\leqq n).$$

We identify V with its dual vector space via the symmetric bilinear form $\langle x, y\rangle=\operatorname{tr} xy$ $(x, y \in V)$. Then the contragredient representation ρ^* is given by $\rho^*(g)y={}^tg^{-1}yg^{-1}$ $(g \in \boldsymbol{G}, x \in V)$. Since $d_n(x)$ is a nondegenerate relative invariant in $K[V]$, $(\boldsymbol{G}, \rho, V)$ satisfies the condition (A.1). Hence the triple $(\boldsymbol{G}, \rho^*, V)$ is also a K-regular prehomogeneous vector space and the singular set $\boldsymbol{S}^*$ is given by

$$\boldsymbol{S}^*=\bigcup_{i=1}^{n}\boldsymbol{S}_i^*, \qquad \boldsymbol{S}_i^*=\{y \in V;\ d_i^*(y)=0\},$$

where $d_i^*(y)$ is the determinant of lower right i by i block of y. The character χ_i^* corresponding to $d_i^*(y)$ is given by

$$\chi_i^*\left(\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}\right)=(a_{n-i+1}\cdots a_n)^{-2}=\chi_n^{-1}\chi_{n-i}.$$

Therefore we have

$$U=\left(\begin{array}{ccc|c} & & 1 & -1 \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ 1 & & & -1 \\ \hline & & & -1 \end{array}\right)\begin{matrix}\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}n-1\\ \\ \end{matrix}, \qquad \lambda=\lambda^*=\left(0, \cdots, 0, \frac{n+1}{2}\right).$$

Lemma 3.1. *$(\boldsymbol{G}, \rho, V)$ and $(\boldsymbol{G}, \rho^*, V)$ satisfy the condition* (A.2) *in Section* 2.

Proof. We prove the condition (A.2)$'$ by induction on n. For $n=1$, the condition (A.2)$'$ is clearly satisfied. Now assume that the condition (A.2)$'$ is satisfied by $(\boldsymbol{G}^{(m)}, \rho, V^{(m)})$ $(m\leqq n-1)$. It is easy to see that under the action of $\rho(\boldsymbol{G}^{(n)})$ every point x in $\boldsymbol{S}$ is equivalent to one of points of the following forms:

$$(\mathrm{I}) \qquad \left(\begin{array}{c|c} 1 & 0\cdots 0 \\ \hline 0 & \\ \vdots & x_0 \\ 0 & \end{array}\right) \qquad (x_0 \in \boldsymbol{S}^{(n-1)}),$$

(II)
$$\left[\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline 0 & \\ \vdots & x_0 \\ 0 & \end{array}\right] \qquad (x_0 \in V^{(n-1)}),$$

(III)
$$\left[\begin{array}{c|c|c|c} 0 & 0\cdots 0 & 1 & 0\cdots 0 \\ \hline 0 & & 0 & \\ \vdots & x_1 & \vdots & x_2 \\ 0 & & 0 & \\ \hline 1 & 0\cdots 0 & 0 & 0\cdots 0 \\ \hline 0 & & 0 & \\ \vdots & {}^t x_2 & \vdots & x_3 \\ 0 & & 0 & \end{array}\right] \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} i-2 \\ \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n-i \end{array} \qquad \left(\begin{pmatrix} x_1 & x_2 \\ {}^t x_2 & x_3 \end{pmatrix} \in V^{(n-2)}\right).$$

By the inductive assumption we see that S is decomposed into a finite number of $\rho(G^{(n)})$-orbits. Therefore it is sufficient to prove the second condition in (A.2)$'$ for all points of the form (I), (II) or (III). If x is of type (I), then

$$G_x = \left\{ \left[\begin{array}{c|c} \pm 1 & 0\cdots 0 \\ \hline 0 & \\ \vdots & g_1 \\ 0 & \end{array}\right] \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n-1;\ g_1 x_0 {}^t g_1 = x_0 \right\}.$$

Hence by the induction hypothesis there exists a character χ of the form $\chi = \prod_{i=2}^{n} \chi_i^{m_i}$ $(m_2, \cdots, m_n \in Z)$ such that $\chi|_{G_x}$ is nontrivial. If x is of type (II), then G_x includes the group $\left\{\begin{pmatrix} a_1 & \\ & I_{n-1} \end{pmatrix};\ a_1 \in GL(1)\right\}$; hence $\chi_1|G_x$ is nontrivial. If x is of type (III), then G_x includes the group

$$\left\{\begin{pmatrix} t & & & \\ & I_{i-2} & & \\ & & t^{-1} & \\ & & & I_{n-i} \end{pmatrix};\ t \in GL(1)\right\};\ \text{hence } \chi_1|_{G_x} \text{ is nontrivial.}$$

This proves the lemma for (G, ρ, V). The proof for (G, ρ^*, V) is quite the same; so we omit it.

Since (G, ρ^*, V) and (G, ρ^*, V) satisfy the conditions (A.1), (A.2) and the singular sets S and S^* are hypersurfaces, the fundamental theorem holds over an arbitrary local field of characteristic 0.

For an $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_n) \in (K^\times/(K^\times)^2)^n$, put

$$V_\varepsilon=\{x \in \boldsymbol{V}(K)-\boldsymbol{S}(K);\ \operatorname{sgn}(d_i(x))=\varepsilon_1\cdots\varepsilon_i\ (1\leqq i\leqq n)\}$$

and

$$V^*_\varepsilon=\{y \in \boldsymbol{V}(K)-\boldsymbol{S}^*(K);\ \operatorname{sgn}(d^*_i(y))=\varepsilon_{n-i+1}\cdots\varepsilon_n\ (1\leqq i\leqq n)\}.$$

Then the $\boldsymbol{G}(K)$-orbit decompositions of $\boldsymbol{V}(K)-\boldsymbol{S}(K)$ and $\boldsymbol{V}(K)-\boldsymbol{S}^*(K)$ are given by

$$\boldsymbol{V}(K)-\boldsymbol{S}(K)=\bigcup_{\varepsilon\in\{K^\times/(K^\times)^2\}^n} V_\varepsilon, \qquad \boldsymbol{V}(K)-\boldsymbol{S}^*(K)=\bigcup_{\varepsilon\in\{K^\times/(K^\times)^2\}^n} V^*_\varepsilon.$$

The zeta distributions are defined by analytic continuations of the following integrals:

$$Z_\omega(\omega)(\phi)=\int_{V_\varepsilon}\prod_{i=1}^n \omega^{(i)}(d_i(x))\phi(x)\frac{|dx|_K}{|\det x|_K^{(n+1)/2}},$$

$$Z^*_\varepsilon(\omega)(\phi)=\int_{V^*_\varepsilon}\prod_{i=1}^n \omega^{(i)}(d^*_i(x))\phi(x)\frac{|dx|_K}{|\det x|_K^{(n+1)/2}}$$

$$(\varepsilon \in \{K^\times/(K^\times)^2\}^n, \quad \omega=(\omega^{(1)}, \cdots, \omega^{(n)}) \in \Omega(K^\times)^n, \quad \phi \in \mathcal{S}(\boldsymbol{V}(K))),$$

where $|dx|_K=\prod_{1\leqq i\leqq j\leqq n}|dx_{ij}|_K$ and $|dx_{ij}|_K$ is the Haar measure on K normalized as in Section 3.1.

Theorem 3.2.

$$\hat{Z}_\varepsilon(\omega)=\sum_{\eta\in\{K^\times/(K^\times)^2\}^n}\Gamma_{\varepsilon\eta}(\omega)Z^*_\eta(\omega^*\omega_\lambda),$$

where

$$\Gamma_{\varepsilon\eta}(\omega)=|2|_K^{-n(n-1)/4}\prod_{1\leqq i<j\leqq n}\gamma(\varepsilon_i\eta_j)\prod_{i=1}^n\Gamma_K(\omega^{(i)}\cdots\omega^{(n)}\omega_{(1-i)/2};\ \varepsilon_i\eta_i)$$

and $\omega^*=(\omega^{(n-1)}, \cdots, \omega^{(1)}, (\omega^{(1)}\cdots\omega^{(n)})^{-1})$. (*For the definitions of* $\gamma(\alpha)$ *and* $\Gamma_K(\omega;\varepsilon)$, *see* § 3.1.)

Proof. By Theorem $\boldsymbol{R}$, Theorem $\boldsymbol{C}$ and the combination of Theorem $k_\mathfrak{p}$ with Lemma 3.1, the functional equation holds for some meromorphic functions $\Gamma_{\varepsilon\eta}(\omega)$. Therefore it is enough to calculate $\Gamma_{\varepsilon\eta}(\omega)$ for $\phi \in \mathcal{S}(\boldsymbol{V}(K))$ with support in $\boldsymbol{V}(K)-\boldsymbol{S}^*(K)$ and for ω with sufficiently large real part. This restriction on ω and ϕ assures the convergence of all integrals appearing in the following calculation. First we note that the formula for $\Gamma_{\varepsilon\eta}(\omega)$ is an immediate consequence of (3.1) and the definition (3.2) of $\Gamma_K(\omega;\varepsilon)$, if $n=1$. Let us prove the formula for general n by induction on n.

Since any $x \in V_\varepsilon$ can be decomposed as

$$x=\left(\begin{array}{c|c} x_1 & y \\ \hline {}^t y & a+{}^t y x_1^{-1} y \end{array}\right)$$

with $x_1 \in V_{\varepsilon'}^{(n-1)}$ ($\varepsilon'=(\varepsilon_1, \cdots, \varepsilon_{n-1})$), $y \in K^{n-1}$, $a \in \varepsilon_n \cdot (K^\times)^2$), we have

$$\begin{aligned}\hat{Z}_\varepsilon(\omega)(\phi) &= Z_\varepsilon(\omega)(\hat{\phi}) \\ &= \int_{V_{\varepsilon'}^{(n-1)}} \prod_{i=1}^{n-2} \omega^{(i)}(d_i(x_1)) \cdot (\omega^{(n-1)}\omega^{(n)}\omega_{-1/2})(d_{n-1}(x_1)) \frac{|dx_1|_K}{|\det x_1|_K^{n/2}} \\ &\quad \times \int_{K^{n-1}} |dy|_K \int_{V_{\varepsilon_n}^{(1)}} (\omega^{(n)}\omega_{(1-n)/2})(a) \hat{\phi}\left(\left(\begin{array}{c|c} x_1 & y \\ \hline {}^t y & a+{}^t y x_1^{-1} y \end{array}\right)\right) \frac{|da|_K}{|a|_K}.\end{aligned}$$

Applying the theorem for $n=1$ to the integral with respect to a, we get

$$\begin{aligned}\hat{Z}_\varepsilon(\omega)(\phi) &= \sum_{\eta_n \in K^\times/(K^\times)^2} \Gamma_K(\omega^{(n)}\omega_{(1-n)/2};\ \varepsilon_n \eta_n) \\ &\quad \times \int_{V_{\varepsilon'}^{(n-1)}} \prod_{i=1}^{n-2} \omega^{(i)}(d_i(x_1)) \cdot (\omega^{(n-1)}\omega^{(n)}\omega_{-1/2})(d_{n-1}(x_1)) \frac{|dx_1|_K}{|\det x_1|_K^{n/2}} \\ &\quad \times \int_{V_{\eta_n}^{(1)}} (\omega_{(1+n)/2}/\omega^{(n)})(a^*) \frac{|da^*|_K}{|a^*|_K} \int_{K^{n-1}} \psi(a^* \, {}^t y x_1^{-1} y) \hat{\phi}_1\left(\left(\begin{array}{c|c} x_1 & y \\ \hline {}^t y & a^* \end{array}\right)\right) |dy|_K,\end{aligned}$$

where

$$\begin{aligned}\hat{\phi}_1\left(\begin{pmatrix} x_1 & y \\ {}^t y & a^* \end{pmatrix}\right) &= \int_{V^{(n-1)}(K) \times K^{n-1}} \phi\left(\begin{pmatrix} x_1^* & y^* \\ {}^t y^* & a^* \end{pmatrix}\right) \\ &\qquad \times \psi(\operatorname{tr} x_1 x_1^* + 2\, {}^t y y^*) |dx_1^*|_K |dy^*|_K.\end{aligned}$$

The integral with respect to y can be calculated by using (3.3) and the formula

$$\gamma(a^* x_1^{-1}) = \gamma(\langle a^* \varepsilon_1^{-1} \rangle \perp \cdots \perp \langle a^* \varepsilon_{n-1}^{-1} \rangle) = \prod_{i=1}^{n-1} \gamma(\varepsilon_i \eta_n).$$

Thus we have

$$\begin{aligned}\hat{Z}_\varepsilon(\omega)(\phi) = |2|_K^{-(n-1)/2} &\sum_{\eta_n \in K^\times/(K^\times)^2} \Gamma_K(\omega^{(n)}\omega_{(1-n)/2};\ \varepsilon_n \eta_n) \prod_{i=1}^{n-1} \gamma(\varepsilon_i \eta_n) \\ &\times \int_{V_{\eta_n}^{(1)}} (\omega_1/\omega^{(n)})(a^*) \frac{|da^*|_K}{|a^*|_K} \int_{K^{n-1}} |dy^*|_K \\ &\times \int_{V_{\varepsilon'}^{(n-1)}} \prod_{i=1}^{n-2} \omega^{(i)}(d_i(x_1)) \cdot (\omega^{(n-1)}\omega^{(n)})(d_{n-1}(x_1)) \\ &\quad \cdot \psi(-a^{*-1}\, {}^t y^* x_1 y^*) \hat{\phi}_2\left(\begin{pmatrix} x_1 & y^* \\ {}^t y^* & a^* \end{pmatrix}\right) \frac{|dx_1|_K}{|\det x_1|_K^{n/2}},\end{aligned}$$

where

$$\hat{\phi}_2\left(\begin{pmatrix} x_1 & y^* \\ {}^t y^* & a^* \end{pmatrix}\right)=\int_{V^{(n-1)}(K)} \phi\left(\begin{pmatrix} x_1^* & y^* \\ {}^t y^* & a^* \end{pmatrix}\right)\psi(\mathrm{tr}\, x_1 x_1^*)|dx_1^*|_K.$$

Since

$$\psi(-a^{*-1}\,{}^t y^* x_1 y^*)\hat{\phi}_2\left(\begin{pmatrix} x_1 & y^* \\ {}^t y^* & a^* \end{pmatrix}\right)$$
$$=\int_{V^{(n-1)}(K)} \phi\left(\left(\begin{array}{c|c} x_1^*+a^{*-1}y^*\,{}^t y^* & y^* \\ \hline {}^t y^* & a^* \end{array}\right)\right)\psi(\mathrm{tr}\, x_1 x_1^*)|dx_1^*|_K,$$

it follows from the induction hypothesis that

$$\hat{Z}_\varepsilon(\omega)(\phi)=|2|_K^{-n(n-1)/4}\sum_{\eta_n\in K^\times/(K^\times)^2}\prod_{i=1}^{n-1}\gamma(\varepsilon_i\eta_n)\Gamma_K(\omega^{(n)}\omega_{(1-n)/2};\,\varepsilon_n\eta_n)$$
$$\times\sum_{\eta'=(\eta_1,\cdots,\eta_{n-1})}\prod_{1\leq i<j\leq n-1}\gamma(\varepsilon_i\eta_j)\prod_{i=1}^{n-1}\Gamma_K(\omega^{(i)}\cdots\omega^{(n)}\omega_{(1-i)/2};\,\varepsilon_i\eta_i)$$
$$\times\int_{K^{n-1}}|dy^*|_K\int_{V^{(1)}_{\eta_n}}(\omega_1/\omega^{(n)})(a^*)\frac{|da^*|_K}{|a^*|_K}$$
$$\times\int_{V^{*(n-1)}_{\eta'}}\prod_{i=1}^{n-2}\omega^{(n-i-1)}(d_i^*(x_1^*))(\omega_{n/2}/\omega^{(1)}\cdots\omega^{(n)})(d_{n-1}^*(x_1^*))$$
$$\cdot\phi\left(\left(\begin{array}{c|c} x_1^*+a^{*-1}y^*\,{}^t y^* & y^* \\ \hline {}^t y^* & a^* \end{array}\right)\right)\frac{|dx_1^*|_K}{|d_{n-1}^*(x_1^*)|_K^{n/2}}$$
$$=|2|_K^{-n(n-1)/4}\sum_{\eta\in\{K^\times/(K^\times)^2\}^n}\prod_{1\leq i<j\leq n}\gamma(\varepsilon_i\eta_j)$$
$$\times\prod_{i=1}^{n}\Gamma_K(\omega^{(i)}\cdots\omega^{(n)}\omega_{(1-i)/2};\,\varepsilon_i\eta_i)Z_\eta^*(\omega^*\omega_\lambda)(\phi).$$

Remark. Theorem 3.2 for $K=\boldsymbol{R}$ was used previously in proving [36, Theorem 8 (3)], where the proof was omitted. Micro-local calculus provides us another method to prove Theorem 3.2 for $K=\boldsymbol{R}$. The calculation of $\Gamma_{\varepsilon\eta}(\omega)$ for $K=\boldsymbol{R}$ based on the method was carried out by T. Miwa several years ago (unpublished, see also [48]).

3.3. (Trig $(n)\times$ Trig (n), $M(n)$): We now consider the group $\boldsymbol{G}=$ **Trig** $(n)\times$ **Trig** (n) and the vector space $\boldsymbol{V}=\boldsymbol{M}(n)$ of n-rowed square matrices. We define a rational representation ρ of $\boldsymbol{G}$ on $\boldsymbol{V}$ by $\rho(g_1, g_2)x=g_1 x\,{}^t g_2$. With respect to the natural K-structure, the triple $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is a prehomogeneous vector space defined over K with the singular set

$$\boldsymbol{S}=\bigcup_{i=1}^{n}\{x\in\boldsymbol{V}:\, d_i(x)=0\},$$

where $d_i(x)$ is the same as in Section 3.2. The character χ_i corresponding to $d_i(x)$ is given by

$$\chi_i\left(\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}, \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ * & & b_n \end{pmatrix}\right) = (a_1 \cdots a_i)\cdot(b_1 \cdots b_i) \qquad (1 \leqq i \leqq n).$$

Since $d_n(x) = \det x$ is a nondegenerate relative invariant, (G, ρ, V) is K-regular.

We identify V with its dual vector space via the symmetric bilinear form $\langle x, y\rangle = \operatorname{tr} {}^t xy$ $(x, y \in V)$. Then the representation ρ^* contragredient to ρ is given by $\rho^*(g_1, g_2)y = {}^t g_1^{-1} y g_2^{-1}$ and the triple (G, ρ^*, V) is a K-regular prehomogeneous vector space with the singular set

$$S^* = \bigcup_{i=1}^{n} \{y \in V;\ d_i^*(y) = 0\}.$$

The character corresponding to $d_i^*(y)$ is given by

$$\chi_i^*\left(\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}, \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ * & & b_n \end{pmatrix}\right) = (a_{n-i+1} \cdots a_n)^{-1}\cdot(b_{n-i+1} \cdots b_n)^{-1}.$$

From the formulas for χ_i and χ_i^* we have

$$U = \left(\begin{array}{ccc|c} & & 1 & -1 \\ & \iddots & & \vdots \\ 1 & & & -1 \\ \hline & & & -1 \end{array}\right) \begin{matrix} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} n-1, \\ \vphantom{1} \end{matrix} \qquad \lambda = \lambda^* = (0, \cdots, 0, n).$$

Lemma 3.3. *(G, ρ, V) and (G, ρ^*, V) satisfy the condition* (A.2) *in* Section 2.

Since the proof is quite similar to that of Lemma 3.1, we omit it.

It is easy to see that $V(K) - S(K)$ (resp. $V(K) - S^*(K)$) is a single $\rho(G(K))$- (resp. $\rho^*(G(K))$-) orbit. The zeta distributions are defined by the analytic continuations of the integrals

$$Z(\omega)(\phi) = \int_{V(K)-S(K)} \prod_{i=1}^{n} \omega^{(i)}(d_i(x))\phi(x) \frac{|dx|_K}{|\det x|_K^n},$$

$$Z^*(\omega)(\phi) = \int_{V(K)-S^*(K)} \prod_{i=1}^{n} \omega^{(i)}(d_i^*(y))\phi(y) \frac{|dy|_K}{|\det y|_K^n}$$

$$(\omega=(\omega^{(1)}, \cdots, \omega^{(n)}) \in \Omega(K^{\times})^n,\ \phi \in \mathscr{S}(V(K))),$$

where $|dx|_K=\prod_{1\leq i,j\leq n}|dx_{ij}|_K$. The normalization of the Haar measure $|dx_{ij}|_K$ on K is the same as in Sections 3.1, 3.2.

Theorem 3.4.

$$\hat{Z}(\omega)=\prod_{i=1}^{n} \Gamma_K(\omega^{(i)}\cdots\omega^{(n)}\omega_{1-i})Z^*(\omega^*\omega_\lambda),$$

where $\omega^*=(\omega^{(n-1)}, \cdots, \omega^{(1)}, (\omega^{(1)}\cdots\omega^{(n)})^{-1})$.

Theorem 3.4 can be proved by an inductive calculation similar to that in the proof of Theorem 3.2. In the present case the functional equation (3.1) is sufficient for the calculation and we need not appeal to the formula for Fourier transforms of quadratic characters.

Remark. One can get an explicit formula for the Γ-matrix for (**Trig** $(2n)$), ρ, **Alt** $(2n)$), where **Alt** $(2n)=\{x \in M(2n);\ x=-{}^t x\}$ and $\rho(g)x=gx\,{}^t g$, by a similar calculation. In [§ 14, § 4], by using the theory of spherical functions on **Alt** $(2n, k_{\mathfrak{p}})$, Hironaka and the author gave an explicit formula for the zeta function $Z(\omega)(\phi)$ for $\omega=(\omega_{s_1}, \cdots, \omega_{s_n})$ and for $\mathrm{ch}_{\mathbf{Alt}(2n,\mathfrak{o})}$, the characteristic function of **Alt** $(2n, \mathfrak{o})$, which gives another method to calculate the Γ-matrix. The formula for the Γ-matrix for (**Trig** $(2n)$, ρ, **Alt** $(2n)$) has an application to calculation of local densities of alternating forms ([15]). The spherical functions introduced in [14] are closely related to the zeta functions associated with ($\boldsymbol{Sp}(n)\times$ **Trig** $(2n-2)$, $\Lambda_1\otimes\Lambda_1$, $M(2n, 2n-2)$); Theorem 6 in [14] combined with Lemma 3.1 in [14] can be viewed as an explicit formula for $Z(\omega)(\phi)$ with $\omega=(\omega_{s_1}, \cdots, \omega_{s_{n-1}})$ and $\phi=\mathrm{ch}_{M(2n,2n-2;\mathfrak{o})}$ for various $\mathfrak{o}$-forms of the prehomogeneous vector space.

3.4. *The vector representation of a maximal parabolic subgroup of an orthogonal group*: Let $Q(x)$ be a nondegenerate isotropic quadratic form in $m+2$ variables with coefficients in K. We assume that $Q(x)$ is of the form

$$Q(x)=x_0x_{m+1}+\sum_{1\leq i,j\leq m} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}).$$

The matrix of Q is given by

$$\begin{pmatrix} & & 1/2 \\ & A & \\ 1/2 & & \end{pmatrix} \qquad \text{with } A=(a_{ij}).$$

Let $\boldsymbol{P}$ be the maximal parabolic subgroup of the orthogonal group $\boldsymbol{O}(Q)=\{g \in \boldsymbol{GL}(m+2); Q(gx)=Q(x)\}$ defined by

$$\boldsymbol{P}(K)=\left\{\begin{pmatrix} a & -2a{}^tuAh & -a{}^tuAu \\ 0 & h & u \\ 0 & 0 & a^{-1} \end{pmatrix} : \begin{array}{l} a \in K^{\times}, \\ u \in K^m, \\ h \in \boldsymbol{O}(A)(K) \end{array}\right\}.$$

We put $\boldsymbol{G}=\boldsymbol{GL}(1)\times\boldsymbol{P}$ and denote by ρ the standard $m+2$-dimensional representation of $\boldsymbol{G}$. The representation space $\boldsymbol{V}$ is identified with the vector space of column vectors with $m+2$ components. The $\boldsymbol{GL}(1)$-part of $\boldsymbol{G}$ acts on $\boldsymbol{V}$ as scalar multiplication. Then the triple $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is a prehomogeneous vector space defined over K with the singular set

$$\bigcup_{i=1}^{2}\{x \in V; P_i(x)=0\}, \quad P_1(x)=x_{m+1}, \quad P_2(x)=Q(x).$$

The character χ_i corresponding to P_i is given by

$$\chi_i\left(t, \begin{pmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & a^{-1} \end{pmatrix}\right)=\begin{cases} ta^{-1} & (i=1), \\ t^2 & (i=2). \end{cases}$$

Since $P_2(x)=Q(x)$ is a nondegenerate relative invariant, $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is K-regular.

Now we identify V with its dual vector space via the symmetric bilinear form $\langle x, y\rangle=x_0y_0+\cdots+x_{m+1}y_{m+1}$. Then the representation ρ^* contragredient to ρ is given by $\rho^*(t, p)y=t^{-1}\,{}^tp^{-1}y$ ($t \in \boldsymbol{GL}(1)$, $p \in \boldsymbol{P}$, $y \in V$). The triple $(\boldsymbol{G}, \rho^*, \boldsymbol{V})$ is a K-regular prehomogeneous vector space with the singular set

$$\boldsymbol{S}^*=\bigcup_{i=1}^{2}\{y \in V; P_i^*(y)=0\}, \quad P_1^*(y)=y_0, \quad P_2^*(y)=Q^*(y),$$

where Q^* is the quadratic form defined by

$$Q^*(y)=y_0y_{m+1}+4^{-1}\sum_{1\leq i,j\leq m} a_{ij}^*y_iy_j \qquad \text{with } A^{-1}=(a_{ij}^*).$$

The character χ_i^* corresponding to $P_i^*(y)$ is given by

$$\chi_i^*\left(t, \begin{pmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & a^{-1} \end{pmatrix}\right)=\begin{cases} (ta)^{-1} & (i=1) \\ t^{-2} & (i=2). \end{cases}$$

From the formulas for χ_i and χ_i^* we have

$$U=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix},\qquad \lambda=\lambda^*=\left(0,\ \frac{m+2}{2}\right).$$

Lemma 3.5. $(\boldsymbol{G}, \rho, \boldsymbol{V})$ and $(\boldsymbol{G}, \rho^*, \boldsymbol{V})$ *satisfies the condition* (A.2) *in* Section 2.

Proof. We give a proof only for $(\boldsymbol{G}, \rho, \boldsymbol{V})$. It is easy to check that $\boldsymbol{V}$ admits the following 6 $\boldsymbol{G}$-orbits:

(1) $\boldsymbol{V}-\boldsymbol{S}$,
(2) $\{x \in \boldsymbol{V};\ x_{m+1}\neq 0,\ P_2(x)=0\}$,
(3) $\{x \in \boldsymbol{V};\ x_{m+1}=0,\ P_2(x)\neq 0\}$,
(4) $\{x \in \boldsymbol{V};\ x_{m+1}=0,\ P_2(x)=0,\ (x_1, \cdots, x_m)\neq 0\}$,
(5) $\{x \in \boldsymbol{V};\ x_1=\cdots=x_{m+1}=0,\ x_0\neq 0\}$,
(6) $\{0\}$.

The group $\boldsymbol{G}^{(\chi_2)}$ (resp. $\boldsymbol{G}^{(\chi_1)}$) acts transitively on the orbits (2), (4), (5), (6) (resp. (3), (6)). This proves the lemma.

For $\varepsilon \in K^\times/(K^\times)^2$, we put

$$V_\varepsilon=\{x \in \boldsymbol{V}(K);\ \operatorname{sgn}(P_2(x))=\varepsilon\},\ V_\varepsilon^*=\{y \in \boldsymbol{V}(K);\ \operatorname{sgn}(P_2^*(y))=\varepsilon\}.$$

Then

$$\boldsymbol{V}(K)-\boldsymbol{S}(K)=\bigcup_{\varepsilon\in K^\times/(K^\times)^2} V_\varepsilon$$

and

$$\boldsymbol{V}(K)-\boldsymbol{S}^*(K)=\bigcup_{\varepsilon\in K^\times/(K^\times)^2} V_\varepsilon^*$$

give the $\boldsymbol{G}(K)$-orbit decompositions of $\boldsymbol{V}(K)-\boldsymbol{S}(K)$ and $\boldsymbol{V}(K)-\boldsymbol{S}^*(K)$, respectively.

The zeta distributions are defined by analytic continuations of the following integrals:

$$Z_\varepsilon(\omega)(\phi)=\int_{V_\varepsilon}\omega^{(1)}(P_1(x))\omega^{(2)}(P_2(x))\phi(x)\frac{|dx|_K}{|P_2(x)|_K^{(m+2)/2}},$$

$$Z_\varepsilon^*(\omega)(\phi)=\int_{V_\varepsilon^*}\omega^{(1)}(P_1^*(x))\omega^{(2)}(P_2^*(x))\phi(x)\frac{|dx|_K}{|P_2^*(x)|_K^{(m+2)/2}},$$

$(\omega=(\omega^{(1)}, \omega^{(2)}) \in \Omega(K^\times)^2,\ \phi \in \mathcal{S}(\boldsymbol{V}(K)))$, where $|dx|_K=\prod_{i=0}^{m+1}|dx_i|_K$ is the normalized Haar measure on $\boldsymbol{V}(K)$.

Theorem 3.6.

$$\hat{Z}_\varepsilon(\omega)=\sum_{\eta\in K^\times/(K^\times)^2}\Gamma_{\varepsilon\eta}(\omega)Z^*_\eta(\omega^*\omega_\lambda),$$

where

$$\Gamma_{\varepsilon\eta}(\omega)=|2|_K^{-m/2}|\det A|_K^{-1/2}\times\sum_{\mu\in K^\times/(K^\times)^2}\gamma(-\mu A)\Gamma_K(\omega^{(2)}\omega_{-m/2};\ \mu\varepsilon)\Gamma_K(\omega^{(1)}\omega^{(2)};\ \mu\eta)$$

and $\omega^*=(\omega^{(1)},(\omega^{(1)}\omega^{(2)})^{-1})$.

Proof. We shall calculate $\Gamma_{\varepsilon\eta}(\omega)$ for $\phi\in\mathscr{S}(V(K))$ with support in $V(K)-S^*(K)$ and ω with sufficiently large real part. Changing the variable x to y defined by

$$\begin{cases} y_0=x_0+{}^ty'Ay'/y_{m+1} & (y'={}^t(y_1,\cdots,y_m))\\ y_i=x_i & (1\leqq i\leqq m+1),\end{cases}$$

we have from Theorem 3.2 for $n=1$ the identity

$$\begin{aligned}\hat{Z}_\varepsilon(\omega)(\phi)&=\int_{\substack{y_{m+1}\neq0\\ \mathrm{sgn}(y_0y_{m+1})=\varepsilon}}(\omega^{(1)}\omega^{(2)}\omega_{-(m+2)/2})(y_{m+1})(\omega^{(2)}\omega_{-(m+2)/2})(y_0)\\ &\quad\times\hat{\phi}(y_0-{}^ty'Ay'/y_{m+1},y',y_{m+1})|dy_0|_K|dy'|_K|dy_{m+1}|_K\\ &=\sum_{\varepsilon_1\in K^\times/(K^\times)^2}\ \sum_{\eta_1\in K^\times/(K^\times)^2}\Gamma_K(\omega^{(2)}\omega_{-m/2};\ \varepsilon_1\eta_1)\\ &\quad\times\int_{\mathrm{sgn}\,y_0^*=\eta_1}(\omega_{m/2}/\omega^{(2)})(y_0^*)|dy_0^*|_K\\ &\quad\times\int_{\mathrm{sgn}\,y_{m+1}=\varepsilon_1\varepsilon}(\omega^{(1)}\omega^{(2)}\omega_{-(m+2)/2})(y_{m+1})|dy_{m+1}|_K\\ &\quad\times\int_{K^m}\psi(-(y_0^*/y_{m+1})\,{}^ty'Ay')\hat{\phi}_1(y_0^*,y',y_{m+1})|dy'|_K,\end{aligned}$$

where

$$\hat{\phi}_1(y_0^*,y',y_{m+1})=\int_{K^{m+1}}\phi(y_0^*,y_1^*,\cdots,y_{m+1}^*)\psi\left(\sum_{i=1}^{m+1}y_iy_i^*\right)\prod_{i=1}^{m+1}|dy_i^*|_K.$$

From (3.3) it follows that

$$\begin{aligned}\hat{Z}_\varepsilon(\omega)(\phi)&=|2|_K^{-m/2}|\det A|_K^{-1/2}\\ &\quad\times\sum_{\varepsilon_1\in K^\times/(K^\times)^2}\ \sum_{\eta_1\in K^\times/(K^\times)^2}\Gamma_K(\omega^{(2)}\omega_{-m/2};\ \varepsilon_1\eta_1)\gamma(-\varepsilon_1\eta_1\varepsilon A)\\ &\quad\times\int_{\mathrm{sgn}\,y_0^*=\eta_1}\omega^{(2)}(y_0^*)^{-1}|dy_0^*|_K\int_{K^m}|dy^{*\prime}|_K\end{aligned}$$

$$\times\int_{\operatorname{sgn} y_{m+1}=\varepsilon_1\varepsilon}(\omega^{(1)}\omega^{(2)}\omega_{-1})(y_{m+1})\psi\left(\frac{y_{m+1}}{4y_0^*}{}^t y^{*\prime}A^{-1}y^{*\prime}\right)$$
$$\cdot\hat{\phi}_2(y_0^*, y^{*\prime}, y_{m+1})|dy_{m+1}|_K,$$

where

$$\hat{\phi}_2(y_0^*, y^{*\prime}, y_{m+1})=\int_K \phi_2(y_0^*, \cdots, y_{m+1}^*)\psi(y_{m+1}y_{m+1}^*)|dy_{m+1}^*|_K.$$

Applying Theorem 3.2 for $n=1$ to the integral with respect to y_{m+1}, we obtain

$$\hat{Z}_\varepsilon(\omega)(\phi)=|2|_K^{-m/2}|\det A|_K^{-1/2}\sum_{\varepsilon_1,\eta_1}\Gamma_K(\omega^{(2)}\omega_{-m/2};\varepsilon_1\eta_1)\gamma(-\varepsilon_1\eta_1\varepsilon A)$$
$$\times\int_{\eta_2\in K^\times/(K^\times)^2}\Gamma_K(\omega^{(1)}\omega^{(2)};\varepsilon_1\eta_2\varepsilon)\int_{\operatorname{sgn} y_0^*=\eta_1}\omega^{(2)}(y_0^*)^{-1}|dy_0^*|_K$$
$$\times\int_{K^m}|dy^{*\prime}|_K\int_{\operatorname{sgn} y_{m+1}^*=\eta_2}(\omega^{(1)}\omega^{(2)})^{-1}(y_{m+1}^*)$$
$$\cdot\phi(y_0^*, y^{*\prime}, y_{m+1}^*-{}^t y^{*\prime}A^{-1}y'/4y_0^*)|dy_{m+1}^*|_K$$
$$=|2|_K^{-m/2}|\det A|_K^{-1/2}$$
$$\times\sum_{\varepsilon_1,\eta_1,\eta_2}\gamma(-\varepsilon_1\eta_1\varepsilon A)\Gamma_K(\omega^{(2)}\omega_{-m/2};\varepsilon_1\eta_1)\Gamma_K(\omega^{(1)}\omega^{(2)};\varepsilon_1\eta_2\varepsilon)$$
$$\times\int_{\substack{\operatorname{sgn} P_1^*(y^*)=\eta_1\\ \operatorname{sgn} P_2^*(y^*)=\eta_1\eta_2}}\omega^{(1)}(P_1^*(y^*))(\omega_{(m+2)/2}/\omega^{(1)}\omega^{(2)})(P_2^*(y^*))\phi(y^*)$$
$$\cdot\frac{|dy^*|_K}{|P_2^*(y^*)|_K^{(m+2)/2}}.$$

In the right hand side of the identity above, we put $\mu=\varepsilon_1\eta_1\varepsilon$ and $\eta=\eta_1\eta_2$. Then it is easy to see that the right hand side is equal to

$$|2|_K^{-m/2}|\det A|_K^{-1/2}\sum_\eta Z_\eta^*(\omega^*\omega_\lambda)(\phi)$$
$$\times\sum_\mu\gamma(-\mu A)\Gamma_K(\omega^{(2)}\omega_{-m/2};\mu\varepsilon)\Gamma_K(\omega^{(1)}\omega^{(2)};\mu\eta).$$

This completes the proof.

Remark. Theorem 3.6 for $K=\boldsymbol{R}$ has been already obtained by Muro (private communication). His calculation is based on the method of micro-local calculus.

3.5. *Observation.* It is remarkable that the entries of the Γ-matrices calculated in Theorems 3.2, 3.4 and 3.6 have the following form in common:

$$a_0 \sum_i c_i \sum_j \Gamma_K(e_j(\omega)\chi_{ij}),$$

where a_0 is a completely elementary constant, the coefficients c_i involve constants of rather delicate nature such as $\gamma(Y)$, the arguments $e_j(\omega)$ of Tate Γ-factors are determined by the b-function of a prehomogeneous vector space under consideration and χ_{ij} are unitary characters of K. Only c_i and χ_{ij} depend on open orbits in $V(K)-S(K)$. It seems to be a fairly general phenomenon that the entries of Γ-matrices can be written in the form above, if independent relative invariants $P_1, \cdots, P_n$ are suitably normalized. The recent results of Muller [25] and Igusa [18] as well as the formula for Γ-matrices over $\boldsymbol{R}$ obtained by the method of micro-local calculus (cf. [19], [22, Theorem 7.10], [28]) support this expectation. In particular it is expected that the exponential polynomials $t_{ij}(s)$ in Theorem $\boldsymbol{R}$ are also under the control of b-functions. In the results of Muller the coefficients c_i are expressed in terms of the coefficients of the Fourier transforms of quadratic characters, on the other hand in the formula based on the micro-local calculus they are expressed in terms of the Maslov indices; it is noteworthy that they are determined by signatures of certain quadratic forms in both cases.

§ 4. Zeta functions as distribution

4.1. Let V be a finite dimensional $\boldsymbol{Q}$-vector space. A complex valued function ϕ on V is called a *Schwartz-Bruhat function*, if there exist two lattices L_1 and L_2 in V such that the support of ϕ is contained in L_1 and $\phi(x)=\phi(x')$ whenever $x-x' \in L_2$. We denote by $\mathscr{S}(V)$ the vector space of all Schwartz-Bruhat functions on V. Let V^* be the vector space dual to V.

Now we fix a lattice L_0 in V and let L_0^* be the lattice in V^* dual to L_0. Define a Fourier transform $\hat{\phi}$ of $\phi \in \mathscr{S}(V)$ by setting

$$\hat{\phi}(y)=\frac{1}{[L_0\colon L]} \sum_{x \in V/L} \phi(x) e^{2\pi i\langle x, y\rangle} \qquad (y \in V^*),$$

where L is a sublattice of L_0 such that y is in the lattice L^* dual to L and $\phi(x)=\phi(x')$ if $x-x' \in L$. It is easy to see that $\hat{\phi}(y)$ is independent of the choice of L and $\hat{\phi}$ defines a Schwartz-Bruhat function on V^*. The Fourier transformation $\phi \mapsto \hat{\phi}$ induces a linear isomorphism of $\mathscr{S}(V)$ onto $\mathscr{S}(V^*)$.

Let $\mathscr{S}(V)'$ be the dual space of $\mathscr{S}(V)$. We call an element in $\mathscr{S}(V)'$ a *distribution on* V. For a distribution $T \in \mathscr{S}(V^*)'$, the Fourier transform $\hat{T}$ ($\in \mathscr{S}(V)'$) is defined by $\hat{T}(\phi)=T(\hat{\phi})$ ($\phi \in \mathscr{S}(V)$).

Put $V(\boldsymbol{R})=V\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ and denote by $\mathscr{S}(V(\boldsymbol{R}))$ the vector space of all rapidly decreasing functions. Let dx be the Euclidean measure on $V(\boldsymbol{R})$ normalized by $\int_{V(\boldsymbol{R})/L_0} dx=1$. We define a Fourier transform $\hat{f}$ of $f\in\mathscr{S}(V(\boldsymbol{R}))$ by

$$\hat{f}(y)=\int_{V(\boldsymbol{R})} f(x)e^{-2\pi i\langle x,y\rangle}dx.$$

The function $\hat{f}$ is a rapidly decreasing function on $V^*(\boldsymbol{R})=V^*\otimes_{\boldsymbol{Q}}\boldsymbol{R}$. For an $f\in\mathscr{S}(V(\boldsymbol{R}))$ and a $\phi\in\mathscr{S}(V)$, the following variant of the Poisson summation formula holds:

$$\sum_{m\in V} f(m)\phi(m)=\sum_{m^*\in V^*}\hat{f}(m)\hat{\phi}(m^*).$$

Remark. The Poisson summation formula above is essentially an interpretation of the Poisson summation formula on the adelized vector space $V(A)$.

4.2. Let $(\boldsymbol{G},\rho,V)$ be a prehomogeneous vector space defined over the rational number field $\boldsymbol{Q}$ and $\boldsymbol{S}$ its singular set. Let $\boldsymbol{S}_1,\cdots,\boldsymbol{S}_n$ be the $\boldsymbol{Q}$-irreducible hypersurfaces contained in $\boldsymbol{S}$ and fix a $\boldsymbol{Q}$-irreducible polynomial P_i defining $\boldsymbol{S}_i$ for each $i=1,\cdots,n$. We assume that

(4.1) *for any $x\in V(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$, the identity component of $\boldsymbol{G}_x=\{g\in\boldsymbol{G};\ \rho(g)x=x\}$ has no nontrivial $\boldsymbol{Q}$-rational character.*

Under this assumption we shall define a relatively $\boldsymbol{G}(\boldsymbol{Q})$-invariant "measure" on $V(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$.

Let G^+ be the connected component of the identity element of $\boldsymbol{G}(\boldsymbol{R})$. Let $V(\boldsymbol{R})-\boldsymbol{S}(\boldsymbol{R})=V_1\cup\cdots\cup V_\nu$ be the G^+-orbit decomposition. Let dg be a right invariant measure on G^+ and $d_V^\times(x)$ be a relatively G^+-invariant measure on $V(\boldsymbol{R})-\boldsymbol{S}(\boldsymbol{R})$ with multiplier $\Delta(h)=d(hg)/dg$. The existence of such a measure is assured by (4.1) and can be normalized as follows:

$$d_V^\times(x)=\prod_{i=1}^n |P_i(x)|^{-\delta_i}dx,$$

where $\delta_1,\cdots,\delta_n$ are some rational numbers (see [35, § 4]).

For any $x\in V_i$, we normalize a Haar measure $d\mu_x$ on $G_x^+=\boldsymbol{G}_x\cap G^+$ by the formula

$$\int_{G^+} f(g)dg=\int_{G^+/G_x^+} d_V^\times(\rho(\dot{g})x)\int_{G_x^+} f(\dot{g}h)d\mu_x(h).$$

Put $x'=\rho(h)x$ $(h \in G^+)$. Then the mapping $g \mapsto hgh^{-1}$ induces an isomorphism of G_x^+ onto $G_{x'}^+$ and the pull back of the measure $d\mu_{x'}$ by the isomorphism coincides with $d\mu_x$.

Let $C^\infty(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$ be the space of all complex valued functions on $\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ invariant under some arithmetic subgroup of $\boldsymbol{G}(\boldsymbol{Q})$ and $\mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$ the subspace of $C^\infty(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$ consisting of all functions whose supports are contained in a finite union of orbits of some arithmetic subgroup of $\boldsymbol{G}(\boldsymbol{Q})$.

We fix an arithmetic subgroup Γ_0 of $\boldsymbol{G}(\boldsymbol{Q})$. For an $f \in \mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$, one can find an arithmetic subgroup Γ of $\boldsymbol{G}(\boldsymbol{Q})$ contained in G^+ and rational points $x_1, \cdots, x_m \in \boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ such that

$$f(x)=\sum_{i=1}^{m} c_i ch_{\Gamma\cdot x_i}(x) \qquad (c_i \in \boldsymbol{C}), \tag{4.2}$$

where $ch_{\Gamma\cdot x_i}$ stands for the characteristic function of $\rho(\Gamma)x_i$.

Define a linear form μ on $\mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$ by

$$\mu(f)=[\Gamma:\Gamma_0]\sum_{i=1}^{m} c_i \int_{G_{x_i}^+/\Gamma_{x_i}} d\mu_{x_i}, \tag{4.3}$$

where $[\Gamma:\Gamma_0]=[\Gamma:\Gamma\cap\Gamma_0]/[\Gamma_0:\Gamma\cap\Gamma_0]$ and $\Gamma_{x_i}=\Gamma\cap G_{x_i}^+$. Note that the right hand side of (4.3) is finite by the assumption (4.1).

In order to see that $\mu(f)$ is independent of the decomposition (4.2) of f, we need the following lemma.

Lemma 4.1. *Let Γ_1 and Γ_2 be two arithmetic subgroups of $\boldsymbol{G}(\boldsymbol{Q})\cap G^+$ with $\Gamma_1\supset\Gamma_2$. For an $x \in \boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$, let*

$$\rho(\Gamma_1)x=\bigcup_{i=1}^{t}\rho(\Gamma_2)x_i$$

be the Γ_2-orbit decomposition. Then

$$[\Gamma_1:\Gamma_0]\int_{G_x^+/\Gamma_{1,x}} d\mu_x=[\Gamma_2:\Gamma_0]\sum_{i=1}^{t}\int_{G_{x_i}^+/\Gamma_{2,x_i}} d\mu_{x_i}.$$

Proof. Fox a $\gamma_i \in \Gamma_1$ such that $x_i=\rho(\gamma_i)x$ for each $i=1, \cdots, t$. Then we have the double coset decomposition

$$\Gamma_1=\bigcup_{i=1}^{t}\Gamma_2\gamma_i\Gamma_{1,x}.$$

Hence

$$\begin{aligned}[\Gamma_1:\Gamma_2]&=\sum_{i=1}^{t}\#(\Gamma_2\backslash\Gamma_2\gamma_i\Gamma_{1,x})\\&=\sum_{i=1}^{t}[\gamma_i\Gamma_{1,x}\gamma_i^{-1}:\Gamma_2\cap\gamma_i\Gamma_{1,x}\gamma_i^{-1}]\\&=\sum_{i=1}^{t}[\Gamma_{1,x_i}:\Gamma_{2,x_i}].\end{aligned}$$

Since x and x_i are equivalent under Γ_1, we have

$$\int_{G_x^+/\Gamma_{1,x}}d\mu_x=\int_{G_{x_i}^+/\Gamma_{1,x_i}}d\mu_{x_i}=[\Gamma_{1,x_i}:\Gamma_{2,x_i}]^{-1}\cdot\int_{G_{x_i}^+/\Gamma_{2,x_i}}d\mu_{x_i}.$$

Therefore

$$\begin{aligned}[\Gamma_2:\Gamma_0]\cdot\sum_{i=1}^{t}\int_{G_{x_i}^+/\Gamma_{2,x_i}}d\mu_{x_i}&=[\Gamma_2:\Gamma_0]\cdot\int_{G_x^+/\Gamma_{1,x}}d\mu_x\cdot\sum_{i=1}^{t}[\Gamma_{1,x_i}:\Gamma_{2,x_i}]\\&=[\Gamma_1:\Gamma_2][\Gamma_2:\Gamma_0]\cdot\int_{G_x^+/\Gamma_{1,x}}d\mu_x\\&=[\Gamma_1:\Gamma_0]\int_{G_x^+/\Gamma_{1,x}}d\mu_x.\end{aligned}$$

In view of (4.3) the identity in Lemma 4.1 implies that

$$\mu(ch_{\Gamma_1\cdot x})=\sum_{i=1}^{t}\mu(ch_{\Gamma_2\cdot x_i});$$

this shows that $\mu(f)$ is independent of the decomposition (4.2) to f. Thus μ determines a positive linear functional on $\mathcal{S}(V(\boldsymbol{Q})-S(\boldsymbol{Q}))$.

For a $g\in G(\boldsymbol{Q})$ we put

$$\begin{aligned}\delta(g)&=[g\Gamma g^{-1}:\Gamma]\\&=[g\Gamma g^{-1}:\Gamma\cap g\Gamma g^{-1}]/[\Gamma:\Gamma\cap g\Gamma g^{-1}].\end{aligned}$$

Then it is easy to check that $\delta(g)$ is independent of the choice of an arithmetic subgronp Γ of $G(\boldsymbol{Q})$ and defines a homomorphism of $G(\boldsymbol{Q})$ into $\boldsymbol{Q}_+^\times$.

For a $g\in G(\boldsymbol{Q})$ and an $f\in\mathcal{S}(V(\boldsymbol{Q})-S(\boldsymbol{Q}))$, put ${}^gf(x)=f(\rho(g^{-1})x)$. Note that ${}^gf$ is also in $\mathcal{S}(V(\boldsymbol{Q})-S(\boldsymbol{Q}))$.

Lemma 4.2. *For any $g\in G(\boldsymbol{Q})$ and any $f\in\mathcal{S}(V(\boldsymbol{Q})-S(\boldsymbol{Q}))$, we have $\mu({}^gf)=\delta(g)\mu(f)$.*

Proof. It is sufficient to prove the lemma for $f=ch_{\Gamma\cdot x}$. Then, by definition, we have

$$\mu({}^g f)=\mu(ch_{g\Gamma g^{-1}\cdot gx})=[g\Gamma g^{-1}:\Gamma_0]\int_{G^+_{gx}/(g\Gamma g^{-1})_{gx}} d\mu_{gx}.$$

Since the pull back of the measure $d\mu_{gx}$ coincides with $d\mu_x$, the right hand side of the identity is equal to

$$[g\Gamma g^{-1}:\Gamma][\Gamma:\Gamma_0]\int_{G^+_x/\Gamma_x} d\mu_x=\delta(g)\mu(ch_{\Gamma\cdot x}).$$

By Lemma 4.2 we may regard μ as a kind of relatively $\boldsymbol{G}(\boldsymbol{Q})$-invariant measure on $\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ with multiplier δ. In this sense we often write

$$\mu(f)=\int_{V(Q)-S(Q)} f(x)d\mu(x) \qquad (f\in\mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))).$$

Moreover for a subset E of $\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ such that the characteristic function ch_E of E is in $C^\infty(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$, we write

$$\mu(f\cdot ch_E)=\int_E f(x)d\mu(x) \qquad (f\in\mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))).$$

If ch_E is in $\mathcal{S}(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$, then the value $\mu(ch_E)$ is called the *volume* (or *density*) of E and is simply written as $\mu(E)$.

For an $f\in C^\infty(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$, take a Γ such that f is $\rho(\Gamma)$-invariant and write

$$f=\sum_{x\in\Gamma\backslash V(Q)-S(Q)} c_x\cdot ch_{\Gamma\cdot x} \qquad (c_x\in\boldsymbol{C}).$$

If the infinite series $\sum_{x\in\Gamma\backslash V(Q)-S(Q)}|c_x|\mu(\Gamma\cdot x)$ is convergent, then we call the function f an *integrable function* (*with respect to* μ) and put

$$\mu(f)=\int_{V(Q)-S(Q)} f(x)d\mu(x)=\sum_{x\in\Gamma\backslash V(Q)-S(Q)} c_x\mu(\Gamma\cdot x).$$

The value $\mu(f)$ is independent of the choice of Γ.

4.3. For $s=(s_1,\cdots,s_n)\in\boldsymbol{C}^n$ and i $(1\leqq i\leqq\nu)$, we define a function $|P|_i^{-s}$ on $\boldsymbol{V}(\boldsymbol{R})-\boldsymbol{S}(\boldsymbol{R})$ by

$$|P(x)|_i^{-s}=\begin{cases}\prod_{j=1}^{n}|P_j(x)|^{-s_j} & (x\in V_i),\\ 0 & (x\notin V_i).\end{cases}$$

Then the restrictions of $|P(x)|_1^{-s},\cdots,|P(x)|_\nu^{-s}$ to $\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q})$ define functions in $C^\infty(\boldsymbol{V}(\boldsymbol{Q})-\boldsymbol{S}(\boldsymbol{Q}))$, which we denote by the same symbols.

The restriction $\phi|_{V(Q)-S(Q)}$ of $\phi \in \mathcal{S}(V(Q))$ also defines a function in $C^\infty(V(Q)-S(Q))$.

We define distributions $Z_i(s)$ ($1 \leqq i \leqq \nu$, $s \in C^n$) on $V(Q)$, which we call the *zeta distributions associated with* (G, ρ, V) *over* Q, by

$$Z_i(s)\colon \mathcal{S}(V(Q)) \ni \phi \longmapsto Z_i(s)(\phi) = \int_{V(Q)-S(Q)} |P(x)|_i^{-s}\phi(x)d\mu(x) \in C.$$

Here we must assume the following condition, which assures the existence of the zeta distributions:

(4.4) *If* $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n$ *are sufficiently large, then the function* $|P(x)|_i^{-s} \cdot \phi|_{V(Q)-S(Q)}$ *is integrable for all* $i=1, \cdots, \nu$ *and all* $\phi \in \mathcal{S}(V(Q))$.

It is a conjecture that the assumption (4.1) implies (4.4).

Let Γ be an arithmetic subgroup of $G(Q)$ such that $|P(x)|_i^{-s}$ ($1 \leqq i \leqq \nu$) and $\phi(x)$ are Γ-invariant. Then by definition we have

$$Z_i(s)(\phi) = \sum_{x \in \Gamma \backslash V(Q) \cap V_i} \phi(x)\mu(\Gamma \cdot x) \Big/ \prod_{j=1}^{n} |P_j(x)|^{s_j}.$$

In particular, if ϕ is the characteristic function of a Γ-invariant lattice L in $V(Q)$, then

$$Z_i(s)(ch_L) = [\Gamma : \Gamma_0] \sum_{x \in \Gamma \backslash L \cap V_i} \left(\int_{G_x^+/\Gamma_x} d\mu_x \right) \Big/ \prod_{j=1}^{n} |P_j(x)|^{s_j}. \tag{4.5}$$

The right hand side of (4.5) is nothing but the zeta functions associated with (G, ρ, V) introduced in [40] and [35, § 4]. Therefore the condition (4.4) can be rephrased into the usual assumption on convergence of zeta functions (cf. [35, (4.3)]). Notice that, if $Z_i(s)(ch_L)$ is absolutely convergent for an L and sufficiently large $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n$, then so is $Z_i(s)(\phi)$ for all $\phi \in \mathcal{S}(V(Q))$.

4.4. The above argument clarifies the meaning of zeta functions. Consider the zeta functions (4.5) and assume that $P_1, \cdots, P_n$ take values in Z on L. For an n-tuple $m=(m_1, \cdots, m_n)$ of positive integers, let $A_i(m; L)$ be the set of all solutions in $L \cap V_i$ of the simultaneous Diophantine equation

$$\begin{cases} P_1(x) = \pm m_1, \\ \quad \vdots \\ P_n(x) = \pm m_n. \end{cases} \tag{4.6}$$

In general the set $A_i(m; L)$ of solutions of (4.6) may be an infinite set.

However we can measure the density of $A_i(m; L)$ by using μ and the zeta function (4.5) can be transformed into the following form:

$$Z_i(s)(ch_L) = \sum_{m_1, \cdots, m_n = 1}^{\infty} \mu(A_i(m; L)) m_{m_1}^{-s_1} \cdots m_{m_n}^{-s_n}.$$

From this expression one sees immediately that the zeta function has an intuitive meaning as a generating function of the densities of solutions of the Diophantine equation (4.6).

Remark 1. The idea of measuring the densities of solutions of Diophantine equations by using integrals of the form $\int_{G_x^+/\Gamma_x} d\mu_x$ goes back to Siegel's work on the arithmetic of indefinite quadratic forms ([45], [46]).

Remark 2. Let $\hat{G}(\boldsymbol{Q})$ be the completion of $G(\boldsymbol{Q})$ with respect to the subgroup topology defined by the family of arithmetic subgroups of $G(\boldsymbol{Q})$. Let

$$X = \lim_{\Gamma} \operatorname{proj} \Gamma \backslash (V(\boldsymbol{Q}) - S(\boldsymbol{Q})),$$

where Γ runs through arithmetic subgroups of $G(\boldsymbol{Q})$. Then X is a totally disconnected locally compact topological space and $\hat{G}(\boldsymbol{Q})$ acts topologically on X. Moreover $\mathscr{S}(V(\boldsymbol{Q}) - S(\boldsymbol{Q}))$ can be identified with the space of locally constant functions with compact support on X. The linear functional μ defined above induces a relatively $\hat{G}(\boldsymbol{Q})$-invariant measure on X and δ: $G(\boldsymbol{Q}) \to \boldsymbol{Q}_+^\times$ coincides with the restriction of the module of $\hat{G}(\boldsymbol{Q})$ to $G(\boldsymbol{Q})$. In this sense μ is really a measure. The space X is closely related to the space $\prod_p' (V(\boldsymbol{Q}_p) - S(\boldsymbol{Q}_p))$ of finite adeles; however X does not coincide with it in general, unless (G, ρ, V) is universally transitive in the sense of [18].

4.5. In the following we assume that

(4.7) *(G, ρ, V) is $\boldsymbol{Q}$-regular*

and

(4.8) *The singular set S is a hypersurface in V,*

as well as the conditions (4.1) and (4.4).

We employ the same notation as in Section 1.1 and Section 1.2 for $K = \boldsymbol{Q}$. Then the prehomogeneous vector spaces (G, ρ^*, V^*) dual to (G, ρ, V) satisfies automatically the assumptions (4.1), (4.7) and (4.8). Recall that the number of $\boldsymbol{Q}$-irreducible components, (all of which are of codimension 1 by (4.8)), of the singular set S^* of (G, ρ^*, V^*) is equal to

n (= the number of $\boldsymbol{Q}$-irreducible components of $\boldsymbol{S}$). Let $\boldsymbol{S}^*=\boldsymbol{S}_1^*\cup\cdots\cup\boldsymbol{S}_n^*$ be the decomposition of $\boldsymbol{S}^*$ into $\boldsymbol{Q}$-irreducible hypersurfaces and take a $\boldsymbol{Q}$-irreducible relative invariant P_i^* defining $\boldsymbol{S}_i^*$ for each $i=1,\cdots,n$. Let $\boldsymbol{V}^*(\boldsymbol{R})-\boldsymbol{S}^*(\boldsymbol{R})=V_1^*\cup\cdots\cup V_\nu^*$ be the G^+-orbit decomposition. Here the number of G^+-orbits in $\boldsymbol{V}^*(\boldsymbol{R})-\boldsymbol{S}^*(\boldsymbol{R})$ coincides with ν (= the number of G^+-orbits in $\boldsymbol{V}(\boldsymbol{R})-\boldsymbol{S}(\boldsymbol{R})$.

We are able to define the zeta distributions $Z_i^*(s)$ $(1\leqq i\leqq\nu)$ on $\boldsymbol{V}^*(\boldsymbol{Q})$ associated with $(\boldsymbol{G},\rho^*,\boldsymbol{V}^*)$ in the same way as for $(\boldsymbol{G},\rho,\boldsymbol{V})$:

$$Z_i^*(s)(\phi^*)=\int_{V^*(Q)-S^*(Q)}|P^*(y)|_i^{-s}\phi^*(y)d\mu^*(y)\qquad(\phi^*\in\mathscr{S}(V^*(\boldsymbol{Q}))),$$

where

$$|P^*(y)|_i^{-s}=\begin{cases}\prod_{j=1}^{n}|P_j^*(y)|^{-s_j}, & \text{if } y\in V_i^*\cap V^*(\boldsymbol{Q}),\\ 0, & \text{otherwise.}\end{cases}$$

We assume the condition (4.4) also for $(\boldsymbol{G},\rho^*,\boldsymbol{V}^*)$.

Let D_0 (resp. D_0^*) be the set of $s\in\boldsymbol{C}^n$, for which $Z_i(s)(\phi)$ (resp. $Z_i^*(s)(\phi^*)$) are absolutely convergent and D (resp. D^*) be the convex hull of $D_0\cup(D_0^*U^{-1}+\lambda)$ (resp. $D_0^*\cup(D_0-\lambda)U$).

Now the main result in the theory of zeta functions associated with prehomogeneous vector spaces can be stated in the language of distributions on $\boldsymbol{V}(\boldsymbol{Q})$ and $\boldsymbol{V}^*(\boldsymbol{Q})$ as follows:

Theorem $\boldsymbol{Q}$. (1) *The distributions $Z_i(s)$ (resp. $Z_i^*(s)$) have analytic continuations to distributions depending meromorphically on s in D (resp. D^*).*

(2) *The following functional equations hold for $s\in D$:*

$$Z_i^*((s-\lambda)U)^\wedge=\sum_{j=1}^{\nu}\Gamma_{ji}(s-\delta)Z_j(s),$$

where $(\Gamma_{ij}(s))$ is the Γ-matrix appearing in Theorem $\boldsymbol{R}$ in Section 1.3 and δ is an n-tuple of rational numbers appearing as exponents of $|P_i(x)|$ in the definition of $d_V^\times(x)$ in Section 4.2.

In the present formulation, the analogy of Theorem $\boldsymbol{Q}$ with Theorem $\boldsymbol{R}$ and Theorem $k_\mathfrak{p}$ is quite obvious.

The proof of Theorem $\boldsymbol{Q}$ is the same as in the proof of [35, § 6 Theorem 2]. The only necessary modification is that the Poisson summation formula should be applied in the form given in Section 4.1. It is known that D and D^* coincide with whole of $\boldsymbol{C}^n$, if $\boldsymbol{G}$ is reductive (see [35, § 6 Corollary 1]).

Remark. Hoffman [16] independently introduced the space $\mathscr{S}(V(\boldsymbol{Q}))$ and studied the zeta function $Z_i(s)(\phi)$. His aim is to establish fundamental properties of zeta function of Hurwitz type associated with prehomogeneous vector spaces of parabolic type whose special values contribute trace formulas for Hecke operators (see also § 4.7).

4.6. Our zeta functions $Z_i(s)(\phi)$ (= the value of the zeta distribution) are more general than the zeta functions treated in [40] and [35], because the consideration in these papers was restricted to characteristic functions of lattices in $V(\boldsymbol{Q})$. As is seen in the following, by specializing $\phi \in \mathscr{S}(V(\boldsymbol{Q}))$ suitably, one can obtain several interesting Dirichlet series.

(A) *Zeta functions of Hurwitz type*: Fix an $a \in V(\boldsymbol{Q})$ and a lattice L. Let ϕ be the characteristic function of $a+L$. Then $Z_i(s)(\phi)$ can be regarded as a generalization of the Hurwitz zeta function $\zeta(s, a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ $(0<a\leqq 1)$ to prehomogeneous vector spaces. Since the Fourier transform of $\phi=ch_{a+L}$ is given by

$$\hat{\phi}(y)=[L: L_0]e^{2\pi i(a, y)}ch_{L^*}(y),$$

the zeta function $Z_j^*(s)(\hat{\phi})$ is a generalization of $\zeta^*(s, a)=\sum_{n=1}^{\infty} e^{2\pi i n a}n^{-s}$.

Aside from the original Hurwitz zeta function, several examples of zeta functions of this type have appeared in literatures. In [3] Bushnell and Reiner proved a functional equation satisfied by "Hurwitz series" associated with a semisimple algebra over $\boldsymbol{Q}$. From our point of view their result can be interpreted as follows.

Let A be a finite dimensional semisimple $\boldsymbol{Q}$-algebra and $A^\times$ the group of invertible elements in A. We consider $A^\times$ as (a set of $\boldsymbol{Q}$-rational points of) an algebraic group defined over $\boldsymbol{Q}$. Then the action of $A^\times$ on A from the left provides us a typical example of prehomogeneous vector spaces defined over $\boldsymbol{Q}$, which we denote by $(A^\times, \rho, A)$. It is easy to see that the triple $(A^\times, \rho, A)$ satisfies all the assumptions in the present section.

Let $A=A_1\times\cdots\times A_r$, where A_i is a simple $\boldsymbol{Q}$-algebra. Let

$$P_i(a_1, \cdots, a_r)=\mathrm{Nr}_{A_i}(a_i) \qquad ((a_1, \cdots, a_r) \in A_1\times\cdots\times A_r),$$

where Nr_{A_i} is the reduced norm of A_i. Then $P_1, \cdots, P_r$ are $\boldsymbol{Q}$-irreducible relative invariants of $(A^\times, \rho, A)$ and the singular set $\boldsymbol{S}$ is given by

$$\boldsymbol{S}=\bigcup_{i=1}^{r}\{x \in A;\, P_i(x)=0\}.$$

Let $Z_1(s_1, \cdots, s_r), \cdots, Z_\nu(s_1, \cdots, s_r)$ (ν = the number of connected component of $A^\times(\boldsymbol{R})=A\otimes\boldsymbol{R}-\boldsymbol{S}(\boldsymbol{R})$) be the zeta distributions associated with

$(A^\times, \rho, A)$. Taking $\phi = ch_{a+L}$ with $a \in A$ and a full $\boldsymbol{Z}$-lattice L in A, we obtain a zeta function $Z(s_1, \cdots, s_r)(ch_{a+L}) = \sum_{i=1}^{\nu} Z_i(s_1, \cdots, s_r)(ch_{a+L})$ of Hurwitz type. Since $A^\times$ is reductive as an algebraic group, the zeta function can be continued meromorphically to whole of $\boldsymbol{C}^r$. The Hurwitz series considered by Bushnell and Reiner is the function $Z(s, \cdots, s)(ch_{a+L})$ obtained by restricting the variable $(s_1, \cdots, s_r)$ to the diagonal $\{(s, \cdots, s); s \in \boldsymbol{C}\}$, In order to make the functional equation explicit, we need the explcit formula for the Γ-matrix $(\Gamma_{ij}(s))$ entering into the functional equation. For the present example the formula for $(\Gamma_{ij}(s))$ is well-known (see e.g. [25, p. 497]).

In their proof of the functional equation in [3], the authors first reduce the things to the case of simple $\boldsymbol{Q}$-algebra, and then two subcases are distinguished according as A satisfies the so-called Eichler condition or not. They posed a problem of finding a method applicable equally to both cases. The theory of prehomogeneous vector spaces supplies the demand.

One can find another discussion on zeta functions of this type in [6].

(B) *L-functions associated with prehomogeneous vector spaces*: Fix a positive integer N and let $\chi_1, \cdots, \chi_n$ be Dirichlet characters defined modulo N. (In the following the characters corresponding to relative invariants do not appear; hence the notation will not cause any confusion.) As usual we set $\chi_i(m)=0$, if $(m, N)\neq 1$.

Let L be a lattice on which $P_1, \cdots, P_n$ take integral values. Consider the function $\phi_\chi = \phi_{\chi_1,\dots,\chi_n} \in \mathscr{S}(\boldsymbol{V}(\boldsymbol{Q}))$ defined by

$$\phi_\chi(x) = \sum_{y \in L/NL} \prod_{i=1}^{n} \chi_i(P_i(y)) ch_{y+NL}(x).$$

We define L-functions $L_i(s; \chi)$ $(1 \leqq i \leqq \nu)$ associated with $(\boldsymbol{G}, \rho, \boldsymbol{V})$ by the formula

$$L_i(s; \chi) = Z_i(s)(\phi_\chi) = \sum_{x \in \Gamma \backslash L \cap V_i} \mu(\Gamma \cdot x) \prod_{j=1}^{n} \chi_j(P_j(x)) \Big/ \prod_{j=1}^{n} |P_j(x)|^{s_j},$$

where Γ is an arithmetic subgroup of $\boldsymbol{G}(\boldsymbol{Q})$ satisfying $\rho(\Gamma)(y+NL) = y+NL$ for all $y \in L/NL$.

To write down the functional equation explicitly, we must calculate the Fourier transform $\hat{\phi}_\chi$ of ϕ_χ. It is easy to check the following identity:

$$\hat{\phi}_\chi(x^*) = [NL: L_0] \sum_{y^* \in N^{-1}L^*/L^*} G_{\chi, P}(y^*) ch_{y^*+L^*}(x^*), \tag{4.9}$$

where

$$G_{\chi\circ P}(y^*)=\sum_{y\in L/NL}\prod_{i=1}^{n}\chi_i(P_i(y))e^{2\pi i(y,y^*)}.$$

Thus the calculation of $\hat{\phi}_\chi$ is reduced to the calculation of a kind of Gaussian sums $G_{\chi\circ P}$.

This is the problem solved by Gyoja and Kawanaka ([10], [11]) under the conditions

(4.10) (i) $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is a regular irreducible prehomogeneous vector space equipped with a natural $\boldsymbol{Z}$-structure,

(ii) $N=p$ is a sufficiently large prime number.

In the rest of this paragraph we assume these two conditions.

To describe the result of Gyoja and Kawanaka, we need some notational preliminaries. By (4.10) (i) the singular set $\boldsymbol{S}$ is an absolutely irreducible hypersurface defined over $\boldsymbol{Q}$; hence $n=1$. Let $P(x)$ (resp. $P^*(y)$) an irreducible relative invariant of $(\boldsymbol{G}, \rho, \boldsymbol{V})$ (resp. $(\boldsymbol{G}, \rho^*, \boldsymbol{V}^*)$), which is unique up to constant multiple, such that $P(x)$ (resp. $P^*(y)$) takes integral values on the lattice L (resp. L^*) defining the $\boldsymbol{Z}$-structure of $(\boldsymbol{G}, \rho, \boldsymbol{V})$ (resp. $(\boldsymbol{G}, \rho^*, \boldsymbol{V}^*)$).

Let $b(s)$ be the b-function of $(\boldsymbol{G}, \rho, \boldsymbol{V})$, namely the polymomial in s defined by the formula

$$P^*(\mathrm{grad}_x)P(x)^{s+1}=b(s)P(x)^s$$

(cf. § 1.3). It is known that $b(s)$ is a polynomial in s of degree $d=\deg P$ and all the roots of $b(s)=0$ are rational numbers.

Let

$$b^{\exp}(t)=\prod_{j=1}^{d}(t-\exp(-2\pi i\alpha_j)),$$

where $\alpha_1, \cdots, \alpha_d$ are the roots of $b(s)=0$. Then $b^{\exp}(t)$ can be expressed as a product of cyclotomic polynomials. We define non-negative integers $m(l)$ $(l\geqq 1)$ by

$$b^{\exp}(t)=\prod_{l=1}^{\infty}\Phi_l(t)^{m(l)},$$

where Φ_l is the l-th cyclotomic polynomial.

Theorem F_p ([11]). *Let χ be a Dirichlet character defined modulo p of order* $\operatorname{ord}\chi$. *Then the following identity holds*:

$$G_{\chi\circ P}(p^{-1}y^*)=\varepsilon(\chi)p^{(\dim V-m(\operatorname{ord}\chi))/2}(\chi^{-1}\otimes\theta)(P^*(y^*))$$
$$(y^*\in L^*,\ P^*(y^*)\not\equiv 0\ (\mathrm{mod}\,p)),$$

where $\varepsilon(\chi)$ *is an algebraic number with absolute value* 1 *and*

$$\theta=\begin{cases}\text{trivial character}, & \text{if } \dim V/d\in \boldsymbol{Z},\\ \text{the Legendre symbol } \left(\dfrac{\cdot}{p}\right), & \text{if } \dim V/d\in \dfrac{1}{2}+\boldsymbol{Z}.\end{cases}$$

In order to see the analogy of Theorem $\boldsymbol{F}_p$ to Theorem $\boldsymbol{R}$, k_p and $\boldsymbol{Q}$, it is enough to note that a Dirichlet character defined modulo p can be regarded as a (quasi-)character of the multiplicative group $\boldsymbol{F}_p^\times$ and the Gaussian sum $G_{\chi\circ P}$ is the Fourier transform of $\chi\circ P$ over $V(\boldsymbol{F}_p)$.

Corollary to Theorem $\boldsymbol{F}_p$ ([11]). *If* $m(\operatorname{ord}\chi)=0$, *then* $G_{\chi\circ P}(p^{-1}y^*)=0$ $(y^*\in L^*,\ P^*(y^*)\equiv 0 \pmod p)$.

Now we denote by $L_i^*(s;\chi^{-1}\otimes\theta)$ the L-function associated with $(\boldsymbol{G},\rho^*,V^*)$.

Theorem $\boldsymbol{L}$. *Let* χ *be a Dirichlet character defined modulo* p *with* $m(\operatorname{ord}\chi)=0$. *Then the L-functions* $L_i(s;\chi)$ *and* $L_j^*(s;\chi^{-1}\otimes\theta)$ *are continued analytically to entire functions of* s *and satisfy the following functional equation*:

$$L_i^*\left(\frac{\dim V}{d}-s,\ \chi^{-1}\otimes\theta\right)=p^{(2ds-\dim V)/2}\varepsilon(\chi)^{-1}[L_0\colon L]\times\sum_{j=1}^{\nu}\Gamma_{ji}\left(s-\frac{\dim V}{d}\right)L_j(s;\chi).$$

Proof. From Theorem $\boldsymbol{F}_p$, Corollary to Theorem $\boldsymbol{F}_p$ and (4.9), it follows that

$$\hat{\phi}_\chi(x^*)=[pL\colon L_0]\varepsilon(\chi)p^{\dim V/2}\sum_{y^*\in L^*/pL^*}(\chi^{-1}\otimes\theta)(P^*(y^*))ch_{y^*+pL^*}(px^*)$$

for any $x^*\in V^*(\boldsymbol{Q})$. Hence by Theorem $\boldsymbol{Q}$ we get the functional equation of the L-functions. The formula for $\hat{\phi}_\chi$ shows that the points belonging to $\boldsymbol{S}^*$ make no contribution to the Poisson summation formula at the end of Section 4.1. Therefore the L-functions can be continued to entire functions.

Theorem L generalizes Stark's result on L-functions of quadratic forms ([47]). His result corresponds to a $\boldsymbol{Z}$-form of the prehomogeneous vector space $(\boldsymbol{GL}(1)\times\boldsymbol{SO}(n),\ \Lambda_1,\ V(n))$ such that the group of real points of $\boldsymbol{SO}(n)$ is compact.

Remark 1. If $m(\mathrm{ord}\,\chi)\neq 0$, $\hat{\phi}_\chi$ does not vanish on $\boldsymbol{S}^*(\boldsymbol{Q})$. Therefore the left hand side of the functional equation in Theorem L does not coincide with $L_i^*\left(\frac{\dim V}{d}-s,\ \chi^{-1}\otimes\theta\right)$, but involves some mysterious terms in addition. Moreover by the same reason the L-function $L_i(s;\chi)$ may have poles, even if χ is a nontrivial character. This explains the observation of Datskovski and Wright that the L-functions of a nontrivial cubic character associated with the vector space of binary cubic forms have a simple pole ([7, p. 30]). Another example of L-functions with poles was studied by Arakawa [1]. He calculated the residues of L-functions attached to the ternary quadratic form x^2-yz and the character $\left(\frac{*}{p}\right)$ given by the Legendre symbol ([1, Theorem 1.11]).

Remark 2. In many cases the constant $\varepsilon(\chi)$ can be expressed in terms of the classical Gaussian sums. For explicit calculation of $\varepsilon(\chi)$, we refer to [4]. In [5], Christian calculated $\varepsilon(\chi)$ for $(\boldsymbol{GL}(n)\times\boldsymbol{SL}(n),\ \Lambda_1\otimes\Lambda_1,\ M(n))$ by using the fact that $\varepsilon(\chi)$ appears as a coefficient of the functional equation satisfied by certain twist of Koecher-Maaß series (= zeta function associated with $(\boldsymbol{SO}(m)\times\boldsymbol{GL}(n),\ \Lambda_1\otimes\Lambda_1,\ M(m,n))$ $(m\geqq n)$).

Remark 3. There exists another application of Theorem $\boldsymbol{F}_p$ due to Weissauer ([51]). From Theorem $\boldsymbol{F}_p$ for $\left(\boldsymbol{GL}(n),\ 2\Lambda_1,\ V\left(\frac{n(n+1)}{2}\right)\right)$ he derived that certain twist of Siegel modular forms by Dirichlet characters are again Siegel modular forms.

Remark 4. The original motivation of Gyoja and Kawanaka for their study of Gaussian sums of prehomogeneous vector spaces was not L-functions but the fact that this kind of Gaussian sums enters into the character formula for certain irreducible representations of finite reductive groups (see Kawanaka [21]).

4.7. In [41] Selberg pointed out that the cusp contribution to the formula for the dimensions of spaces of automorphic forms (based upon his celebrated trace formula) can be evaluated as special values of certain Dirichlet series at integer argument. The theory of prehomogeneous vector spaces might provide us a convenient tool of formulating Selberg's observation in a more concrete form. Roughly speaking one can expect that the parabolic contributions to the dimension formula for a space of automorphic forms on a real reductive Lie group with respect to a lattice can be expressed in terms of special values of zeta functions of Hurwitz type associated with prehomogeneous vector spaces of parabolic type (in the sense of Rubenthaler [33]). This expectation has been confirmed by several authors mainly in two special cases, namely the case of Siegel cusp forms and the case of lattices of rank 1.

1) *Siegel cusp forms*: In [43] Shintani calculated the the purely unipotent contributions to the dimension of the space of Siegel cusp forms of degree n and of weight k ($k \geqq 2n+3$) and showed that the contributions coincide with certain special values of zeta functions associated with the prehomogeneous vector spaces $\left(GL(m),\, 2\Lambda_1,\, V\left(\frac{m(m+1)}{2}\right)\right)$ $(1 \leqq m \leqq n)$ up to elementary factors. Hashimoto [12] made a detailed investigation in the case of degree 2 and found that the contributions of (not necessarily unipotent) conjugacy classes can also be expressed in terms of special values of certain zeta functions (the Riemann zeta function, the Hurwitz zeta function).

Recently Shintani's calculation was extended in two directions. Arakawa considered in [1] the representation μ_k of the finite symplectic group $Sp(2n, F_p)$ on the space of Siegel cusp forms of degree n and weight k with respect to the principal congruence subgroup $\Gamma_{2n}(p)$, p being a odd prime. For some unipotent elements α in $Sp(2n, F_p)$, he obtained an expression of the purely unipotent contribution to tr $\mu_k(\alpha)$ in terms of special values of L-functions associated with $\left(GL(m),\, 2\Lambda_1,\, V\left(\frac{m(m+1)}{2}\right)\right)$. In this Arakawa's work we encounter the L-function we have studied in Section 4.6 (B) and some other type of L-functions.

In [27], Murase and Sugano extended Shintani's calculation in another direction, namely to the dimension formula for the space of Jacobi cusp forms. This time in the purely unipotent contribution there appear special values of zeta functions associated with the following prehomogeneous vector space:

$$G=\left\{g(h, x)=\left(\begin{array}{c|c} I_m & x \\ \hline 0 & h \end{array}\right) \in GL(m+n);\ h \in GL(n),\ x \in M(m, n)\right\},$$

$$V=\left\{v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix};\ v_1 \in M(m, n),\ v_2={}^t v_2 \in M(n)\right\},$$

$$\rho(g(h, x))v=g(h, x)v\,{}^t h.$$

What is remarkable here is that (G, ρ, V) is not a regular prehomogeneous vector space. Murase and Sugano succeeded in proving the existence of analytic continuations and functional equations of the associated zeta functions. Since the existing theory of zeta functions can be applied only to regular prehomogeneous vector spaces, their result goes beyond the general theory and seems to suggest a possibility of further generalization of the theory of prehomogeneous vector spaces.

2) *Lattices of rank* 1: Let G be an admissible reductive Lie group and Γ a lattice of rank 1. In [30] Osborne and Warner established the trace formula for $L^{\mathrm{disc}}_{G/\Gamma}(\alpha)$ $(\alpha \in C_0^\infty(G))$; however in their result it was not clear what kind of zeta functions appear in the parabolic contributions to the trace formula. Previous result of Warner ([49]) showed that the Epstein zeta function is sufficient to describe the parabolic contribution, if G is of $\boldsymbol{R}$-rank 1 (see also Osborne and Warner [29]). Recently Hoffmann [16] extended the work of Osborne and Warner [30] to the trace formula for Hecke operators. Moreover he succeeded in expressing the parabolic contributions, which were still mysterious in [30], in terms of orbital integrals and special values of zeta functions associated with prehomogeneous vector spaces of parabolic type obtained from split parabolic subgroups of centralizers of semisimple elements.

Kato [20] gave an example of explicit determination of the dimensions of automorphic forms on a group of $\boldsymbol{R}$-rank 1, namely on $SU(p, 1)$.

It is quite desirable to develop a method of calculating special values of zeta functions associated with prehomogeneous vector spaces. As for the progress made in this direction, we refer the reader to Satake [34], Kurihara [24] and Arakawa [1]; these works were stimulated by the pathbreaking work of Shintani [44].

Finally we note that the appearence of zeta functions associated with prehomogeneous vector spaces of parabolic type in trace formulas has a strong resemblance to the appearence of Gaussian sums (zeta functions over finite fields) associated with prehomogeneous vector spaces of parabolic type in the formulas for characters (evaluated at unipotent elements) of certain irreducible representations of finite reductive groups (cf. Kawanaka [21] and Remark 4 at the end of Section 4.6).

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