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Invariants and Hodge Cycles

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Dedications

I would like to dedicate this paper to Professor I. Satake and Professor F. Hirzebruch on their birthdays, whose mathematics greatly influenced me.

I would like to thank Professor Parry, who pointed out an initial miscalculation in this work, and the decomposition of $P=P_0+P_1$ (which looks trivial now, but initially was not), which made later calculations easier.

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§ 1. Introduction

Let F be an even $2N$ dimensional vector space over $\boldsymbol{Q}$, and β be a non degenerate alternating bilinear form on F. We put

$$F_{\boldsymbol{R}}=F\otimes\boldsymbol{R}\cong\boldsymbol{R}^{2N},$$
$$Sp(F,\beta)=\{g\in\mathrm{aut}(F)\,|\,\beta(gu,gv)=\beta(u,v)\},$$

and

$$\mathfrak{H}(F,\beta)=\left\{J\in\mathrm{aut}(F_{\boldsymbol{R}})\;\middle|\;\begin{array}{l}J^2=-1\\ \beta(u,Jv)=\text{symmetric on } u,v\\ \beta(u,Ju)>0 \text{ for } 0\neq{}^\forall u\in F\end{array}\right\};$$

and we call $\mathfrak{H}(F,\beta)$ the Siegel half space.

An element J of $\mathfrak{H}(F,\beta)$ is a complex structure of the real vector space $F_{\boldsymbol{R}}$, therefore it defines a complex vector space $(F_{\boldsymbol{R}},J)$ of N dimension, which we denote by E or E_J. The group $Sp(F,\beta)$ operates on $\mathfrak{H}(F,\beta)$ by

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$$\left.\begin{array}{r} Sp(F, \beta) \ni g \\ \mathfrak{H}(F, \beta) \ni J \end{array}\right\} \longrightarrow g(J) = g^{-1}Jg$$

and $\mathfrak{H}(F, \beta)$ is actually a symmetric space of the simple Lie group $Sp(F, \beta)(\boldsymbol{R})$ of R-points of $Sp(F, \beta)$ modulo a maximal compact subgroup.

A lattice $L \subset F$ is said to be integral, iff $\beta(L, L) \subset Z$. If an integral lattice L is given, $(F_{\boldsymbol{R}}, L, J, \beta)$ is a polarized abelian variety for all J in $\mathfrak{H}(F, \beta)$. Put

$$Sp(L, \beta) = \{\gamma \in Sp(F, \beta) \mid \gamma(L) = L\}.$$

$Sp(L, \beta)$ acts on $\mathfrak{H}(F, \beta)$ properly discontinuously and two polarized abelian varieties $(F_{\boldsymbol{R}}, L, J, \beta)$ and $(F_{\boldsymbol{R}}, L, J', \beta)$ with two elements J, J' of $\mathfrak{H}(F, \beta)$, are isomorphic iff J', J' are $Sp(L, \beta)$ equivalent.

The symmetric space $\mathfrak{H}(F, \beta)$ is actually a symmetric domain. Namely $\mathfrak{H}(F, \beta)$ has complex structure, which is invariant under the action of g for every $g \in Sp(F, \beta)(\boldsymbol{R})$. There are exactly 2 such complex structures of $\mathfrak{H}(F, \beta)$; one is complex conjugate of the other. Among these two, one satisfies the following condition.

A system of complex manifolds

$$A \xrightarrow{\pi} D$$

is called 1 dimensional family of abelian varieties iff

1) A, D are complex manifolds; π is smooth surjective holomorphic map;
2) D is a disc and π is proper;
3) $\pi^{-1}(\lambda)$ $(\lambda \in D)$ are all abelian varieties.

When such a 1-dimensional family $A \xrightarrow{\pi} D$ of abelian varieties is given, there exist a polarization of $\pi^{-1}(\lambda)$, (F, β, L), and a map $t: D \to \mathfrak{H}(F, \beta)$ such that

$$\pi^{-1}(\lambda) \cong (F_{\boldsymbol{R}}, L, \beta, t(\lambda)) \quad \text{(as polarized abelian varieties)}$$

for all $\lambda \in$ small neighborhood of origin of D. Such t is called a moduli-map of $A \xrightarrow{\pi} D$.

$\mathfrak{H}(F, \beta)$ is a complex manifold. The complex structure in $\mathfrak{H}(F, \beta)$ is characterized as the unique one such that the moduli-map t is holomorphic in a small neighborhood of origin of D for every 1-dimensional family $A \xrightarrow{\pi} D$ of abelian varieties (belonging to β).

If a symplectic representation $\rho: G \to Sp(F, \beta)$ of a semi-simple connected hermitian algebraic group $G/\boldsymbol{Q}$, not containing trivial representation, together with an holomorphic map $\tau: X \to \mathfrak{H}(F, \beta)$ of the corresponding hermitian symmetric domain $X=G(\boldsymbol{R})^0$/maximal compact into the Siegel half space $\mathfrak{H}(F, \beta)$ of (F, β) commuting with symmetries, satisfies the compatibility condition:

$$\rho(g)[\tau(x)]=\tau[g(x)] \qquad \forall g \in G \quad \forall x \in X$$

the pair (ρ, τ) defines so-called GTAS A over an arithmetic variety $V=\Gamma\backslash X$, if $\rho(\Gamma)\subset Sp(L, \beta)$ for a cocompact subgroup Γ of G without torsion. Here L is an integral lattice.

A is defined as follows; The semi direct product $\Gamma \ltimes_\rho L$, with respect to the representation $\rho : \Gamma \to \mathrm{aut}(L)$, operates on the product manifold $X \times F_{\boldsymbol{R}}$ properly discontinuously, so that it produces the quotient space $\Gamma \ltimes L \backslash X \times F_{\boldsymbol{R}}$, which is denoted by A. And the natural map of division by $\Gamma \ltimes_\rho L$, from $X \times F_{\boldsymbol{R}}$ to A, is denoted by p. From A to $V=\Gamma\backslash X$ taking the left factor, we have a natural map π. $A \xrightarrow{\pi} V$ is obviously a fibre bundle over the arithmetic variety V with torus as fibres A_λ ($\lambda \in V$) (associated to the fundamental group).

Take a point $\lambda_0 \in X$, we use the same symbol λ_0 for the point in V which is covered by $\lambda_0 \in X$. Identify $\pi_1(V, \lambda_0)$ with Γ, then the action of $\pi_1(V, \lambda_0)=\Gamma$ on the homology group $H_1(A_{\lambda_0}, \boldsymbol{Q})=F$ of the fibre is nothing else than ρ.

Therefore the representation of $\pi_1(V, \lambda_0)=\Gamma$ on the cohomology group $H^1(A_{\lambda_0}, \boldsymbol{Q})={}^tF$ (dual space of F) of the fibre is the dual representation ρ^* of ρ.

The representation of $\pi_1(V, \lambda_0)=\Gamma$ on the higher cohomology group $H^r(A_{\lambda_0}, \boldsymbol{Q})=\Lambda^r({}^tF)$ is therefore $\Lambda^r(\rho^*)$.

Since the representation ρ is always self-dual ($\because$ existence of β), the difference between ρ or ρ^* is not essential.

In the manifold A, we can define a complex structure; and A becomes a complex manifold. The complex structure of A is canonical by the following properties.

(1) $A \xrightarrow{\pi} V$ is holomorphic

(2) The map of $E_{\tau(x)}$ to A defined by $p \circ \varphi_x$ is holomorphic for every $x \in X$.

$$\begin{array}{ccc} E_{\tau(x)}=(F_R, \tau(x)) & \xrightarrow{\varphi_x} & X \times F_R \\ & {\scriptstyle p \circ \varphi_x} \searrow & \downarrow {\scriptstyle p} \\ & & A \end{array}$$

where φ_x is defined by

$$\varphi_x(u)=x \times u \qquad \text{for all } u \in F_{\boldsymbol{R}}.$$

(3) The complex manifold $\tilde{A}=\Gamma\backslash X\times F_{\boldsymbol{R}}$, (which is a covering of $A=\Gamma\ltimes L\backslash X\times F_{\boldsymbol{R}}$) is a complex vector bundle $\boldsymbol{E}$ over V, where Γ is identified with the subgroup $\Gamma\ltimes\{0\}$ of $\Gamma\ltimes L$.

A, as complex manifold with this complex structure, is actually a projective algebraic variety, and π is a rational map. Moreover addition $A\underset{V}{\times}A\to A$, and inverse $A\to A$ are also algebraic.

Therefore, fibres $A_\lambda=\pi^{-1}(\lambda)$ are Abelian varieties. Thus such variety A is called GTAS over V. (Group theoretical abelian scheme over V). It has the following properties:

(1) $A\xrightarrow{\pi}V$ are projective algebraic varieties and π is everywhere defined surjective smooth rational map.

(2) $A\underset{V}{\times}A\xrightarrow{+}A,\quad A\xrightarrow{-1}A$ are also algebraic.

(3) As C^∞-manifold $A\xrightarrow{\pi}V$ is a fibre bundle, whose fibres are tori, and associated to the fundamental group.

(4) Fibres $A_\lambda=\pi^{-1}(\lambda)$ $(\lambda\in V)$ are abelian varieties.

(5) The base V is an arithmetic variety, i.e., $V=\Gamma\backslash X$; $X=G(\boldsymbol{R})^0$/maximal compact to $\Gamma\subset G$, cocompact and no torsion.

(6) The action ρ of $\pi_1(V,\lambda)=\Gamma$ on $F=H_1(A_\lambda,\boldsymbol{Q})$ is extendable to an algebraic representation $\rho: G\to\mathrm{aut}(F)$ of G defined over $\boldsymbol{Q}$.

Conversely, does the properties (1)~(6) characterize GTAS? No, perhaps not. Actually from (1)~(6), we can prove that there exist (F,L,β), and holomorphic map $\tau: X\to\mathfrak{H}(F,\beta)$ and $\rho: G\to S_P(F,\beta)$, such that $\rho(\gamma)[\tau(x)]=\tau[\gamma(x)]$ for all $\gamma\in\Gamma$.
But we cannot prove

$$(*)\quad \rho(g)[\tau(x)]=\tau[g(x)]\qquad \forall g\in G\quad \forall x\in X$$

and

(**) the commutativity with symmetries.

If this (*) (**) be true, our A must be GTAS. In order to have (*), perhaps we must discuss Hecke-operators, densely distributed, beside Γ. Anyway, in this note, we use only those properties (1)~(6) of GTAS.

Satake classified all G and most ρ which admits some GTAS [6]. Susan Addington, inherited Satake, (roughly) classified all ρ of G which admits some GTAS when G is the quaternion type. [2], [3]. For that purpose, she invented some combinatorix in a finite group $\mathscr{G}$, called "*chemistry*". For that, see [2] or [3] [4].

When $A\xrightarrow{\pi}V$ is a rigid GTAS over an arithmetic variety $V=\Gamma\backslash X$, the space of Hodge cycles in a generic fibre $A_\lambda=\pi^{-1}(\lambda)$ $(\lambda\in V)$ is denoted by

$$HH^r(A_\lambda,\boldsymbol{Q})\qquad (HH^r=0 \text{ if } r=\text{odd})$$

and is equal to

$$H^r(A_\lambda, Q)^G = \wedge^r({}^tF)^G$$

where, $F = H_1(A_\lambda, Q)$ on which $\Gamma = \pi_1(V, \lambda)$ operates; and the operation is extendable to an action of G on F. Here, $\wedge^r({}^tF)$ is identified canonically with $H^r(A_\lambda, Q)$: in which Γ-invariant part is equal to the G-invariant part, and coincides with the space of Hodge cycles. (See [1], [4]).

Thus the problem of determining the space of Hodge cycles is reducible to the problem of determining the space of group invariants. But unlike the ordinary invariants-theory, this invariants-theory is a bit complicated, because it relates also to a combinatrix of the Galois-group ("*chemistry*").

In this paper, the author tries to calculate $\dim HH^r(A_\lambda, Q)$ in some examples with simple "*chemistries*". We put $\dim HH^r(A_\lambda, Q) = \dim \wedge^r({}^tF)^G = b_r$ also we put the polynomial $\sum_{r=0}^N b_r t^r = F(t)$ where $N = 2\dim_C A_\lambda = \dim \rho_{A_\lambda} = \dim P$. See [5].

In the most of the following cases, G is of the quaternion type. In this case, the assumption that ρ which admits a rigid GTAS does not contain 1 implies that $\wedge^{2r+1} \circ \rho^*$ is also trivial part free; i.e. $b_{2r+1} = 0$ for odd $2r+1$.

Let k be a totally real number field with $[k:Q] = m$; K be a Galois extension of Q such that $K \supset k$ with $\mathscr{G} = \mathrm{Gal}(K/Q)$, S = the set of ∞-places of k on which $\mathscr{G}$ acts transitively. Let B be a quaternion-algebra over k; and $S_0 (\subset S)$ be the set of ∞-places of k which splits in B. We assume that $S_0 \neq \phi$. Then $(\mathscr{G}, S, S_0)$ is a *chemistry* ([2] [4]). Take a (maximal) order $\mathfrak{o}$ of B, and put

$$\Gamma(1) = \{\gamma \in \mathfrak{o}^X \mid \nu(\gamma) = 1\}$$

where $\nu: B^\times \to k^\times$ is the reduced norm. If $\Gamma \subset \Gamma(1)$, $\Gamma \sim \Gamma(1)$, (commensurable), $V = \Gamma \backslash \mathfrak{h}^{|S_0|}$ is an arithmetic variety, where $\mathfrak{h}$ = the upper half plain, of which product $\mathfrak{h}^{|S_0|}$, Γ acts as usual. Susan Addington found a functor

$$\begin{array}{c} A \\ \big| \\ V \end{array} \rightsquigarrow P = \sum X_i$$

associating a $\mathscr{G}$-invariant stable polymer P in the *chemistry* $(\mathscr{G}, S, S_0)$ to each group-theoretical abelian scheme A over V. Moreover if this P is furthermore rigid, then A can not have deformations [1]. Conversely, for any $\mathscr{G}$-invariant stable polymer P in $(\mathscr{G}, S, S_0)$ there exists a group theoretical scheme A over V and a positive integer μ such that

$$A \xrightarrow{\quad} V \rightsquigarrow \mu P.$$

Moreover Salmon Abdulali [1] showed that $H^r(A_\lambda, Q)^G = \wedge^r({}^tF)^G$ coincides with the space $HH^r(A_\lambda, Q)$ of Hodge cycles in the generic fibre A_λ if the polymer is rigid.

So, in this paper, we are going to calculate

$$\dim \wedge^{2r}({}^tF)^G$$

for our rigid polymer representation space ${}^tF$ of $\mathscr{G}$, in some special cases. As notation and notions, we use the same notation and notions as in [2], [4] except group theoretical abelian scheme.

We called group theoretical abelian scheme a GTFabV there and denoted it by the notation $V \xrightarrow{\pi} U$ there instead of the notation $A \xrightarrow{\pi} V$ here.

§ 2. The character-algebra of $SL(2, C)$

The trivial representation of $SL(2, C)$ is denoted by 1. The class of the identity representation

$$x \longmapsto x \in SL(2, C)$$

of $SL(2, C)$ is denoted by X. The class of the symmetric tensor representation of $SL(2, C)$ of the degree ν is denoted by X_ν. Hence $X = X_1$, $1 = X_0$.

For 2 classes P, Q of representation of $SL(2, C)$, the class $P \oplus Q$ of representation is also denoted by $P+Q$ the class $P \otimes Q$ of representation is also denoted by $P \cdot Q$. Thus the all virtual classes of representation of $SL(2, C)$ generates a ring $\mathscr{R}$ which is generated by X, and actually isomorphic to the polynomial ring $Z[t]$ of 1 variable over Z. The "multiplicity" of 1 in $P \in \mathscr{R}$ is denoted $(P, 1)$. $\mathscr{R}$ contains $1, X_1, X_2, X_3, \cdots$, and all the elements P of $\mathscr{R}$ are expressable uniquely as linear combinations of them with Z as coefficients; and the coefficient of 1 in that linear expression of P is nothing else than $(P, 1)$.

The multiplicative structure of $\mathscr{R}$ with respect to the above additive structure is given by the Clebsch-Gordan formula: $X_\nu \cdot X_\mu = X_{\nu+\mu} + X_{\nu+\mu-2} + X_{\nu+\mu-4} + \cdots + X_{|\nu-\mu|}$.

For example:

$$X^2 = X \cdot X = X_2 + 1$$

$$X^3 = X^2 \cdot X = (X_2+1)X = X_2X + X = X_3 + X + X = X_3 + 2X$$

$$X^4 = X^3 \cdot X = (X_3 + 2X)X = X_3X + 2XX = X_4 + X_2 + 2(X_2+1)$$
$$= X_4 + 3X_2 + 2$$

$$X^5=X^4\cdot X=(X_4+3X_2+2)X=X_4X+3X_2X+2X$$
$$=X_5+X_3+3(X_3+X)+2X=X_5+4X_3+5X$$
$$\vdots$$
$$X_2^2=X_2\cdot X_2=X_4+X_2+1$$
$$X_2^3=X_2^2\cdot X_2=(X_4+X_2+1)X_2=X_4X_2+X_2X_2+X_2$$
$$=X_6+X_4+X_2+X_4+X_2+1+X_2=X_6+2X_4+3X_2+1$$
$$X_2^4=X_2^3\cdot X_2=(X_6+2X_4+3X_2+1)X_2=X_8+X_6+X_4+2(X_6+X_4+X_2)$$
$$+3(X_4+X_2+1)+X_2=X_8+3X_6+6X_4+6X_2+3$$
$$\vdots$$
$$XX_2=X_3+X$$
$$X^2X_2=X(X_3+X)=XX_3+XX=X_4+X_2+X_2+1=X_4+2X_2+1$$
$$\vdots$$
$$XX_2^2=X(X_4+X_2+1)=X_5+X_3+X_3+X+X=X_5+2X_3+2X$$
$$X^2X_2^2=X(X_5+2X_3+2X)=X_6+X_4+2(X_4+X_2)+2(X_2+1)$$
$$=X_6+3X_4+4X_2+1$$
$$\vdots$$

Lemma. *If $n>0$ is an even integer X^n is a linear combination of $1, X_2, X_4, \cdots, X_{2k}, \cdots, X_n$ of even suffixes $2k$, with positive integers a_{2k} as coefficients, and $a_n=1$; i.e.*

$$X^n=a_0 1+a_2X_2+\cdots+a_{2k}X_{2k}+\cdots+X_n \qquad Z\ni a_{2k}>0,\ a_n=1$$

If $n>0$ is an odd integer X^n is a linear combination of $X, X_3, \cdots, X_{2k+1}, \cdots, X_n$ of odd suffixes $2k+1$, with positive integers a_{2k+1} as coefficients, and $a_n=1$; i.e.

$$X^n=a_1X+a_3X_3+\cdots+a_{2k+1}X_{2k+1}+\cdots+X_n \quad Z\ni a_{2k+1}>0,\ a_n=1$$

Proof. By the induction on n. □

Definition. If a representation P of $SL(2, C)$ is a linear combination of $1, X_2, X_4, \cdots$ with non-negative coefficients, P is said to be *even*.

Lemma. *For $m, n\in Z$, $n>0, m>0$,*

$$X^n \text{ is even} \rightleftarrows n \text{ even}$$
$$X^nX_2^m \text{ is even} \rightleftarrows n \text{ even}.$$

Lemma. *For $n, m\in Z$, $n>0, m>0$,*

$$(X^n, 1)>0 \rightleftarrows n \text{ even}$$
$$(X^n X_2^m, 1)>0 \rightleftarrows n \text{ even}.$$

Easy.

Lemma. *For integers* $\nu \geqq 0$, $\mu \geqq 0$,

$$(X_\nu \cdot X_\mu; 1)=\begin{cases}1 & \nu=\mu \\ 0 & \nu \neq \mu.\end{cases}$$

§ 3. The character-algebra of (a product of) $SL_2(C)$

For each atom $\alpha \in S$, we prepare a Lie group $G_\alpha \cong SL_2(C)$. The product $G_S=\prod_{\alpha \in S} G_\alpha$, is isomorphic to the complex form $G(C)$ of G. By this isomorphism, G is considered as a subgroup of G_S. i.e.

$$G \xrightarrow{\iota} G_S$$

For an atom $\alpha \in S$, the projection to the α-th component G_α of $G_S=\prod G_\alpha$ is a 2 dimensional representation of G_S. The class of this representation, we denote that by ρ_α or α. The corresponding representation space, we denote by W_α or by α. For a molecule $X=\langle \alpha, \beta, \cdots, \omega \rangle$ the representation (class) $\alpha \otimes \beta \otimes \cdots \otimes \omega$ of G_S is denoted by ρ_X or by X. The corresponding representation space, is denoted by W_X or by X. For a polymer $P=\sum_{i=1} X_i$ the representation class $X_1 \oplus \cdots \oplus X_k$ is denoted by ρ_P or P. The corresponding representation space we denote by W_P or P. Finally symbol $\oplus$ is often abbreviated as $+$; and symbol $\otimes$ is often abbreviated as $\cdot$. For each non-negative integer ν, and for each atom $\alpha \in S$, we put $\alpha_\nu = X_\nu \circ \alpha$: $G \xrightarrow{\alpha} G_\alpha = SL_2(2, C) \xrightarrow{X_\nu} SL(\nu+1, C)$. In particular, $\alpha_1=\alpha$, $\alpha_0=1$ for all α. Then $1, \alpha_1, \alpha_2, \alpha_3, \cdots, \beta_1, \beta_2, \beta_3, \cdots, \gamma_1, \gamma_2, \gamma_3, \cdots$ are in the commutative ring $\mathscr{R}(S)$ of the all virtual representation-classes of G_S; $\mathscr{R}(S)$ is generated by them, in fact, they are spanned by the product: $\alpha_{\nu_\alpha}\beta_{\nu_\beta}\gamma_{\nu_\gamma}\cdots$ of them: i.e.

$$\mathscr{R}(S) \ni P; \; P=\sum_{(\nu)} a_{\nu_\alpha \nu_\beta \cdots \nu_\omega} \alpha_{\nu_\alpha} \beta_{\nu_\beta} \cdots \omega_{\nu_\omega} \qquad a_{(\nu)} \in Z$$

This linear expression is unique, and $(P, 1)=a_{0,\ldots,0}$ is obviously the multiplicty of 1 in P.

The function:

$$P \rightsquigarrow (P, 1)$$

is an additive homomorphism of $\mathscr{R}(S)$ to Z, but unfortunately, it is not multiplicative, but we have

Lemma. *If $P(\alpha, \beta, \cdots, \omega)$, $Q(\xi, \eta, \cdots, \zeta)$ has no letter in common, then*

$$(P(\alpha, \cdots, \omega) Q(\xi, \cdots, \zeta), 1) = (P(\alpha, \cdots, \omega), 1)(Q(\xi, \cdots, \zeta), 1).$$

Easy.

The multiplicity $(P, 1)$ is also denoted by

$$(P, 1) = \int \cdots \int P(\alpha, \cdots, \omega) d\alpha \cdots d\omega.$$

If we take the maximal compact subgroup $G_u = SU(2)^m$ of $G = SL(2, C)^m$ and if we write the Haar measure of G_u as $d\alpha \cdots d\omega$, normalized with $\int \cdots \int_{G_u} d\alpha \cdots d\omega = 1$, the above symbol of the multiplicity acquaires reality. I.e., identifying the representation P with its character functions $\operatorname{tr} P$,

$$(P, 1) = \int \cdots \int P(\alpha, \cdots, \omega) d\alpha \cdots d\omega$$

is just the orthogonality relation via the unitary trick.

If P involves variables $\alpha, \beta, \cdots, \omega; \xi, \cdots, \eta$, obviously we have

Lemma.

$$\int \cdots \int \left(\int \cdots \int P(\alpha, \beta \cdots \omega, \xi \cdots \eta) d\alpha \cdots d\omega \right) d\xi \cdots d\eta$$
$$= \int \cdots \int \left(\int \cdots \int P(\alpha, \beta \cdots \omega, \xi \cdots \eta) d\xi \cdots d\eta \right) d\alpha \cdots d\omega$$
$$= \int \cdots \int P(\alpha \cdots \eta) d\alpha \cdots d\eta.$$

This is because $\operatorname{tr}(P \otimes Q) = \operatorname{tr} P \cdot \operatorname{tr} Q$, $\operatorname{tr}(P+Q) = \operatorname{tr} P + \operatorname{tr} Q$ and Fubini's theorem.

§ 4. Chemistry and abelian scheme

The fact that a G.T. abelian scheme $A \xrightarrow{\pi} V = \Gamma \backslash \mathfrak{H}^{|S_0|}$ over V, corresponds to a polymer P implies the fact that the action of $\Gamma = \pi_1(V, \lambda)$, ($\lambda$ is a generic point of V) on the cohomology group $H^{2r}(A_\lambda, Q) = \Lambda^{2r}({}^tF)$, (${}^tF = H_0^1(A_\lambda, Q)$), is extendable to a Q-representation $\rho_A^{(r)}$ of G, which is C-isomorphic to $\Lambda^r(\rho_P)$.

Now, therefore, if

$$P = \sum_{i=0}^{k} X_i \qquad X_i = \{\alpha_i, \beta_i, \cdots, \omega_i\}$$

then ${}^tF\underset{Q}{\otimes}C$ has a unique subspaces $F_{X_1}, F_{X_2}, \cdots, F_{X_k}$ such that

$$ {}^tF\otimes C=\bigoplus_{i=0}^{k} F_{X_i} \qquad F_{X_i}\cong W_{X_i}\ (=X_i) $$

Therefore:

$$ \begin{aligned} \wedge^r({}^tF)\otimes C &= \wedge^r({}^tF\otimes C)=\wedge^r(F_{X_i}\oplus\cdots\oplus F_{X_k}) \\ &= \bigoplus_{r_1+r_2+\cdots+r_k=r} \wedge^{r_1}(F_{X_1})\otimes\wedge^{r_2}(F_{X_2})\otimes\cdots\otimes\wedge^{r_k}(F_{X_k}) \end{aligned} $$

Now if $X=\{\alpha, \beta, \cdots, \omega\}$, there exist two-dimensional C-linear G_S-spaces $F_{X,\alpha}, F_{X,\beta}, \cdots, F_{X,\omega}$ such that

$$ F_X=F_{X,\alpha}\otimes F_{X,\beta}\otimes\cdots\otimes F_{X,\omega}; \quad F_{X,\alpha}\cong W_\alpha, \cdots, F_{X,\omega}\cong W_\omega $$

as G_S-spaces. Now, in order to compute

$$ \begin{aligned} F(t) &= \sum_{r=0}^{N} b_r t^r=\sum \dim_Q[\wedge^r({}^tF)^G]t^r=\sum \dim_C[\wedge^r({}^tF)^G\otimes C]t^r \\ &= \sum \dim_C[\wedge^r({}^tF\otimes C)^{G_S}]t^r, \end{aligned} $$

we define:

$$ \begin{aligned} G(t) &= \sum_r \dim_C \wedge^r({}^tF\otimes C)t^r \\ &= \sum \dim\wedge^{r_1}(F_{X_1})t^{r_1}\dim\wedge^{r_2}(F_{X_2})t^{r_2}\cdots\dim\wedge^{r_k}(F_{X_k})t^{r_k} \end{aligned} $$

and

$$ H(t)=\bigoplus_{r_1\cdots r_k} \wedge^{r_1}(F_{X_1})t^{r_1}\otimes\cdots\otimes\wedge^{r_k}(F_{X_k})t^k=f(F_{X_1})\otimes\cdots\otimes f(F_{X_k}) $$

where $f(F_X)=\sum_{r=0}^{\dim F_X}\wedge^r(F_X)t^r$ is a formal polynomial of t with the G vector spaces as coefficients.

§5. $|X|=2$

Lemma. *For two linear spaces A, B over a same field k.*

$$ \wedge^2(A\otimes B)\cong[\wedge^2(A)\otimes S^2(B)]\oplus[S^2(A)\otimes\wedge^2(B)] $$

where $S^2(V)$, $\wedge^2(V)$ are the spaces of symmetric and alternating 2-tensors in V.

∵ Well known. □

Putting $X=\{\alpha, \beta\}$, since $F_X\cong W_X=\alpha\beta$, we have

Lemma. $\wedge^0(F_X)\cong \boldsymbol{C}$

$\wedge^1(F_X)\cong F_X\cong \alpha\beta$

$\wedge^2(F_X)\cong \alpha_2+\beta_2$

$\wedge^3(F_X)\cong \alpha\beta$

$\wedge^4(F_X)\cong \boldsymbol{C}$

The first, second, fourth, fifth formulas are obvious. Because of the last lemma,

$$\wedge^2(F_X)\cong \wedge^2(\alpha\beta)=\wedge^2(\alpha\otimes\beta)\cong(\wedge^2(\alpha)\otimes S^2(\beta))\oplus(S^2(\alpha)\otimes\wedge^2(\beta))=\beta_2+\alpha_2.$$

Therefore

$$\begin{aligned} f(F_X)&=\sum_{r=0}^{4}\wedge^r(F_X)t^r\cong \boldsymbol{C}+\alpha\beta t+(\alpha_2+\beta_2)t^2+\alpha\beta t^3+\boldsymbol{C}t^4\\ &=(1+t^4)1+(t+t^3)\alpha\beta+t^2\alpha_2+t^2\beta_2 \end{aligned}$$

where 1 means the trivial vector space $\boldsymbol{C}$ over $\boldsymbol{C}$. We write this by $f(\alpha, \beta)$: i.e.

$$\begin{aligned} f(F_X)=f(\alpha,\beta)&=(1+t^4)1+(t+t^3)\alpha\beta+t^2\alpha_2+t^2\beta_2\\ &=(1,\alpha,\alpha_2)\begin{pmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ \beta\\ \beta_2\end{pmatrix}={}^t\hat{\alpha}P\hat{\beta}, \end{aligned}$$

where

$$\hat{\alpha}=\begin{pmatrix}1\\ \alpha\\ \alpha_2\end{pmatrix}\qquad \hat{\beta}=\begin{pmatrix}1\\ \beta\\ \beta_2\end{pmatrix}\quad \text{etc. and}$$

$$P=\begin{pmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{pmatrix}.$$

For the later purpose we put

$$M(\alpha)=\begin{pmatrix}1\\ \alpha\\ \alpha_2\end{pmatrix}(1,\alpha,\alpha_2)=\begin{pmatrix}1 & \alpha & \alpha_2\\ \alpha & \alpha\alpha & \alpha\alpha_2\\ \alpha_2 & \alpha_2\alpha & \alpha_2\alpha_2\end{pmatrix}.$$

Because of the Lemma we have

$$\int M(\alpha)d\alpha=E \qquad (3\times 3 \text{ identity matrix}).$$

We put for a non-negative integer ν

$$f(\alpha,\beta)^{\nu}=[(1+t^{4})1+(t+t^{3})\alpha\beta+t^{2}\alpha_{2}+t^{2}\beta_{2}]^{\nu}={}^{t}\hat{\alpha}_{\nu}P_{\nu}\hat{\beta}_{\nu}$$

where

$$\hat{\alpha}_{\nu}=\begin{bmatrix}1\\ \alpha\\ \alpha_{2}\\ \vdots\\ \alpha_{2\nu}\end{bmatrix}\qquad \hat{\beta}_{\nu}=\begin{bmatrix}1\\ \beta\\ \vdots\\ \beta_{2\nu}\end{bmatrix}$$

and P_{ν} is $(2\nu+1)\times(2\nu+1)$ matrix, of which entries are polynomials in t, of degree 4ν. Similarly to $M(\alpha)$, we define for non-negative ν

$$M_{\nu}(\alpha)={}^{t}\hat{\alpha}_{\nu}\cdot\hat{\alpha}_{\nu}=\begin{bmatrix}1 & \alpha & \alpha_{2}\cdots\cdots & \alpha_{2\nu}\\ \alpha & & & \\ \vdots & & \alpha_{1}\alpha_{j} & \\ \vdots & & & \ddots \\ \alpha_{2\nu} & & & \alpha_{2\nu}\alpha_{2\nu}\end{bmatrix}.$$

This is a $(2\nu+1)\times(2\nu+1)$ matrix, and

$$\int M_{\nu}(\alpha)d\alpha=E\qquad ((2\nu+1)\times(2\nu+1)\text{ unit matrix}).$$

Also we define $P_{\nu,0}$, the $(\nu+1)\times(\nu+1)$ matrix which consists of even-th rows and even-th columns of P_{ν}, and define $P_{\nu,1}$ the $\nu\times\nu$ matrix which consists of odd-th rows and odd-th columns of P_{ν}.

Then

$$P_{\nu}\sim P_{\nu,0}+P_{\nu,1}$$

and

$$P_{\nu}^{n}\sim P_{\nu,0}^{n}+P_{\nu,1}^{n}\qquad \operatorname{tr}P_{\nu}^{n}=\operatorname{tr}P_{\nu,0}^{n}+\operatorname{tr}P_{\nu,1}^{n}.$$

For small $\mu'S$, P_{μ}, $P_{\mu,0}$, $P_{\mu,1}$ are

$$P_{1}=P=\begin{pmatrix}1+t^{4} & 0 & t^{2}\\ 0 & t+t^{3} & 0\\ t^{2} & 0 & 0\end{pmatrix}$$

$$P_{1,0}=\begin{pmatrix}1+t^{4} & t^{4}\\ t^{2} & 0\end{pmatrix},\quad P_{1,1}=(t+t^{3})$$

$$P_2=\begin{bmatrix} 1+t^2+6t^4+t^6+t^8 & 0 & 3t^2+3t^4+3t^6 & 0 & t^4 \\ 0 & 2t+6t^3+6t^5+2t^7 & 0 & 2t^3+2t^5 & 0 \\ 3t^2+3t^4+3t^6 & 0 & t^2+4t^4+t^6 & 0 & 0 \\ 0 & 2t^3+2t^5 & 0 & 0 & 0 \\ t^4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{2,0}=\begin{pmatrix} 1+t^2+6t^4+t^6+t^8 & 3t^2+3t^4+3t^6 & t^4 \\ 3t^2+3t^4+3t^6 & t^2+4t^4+t^6 & 0 \\ t^4 & 0 & 0 \end{pmatrix}$$

$$P_{2,1}=\begin{pmatrix} 2t+6t^3+6t^5+2t^7 & 2t^3+2t^5 \\ 2t^3+2t^5 & 0 \end{pmatrix}.$$

§ 6. Cyclic case, $\mu=1$

Let $\mathscr{G}=Z_n=\{0, 1, \cdots, n-1\}$, $P=\sum gX$, $X=\langle 0, 1\rangle$. Put $0=\alpha$, $1=\beta$, $2=\gamma$, $3=\delta$, $\cdots$, $n-2=\eta$, $n-1=\omega$, $X_i=\langle i, i+1\rangle$, $n=0$: then

$$\begin{aligned} P&=\sum_g gX=X_1+\cdots+X_{n-1}=\{0, 1\}+\{1, 2\}+\{2, 3\}+\cdots+\{n-1, 0\} \\ &=\{\alpha, \beta\}+\{\beta, \gamma\}+\cdots+\{\eta, \omega\}+\{\omega, \alpha\} \\ H(t)&=f(F_{X_1})f(F_{X_2})\cdots f(F_{X_{n-1}})=f(\alpha, \beta)f(\beta, \gamma)f(\gamma, \delta)\cdots f(\eta, \omega)f(\omega, \alpha) \\ &={}^t\hat{\alpha}P\hat{\beta}{}^t\hat{\beta}P\hat{\gamma}{}^t\hat{\gamma}P\hat{\delta}\cdots P\hat{\omega}{}^t\hat{\omega}P=\mathrm{tr}\,({}^t\hat{\alpha}P\hat{\beta}{}^t\hat{\beta}P\cdots P\hat{\omega}{}^t\hat{\omega}P\hat{\alpha}) \\ &=\mathrm{tr}(P\hat{\beta}{}^t\hat{\beta}P\hat{\gamma}{}^t\hat{\gamma}\cdots P\hat{\omega}{}^t\hat{\omega}P\hat{\alpha}{}^t\hat{\alpha}). \end{aligned}$$

Now put $\hat{\beta}{}^t\hat{\beta}=M(\beta)$, $\cdots$, $\hat{\alpha}{}^t\hat{\alpha}=M(\alpha)$, then

$$M(\alpha)=\hat{\alpha}{}^t\hat{\alpha}=\begin{pmatrix} 1 \\ \alpha \\ \alpha_2 \end{pmatrix}(1, \alpha, \alpha_2)=\begin{pmatrix} 1 & \alpha & \alpha_2 \\ \alpha & \alpha\alpha & \alpha\alpha_2 \\ \alpha_2 & \alpha_2\alpha & \alpha_2\alpha_2 \end{pmatrix}$$

and

$$H(t)=\mathrm{tr}\,(PM(\beta)PM(\gamma)\cdots PM(\omega)PM(\alpha)),$$

therefore

$$\begin{aligned} F(t)&=(H(t), 1)=\int\cdots\int H(t)d\alpha\cdots d\omega \\ &=\mathrm{tr}\Big(P\Big(\int M(\beta)d\beta\Big)P\int M(\gamma)d\gamma\cdots P\int M(\omega)d\omega P\int M(\alpha)d\alpha\Big) \\ &=\mathrm{tr}\,(PE\;PE\cdots PE\;PE)=\mathrm{tr}\,(P^n). \end{aligned}$$

Now since:
$$P \sim \begin{pmatrix} 1+t^4 & t^2 \\ t^2 & 0 \end{pmatrix} \oplus (t+t^3),$$

the eigen polynomial of $P = \{Z^2-(1+t^4)Z-t^4\}\{Z-(t+t^3)\}$. Eigenvalues of P are

$$\lambda = \frac{(1+t^4)+\sqrt{1+6t^4+t^8}}{2}, \qquad \mu = \frac{(1+t^4)-\sqrt{1+6t^4+t^8}}{2}, \qquad \rho = t+t^3.$$

Therefore for even $n=2m$,

$$\begin{aligned} \operatorname{tr}(P^n) &= \lambda^n+\mu^n+\rho^n \\ &= \left(\frac{(1+t^4)+\sqrt{1+6t^4+t^8}}{2}\right)^n + \left(\frac{(1+t^4)-\sqrt{1+6t^4+t^8}}{2}\right)^n + (t+t^3)^n \\ &= \frac{2}{2^n}\left[\sum_{k=0}^{m}\binom{n}{2k}(1+t^4)^{2k}(1+6t^4+t^8)^{m-k}\right] + (t+t^3)^n. \end{aligned}$$

It is also determinable recursively by:

$$f^{(n)}+\sigma_1 f^{(n-1)}+\sigma_2 f^{(n-2)}+\sigma_3 f^{(n-3)}=0 \quad (n=3, 4, \cdots).$$

from $f^{(2)}, f^{(1)}, f^{(0)}$, where

$$\begin{aligned} f^{(n)} &= \operatorname{tr}(P^n) \in Z[t] \\ f^{(2)}(t) &= 1+t^2+10t^4+t^5+t^8 \\ f^{(1)}(t) &= 1+t^4 \\ f^{(0)}(t) &= 3 \end{aligned}$$

where

$$\sigma_i = \sigma_i(t) \in Z[t] \qquad (i=1, 2, 3)$$

are coefficients of the characteristic polynomial

$$\begin{aligned} Z^3+\sigma_1(t)Z^2+\sigma_2(t)Z+\sigma_3(t) &= \det(ZI_3-P) \\ &= Z^3-(1+t+t^3+t^4)Z^4+(t+t^3-t^4+t^5+t^7)Z+(t^5+t^7). \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_1(t) &= -1-t-t^3-t^4, \\ \sigma_2(t) &= t+t^3-t^4+t^5+t^7, \\ \sigma_3(t) &= t^5+t^7. \end{aligned}$$

For example:

$$\begin{aligned} f^{(3)}(t) &= -\sigma_1(t)f^{(2)}(t)-\sigma_2(t)f^{(1)}(t)-\sigma_3(t)f^{(0)}(t) \\ &=(1+t+t^3+t^4)(1+t^2+10t^4+t^6+t^8) \\ &\quad -(t+t^3-t^4+t^5+t^7)(1+t^4)-(t^5+t^7)3 \\ &=1+t^2+t^3+12t^4+6t^5+2t^6+6t^7+12t^8+t^9+t^{10}+t^{12} \\ f^{(4)}(t) &= -\sigma_1\cdot f^{(3)}-\sigma_2\cdot f^{(2)}-\sigma_3\cdot f^{(1)} \\ &=(1+t+t^3+t^4)(1+t^2+t^3+\cdots+t^{12}) \\ &\quad -(t+t^3-t^4+t^5+t^7)(1+t^2+10t^4+t^6+t^8)-(t^5+t^7)(1+t^4) \\ &=1+t^2+t^3+15t^4+6t^5+11t^6+7t^7+41t^8+7t^9+11t^{10} \\ &\quad +6t^{11}+15t^{12}+t^{13}+t^{14}+t^{16}. \end{aligned}$$

§ 7. Cyclic case, $\mu=2$ or $\mu\geqq 2$

If $\mathscr{G}=Z_n=\{0, 1, \cdots, n-1\}$, $X=\{0, 1\}$, $P=2\sum_g gX=2\{0, 1\}+2\{1, 2\}+\cdots+2\{n-1, 0\}$, it is treatable as if $\mu=1$ if we count the same molecule twice in P i.e.; put $X_0=X_1=\{0, 1\}$, $X_2=X_3=\{1, 2\}$, $X_4=X_5=\{2, 3\}$, $\cdots$, $X_{2n-2}=X_{2n-1}=\{n-1, 0\}$

$$P=X_0+X_1+X_2+\cdots+X_{2n-2}+X_{2n-1}.$$

Therefore: by putting $\alpha=0$, $\beta=1$, $\cdots$, $\omega=n-1$,

$$\begin{aligned} H(t) &= f(\alpha, \beta)f(\alpha, \beta)f(\beta, \gamma)f(\beta, \gamma)f(\gamma, \delta)f(\gamma, \delta) \\ &\qquad \cdots f(\eta, \omega)f(\eta, \omega)f(\omega, \alpha)f(\omega, \alpha) \\ &= f(\alpha, \beta)^2 f(\beta, \gamma)^2 f(\gamma, \delta)^2\cdots f(\eta, \omega)^2 f(\omega, \alpha)^2 \\ &= {}^t\hat{\alpha}_2P_2\hat{\beta}_2{}^t\hat{\beta}_2P_2\hat{\gamma}_2{}^t\hat{\gamma}_2P_2\hat{\delta}_2\cdots{}^t\hat{\eta}_2P_2\hat{\omega}_2{}^t\hat{\omega}_2P_2\hat{\alpha}_2 \\ &= {}^t\hat{\alpha}_2P_2M_2(\beta)P_2M_2(\gamma)P_2M_2(\delta)\cdots P_2M_2(\omega)P_2\hat{\alpha}_2 \\ &= \mathrm{tr}(P_2M_2(\beta)P_2M_2(\gamma)\cdots P_2M_2(\omega)P_2M_2(\alpha)) \end{aligned}$$

Therefore:

$$\begin{aligned} F(t) &= (H(t), 1) \\ &= \int\cdots\int \mathrm{tr}(P_2M_2(\beta)P_2M_2(\gamma)\cdots P_2M_2(\alpha)d\alpha d\beta\cdots d\omega \\ &= \mathrm{tr}\Big(P_2\Big(\int M_2(\beta)d\beta\Big)P_2\Big(\int M(\gamma)d\gamma\Big)\cdots P_2\Big(\int M(\alpha)d\alpha\Big)\Big) \\ &= \mathrm{tr}(P_2EP_2E\cdots P_2E)=\mathrm{tr}(P_2^n) \end{aligned}$$

Similarly for general μ;

$$F(t)=\mathrm{tr}(P_\mu^n).$$

Therefore, they are determinable recursively by

$$f_\mu^{(n)}+\sigma_1 f_\mu^{(n-1)}+\sigma_2 f_\mu^{(n-2)}+\cdots+\sigma_m f_\mu^{(n-m)}=0$$

for $n\geqq m$ from $f_\mu^{(0)},\cdots,f_\mu^{(m-1)}$ where

$$f_\mu^{(n)}=f_\mu^{(n)}(t)=\mathrm{tr}(P_\mu^n)\in Z[t]$$
$$f_\mu^{(0)}=m\in Z$$

and

$$\sigma_1=\sigma_1(t),\ \cdots,\ \sigma_m=\sigma_m(t)\in Z[t]$$

are coefficients of the characteristic polynomial

$$Z^m+\sigma_1(t)Z^{m-1}+\sigma_2(t)Z^{m-2}+\cdots+\sigma_m(t)$$

of the matrix P_μ, where $m=2\mu+1$.

For small μ and n, the values of $\mathrm{tr}\,P_\mu^n$ are

$$\mathrm{tr}\,P_2^1=1+2t+2t^2+6t^3+10t^4+6t^5+2t^6+2t^7+t^8$$
$$\mathrm{tr}\,P_2^2=1+6t^2+56t^4+126t^6+210t^8+126t^{10}+56t^{12}+6t^{14}+t^{16}$$
$$\begin{aligned}\mathrm{tr}\,P_2^4=&1+4t^2+82t^4+452t^6+2600t^8+8208t^{10}+20574t^{12}+33224t^{14}\\&+40790t^{16}+33224t^{18}+20574t^{20}+8208t^{22}+2600t^{24}+452t^{26}\\&+82t^{28}+4t^{30}+t^{32}.\end{aligned}$$

We write here the characteristic polynomial of P_2:

$$P_2=\begin{pmatrix}1+t^2+6t^4+t^6+t^8 & 0 & 3t^2+3t^4+3t^6 & 0 & t^4\\ 0 & 2t+6t^3+6t^5+2t^7 & 0 & 2t^3+2t^5 & 0\\ 3t^2+3t^4+3t^6 & 0 & t^2+4t^4+t^6 & 0 & 0\\ 0 & 2t^3+2t^5 & 0 & 0 & 0\\ t^4 & 0 & 0 & 0 & 0\end{pmatrix}$$
$$\sim P_{2,0}\oplus P_{2,1}$$

where

$$P_{2,0}=\begin{pmatrix}1+t^2+6t^4+t^6+t^8 & 3t^2+3t^4+3t^6 & t^4\\ 3t^2+3t^4+3t^6 & t^2+4t^4+t^6 & 0\\ t^4 & 0 & 0\end{pmatrix}$$

$$P_{1,0}=\begin{pmatrix} 2+6t^3+6t^5+2t^7 & 2t^3+2t^5 \\ 2t^3+2t^5 & 0 \end{pmatrix}.$$

Therefore the characteristic polynomial of P_2 = the characteristic polynomial of $P_{2,0}\times$ the characteristic polynomial of $P_{2,1}$.

The characteristic polynomial of

$$P_{2,0}=\begin{vmatrix} Z-(1+t^2+6t^4+t^6+t^8) & -(3t^2+3t^4+3t^6) & -t^4 \\ -(3t^2+3t^4+3t^6) & Z-(t^2+4t^4+t^6) & 0 \\ -t^4 & 0 & Z \end{vmatrix}$$

$$\begin{aligned}
&=[Z-(1+t^2+6t^4+t^6+t^8)][Z-(t^2+4t^4+t^6)]Z \\
&\quad -t^8(Z-(t^2+4t^4+t^6))-(3t^2+3t^4+3t^6)^2Z \\
&=Z^3-(1+2t^2+10t^4+2t^6+t^8)Z^2 \\
&\quad +(t^2-4t^4-7t^6-2t^8-7t^{10}-4t^{12}+t^{14})Z+(t^{10}+4t^{12}+t^{14})
\end{aligned}$$

The characteristic polynomial of

$$P_{2,1}=\begin{vmatrix} Z-(2t+6t^3+6t^5+2t^7) & -(2t^3+2t^5) \\ -(2t^3+2t^5) & Z \end{vmatrix}$$

$$=Z^2-(2t+6t^3+6t^5+2t^7)Z-(4t^6+8t^8+4t^{10})$$

Therefore the characteristic polynomial of

$$\begin{aligned}
P_2&=[Z^3-(1+2t^2+10t^4+2t^6+t^8)Z^2 \\
&\quad +(t^2-4t^4-7t^6-2t^8-7t^{10}-4t^{12}+t^{14})Z+(t^{10}+4t^{12}+t^{14})] \\
&\quad \times[Z^2-(2t+6t^3+6t^5+2t^7)Z-(4t^6+8t^8+4t^{10})] \\
&=Z^5-(1+2t+2t^2+6t^3+10t^4+6t^5+2t^6+2t^7+t^8)Z^4 \\
&\quad +(2t+t^2+10t^3-4t^4+38t^5-11t^6+78t^7-10t^8+78t^9-11t^{10} \\
&\quad +38t^{11}-4t^{12}+10t^{13}+t^{14}+2t^{15})Z^3 \\
&\quad +(-2t^3+2t^5+4t^6+32t^7+16t^8+68t^9+61t^{10}+76t^{11}+100t^{12} \\
&\quad +76t^{13}+61t^{14}+68t^{15}+16t^{16}+32t^{17}+4t^{18}+2t^{19}-2t^{21})Z^2 \\
&\quad +(-4t^8+8t^{10}-2t^{11}+56t^{12}-14t^{13}+80t^{14}-32t^{17}+72t^{16} \\
&\quad -32t^{17}+80t^{18}-14t^{19}+56t^{20}-2t^{21}+8t^{22}-4t^{24})Z \\
&\quad -(4t^{16}+24t^{18}+40t^{20}+24t^{22}+4t^{24}).
\end{aligned}$$

Let $\mathscr{G}$ be an arbitral finite group. Assume in this section, that $S=\mathscr{G}$ and the action of $\mathscr{G}$ on $S(=\mathscr{G})$ is the left multiplication. We are still assuming that $|X|=2$, and put

$$X=\{\alpha, \xi\}.$$

Take the unique element $h \in \mathscr{G}$ such that $\xi=h\alpha$. Then it is easy

$$P=\sum gX=\{\alpha, h\alpha\}+\{h\alpha, h^2\alpha\}+\cdots+\{h^{n-1}\alpha, \alpha\}+\{\beta, h_2\beta\}$$
$$+\{h_2\beta, h_2^2\beta\}+\cdots+\{h^{n-1}\beta, \beta\}+\{\gamma, h_3\gamma\}+\cdots$$
$$+\{h_3^{n-1}\gamma, \gamma\}+\cdots+\{\omega, h_k\omega\}+\cdots\{h_k^{n-1}\omega, \omega\}$$

where:

$n=$the order of h in $\mathscr{G}$ (=the smallest positive integer n such that $h^n=1$

$k=|\mathscr{G}/H|$

$g_1=1, g_2, g_3, \cdots, g_k=$representatives of $\mathscr{G}/H$

$\beta=g_2\alpha, \gamma=g_3\alpha, \cdots, \omega=g_k\alpha, h_2=g_2hg_2^{-1}, h_3=g_3hg_3^{-1}, \cdots, h_k=g_khg_k^{-1}$

$H=\{1, h, h^2, \cdots, h^{n-1}\}=$the cyclic subgroup generated by h

And, in this case, by the same calculation as before,

$$F(t)=\sum_{j=0}^{2\dim A_X} \dim HH^r(A_\lambda, Q)t^r=(\mathrm{tr}\, P^n)^k.$$

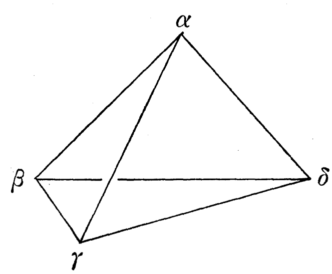
Where $A\xrightarrow{\pi}V$ is an abelian scheme belonging to the polymer $P=\sum gX$ and A_λ is a generic fibre in that. In the same way, if we taken an abelian scheme belonging to $P=\mu\sum g_X$ with higher $\mu>0$, $F(t)$ becomes:

$$F(t)=(\mathrm{tr}(P_\mu^n))^k$$

§ 8.

If $\mathscr{G}\neq S$, the calculation will become difficult, and only for some examples we can calculates the dimension of G-invariant cycles.

I) Edges of a tetrahedron.



$\mathcal{G}$ = tetrahedral group $\cong A_4$

$$P=\{\alpha,\beta\}+\{\alpha,\gamma\}+\{\alpha,\delta\}+\{\beta,\gamma\}+\{\beta,\delta\}+\{\gamma,\delta\}$$
$$F(t)=(f(\alpha,\beta)f(\alpha,\gamma)f(\alpha,\delta)f(\beta,\gamma)f(\beta,\delta)f(\gamma,\delta),\ 1).$$

Lemma. $((a+bX+cX_2)(a'+b'X+c'X_2)(a''+b''X+c''X_2),\ 1)$
$$=aa'a''+(ab'b''+ba'b''+bb'a'')+(ac'c''+ca'c''+cc'a'')$$
$$+(bc'c''+cb'c''+cc'b'')+cc'c''$$

∵ Calculation □

We put this 3-linear form

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}=aa'a''+ab'b''+ba'b''+bb'a''+ac'c''+ca'c''$$
$$+cc'a''+bc'c''+cb'c''+cb'c''+cc'b''+cc'c''.$$

We put furthermore,

Definition.

$$\{A,B,C\}=\mathrm{tr}(PAPAPA)+\mathrm{tr}(PAPBPB)\times 3+\mathrm{tr}(PAPCPC)$$
$$\times 3+\mathrm{tr}(PBPCPC)\times 3+\mathrm{tr}(PCPCPC)$$

for three 3×3 matrices A, B, C.

$$F(t)=\iiiint f(\beta,\gamma)f(\gamma,\delta)f(\delta,\beta)\cdot f(\alpha,\beta)f(\alpha,\gamma)f(\alpha,\delta)\,d\alpha\, d\beta\, d\gamma\, d\delta$$
$$=\iiint \mathrm{tr}(PM(\gamma)PM(\delta)PM(\beta))\Big[\int \{(a+c\beta_2)+b\beta\alpha+c\alpha_2\}$$
$$\times\{(a+c\gamma_2)+b\gamma\alpha+c\alpha_2\}\{(a+c\delta_2)+b\delta\alpha+c\alpha_2\}d\alpha\Big]d\beta\, d\gamma\, d\delta$$
$$=\iiint \mathrm{tr}(PM(\gamma)PM(\delta)PM(\beta)\begin{pmatrix} a+c\beta_2 & b\beta & c \\ a+c\gamma_2 & b\gamma & c \\ a+c\delta_2 & b\delta & c \end{pmatrix}d\beta\, d\gamma\, d\gamma,$$

where $a=1+t^4$
$b=t+t^3$
$c=t^2$

$$=\iiint \mathrm{tr}(PM(\beta)PM(\gamma)PM(\delta))\times[(a+c\beta_2)(a+c\gamma_2)(a+c\delta_2)$$

$$+\{(x+c\beta_2)b\gamma b\delta+b\beta(a+c\gamma_2)b\delta+b\beta b\gamma(a+c\delta_2)\}$$
$$+\{(a+c\beta_2)cc+c(a+c\gamma_2)c+cc(a+c\delta_2)\}$$
$$+\{b\beta cc+cb\gamma c+ccb\delta\}+ccc]d\beta d\gamma d\delta$$
$$=\iiint \mathrm{tr}(P(a1+c\beta_2M(\beta))P(a1+c\gamma_2M(\gamma))P(a1+c\delta_2M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}(P(a1+c\beta_2M\beta))P(\gamma M(\gamma))P(\delta M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}(P(b\beta M(\beta))P(a+c\gamma_2M(\gamma))P(b\delta M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}(P(b\beta M(\beta))P(b\gamma M(\gamma))P(a1+c\delta_2M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(a1+c\beta_2M(\beta))P(cM(\gamma))P(cM(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(cM(\beta))P(a1+c\gamma_2M(\gamma))P(cM(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(cM(\beta))P(cM(\gamma))P(a1+c\delta_2M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(b\beta M(\beta))P(cM(\gamma))P(cM(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(cM(\beta))P(b\gamma M(\gamma))P(cM(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(cM(\beta))P(cM(\gamma))P(b\delta M(\delta))d\beta d\gamma d\delta$$
$$+\iiint \mathrm{tr}\,P(cM(\beta))P(cM(\gamma))P(cM(\delta))d\beta d\gamma d\delta.$$

∴ Defining matrices A, B, C, by

$$\int (a1+cX_2M(X))dX=A$$
$$\int bXM(X)dX=B$$
$$\int cM(X)dX=C,$$

we have

$$F(t)=\mathrm{tr}(PAPAPA)+\mathrm{tr}(PAPBPB)\times 3+\mathrm{tr}(PAPCPC)\times 3$$
$$+\mathrm{tr}(PBPCPC)\times 3+\mathrm{tr}(PCPCPC)$$
$$=\{ABC\}$$

In order to compute these matrices A, B, C, we put:

Lemma.

$$\int XM(X)dX=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}=K$$

$$\int X_2M(X)dX=\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}=J$$

then obviously

$$A=a1+cJ=\begin{pmatrix}a&0&c\\0&a+c&0\\c&0&a+c\end{pmatrix}$$

$$B=bk=\begin{pmatrix}0&b&0\\b&0&b\\0&b&0\end{pmatrix}$$

$$C=c1=\begin{pmatrix}c&0&0\\0&c&0\\0&0&c\end{pmatrix},$$

where $a=1+t^4$, $b=t+t^3$, $c=t^2$.

Now

$$\operatorname{tr}(PAPAPA)=\operatorname{tr}(((1+t^4)P+t^2PJ)^3),\ (A=(1+t^4)I_3+t^2J)$$

$$=\operatorname{tr}\left(\left((1+t^4)\begin{pmatrix}1+t^4&0&t^2\\0&t+t^3&0\\t^2&0&0\end{pmatrix}+t^2\begin{pmatrix}1+t^4&0&t^2\\0&t+t^3&0\\t^2&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}\right)^3\right)$$

$$=\operatorname{tr}\left|\left(\begin{pmatrix}1+3t^2+t^4&2t^2+t^4+2t^6\\t^2+t^6&t^4\end{pmatrix}\oplus(t+2t^3+2t^5+t^7)\right)^3\right|$$

$$=\operatorname{tr}\begin{bmatrix}\begin{matrix}1+13t^4+2t^6+52t^8\\+9t^{10}+81t^{12}+9t^{14}\\+52t^{16}+2t^{18}+13t^{20}\\+t^{24}\end{matrix}&\begin{matrix}2t^2+t^4+20t^6+11t^8\\+57t^{10}+23t^{12}+57t^{14}\\+11t^{16}+20t^{18}+t^{20}\\+2t^{22}\end{matrix}\\ \begin{matrix}t^2+10t^6+t^8+28t^{10}\\+2t^{12}+28t^{14}+t^{16}\\+10t^{18}+t^{22}\end{matrix}&\begin{matrix}2t^4+t^6+14t^8+6t^{10}\\+25t^{12}+6t^{14}+14t^{16}\\+t^{18}+2t^{24}\end{matrix}\end{bmatrix}+(t+2t^3+2t^5+t^7)^3$$

$$=1+t^3+15t^4+6t^5+3t^6+18t^7+66t^8+35t^9+15t^{10}+48t^{11}+106t^{12}$$
$$+48t^{13}+15t^{14}+35t^{15}+66t^{16}+18t^{17}+3t^{18}+6t^{19}+15t^{20}+t^{21}+t^{24}.$$

On the other hand

Lemma.

$$PK=\begin{pmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}=\begin{pmatrix}0 & 1+t^2+t^4 & 0\\ t+t^3 & 0 & t+t^3\\ 0 & t^2 & 0\end{pmatrix}$$

$$PJ=\begin{pmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}=\begin{pmatrix}t^2 & 0 & 1+t^2+t^4\\ 0 & t+t^3 & 0\\ 0 & 0 & t^2\end{pmatrix}$$

$\because$ Calculation.

Now, since $A=aI+bJ$, $B=bK$,

$$\begin{aligned}\mathrm{tr}(PAPBPB)&=a\,\mathrm{tr}(PPBPB)+c\,\mathrm{tr}(PJPBPB)\\&=ab^2\mathrm{tr}(PPKPK)+cb^2\mathrm{tr}(PJPKPK)\\&=ab^2\mathrm{tr}(PKPKP)+cb^2\mathrm{tr}(JPKPKP)\\&=\mathrm{tr}((ab^2I_3+cb^2J)(PKPKP))\end{aligned}$$

and

$$\begin{aligned}ab^2I_3+cb^2J&=(1+t^4)(t+t^3)^2I_3+t^2(t+t^3)^2J\\&=\begin{bmatrix}(1+t^4)(t+t^3)^2 & 0 & t^2(t+t^3)^2\\ 0 & (1+t^2+t^4)(t+t^3)^2 & 0\\ t^2(t+t^3)^2 & 0 & (1+t^2+t^4)(t+t^3)^2\end{bmatrix}\\&=\begin{bmatrix}t^2+2t^4+2t^6+2t^8+t^{10} & 0 & t^4+2t^6+t^8\\ 0 & t^2+3t^4+4t^6+3t^8+t^{10} & 0\\ t^4+2t^6+t^8 & 0 & t^2+3t^4+4t^6+3t^8+t^{10}\end{bmatrix}=Q\end{aligned}$$

and

$$\begin{aligned}PKPKP&=\begin{bmatrix}0 & 1+t^2+t^4 & 0\\ t+t^3 & 0 & t+t^3\\ 0 & t^2 & 0\end{bmatrix}\begin{bmatrix}0 & 1+t^2+t^4 & 0\\ t+t^3 & 0 & t+t^3\\ 0 & t^2 & 0\end{bmatrix}\begin{bmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{bmatrix}\\&=\begin{bmatrix}(1+t^2+t^4)\times(t+t^3) & 0 & (1+t^2+t^4)\times(t+t^3)\\ 0 & (t+t^3)(1+t^2+t^4)+(t+t^3)t^2 & 0\\ t^2(t+t^3) & 0 & t^2(t+t^3)\end{bmatrix}\begin{bmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{bmatrix}\end{aligned}$$

$$=\begin{pmatrix} t+3t^3+5t^5+5t^7 \\ +3t^9+t^{11} & 0 & t^3+2t^5+2t^7+t^9 \\ 0 & t^2+4t^4+6t^6+4t^8 \\ & +t^{11} & 0 \\ t^3+2t^5+2t^7+t^9 & 0 & t^5+t^7 \end{pmatrix}$$

hence

$$Q(PKPKP)=\begin{pmatrix} t^2+2t^4+2t^6+2t^8 \\ +t^{10} & 0 & t^4+2t^6+t^8 \\ 0 & t^2+3t^4+4t^6+3t^8 \\ & +t^{10} & 0 \\ t^4+2t^6+t^8 & 0 & t^2+3t^4+4t^6+3t^8 \\ & & +t^{10} \end{pmatrix}$$

$$\times\begin{pmatrix} t+3t^2+5t^5+5t^7 \\ +3t^9+t^{11} & 0 & t^3+2t^5+2t^7+t^9 \\ 0 & t^2+4t^4+6t^6+4t^8 \\ & +t^{10} & 0 \\ t^3+2t^5+2t^7+t^9 & 0 & t^5+t^7 \end{pmatrix}$$

$$=\begin{pmatrix} t^3+5t^5+14t^7+27t^9 & & t^5+4t^7+9t^9+14t^{11} \\ +37t^{11}+37t^{13}+27t^{15} & 0 & +14t^{13}+9t^{15}+4t^{17} \\ +14t^{17}+5t^{19}+21t^{21} & & +t^{19} \\ & t^4+7t^6+22t^8 & \\ 0 & +41t^{10}+50t^{12} & 0 \\ & +41t^{14}+22t^{16} & \\ & +7t^{18}+t^{20} & \\ 2t^5+10t^7+24t^9 & & 2t^7+8t^9+14t^{11} \\ +36t^{11}+36t^{13}+24t^{15} & 0 & +14t^{13}+8t^{15}+2t^{17} \\ +10t^{17}+2t^{19} & & \end{pmatrix}$$

Therefore

$$\begin{aligned}\mathrm{tr}(QPKPKP)=t^3&+t^4+5t^5+7t^6+16t^7+22t^8+35t^9+41t^{10}+51t^{11}\\&+50t^{12}+51t^{13}+41t^{14}+35t^{15}+22t^{16}+16t^{17}+7t^{18}\\&+5t^{19}+t^{20}+t^{21}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{tr}(PAPBPB)\times 3=\mathrm{tr}(QPKPKP)\times 3=3t^3+3t^4+15t^5+21t^6+48t^7\\+66t^8+105t^9+123t^{10}+153t^{11}+150t^{12}+153t^{13}\cdots.\end{aligned}$$

Lastly

$$\begin{aligned}&\mathrm{tr}(PAPCPC)\times 3+\mathrm{tr}(PBPCPC)\times 3+\mathrm{tr}(PCPCPC)\\&\quad=\mathrm{tr}(3APCPCP)+\mathrm{tr}(3BPCPCP)+\mathrm{tr}(PCPCPC)\\&\quad=\mathrm{tr}(3t^4AP^3)+\mathrm{tr}(3t^4BP^3)+\mathrm{tr}(t^6P^3)\\&\quad=\mathrm{tr}((3t^4A+3t^4D+t^6I_3)P^3).\end{aligned}$$

Now we put

$$3t^4A+3t^3B+t^6I_3=R,$$

then our 3-ple sum is equal to $\mathrm{tr}(RP^3)$, and since

$$R=\begin{bmatrix} 3t^4+t^6+3t^8 & 3t^5+3t^7 & 3t^6 \\ 3t^5+3t^7 & 3t^5+4t^6+3t^8 & 3t^5+3t^7 \\ 3t^6 & 3t^5+3t^7 & 3t^4+4t^6+3t^8 \end{bmatrix}$$

$$P^3=\begin{bmatrix} 1+t^4 & 0 & t^2 \\ 0 & t+t^3 & 0 \\ t^2 & 0 & 0 \end{bmatrix}^3$$

$$=\begin{bmatrix} 1+5t^4+5t^8+t^{12} & 0 & t^2+3t^6+t^{10} \\ 0 & t^3+3t^5+3t^7+t^9 & 0 \\ t^2+3t^6+t^{10} & 0 & t^4+t^8 \end{bmatrix},$$

$\therefore$ We have

$$\mathrm{tr}(RP^3)=\mathrm{tr}\begin{bmatrix} 3t^4+t^6+3t^8 & 3t^5+3t^7 & 3t^6 \\ 3t^5+3t^7 & 3t^4+4t^6+3t^8 & 3t^5+3t^7 \\ 3t^6 & 3t^5+3t^7 & 3t^4+4t^6+3t^8 \end{bmatrix}$$

$$\times\begin{bmatrix} 1+5t^4+5t^8+t^{12} & 0 & t^2+3t^6+t^{10} \\ 0 & t^3+3t^5+3t^7+t^9 & 0 \\ t^2+3t^6+t^{10} & 0 & t^4+t^8 \end{bmatrix}$$

$$=\mathrm{tr}\begin{bmatrix} \begin{matrix}3t^4+t^6+21t^8+5t^{10}\\+39t^{12}+5t^{14}+21t^{16}\\+t^{18}+3t^{20}\end{matrix} & \begin{matrix}3t^6+12t^{10}+18t^{12}\\+12t^{14}+3t^{16}\end{matrix} & \begin{matrix}3t^6+t^8+15t^{10}\\+3t^{12}+15t^{14}\\+t^{16}+3t^{18}\end{matrix} \\ ? & \begin{matrix}3t^7+13t^9+24t^{11}\\+24t^{13}+13t^{15}\\+3t^{17}\end{matrix} & ? \\ ? & ? & \begin{matrix}6t^8+4t^{10}+15t^{12}\\+4t^{14}+6t^{16}\end{matrix} \end{bmatrix},$$

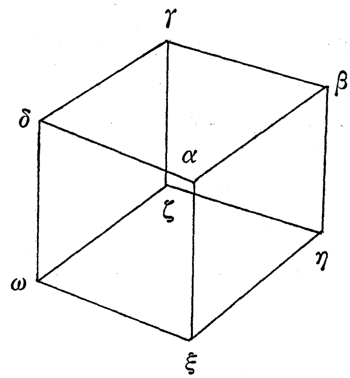
$$=3t^4+t^6+3t^7+27t^8+13t^9+9t^{10}+24t^{11}+54t^{12}+24t^{13}+9t^{14}$$
$$+13t^{15}+27t^{16}+3t^{17}+t^{18}+3t^{20}.$$

Therefore

$$\mathrm{tr}(PA)^3+\mathrm{tr}(PAPBPB)\times 3+\mathrm{tr}(RP^3)=(1+4t^3+21t^4+21t^5+25t^6$$
$$+69t^7+159t^8+153t^9+147t^{10}+225t^{11}+310t^{12}+225t^{13}+147t^{14}$$
$$+153t^{15}+159t^{16}+69t^{17}+25t^{18}+21t^{19}+21t^{20}+4t^{21}+t^{24}).$$

This is the $F(t)$. This polymer is not rigid for any S.

II) Hexahedron



$\mathscr{G}=$ hexahedral group $\cong S_4$

$$P=\{\alpha, \beta\}+\{\beta, \gamma\}+\{\gamma, \delta\}+\{\delta, \alpha\}+\{\alpha, \xi\}+\{\beta, \eta\}+\{\gamma, \zeta\}+\{\delta, \omega\} +\{\xi, \eta\}+\{\eta, \zeta\}+\{\zeta, \omega\}+\{\omega, \xi\}$$

$$\begin{aligned} F(t)=&(f(\alpha, \beta)f(\beta, \gamma)f(\gamma, \delta)f(\delta, \alpha)f(\alpha, \xi)f(\beta, \eta)f(\gamma, \zeta)f(\delta, \omega)f(\xi, \eta) \\ &\times f(\eta, \zeta)f(\zeta, \omega)f(\omega, \xi), 1) \\ =&([\mathrm{tr}(PM(\alpha)PM(\beta)PM(\gamma)PM(\delta))][\mathrm{tr}(PM(\xi)PM(\eta)PM(\zeta)PM(\omega))] \\ &\times f(\alpha, \xi)f(\beta, \eta)f(\gamma, \zeta)f(\delta, \omega), 1) \end{aligned}$$

Lemma.

$$\begin{aligned} \mathrm{tr}(ACB)\mathrm{tr}(A'C'B')&=\mathrm{tr}[(ACB)\otimes(A'C'B')] \\ &=\mathrm{tr}[(A\otimes A')(C\otimes C')(B\otimes B')]. \end{aligned}$$

∵ Obvious □

Therefore

$$\begin{aligned} F(t)=&(\mathrm{tr}(P^{(2)}M(\alpha, \xi)P^{(2)}M(\beta, \eta)P^{(2)}M(\gamma, \zeta)P^{(2)}M(\delta, \omega)) \\ &\times f(\alpha, \xi)f(\beta, \eta)f(\gamma, \zeta)f(\delta, \omega), 1) \end{aligned}$$

where:

$$P^{(2)}=P\otimes P$$
$$M(X, Y)=M(X)\otimes M(Y).$$

Now

Lemma.

$$\iint AM(X, Y)Bf(X, Y)dXdY=A[a1\otimes 1+bK\otimes K+c(J\otimes 1+1\otimes J)]B$$

where $a=1+t^4$, $b=t+t^3$, $c=t^2$ *and* A, B *are any* 1×9 *or* 9×1 *constant matrices and*

$$M(X, Y)=M(X)\otimes M(Y)=\begin{pmatrix} 1 & X & X_2 \\ X & X^2 & XX_2 \\ X_2 & X_2X & X_2^2 \end{pmatrix}\otimes\begin{pmatrix} 1 & Y & Y_2 \\ Y & Y^2 & YY^2 \\ Y_2 & Y_2Y & Y_2^2 \end{pmatrix}.$$

$\because$ Since $f(X, Y)=a+bXY+c(X_2+Y_2)$,

$$\int M(X)dX=1_3$$

$$\int XM(X)dX=K=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\int X_2M(X)dX=J=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\iint AM(X, Y)Bf(X, Y)dXdY$$

$$=A\left(\iint (a+bXY+dX_2+Y_2)M(X)\otimes M(Y)dXdY\right)B$$

$$=A\Big[a\int M(X)dX\otimes\int M(Y)dY+b\int XM(X)dX\otimes\int YM(Y)dY$$

$$+c\int X_2M(X)dX\otimes\int M(Y)dY+c\int M(X)dX\otimes\int Y_2M(Y)dY\Big]B$$

$$=A[a1_3\otimes 1_3+bK\otimes K+c(J\otimes 1_3+1_3\otimes J)]B.$$

Q.E.D.

Therefore

$$F(t)=\mathrm{tr}(P^{(2)}QP^{(2)}QP^{(2)}QP^{(2)}Q)$$

where

$$Q=a1\otimes 1+bK\otimes K+c(J\otimes 1+1\otimes J)$$

$$=\begin{bmatrix} 1+t^4 & & & & & & & & \\ & 1+t^4 & & & & & & 0 & \\ & & 1+t^4 & & & & & & \\ & & & 1+t^4 & & & & & \\ & & & & 1+t^4 & & & & \\ & & & & & 1+t^4 & & & \\ & 0 & & & & & 1+t^4 & & \\ & & & & & & & 1+t^4 & \\ & & & & & & & & 1+t^4 \end{bmatrix}$$

$$
+(t+t^3)\left(\begin{array}{ccc|ccc|ccc}
 & & & 0 & 1 & 0 & & & \\
 & & & 1 & 0 & 1 & & & \\
 & & & 0 & 1 & 0 & & & \\
\hline
0 & 1 & 0 & & & & 0 & 1 & 0 \\
1 & 0 & 1 & & & & 1 & 0 & 1 \\
0 & 1 & 0 & & & & 0 & 1 & 0 \\
\hline
 & & & 0 & 1 & 0 & & & \\
 & & & 1 & 0 & 1 & & & \\
 & & & 0 & 1 & 0 & & &
\end{array}\right)
$$

$$
+t^2\left[\left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 1 & & & & & & \\
0 & 1 & 0 & & & & & & \\
1 & 0 & 1 & & & & & & \\
\hline
 & & & 0 & 0 & 1 & & & \\
 & & & 0 & 1 & 0 & & & \\
 & & & 1 & 0 & 1 & & & \\
\hline
 & & & & & & 0 & 0 & 1 \\
 & & & & & & 0 & 1 & 0 \\
 & & & & & & 1 & 0 & 1
\end{array}\right)+\left(\begin{array}{ccc|ccc|ccc}
 & & & & & & 1 & 0 & 0 \\
 & 0 & & & 0 & & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & 1 \\
\hline
 & & & 1 & 0 & 0 & & & \\
 & & & 0 & 1 & 0 & & & \\
 & & & 0 & 0 & 1 & & & \\
\hline
1 & 0 & 0 & & & & 1 & 0 & 0 \\
0 & 1 & 0 & & 0 & & 0 & 1 & 0 \\
0 & 0 & 1 & & & & 0 & 0 & 1
\end{array}\right)\right]
$$

$$
=\left(\begin{array}{ccc|ccc|ccc}
1+t^4 & 0 & t^2 & 0 & t+t^3 & 0 & t^2 & 0 & 0 \\
0 & 1+t^2+t^4 & 0 & t+t^3 & 0 & t+t^3 & 0 & t^3 & 0 \\
t^2 & 0 & 1+t^2+t^4 & 0 & t+t^3 & 0 & 0 & 0 & t^2 \\
\hline
0 & t+t^3 & 0 & 1+t^2+t^4 & 0 & t^2 & 0 & t+t^2 & 0 \\
t+t^3 & 0 & t+t^3 & 0 & 1+2t^2+t^4 & 0 & t+t^3 & 0 & t+t^3 \\
0 & t+t^3 & 0 & t^2 & 0 & 1+2t^2+t^4 & 0 & t+t^3 & 0 \\
\hline
t^2 & 0 & 0 & 0 & t+t^3 & 0 & 1+t^2+t^4 & 0 & t^2 \\
0 & t^2 & 0 & t+t^3 & 0 & t+t^3 & 0 & 1+2t^2+t^4 & 0 \\
0 & 0 & t^2 & 0 & t+t^3 & 0 & t^2 & 0 & 1+2t^2+t^4
\end{array}\right).
$$

Now

$$P^{(2)}=P\otimes P=\begin{bmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix}1+t^4 & 0 & t^2\\ 0 & t+t^3 & 0\\ t^2 & 0 & 0\end{bmatrix}$$

$$=\left[\begin{array}{ccc|ccc|ccc}
(1+t^4)^2 & 0 & (1+t^4)t^2 & & & & t^2(1+t^4) & 0 & t^4\\
0 & (1+t^4)\times(t+t^3) & 0 & & 0 & & 0 & t^2(t+t^3) & 0\\
(1+t^4)t^2 & 0 & 0 & & & & t^4 & 0 & 0\\ \hline
 & & & (t+t^3)\times(1+t^4) & 0 & (t+t^3)t^2 & & & \\
 & 0 & & 0 & (t+t^3)^2 & 0 & & 0 & \\
 & & & (t+t^3)t^2 & 0 & 0 & & & \\ \hline
t^2(1+t^4) & 0 & t^4 & & & & & & \\
0 & t^2(t+t^3) & 0 & & 0 & & & 0 & \\
t^4 & 0 & 0 & & & & & &
\end{array}\right]$$

$$=\left[\begin{array}{ccc|ccc|ccc}
1+2t^4+t^8 & 0 & t^2+t^6 & & & & t^2+t^6 & 0 & t^4\\
0 & t+t^3+t^5+t^7 & 0 & & 0 & & 0 & t^3+t^5 & 0\\
t^2+t^6 & 0 & 0 & & & & t^4 & 0 & 0\\ \hline
 & & & t+t^3+t^5+t^7 & 0 & t^3+t^5 & & & \\
 & 0 & & 0 & t^2+2t^4+t^6 & 0 & & 0 & \\
 & & & t^3+t^5 & 0 & 0 & & & \\ \hline
t^2+t^6 & 0 & t^4 & & & & & & \\
0 & t^3+t^5 & 0 & & 0 & & & 0 & \\
t^4 & 0 & 0 & & & & & &
\end{array}\right]$$

$$
P^{(2)}Q=\left(\begin{array}{ccc|ccc|ccc}
1+5t^4+5t^8+t^{12} & & 2t^2+t^4+5t^6+t^8+2t^{10} & & t+3t^3+5t^5+5t^7+3t^9+t^{11} & & 2t^2+t^4+5t^6+t^8+2t^{10} & & 3t^4+4t^6+3t^8 \\
 & t+2t^3+4t^5+4t^7+2t^9+t^{11} & & t^2+3t^4+4t^6+3t^8+t^{10} & & t^2+3t^4+4t^6+3t^8+t^{10} & & 2t^3+4t^5+4t^7+2t^9 & \\
t^2+3t^6+t^{10} & & t^4+t^8 & & t^3+2t^5+2t^7+t^9 & & 2t^4+t^6+2t^8 & & t^6 \\
\hline
 & t^2+3t^4+4t^6+3t^8+t^{10} & & t+2t^3+4t^5+4t^7+2t^9+t^{11} & & 2t^3+4t^5+4t^7+2t^9 & & t^2+3t^4+4t^6+3t^8+t^{10} & \\
t^3+3t^5+3t^7+t^9 & & t^3+3t^5+3t^7+t^9 & & t^2+4t^4+6t^6+4t^8+t^{10} & & t^3+3t^5+3t^7+t^9 & & t^3+3t^5+3t^7+t^9 \\
 & t^4+2t^6+t^8 & & t^3+2t^5+2t^7+t^9 & & t^5+t^7 & & t^4+2t^6+t^8 & \\
\hline
t^2+3t^6+t^{10} & 0 & 2t^4+t^6+2t^8 & 0 & t^3+2t^5+2t^7+t^9 & 0 & t^4+t^8 & 0 & t^6 \\
0 & t^3+2t^5+2t^7+t^9 & 0 & t^4+2t^6+t^8 & 0 & t^3+2t^5+t^8 & 0 & t^4+t^8 & 0 \\
t^4+t^8 & 0 & t^6 & 0 & t^5+t^7 & 0 & t^6 & 0 & 0
\end{array}\right)
$$

We put this $P^{(2)}Q=A$, then

$$F(t)=\mathrm{tr}(A^4).$$

Since $A\sim A_0+A_1$, where

$$A_0=\begin{pmatrix} 1+5t^4+5t^8+t^{12} & 2t^2+t^4+5t^6+t^8+2t^{10} & t+3t^3+5t^5+5t^7+3t^9+t^{11} & 2t^2+t^4+5t^6+t^8+2t^{10} & 3t^4+4t^6+3t^8 \\ t^2+3t^6+t^{10} & t^4+t^8 & t^3+2t^5+2t^7+t^9 & 2t^4+t^6+2t^8 & t^6 \\ t^3+3t^5+3t^7+t^9 & t^3+3t^5+3t^7+t^9 & t^2+4t^4+6t^6+4t^8+t^{10} & t^3+3t^5+3t^7+t^9 & t^3+3t^5+3t^7+t^9 \\ t^2+3t^6+t^{10} & 2t^4+t^6+2t^8 & t^3+2t^5+2t^7+t^9 & t^4+t^8 & t^6 \\ t^4+t^8 & t^6 & t^5+t^7 & t^6 & 0 \end{pmatrix}$$

$$A_1=\begin{pmatrix} t+2t^3+4t^5+4t^7+2t^9+t^{11} & t^2+3t^4+4t^6+3t^8+t^{10} & t^2+3t^4+4t^6+3t^8+t^{10} & 2t^3+4t^5+4t^7+2t^9 \\ t^2+3t^4+4t^6+3t^8+t^{10} & t+2t^3+4t^5+4t^7+2t^9+t^{11} & 2t^3+4t^5+4t^7+2t^9 & t^2+3t^4+4t^6+3t^8+t^{10} \\ t^4+2t^6+t^8 & t^3+2t^5+2t^7+t^9 & t^5+t^7 & t^4+2t^6+t^8 \\ t^3+2t^5+2t^7+t^9 & t^4+2t^6+t^8 & t^4+2t^6+t^8 & t^5+t^7 \end{pmatrix}$$

Therefore

$$F(t)=\mathrm{tr}(A_0^4)+\mathrm{tr}(A_1^4).$$

We know that for odd power t^ν the coefficient C_ν in $F(t)=\mathrm{tr}(A^4)=\sum C_\nu t^\nu$ is $0\leqq C_\nu=\dim \wedge^\nu(F)^G\leqq \dim \wedge^\nu(\mu F)^G=\dim H^\nu(A_\lambda Q)=0$, for a large μ. So we calculate $\mathrm{tr}\, A^4=\sum C_\nu t^\nu$ only for even power.

We put $A_0^+=$ the even power polynomial part of A_0.

$$=\begin{pmatrix} 1+5t^4+5t^8+t^{10} & 2t^2+t^4+5t^6+t^8+2t^{10} & 0 & 2t^2+t^4+5t^6+t^8+2t^{10} & 3t^4+4t^6+3t^8 \\ t^2+3t^6+t^{10} & t^4+t^8 & 0 & 2t^4+t^6+2t^8 & t^6 \\ 0 & 0 & t^2+4t^4+6t^6+4t^8+t^{10} & 0 & 0 \\ t^2+3t^6+t^{10} & 2t^4+t^6+2t^8 & 0 & t^4+t^8 & t^6 \\ t^4+t^8 & t^6 & 0 & t^6 & 0 \end{pmatrix}$$

We put A_0^- = the odd power polynomial part of A_0

$$
= \begin{pmatrix}
0 & 0 & \begin{matrix} t+3t^3+5t^5 \\ 5t^7+3t^9+t^{11} \end{matrix} & 0 & 0 \\
0 & 0 & \begin{matrix} t^3+2t^5+2t^7 \\ +t^9 \end{matrix} & 0 & 0 \\
\begin{matrix} t^3+3t^5+3t^7 \\ +t^9 \end{matrix} & \begin{matrix} t^3+3t^5+3t^7 \\ +t^9 \end{matrix} & 0 & \begin{matrix} t^3+3t^5+3t^7 \\ +t^9 \end{matrix} & \begin{matrix} t^3+3t^5+3t^7 \\ +t^9 \end{matrix} \\
0 & 0 & \begin{matrix} t^3+2t^5+2t^7 \\ +t^9 \end{matrix} & 0 & 0 \\
0 & 0 & t^5+t^7 & 0 & 0
\end{pmatrix}
$$

$$
\begin{aligned}
A_0 &= A_0^+ + A_0^- \\
A_0^2 &= A_0^{+2} + A_0^+A_0^- + A_0^-A_0^+ + A_0^{-2} \\
A_0^4 &= A_0^{+4} + A_0^{+3}A_0^- + A_0^{+2}A_0^-A_0^+ + A_0^{+2}A_0^{-2} + A_0^+A_0^-A_0^{+2} + A_0^+A_0^-A_0^+A_0^- \\
&\quad + A_0^+A_0^-A_0^-A_0^+ + A_0^+A_0^-A_0^-A_0^- + A_0^-A_0^+A_0^+A_0^+ + A_0^-A_0^+A_0^+A_0^- \\
&\quad + A_0^-A_0^+A_0^-A_0^+ + A_0^-A_0^+A_0^-A_0^- + A_0^-A_0^-A_0^+A_0^+ + A_0^-A_0^-A_0^+A_0^- \\
&\quad + A_0^-A_0^-A_0^-A_0^+ + A_0^-A_0^-A_0^-A_0^- \\
&\equiv A_0^{+4} + A_0^{-4} + A_0^+A_0^-A_0^+A_0^- + A_0^+A_0^-A_0^-A_0^+ + A_0^-A_0^+A_0^+A_0^- \\
&\quad + A_0^-A_0^+A_0^-A_0^+ + A_0^-A_0^-A_0^+A_0^+ + A_0^+A_0^+A_0^-A_0^- \\
&\qquad\qquad \text{(mod odd power of } t\text{)},
\end{aligned}
$$

$$
\operatorname{tr} A_0^4 \equiv \operatorname{tr} A_0^{+4} + \operatorname{tr} A_0^{-4} + \operatorname{tr}(A_0^+A_0^-A_0^+A_0^-)\times 2 + \operatorname{tr}(A_0^{+2}A_0^{-2})\times 4.
$$

Now since

$$
A_0^+ = \begin{pmatrix}
a(1,1) & a(1,2) & 0 & a(1,4) & a(1,5) \\
a(2,1) & a(2,2) & 0 & a(2,4) & a(2,5) \\
0 & 0 & a(3,3) & 0 & 0 \\
a(4,1) & a(4,2) & 0 & a(4,4) & a(4,5) \\
a(5,1) & a(5,2) & 0 & a(5,4) & a(5,5)
\end{pmatrix},
$$

$$
A_0^- = \begin{pmatrix}
0 & 0 & b(1,3) & 0 & 0 \\
0 & 0 & b(2,3) & 0 & 0 \\
b & b & 0 & b & b \\
0 & 0 & b(4,3) & 0 & 0 \\
0 & 0 & b(5,3) & 0 & 0
\end{pmatrix}
$$

$$a(1,1)=1+5t^4+5t^8+t^{12} \qquad a(1,2)=2t^2+t^4+5t^6+t^8+2t^{10}$$
$$a(2,1)=t^2+3t^6+t^{10} \qquad a(2,2)=t^4+t^8$$
$$a(4,1)=t^2+3t^6+t^{10} \qquad a(4,2)=2t^4+t^6+2t^8$$
$$a(5,1)=t^4+t^8 \qquad a(5,2)=t^6$$
$$a(3,3)=t^2+4t^4+6t^6+4t^8+t^{10}$$
$$b(1,3)=t+3t^3+5t^5+5t^7+3t^9+t^{11}$$
$$b(2,3)=t^3+2t^5+2t^7+t^9$$
$$b(4,3)=t^3+2t^5+2t^7+t^9$$
$$b(5,3)=t^5+t^7$$
$$b \quad =t^3+3t^5+3t^7+t^9.$$

$$\therefore \quad A_0^+A_0^-=\begin{bmatrix} 0 & 0 & c(1,3) & 0 & 0 \\ 0 & 0 & c(2,3) & 0 & 0 \\ a(3,3)b & a(3,3)b & 0 & a(3,3)b & a(3,3)b \\ 0 & 0 & c(4,3) & 0 & 0 \\ 0 & 0 & c(5,3) & 0 & 0 \end{bmatrix}=B$$

where
$$c(1,3)=\sum a(1,k)b(k,3)$$
$$c(2,3)=\sum a(2,k)b(k,3)$$
$$c(4,3)=\sum a(4,k)b(k,3)$$
$$c(5,3)=\sum a(5,k)b(k,3).$$

We put this matrix B.

$$\begin{aligned}\therefore \quad \operatorname{tr}(B^2) &= \operatorname{tr} A_0^+A_0^-A_0^+A_0^- \\ &= c(1,3)a(3,3)b+c(2,3)a(3,3)b+a(3,3)c(1,3)b \\ &\quad +a(3,3)c(2,3)b+a(3,3)c(4,3)b+a(3,3)c(5,3)b \\ &\quad +c(4,3)a(3,3)b+c(5,3)a(3,3)b \\ &= 2[c(1,3)+c(2,3)+c(4,3)+c(5,3)]a(3,3)b \\ &= 2[\textstyle\sum_k a(1,k)b(k,3)+\sum a(2,k)b(k,3) \\ &\quad +\textstyle\sum a(4,k)b(k,3)+\sum a(5,k)b(k,3)]a(3,3)b \\ &= 2[\textstyle\sum_k (a(1,k)+a(2,k)+a(4,k)+a(5,k))b(k,3)]a(3,3)b \\ &= 2[(1+2t^2+6t^4+6t^6+6t^8+2t^{10}+t^{12}) \\ &\quad \times(t+3t^3+5t^5+5t^7+3t^9+t^{11}) \\ &\quad +(2t^2+4t^4+7t^6+4t^8+2t^{10})(t^3+2t^5+2t^7+t^9) \\ &\quad +(2t^2+4t^4+7t^6+4t^8+2t^{10})(t^3+2t^5+2t^7+t^9)\end{aligned}$$

$$+(3t^4+6t^6+3t^8)(t^5+t^7)](t^2+4t^4+6t^6+4t^8+t^{10})$$
$$\times(t^3+3t^5+3t^7+t^9)$$
$$=2\times(t^6+12t^8+77t^{10}+342t^{12}+1144t^{14}+2994t^{16}+6256t^{18}$$
$$+10546t^{20}+14410t^{22}+15983t^{24}+14410t^{26}+10546t^{28}$$
$$+6256t^{30}+2994t^{32}+1144t^{34}+342t^{36}+77t^{38}+12t^{40}+t^{42}).$$

Now

$$A_0^{-2}=\begin{bmatrix} b(1,3)b & b(1,3)b & 0 & b(1,3)b & b(1,3)b \\ b(2,3)b & b(2,3)b & 0 & b(2,3)b & b(2,3)b \\ 0 & 0 & \triangle & 0 & 0 \\ b(4,3)b & b(4,3)b & 0 & b(4,3)b & b(4,3)b \\ b(5,3)b & b(5,3)b & 0 & b(5,3)b & b(5,3)b \end{bmatrix}$$

where $\triangle=b\cdot b(1,3)+b\cdot b(2,3)+b\cdot b(4,3)+b\cdot b(5,3)$,

$$\therefore\quad A_0^{-2}=\begin{bmatrix} b(1,3) & b(1,3) & 0 & b(1,3) & b(1,3) \\ b(2,3) & b(2,3) & 0 & b(2,3) & b(2,3) \\ 0 & 0 & \square & 0 & 0 \\ b(4,3) & b(4,3) & 0 & b(4,3) & b(4,3) \\ b(5,3) & b(5,3) & 0 & b(5,3) & b(5,3) \end{bmatrix}\times b$$

where $\square=b(1,3)+b(2,3)+b(4,3)+b(5,3)$.

$$\therefore\quad A_0^{-4}=\begin{bmatrix} b_1^2+b_1b_2+b_1b_4+b_1b_5 & // & 0 & // & // \\ // & b_2b_1+b_2^2+b_2b_4+b_2b_5 & 0 & // & // \\ 0 & 0 & \square^2 & 0 & 0 \\ // & // & 0 & b_4b_1+b_4b_2+b_4^2+b_4b_5 & // \\ // & // & 0 & // & b_5b_1+b_5b_2+b_5b_4+b_5^2 \end{bmatrix} b^2$$

where

$$b_i=b(1,3)\quad(i=1,2,4,5).$$

$$\therefore\quad \mathrm{tr}(A_0^{-4})=(b_1^2+b_1b_2+\cdots+\square^2+\cdots+b_5b_1+\cdots+b_5^2)b^2$$
$$=\{(b_1+b_2+b_4+b_5)^2+\square^2\}b^2=2\times(b_1+b_2+b_4+b_5)^2\times b^2.$$

Therefore

$$\begin{aligned}\operatorname{tr}((A_0^-)^4)&=2\times(t+5t^3+10t^5+10t^7+5t^9+t^{11})^2+(t^3+3t^5+3t^7+t^9)^2\\&=2\times(t^8+16t^{10}+120t^{12}+560t^{14}+1820t^{16}+4368t^{18}+8008t^{20}\\&\quad+11440t^{22}+12878t^{24}+11440t^{26}+8008t^{28}+4368t^{30}\\&\quad+1820t^{32}+560t^{34}+120t^{36}+16t^{38}+t^{40}).\end{aligned}$$

Next put

$$A_0^{+2}=4(B(ij))\oplus(b)$$
$$A^{-2}\sim(C(ij))\oplus(\varDelta).$$

We know $B(i,j)$, $C(i,j)$; b, $\varDelta$ are polynomial of t. Now

$$(C(i,j))=\left[\begin{array}{cc|cc} b(1,3) & b(1,3) & b(1,3) & b(1,3)\\ b(2,3) & b(2,3) & b(2,3) & b(2,3)\\ \hline b(4,3) & b(4,3) & b(4,3) & b(4,3)\\ b(5,3) & b(5,3) & b(5,3) & b(5,3)\end{array}\right]b$$

and $(B(i,j))$ is known by a computer.

Namely, if we put,

$$(A_0^+)^2\sim\left[\begin{array}{cc|cc} f_{11}(t) & f_{12}(t) & f_{14}(t) & f_{15}(t)\\ f_{21}(t) & f_{22}(t) & f_{24}(t) & f_{25}(t)\\ \hline f_{41}(t) & f_{42}(t) & f_{44}(t) & f_{45}(t)\\ f_{51}(t) & f_{52}(t) & f_{54}(t) & f_{55}(t)\end{array}\right]\oplus(f(t))$$

and put

$$\sum_i f_{ij}(t)=F_j(t).$$

Then

$$\begin{aligned}F_1(t)&=4+0t^2+56t^4+8t^6+240t^8+48t^{10}+144t^{12}+48t^{14}+240t^{16}\\&\quad+8t^{16}+56t^{20}+0t^{22}+4t^{24}\\ F_2(t)&=0+8t^2+4t^4+84t^6+33t^8+248t^{10}+100t^{12}+248t^{14}+44t^{16}\\&\quad+84t^8+4t^{20}+8t^{22}+0t^{24}\\ F_4(t)&=0+8t^2+4t^4+84t^6+44t^8+248t^{10}+100t^{12}+248t^{14}+44t^{16}\\&\quad+84t^{18}+4t^{20}+8t^{22}+0t^{24}=F_2(t)\end{aligned}$$

$$F_5(t)=0+0t^2+12t^4+16t^6+88t^8+88t^{10}+160t^{12}+88t^{14}+88t^{16}$$
$$+16t^{18}+12t^{20}+0t^{22}+0t^{24}.$$

Since

$$A_0^{-2}=(C(ij))+(\varDelta)=\begin{bmatrix} b(1,3) & b(1,3) & b(1,3) & b(1,3) \\ b(2,3) & b(2,3) & b(2,3) & b(2,3) \\ b(4,3) & b(4,3) & b(4,3) & b(4,3) \\ b(5,3) & b(5,3) & b(5,3) & b(5,3) \end{bmatrix} b\oplus(\varDelta),$$

$$\mathrm{tr}[(A_0^{+2})(A_0^{-2})]-\varDelta f$$
$$=\mathrm{tr}(f(ij))(C(ij))=\sum_{i,j} f(i,j)b(j,3)b=\sum_j F_j\cdot b(j,3)\cdot b$$
$$=[F_1(t)b(1,3)+F_2(t)b(2,3)+F_4 b(4,3)+F_5 b(5,3)]\times b$$
$$=(4t+12t^3+92t^5+236t^7+784t^9+1376t^{11}+2816t^{13}+3712t^{15}$$
$$+4296t^{17}+4296t^{19}+3712t^{21}+2816t^{23}+1376t^{25}+784t^{27}$$
$$+236t^{29}+92t^{31}+12t^{33}+4t^{35})(t^3+3t^5+3t^7+t^9)$$
$$=4t^4+24t^6+140t^8+552t^{10}+1780t^{12}+4528t^{14}+11532t^{16}$$
$$+17072t^{18}+25256t^{20}+31136t^{22}+33200t^{24}+31136t^{26}$$
$$+25256t^{28}+17072t^{30}+11532t^{32}+4528t^{34}+1780t^{36}$$
$$+552t^{38}+140t^{40}+24t^{42}+4t^{44}$$
$$=X.$$
$$\therefore\quad \mathrm{tr}(A_0^{+2})(A_0^{-2})=X+\varDelta f.$$

$$\varDelta=\Box b=b(b(1,3)+b(2,3)+b(4,3)+b(5,3))$$
$$=(t^3+3t^5+3t^7+t^9)(t+5t^3+10t^5+10t^7+5t^9+t^{11})$$
$$=(t^2+4t^4+6t^6+4t^8+t^{10})^2=\{(t^2(1+t^2)^2)\}$$
$$\varDelta=t^4+8t^6+28t^8+56t^{10}+70t^{12}+56t^{14}+28t^{16}+8t^{18}+t^{20}$$
$$f=t^4+8t^6+28t^8+56t^{10}+70t^{12}+56t^{14}+28t^{16}+8t^{18}+t^{20}=\varDelta$$
$$\varDelta f=(t^2(1+t^2)^4)^4=t^8(1+t^2)^{16}$$
$$=t^8+16t^{10}+120t^{12}+560t^{14}+1820t^{16}+4368t^{18}+8008t^{20}$$
$$+11440t^{22}+12870t^{24}+11440t^{26}+8008t^{28}+4368t^{30}$$
$$+1820t^{32}+560t^{34}+120t^{36}+16t^{38}+t^{40}.$$

Now by a computer,

$$\mathrm{tr}(A_0^+)^4-g^4=1+36t^4+8t^6+510t^8+200t^{10}+3692t^{12}+1816t^{14}$$
$$+14899t^{16}+7576t^{18}+34452t^{20}+15400t^{22}+45622t^{24}$$

$$+15400t^{26}+34452t^{28}+7576t^{30}+14899t^{32}+1816t^{34}$$
$$+3692t^{36}+200t^{38}+510t^{40}+8t^{42}+36t^{44}+t^{46}$$

with $g=t^2(1+t^2)4=t^2+4t^4+6t^6+4t^8+t^{10}$

$$g^4=t^8(1+t^2)^{16}=f^2$$
$$=t^8+16t^{10}+120t^{12}+560t^{14}+1820t^{16}+4368t^{18}+8008t^{20}$$
$$+11440t^{22}+12870t^{24}+11440t^{26}+8008t^{28}+4368t^{30}+1820t^{32}$$
$$+560t^{34}+120t^{36}+16t^{38}+t^{40}.$$

Now

$$\operatorname{tr} A_0^{+4}=1+36t^4+8t^6+510t^8+200t^{10}+3692t^{12}+1816t^{14}$$
$$+14899t^{16}+7976t^{18}+34452t^{20}+15400t^{22}+45622t^{24}$$
$$+15400t^{26}+34452t^{28}+7976t^{30}+14899t^{32}+1816t^{34}$$
$$+3692t^{36}+200t^{38}+510t^{40}+8t^{42}+36t^{44}+t^{48}+g^4$$

$$\operatorname{tr} A_0^{-4}=2t^8+32t^{10}+240t^{12}+1120t^{14}+3640t^{16}+8736t^{18}$$
$$+16016t^{20}+22880t^{22}+25756t^{24}+22880t^{26}+16016t^{28}$$
$$+8736t^{30}+3640t^{32}+1120t^{34}+240t^{36}+32t^{38}+2t^{40}$$

$$\operatorname{tr}(A_0^{+2}A_0^{-2})\times 4=16t^4+96t^6+560t^8+2208t^{10}+7120t^{16}+18112t^{18}$$
$$+46128t^{16}+68288t^{18}+101024t^{20}+124544t^{22}+132800t^{24}$$
$$+124544t^{26}+101024t^{28}+68288t^{30}+46128t^{32}+18112t^{34}$$
$$+7120t^{36}+2208t^{38}+560t^{40}+96t^{42}+16t^{44}+4\Delta f$$

$$\operatorname{tr}(A_0^{+}A_0^{-})^2\times 2=4t^6+48t^8+308t^{10}+1368t^{12}+4576t^{14}+11976t^{16}+25024t^{18}$$
$$+42184t^{20}+57640t^{22}+63952t^{24}+57640t^{26}+42184t^{28}$$
$$+25024t^{30}+11976t^{32}+4576t^{34}+1368t^{36}+308t^{38}+48t^{40}$$
$$+4t^{42}$$

$$g^4+4\Delta f=5g^4=5t^8+80t^{10}+600t^{12}+2800t^{14}+9100t^{16}+21840t^{18}$$
$$+40040t^{20}+57200t^{22}+64350t^{24}+57200t^{26}+40040t^{28}$$
$$+21840t^{30}+9100t^{32}+2800t^{34}+600t^{36}+80t^{38}+5t^{40}$$

$$\operatorname{tr} A_0^4\equiv\operatorname{tr}(A_0^+)^4+\operatorname{tr}(A_0^-)^4+[\operatorname{tr}(A_0^{+2}A_0^{-2})]\times 4+[\operatorname{tr}(A_0^+A_0^-A_0^+A_0^-)]\times 2$$
$$\text{(mod odd powers of } t)$$
$$=1+52t^4+108t^6+1125t^8+2828t^{10}+13020t^{12}+28424t^{14}$$
$$+85743t^{16}+131864t^{18}+233720t^{20}+277664t^{22}+332480t^{24}$$
$$+277664t^{26}+233720t^{28}+131864t^{30}+85743t^{32}+28424t^{34}$$
$$+13020t^{36}+2828t^{38}+1125t^{40}+108t^{42}+52t^{44}+t^{48}.$$

Since

$$A_1^+=\begin{pmatrix} 0 & t^2+3t^4+4t^6+3t^8+t^{10} & t^2+3t^4+4t^6+3t^8+t^{10} & 0 \\ t^2+3t^4+4t^6+3t^8+t^{10} & 0 & 0 & t^2+3t^4+4t^6\ 3t^8+t^{10} \\ t^4+2t^6+t^8 & 0 & 0 & t^4+2t^6+t^8 \\ 0 & t^4+2t^6+t^8 & t^4+2t^6+t^8 & 0 \end{pmatrix}$$

$$A_1^+\sim\begin{pmatrix} a & a \\ b & b\end{pmatrix}\oplus\begin{pmatrix} a & a \\ b & b\end{pmatrix}$$

with $a=t^2+3t^4+4t^6+3t^8+t^{10}$, $b=t^4+2t^6+t^8$.

$$\therefore\quad \mathrm{tr}(A_1^+)^4=2\,\mathrm{tr}\begin{pmatrix} a & a \\ b & b\end{pmatrix}^4=2\,\mathrm{tr}\begin{pmatrix} a^2+ab & a^2+ab \\ ba+b^2 & ba+b^2\end{pmatrix}^2=2\,\mathrm{tr}\begin{pmatrix} c & c \\ d & d\end{pmatrix}\begin{pmatrix} c & c \\ d & d\end{pmatrix}$$

$$\begin{aligned} &=2[(c^2+cd)+(cd+d^2)]=2(c+d)^2 \\ &=2(a+b)^4 && c=a^2+ab \\ &=2(t^2+4t^4+6t^6+4t^8+t^{10})^4 && d=b^2+ab \\ &=2[t^2(1+t^2)^4]^4 && c+d=(a+b)^2 \\ &=2g^4 \\ &=2(t^8+16t^{10}+120t^{12}+560t^{14}+1820t^{16}+4368t^{18} \\ &\quad +8008t^{20}+11440t^{22}+12870t^{24}+11440t^{26}+8008t^{28} \\ &\quad +4368t^{30}+1820t^{32}+560t^{34}+120t^{36}+16t^{38}+t^{40}). \end{aligned}$$

Similarly

$$A_1^-\sim\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}+\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}$$

with

$$\begin{aligned} \alpha&=t+2t^3+4t^5+4t^7+2t^9+t^{11} \\ \beta&=t^3+2t^5+2t^7+t^9 \\ \gamma&=2t^3+4t^5+4t^7+2t^9 \\ \delta&=t^5+t^7. \end{aligned}$$

Putting λ_1, λ_2 eigenvalues of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}$,

$$\begin{aligned} \mathrm{tr}[(A_1^-)^4]&=2\,\mathrm{tr}\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}^4\right] \\ &=2(\lambda_1^4+\lambda_2^4)=2[(\lambda_1+\lambda_2)^4-(4\lambda_1^3\lambda_2+6\lambda_1^2\lambda_2^2+4\lambda_1\lambda_2^3)] \\ &=2[(\alpha+\delta)^4-\lambda_1\lambda_2(4\lambda_1^2+8\lambda_1\lambda_2+\lambda_2^2-\lambda_1\lambda_2)] \end{aligned}$$

$$
\begin{aligned}
&=2[(\alpha+\delta)^4-(\alpha\cdot\delta-\beta\cdot\gamma)[4(\lambda_1+\lambda_2)^2-2(\alpha\cdot\delta-\beta\cdot\gamma)]] \\
&=2[(\alpha+\delta)^4-(\alpha\cdot\delta-\beta\cdot\gamma)[4(\alpha+\delta)^2-2(\alpha\cdot\delta-\beta\cdot\gamma)]] \\
&\qquad \alpha+\delta=t+2t^3+5t^5+5t^7+2t^9+t^{11} \\
&\qquad \alpha\cdot\delta-\beta\cdot\gamma=-t^6-5t^8-10t^{10}-12t^{12}-10t^{14}-5t^{16}-t^{18} \\
&=2[(t+2t^3+5t^5+5t^7+2t^9+t^{11})^4 \\
&\quad +(t^6+5t^8+10t^{10}+12t^{12}+10t^{14}+5t^{16}+t^{18}) \\
&\quad \times(4t^2+16t^4+57t^6+125t^8+206t^{10}+252t^{12}+206t^{14} \\
&\quad +125t^{16}+57t^{18}+16t^{20}+4t^{22})] \\
&=2(t^4+8t^6+48t^8+208t^{10}+711t^{12}+1970t^{14}+4483t^{16} \\
&\quad +8468t^{18}+13271t^{20}+17478t^{22}+19136t^{24}+17478t^{26} \\
&\quad +13271t^{28}+8468t^{30}+4483t^{32}+1970t^{34}+711t^{36} \\
&\quad +208t^{38}+48t^{40}+8t^{42}+t^{44}).
\end{aligned}
$$

Now

$$
A^+A^-=\begin{pmatrix} 0 & a & a & 0 \\ a & 0 & 0 & a \\ b & 0 & 0 & b \\ 0 & b & b & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 & \gamma \\ 0 & \alpha & \gamma & 0 \\ 0 & \beta & \delta & 0 \\ \beta & 0 & 0 & \delta \end{pmatrix}
$$

$$
=\begin{pmatrix} 0 & a\alpha+a\beta & a\gamma+a\delta & 0 \\ a\alpha+a\beta & 0 & 0 & a\gamma+a\delta \\ b\alpha+b\beta & 0 & 0 & b\gamma+b\delta \\ 0 & b\alpha+b\beta & b\gamma+b\delta & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\mathrm{tr}(A_1^+A_1^-)^2&=(a\alpha+a\beta)^2+(a\gamma+a\delta)(b\alpha+b\beta) \\
&\quad +(a\alpha+a\beta)^2+(a\gamma+a\delta)(b\alpha+b\beta) \\
&\quad +(b\alpha+b\beta)(a\gamma+a\delta)+(b\gamma+b\delta)^2 \\
&\quad +(b\alpha+b\beta)(a\gamma+a\delta)+(b\gamma+b\delta)^2 \\
&=2a^2(\alpha+\beta)^2+2ab(\gamma+\delta)(\alpha+\beta) \\
&\quad +2ab(\alpha+\beta)(\gamma+\delta)+2b^2(\gamma+\delta)^2 \\
&=2a^2(\alpha+\beta)^2+2b^2(\gamma+\delta)^2+4ab(\alpha+\beta)(\gamma+\delta) \\
&=2(aX)^2t^2+2(bY)^2t^2+4abXYt^2, \\
&\qquad a=t^2+3t^4+4t^6+3t^8+t^{10} \\
&\qquad b=t^4+2t^6+t^8 \\
&\qquad tX=\alpha+\beta=t+3t^3+6t^5+6t^7+3t^9+t^{11}
\end{aligned}
$$

$$tY=\gamma+\delta=2t^3+5t^5+5t^7+2t^9$$
$$X=1+3t^2+6t^4+6t^6+3t^8+t^{10}$$
$$Y=2t^2+5t^4+5t^6+2t^9.$$

By a computer,

$$\begin{aligned}\operatorname{tr}(A_1^+A_1^-)^2=2(&t^6+12t^8+78t^{10}+348t^{12}+1161t^{14}+3024t^{16}+6288t^{18}\\&+10554t^{20}+14376t^{22}+15932t^{24}+14376t^{26}+10554t^{28}\\&+6288t^{30}+3024t^{32}+1161t^{34}+348t^{38}+78t^{38}+12t^{40}+t^{42}).\end{aligned}$$

Now

$$A_1^{+2}=\begin{bmatrix}0&a&a&0\\a&0&0&a\\b&0&0&b\\0&b&b&0\end{bmatrix}\begin{bmatrix}0&a&a&0\\a&0&0&a\\b&0&0&b\\0&b&b&0\end{bmatrix}$$
$$=\begin{bmatrix}a^2+ab&0&0&a^2+ab\\0&a^2+ab&a^2+ab&0\\0&ab+b^2&ba+b^2&0\\ba+b^2&0&0&ba+b^2\end{bmatrix}$$

$$A_1^{-2}=\begin{bmatrix}\alpha&0&0&\gamma\\0&\alpha&\gamma&0\\0&\beta&\delta&0\\\beta&0&0&\delta\end{bmatrix}\begin{bmatrix}\alpha&0&0&\gamma\\0&\alpha&\gamma&0\\0&\beta&\delta&0\\\beta&0&0&\delta\end{bmatrix}$$
$$=\begin{bmatrix}\alpha^2+\gamma\beta&0&0&\alpha\gamma+\gamma\delta\\0&\alpha^2+\gamma\beta&\alpha\gamma+\gamma\delta&0\\0&\alpha\beta+\delta\beta&\beta\gamma+\delta^2&0\\\beta\alpha+\delta\beta&0&0&\beta\gamma+\delta^2\end{bmatrix}$$

$$A_1^{+2}=\begin{bmatrix}A&0&0&A\\0&A&A&0\\0&B&B&0\\B&0&0&B\end{bmatrix}\qquad\begin{aligned}A&=a^2+ab\\B&=ab+b^2\\C&=\alpha^2+\alpha\beta\\D&=\alpha\gamma+\gamma\delta\\E&=\alpha\beta+\beta\delta\\F&=\beta\gamma+\delta^2\end{aligned}$$

$$A_1^{-2}=\begin{bmatrix}C&0&0&D\\0&C&D&0\\0&E&F&0\\E&0&0&F\end{bmatrix}$$

$$A_1^{+2}A_1^{-2}=\begin{pmatrix} A & 0 & 0 & A \\ 0 & A & A & 0 \\ 0 & B & B & 0 \\ B & 0 & 0 & B \end{pmatrix}\begin{pmatrix} C & 0 & 0 & D \\ 0 & C & D & 0 \\ 0 & E & F & 0 \\ E & 0 & 0 & F \end{pmatrix}$$

$$=\begin{pmatrix} AC+AE & & & X \\ & AC+AE & & \\ & & BD+BF & \\ X & & & BD+BF \end{pmatrix}$$

$$\begin{aligned}
\operatorname{tr}(A_1^{+2}A_1^{-2})&=2A(C+E)+2B(D+F)\\
&=2(a^2+ab)(\alpha^2+\gamma\beta+\alpha\beta+\delta\beta)+2(ab+b^2)(\alpha\gamma+\gamma\delta+\beta\gamma+\delta^2)\\
&=2a(a+b)(\alpha(\alpha+\beta)+\beta(\gamma+\delta))+2b(2+b)(\delta(\gamma+\delta)+\gamma(\alpha+\beta))\\
&=2a(a+b)(\alpha_0X+\beta_0Y)t^2+2b(a+b)(\gamma_0X+\delta_0Y)t^2\\
&=2a(a+b)\xi+2b(a+b)\eta
\end{aligned}$$

$$a+b=t^2+4t^4+6t^6+4t^8+t^{10}=t^2(1+t^2)^4$$

$$\alpha+\beta=t+3t^3+6t^5+6t^7+3t^9+t^{11}=tX$$

$$\gamma+\delta=2t^3+5t^5+5t^7+2t^9=tY,$$

$$a=t^2+3t^4+4t^6+3t^8+t^{10}$$

$$b=t^4+2t^6+t^8$$

$$\alpha=t+2t^3+4t^5+4t^7+2t^9+t^{11}=\alpha_0t$$

$$\beta=t^3+2t^5+2t^7+t^9=\beta_0t$$

$$\gamma=2t^3+4t^5+4t^7+2t^9=\gamma_0t$$

$$\delta=t^5+t^7=\delta_0t$$

where

$$\begin{aligned}
\xi&=(\alpha_0X+\beta_0Y)t^2\\
&=t^2+5t^4+18t^6+43t^8+72t^{10}+86t^{12}+72t^{14}+43t^{16}+18t^{18}\\
&\quad+5t^{20}+t^{22}\\
\eta&=(\gamma_0X+\delta_0Y)t^2\\
&=2t^4+10t^6+30t^8+57t^{10}+70t^{12}+57t^{14}+30t^{16}+10t^{16}+2t^{20}.
\end{aligned}$$

By a computer,

$$\begin{aligned}
\operatorname{tr}(A^{+2}A_1^{-2})=2(&t^6+12t^8+77t^{10}+342t^{12}+1144t^{14}+2992t^{16}+6240t^{18}\\
&+10490t^{20}+14298t^{22}+15848t^{24}+14298t^{26}+10490t^{28}\\
&+6240t^{30}+2992t^{32}+1144t^{34}+342t^{36}+77t^{38}+12t^{40}+t^{44})
\end{aligned}$$

Now

$$\begin{aligned}\operatorname{tr} A_1^4 &\equiv \operatorname{tr}(A_1^{+4})^4 + \operatorname{tr}(\Delta_1^{-4}) + [\operatorname{tr}(A_1^+ A_1^- A_1^+ A_1^-)] \times 2 + [\operatorname{tr}(A_1^{+2} A_1^{-2})] \times 4 \\ &\qquad\qquad \text{(mod odd powers of } t\text{)} \\ &= 2 \times (t^4 + 14t^6 + 121t^8 + 688t^{10} + 2955t^{12} + 9428t^{14} + 24319t^{16} \\ &\quad + 25412t^{18} + 84347t^{20} + 114862t^{22} + 127262t^{24} + 114862t^{26} \\ &\quad + 84347t^{28} + 25412t^{30} + 24319t^{32} + 9428t^{34} + 2955t^{36} + 688t^{38} \\ &\quad + 121t^{40} + 14t^{42} + t^{44})\end{aligned}$$

$$\begin{aligned}\operatorname{tr} A^4 &= \operatorname{tr} A_0^4 + \operatorname{tr} A_1^4 \\ &= 1 + 54t^4 + 136t^6 + 1367t^8 + 4204t^{10} + 18930t^{12} + 47280t^{14} \\ &\quad + 134381t^{16} + 182688t^{18} + 402414t^{20} + 507388t^{22} + 587004t^{24} \\ &\quad + 507388t^{26} + 402414t^{28} + 182688t^{30} + 134381t^{32} + 47280t^{34} \\ &\quad + 18930t^{36} + 4204t^{38} + 1367t^{40} + 136t^{42} + 54t^{44} + t^{48}.\end{aligned}$$

This is the wanted $F(t)$. This polymer is rigid for $S_0 = \{\alpha, \beta, \xi, \zeta\}$.

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