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A Formula for the Dimension of Spaces of Cusp Forms of Weight 1

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Dedicated to Prof. Ichiro Satake on his sixtieth birthday

Introduction

Let Γ be a fuchsian group of the first kind and denote by d_1 the space of cusp forms of weight 1 on the group Γ. It would be interesting to have a certain formula for d_1. But it is not effective to compute the dimension d_1 by means of the Riemann-Roch theorem. The purpose of this paper is to give some formula of d_1 by making use of the Selberg trace formula ([4], [6], [7]).

§ 1. The Selberg eigenspace

Let S denote the complex upper half-plane and we put $G=SL(2, \boldsymbol{R})$. Consider direct products

$$\tilde{S}=S\times T,\ \tilde{G}=G\times T,$$

where T denotes the real torus. The operation of an element (g, α) of $\tilde{G}$ on $\tilde{S}$ is represented as follows:

$$\tilde{S}\ni(z, \phi)\longrightarrow(g, \alpha)(z, \phi)=\left(\frac{az+b}{cz+d},\ \phi+\arg(cz+d)-\alpha\right)\in\tilde{S},$$

where $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G$. The space $\tilde{S}$ is a weakly symmetric Riemannian space with the $\tilde{G}$-invariant metric

$$ds^2=\frac{dx^2+dy^2}{y^2}+\left(d\phi-\frac{dx}{2y}\right)^2,$$

and with the isometry μ defined by $\mu(z, \phi)=(-\bar{z}, -\phi)$. The $\tilde{G}$-invariant measure $d(z, \phi)$ associated to the $\tilde{G}$-invariant metric is given by

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$$d(z, \phi)=d(x, y, \phi)=\frac{dx\wedge dy\wedge d\phi}{y^2}.$$

The ring of $\tilde{G}$-invariant differential operators on $\tilde{S}$ is generated by $\partial/\partial\phi$ and

$$\tilde{\triangle}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{5}{4}\frac{\partial^2}{\partial\phi^2}+y\frac{\partial}{\partial\phi}\frac{\partial}{\partial x},$$

which we call the Casimir operator of $\tilde{S}$. By the correspondence

$$G \ni g \longleftrightarrow (g, 0) \in \tilde{G},$$

we identify the group G with a subgroup $G\times\{0\}$ of $\tilde{G}$, and so the subgroup Γ of G with a subgroup $\Gamma\times\{0\}$ of $\tilde{G}$. For an element $(g, \alpha)\in\tilde{G}$, we define a mapping $T_{(g,\alpha)}$ of $L^2(\tilde{S})$ into itself by $(T_{(g,\alpha)}f)(z, \phi)=f((g, \alpha)(z, \phi))$. For an element $g\in G$, we put $T_{(g,0)}=T_g$. Then we have

$$(T_g f)(z, \phi)=f\left(\frac{az+b}{cz+d}, \phi+\arg(cz+d)\right),$$

where $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let Γ be a fuchsian group of the first kind not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}(=-I)$. We denote by $\mathfrak{m}_\Gamma(k, \lambda)=\mathfrak{m}(k, \lambda)$ the set of all functions $f(z, \phi)$ satisfying the following conditions:

(i) $f(z, \phi)\in L^2(\Gamma\backslash\tilde{S})$,

(ii) $\tilde{\triangle}f(z, \phi)=\lambda f(z, \phi)$, $\dfrac{\partial}{\partial\phi}f(z, \phi)=-\sqrt{-1}\,kf(z, \phi)$.

We call $\mathfrak{m}(k, \lambda)$ the Selberg eigenspace of Γ.

We denote by $S_1(\Gamma)$ the space of cusp forms of weight 1 for the above fuchsian group Γ and put

$$d_1=\dim S_1(\Gamma).$$

Then the following equality holds:

Lemma ([1], [3]). *The notation and the assumption being as above, we have*

$$\mathfrak{m}\left(1, -\frac{3}{2}\right)=\{e^{-\sqrt{-1}\phi}y^{1/2}F(z) : F(z)\in S_1(\Gamma)\},$$

and hence

$$d_1=\dim\mathfrak{m}\left(1, -\frac{3}{2}\right). \tag{1}$$

§ 2. The compact case

In this section we suppose that the group Γ has a compact fundamental domain in the upper half-plane S.

It is well known that every eigenspace $\mathfrak{m}(k, \lambda)$ defined in Section 1 is finite dimensional and orthogonal to each other, and also the eigenspaces span together the space $L^2(\Gamma\backslash\tilde{S})$. We put $\boldsymbol{\lambda}=(k, \lambda)$. For every invariant integral operator with a kernel function $k(z, \phi; z', \phi')$ on $\mathfrak{m}(k, \lambda)$, we have

$$\int_{\tilde{S}} k(z, \phi; z', \phi')f(z', \phi')d(z', \phi')=h(\boldsymbol{\lambda})f(z, \phi),$$

for $f\in\mathfrak{m}(k, \lambda)$. Note that $h(\boldsymbol{\lambda})$ does not depend on f as far as f is in $\mathfrak{m}(k, \lambda)$. We also know that there is a basis $\{f^{(n)}\}_{n=1}^{\infty}$ of the space $L^2(\Gamma\backslash\tilde{S})$ such that each $f^{(n)}$ satisfies the condition (ii) in Section 1. Then we put $\boldsymbol{\lambda}^{(n)}=(k, \lambda)$ for such spectra. We now obtain the following Selberg trace formula for $L^2(\Gamma\backslash\tilde{S})$:

$$(2) \qquad \sum_{n=1}^{\infty} h(\boldsymbol{\lambda}^{(n)})=\sum_{M\in\Gamma}\int_{\tilde{D}} k(z, \phi; M(z, \phi))d(z, \phi),$$

where $\tilde{D}$ denotes a compact fundamental domain of Γ in $\tilde{S}$ and $k(z, \phi; z', \phi')$ is a point-pair invariant kernel of (a)-(b) type in the sense of Selberg such that the series on the left-hand side of (2) is absolutely convergent ([6]). Denote by $\Gamma(M)$ the centralizer of M in Γ and put $\tilde{D}_M=\Gamma(M)\backslash\tilde{S}$. Then

$$(3) \quad \sum_{M\in\Gamma}\int_{\tilde{D}} k(z, \phi; M(z, \phi))d(z, \phi)=\sum_{l}\int_{\tilde{D}_{M_l}} k(z, \phi; M_l(z, \phi))d(z, \phi),$$

where the sum over $\{M_l\}$ is taken over the distinct conjugacy classes of Γ.

We consider an invariant integral operator on the Selberg eigenspace $\mathfrak{m}(k, \lambda)$ defined by

$$\omega_\delta(z, \phi; z', \phi')=\left|\frac{(yy')^{1/2}}{(z-\bar{z}')/2\sqrt{-1}}\right|^{\delta}\frac{(yy')^{1/2}}{(z-\bar{z}')/2\sqrt{-1}}e^{-\sqrt{-1}(\phi-\phi')},\ (\delta>1).$$

It is easy to see that our kernel ω_δ is a point-pair invariant kernel of (a)-(b) type under the condition $\delta>1$ and vanishes on $\mathfrak{m}(k, \lambda)$ for all $k\neq 1$. Since $\Gamma\backslash\tilde{G}$ is compact, the distribution of spectra (k, λ) is discrete and we put

$$\mu_1=-\frac{3}{2}, \mu_2, \mu_3, \cdots,$$

$$d_\beta = \dim \mathfrak{m}(1, \mu_\beta), \ (\beta = 1, 2, 3, \cdots).$$

Then the left-hand side of the trace formula (2) equals $\sum_{\beta=1}^{\infty} d_\beta \Lambda_\beta$, where Λ_β denotes the eigenvalue of ω_δ in $\mathfrak{m}(1, \mu_\beta)$. As for the eigenvalue Λ_β, using the special eigenfunction

$$f(z, \phi) = e^{-\sqrt{-1}\phi} y^{v_\beta}, \ \mu_\beta = v_\beta(v_\beta - 1) - \frac{5}{4},$$

for a spectrum $(1, \mu_\beta)$ in $L^2(\tilde{S})$, we obtain

$$\Lambda_\beta = 2^{2+\delta}\pi \frac{\Gamma(1/2)\Gamma((1+\delta)/2)}{\Gamma(\delta)/\Gamma(1+(\delta/2))} \Gamma\left(\frac{\delta-1}{2} + v_\beta\right)\Gamma\left(\frac{\delta+1}{2} - v_\beta\right).$$

If we put $v_\beta = 1/2 + \sqrt{-1}\, r_\beta$, then

$$(4) \quad \Lambda_\beta = 2^{2+\delta}\pi \frac{\Gamma(1/2)\Gamma((1+\delta)/2)}{\Gamma(\delta)\Gamma(1+(\delta/2))} \Gamma\left(\frac{\delta}{2} + \sqrt{-1}\, r_\beta\right)\Gamma\left(\frac{\delta}{2} - \sqrt{-1}\, r_\beta\right).$$

In general, it is known that the series $\sum_{\beta=1}^{\infty} d_\beta \Lambda_\beta$ is absolutely convergent for $\delta > 1$. By the Stirling formula, we see that the above series is also absolutely and uniformly convergent for all bounded δ except $\delta = \pm(2v_\beta - 1)$.

Now we shall calculate the components of trace appearing in the right-hand side of (3) ([2]).

1) Unit class: $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It is clear that $\omega_\delta(z, \phi; M(z, \phi)) = 1$, and hence

$$J(I) = \int_{\tilde{D}_M} d(z, \phi) = \int_{\tilde{D}} d(z, \phi) < \infty.$$

2) Hyperbolic conjugacy classes.

For the primitive hyperbolic element P, we put

$$g^{-1}Pg = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \qquad (g \in G), \ |\lambda_0| > 1$$

and $\Gamma' = g^{-1}\Gamma g$. Then

$$\Gamma'\left(\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}\right) = g^{-1}\Gamma(P)g.$$

The hyperbolic component is calculated as follows:

$$J(P^k)=\int_{\tilde{D}_P}\omega_\delta(z,\phi;P^k(z,\phi))d(z,\phi)$$
$$=\int_{g^{-1}\tilde{D}_P}\omega_\delta(g(z,\phi);P^kg(z,\phi))d(z,\phi)$$
$$=\int_{g^{-1}\tilde{D}_P}\omega_\delta(z,\phi;g^{-1}P^kg(z,\phi))d(z,\phi)$$
$$=(2\pi)(2^{\delta+1}\sqrt{-1})|\lambda_0^k|^{\delta+1}(\mathrm{sgn}\,\lambda_0)^k\int_{g^{-1}D_P}\frac{y^{\delta-1}}{(z-\lambda_0^{2k}\bar{z})|z-\lambda_0^{2k}\bar{z}|^\delta}dxdy,$$

where $g^{-1}D_P$ is a fundamental domain of $\Gamma'\left(\begin{pmatrix}\lambda_0 & 0\\ 0 & \lambda_0^{-1}\end{pmatrix}\right)$ in S. Thus,

$$J(P^k)=(2^{3+\delta}\pi)\frac{\Gamma(1/2)\Gamma((\delta+1)/2)}{\Gamma((\delta+2)/2)}\frac{(\mathrm{sgn}\,\lambda_0)^k\log|\lambda_0|}{|\lambda_0^{-k}-\lambda_0^k||\lambda_0^{-k}+\lambda_0^k|^\delta}.$$

Let $\{P_\alpha\}$ be a complete system of representatives of the primitive hyperbolic conjugacy classes in Γ and let $\lambda_{0,\alpha}$ be the eigenvalue ($|\lambda_{0,\alpha}|>1$) of representative P_α. Then, the hyperbolic component $J(P)$ is expressed by the following

$$J(P)=\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}J(P_\alpha^k)$$
$$=\frac{2^{3+\delta}\pi^{3/2}\Gamma((\delta+1)/2)}{\Gamma((\delta+2)/2)}\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}\frac{(\mathrm{sgn}\,\lambda_{0,\alpha})^k\log|\lambda_{0,\alpha}|}{|\lambda_{0,\alpha}^k-\lambda_{0,\alpha}^{-k}|}|\lambda_{0,\alpha}^k+\lambda_{0,\alpha}^{-k}|^{-\delta}.$$

3) Elliptic conjugacy classes.

Let ρ, $\bar{\rho}$ be the fixed points of an elliptic element $M(\rho\in S)$ and ζ, $\bar{\zeta}$ be the eigenvalues of M. We denote by Φ a linear transformation which maps S into a unit disk:

$$w=\Phi(z)=\frac{z-\rho}{z-\bar{\rho}}.$$

Then we have $\Phi M\Phi^{-1}=\begin{pmatrix}\zeta & 0\\ 0 & \bar{\zeta}\end{pmatrix}$ and

$$\frac{Mz-\rho}{Mz-\bar{\rho}}=\frac{\zeta}{\bar{\zeta}}\frac{z-\rho}{z-\bar{\rho}}.$$

The elliptic component is calculated as follows:

$$J(M)=\int_{\tilde{D}_M}\omega_\delta(z,\phi;M(z,\phi))d(z,\phi)$$
$$=\frac{2^{\delta+1}\sqrt{-1}}{[\Gamma(M):1]}\int_{\tilde{S}}\frac{(yy')^{(\delta+1)/2}}{(z-\bar{z}')|z-\bar{z}'|^\delta}e^{-\sqrt{-1}(\phi-\phi')}d(z,\phi)\quad((z',\phi')=M(z,\phi))$$

$$=\frac{8\pi\bar{\zeta}}{[\Gamma(M):1]}\int_{|w|<1}\frac{(1-w\bar{w})^{\delta-1}}{(1-\bar{\zeta}^2w\bar{w})|1-\bar{\zeta}^2w\bar{w}|^{\delta}}dudv \quad (w=u+\sqrt{-1}v)$$

$$=\frac{16\pi^2\bar{\zeta}}{[\Gamma(M):1]}\int_0^1\frac{(1-r^2)^{\delta-1}}{(1-\bar{\zeta}^2r^2)|1-\bar{\zeta}^2r^2|^{\delta}}dr.$$

We put

$$I(\delta)=\int_0^1\frac{(1-r^2)^{\delta-1}r}{(1-\bar{\zeta}^2r^2)|1-\bar{\zeta}^2r^2|^{\delta}}dr.$$

Then, under the condition $\delta>0$, the function $\frac{\delta(1-r^2)^{\delta-1}r}{1-\bar{\zeta}^2r^2}$ is Lebesgue-integrable on [0, 1]. Hence

$$\lim_{\delta\to 0}\delta I(\delta)=\lim_{\delta\to 0}\int_0^1\frac{\delta(1-r^2)^{\delta-1}r}{1-\bar{\zeta}^2r^2}dr$$

$$=\lim_{\delta\to 0}\left\{\left[-\frac{(1-t)^{\delta}}{2}\frac{1}{1-\bar{\zeta}^2t}\right]_0^1+\int_0^1(1-t)^{\delta}\left(\frac{1}{1-\bar{\zeta}^2t}\right)'\frac{dt}{2}\right\}$$

$$=\frac{1}{2(1-\bar{\zeta}^2)}.$$

Therefore we obtain

$$\lim_{\delta\to 0}\delta J(M)=\frac{8\pi^2}{[\Gamma(M):1]}\frac{\bar{\zeta}}{1-\bar{\zeta}^2}.$$

Since M and M^{-1} are not conjugate and $\bar{\zeta}/(1-\bar{\zeta}^2)$ is pure imaginary, we have

$$\lim_{\delta\to 0}\delta J(M)+\lim_{\delta\to 0}\delta J(M^{-1})=0.$$

We conclude that the contribution from elliptic classes to d_1 vanishes.

Now we put

$$(5)\qquad \zeta_1^*(\delta)=\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}\frac{(\operatorname{sgn}\lambda_{0,\alpha})^k\log|\lambda_{0,\alpha}|}{|\lambda_{0,\alpha}^k-\lambda_{0,\alpha}^{-k}|}|\lambda_{0,\alpha}^k+\lambda_{0,\alpha}^{-k}|^{-\delta}.$$

Then, by the trace formula (2), the Dirichlet series (5) extends to a meromorphic function on the whole δ-plane and has a simple pole at $\delta=0$ whose residue will appear in (6) below. Finally, *multiply the both sides of* (2) *by* δ *and let* δ *tend to zero,* then the limit is expressed, by the above 1), 2) and 3), as follows:

$$\dim \mathfrak{m}\left(1,-\frac{3}{2}\right)=\frac{1}{2}\operatorname*{Res}_{\delta=0}\zeta_1^*(\delta),$$

namely, by (1) we have

$$d_1=\frac{1}{2}\operatorname*{Res}_{\delta=0}\zeta_1^*(\delta). \tag{6}$$

Remark 1. Let Γ be a fuchsian group of the first kind which contains the element $-I$, and χ a unitary representation of Γ of degree 1 such that $\chi(-I)=-1$. Let $S_1(\Gamma, \chi)$ be the linear space of cusp forms of weight 1 on the group Γ with character χ, and denote by d_1 the dimension of the linear space $S_1(\Gamma, \chi)$. When the group Γ has a compact fundamental domain in the upper half-plane S, we have the following dimension formula in the same way as in the case $\Gamma \not\ni -I$:

$$d_1=\frac{1}{2}\sum_{\{M\}}\frac{\chi(M)}{[\Gamma(M):\pm I]}\frac{\bar{\zeta}}{1-\bar{\zeta}^2}+\frac{1}{2}\operatorname*{Res}_{s=0}\zeta_2^*(s), \tag{7}$$

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of $\Gamma/\{\pm I\}$, $\Gamma(M)$ denotes the centralizer of M in Γ, $\bar{\zeta}$ is one of the eigenvalues of M, and $\zeta_2^*(s)$ denotes the Selberg type zeta-function defined by

$$\zeta_2^*(s)=\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(P_\alpha)^k\log\lambda_{0,\alpha}}{\lambda_{0,\alpha}^k-\lambda_{0,\alpha}^{-k}}|\lambda_{0,\alpha}^k+\lambda_{0,\alpha}^{-k}|^{-s}. \tag{8}$$

Here $\lambda_{0,\alpha}$ denotes the eigenvalue $(\lambda_{0,\alpha}>1)$ of representative P_α of the primitive hyperbolic conjugacy classes $\{P_\alpha\}$ in $\Gamma/\{\pm I\}$.

§ 3. The finite case 1 ($:\Gamma \not\ni -I$)

Let Γ be a fuchsian group of the first kind not containing the element $-I$, and suppose that Γ has a non-compact fundamental domain $\tilde{D}$ in the space $\tilde{S}$. Then, we see that the integral

$$\int_{\tilde{D}}\sum_{M\in\Gamma}\omega_\delta(z, \phi; M(z, \phi))d(z, \phi)$$

is uniformly bounded at a neighborhood of each irregular cusp of Γ, and that by the Riemann-Roch theorem, the number of regular cusps of Γ is even. Therefore we assume for simplicity that $\{\kappa_1, \kappa_2\}$ is a maximal set of cusps of Γ which are regular cusps and not equivalent with respect to Γ. Let Γ_i be the stabilizer in Γ of κ_i, and fix an element $\sigma_i \in SL(2, \boldsymbol{R})$ such that $\sigma_i\infty=\kappa_i$ and such that $\sigma_i^{-1}\Gamma_i\sigma_i$ is equal to the group $\left\{\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} : m \in \boldsymbol{Z}\right\}$. Then the Eisenstein series attached to the regular cusp κ_i is defined by

$$E_i(z, \phi; s)=\sum_{\substack{\sigma\in\Gamma_i\backslash\Gamma \\ \sigma_i^{-1}\sigma=\binom{*\,*}{c\,d}}}\frac{y^s}{|cz+d|^{2s}}e^{-\sqrt{-1}(\phi+\arg(cz+d))} \quad (i=1, 2), \tag{9}$$

where $s=t+\sqrt{-1}r$ with $t>1$. The series (9) has the Fourier expansion at κ_j in the form

$$E_i(\sigma_j(z, \phi); s)=\sum_{m=-\infty}^{\infty} a_{ij,m}(y, \phi; s)e^{2\pi\sqrt{-1}mx}.$$

The constant term $a_{ij,0}(y, \phi; s)$ is given by

$$\begin{aligned} e^{\sqrt{-1}\phi}a_{ij,0}(y, \phi; s)&=a_{ij,0}(y; s)\\ &=\delta_{ij}y^s+\psi_{ij}(s)y^{1-s} \end{aligned}$$

with Kronecker's δ, and

$$\psi_{ij}(s)=-\sqrt{-1}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+(1/2))}\sum_{c\neq 0}\frac{(\operatorname{sgn} c)\cdot N_{ij}(c)}{|c|^{2s}},$$

where $N_{ij}(c)=\#\left\{0\leqq d<|c|: \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j\right\}$. We put

$$\Phi(s)=(\psi_{ij}(s)).$$

Then it is easy to see that the Eisenstein matrix $\Phi(s)$ is a skew-symmetric matrix.

Since Γ is of finite type, the integral operator defined by ω_δ is not completely continuous on $L^2(\Gamma\backslash\tilde{S})$ in general and the space $L^2(\Gamma\backslash\tilde{S})$ has the following spectral decomposition

$$L^2(\Gamma\backslash\tilde{S})=L_0^2(\Gamma\backslash\tilde{S})\oplus L_{Sp}^2(\Gamma\backslash\tilde{S})\oplus L_{\mathrm{cont}}^2(\Gamma\backslash\tilde{S}),$$

where L_0^2 is the space of cusp forms and is discrete, L_{Sp}^2 is the discrete part of the orthogonal complement of L_0^2 and L_{cont}^2 is the continuous part of the spectra. We put

$$\begin{aligned} \tilde{H}_\delta(z, \phi; z', \phi')=\frac{1}{8\pi^2}\sum_{i=1}^{2}\int_{-\infty}^{\infty} h(r)E_i\left(z, \phi; \frac{1}{2}+\sqrt{-1}r\right) \\ \times\overline{E_i\left(z', \phi'; \frac{1}{2}+\sqrt{-1}r\right)}dr. \end{aligned}$$

Here $h(r)$ denotes the eigenvalue of ω_δ in $\mathfrak{m}(1, \lambda)$ given by (4):

$$h(r)=2^{2+\delta}\pi\frac{\Gamma(1/2)\Gamma((1+\delta)/2)}{\Gamma(\delta)\Gamma(1+(\delta/2))}\Gamma\left(\frac{\delta}{2}+\sqrt{-1}r\right)\Gamma\left(\frac{\delta}{2}-\sqrt{-1}r\right) \tag{10}$$

with $\lambda=s(s-1)-\dfrac{5}{4}$ and $s=\dfrac{1}{2}+\sqrt{-1}r$. We put

$$\kappa_\delta(z, \phi; z', \phi') = \sum_{M \in \Gamma} \omega_\delta(z, \phi; M(z', \phi'))$$

and

$$\tilde{\kappa}_\delta = \kappa_\delta - \tilde{H}_\delta.$$

Then the integral operator $\tilde{\kappa}_\delta$ is now complete continuous on $L^2(\Gamma\backslash\tilde{S})$ and has all discrete spectra of κ_δ. Furthermore, an eigenvalue of $f(z, \phi)$ in $L_0^2(\Gamma\backslash\tilde{S}) \oplus L_{Sp}^2(\Gamma\backslash\tilde{S})$ for $\tilde{\kappa}_\delta$ is equal to that for κ_δ and the image of $\tilde{\kappa}_\delta$ on it is contained in $L_0^2(\Gamma\backslash\tilde{S})$. Considering the trace of $\tilde{\kappa}_\delta$ on $L_0^2(\Gamma\backslash\tilde{S})$, we now obtain the following modified trace formula ([4], [7]):

$$\begin{aligned}\sum_{n=1}^{\infty} h(\lambda^{(n)}) &= \int_{\tilde{D}} \tilde{\kappa}_\delta(z, \phi; z, \phi) d(z, \phi) \\ &= \int_{\tilde{D}} \{\sum_{M \in \Gamma} \omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\} d(z, \phi),\end{aligned}$$

where each of $\lambda^{(n)}$ denotes an eigenvalue corresponding to an orthogonal basis $\{f^{(n)}\}$ for $L_0^2(\Gamma\backslash\tilde{S})$. We put

$$\begin{aligned}&\int_{\tilde{D}} \{\sum_{M \in \Gamma} \omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\} d(z, \phi) \\ &\quad = J(I) + J(P) + J(R) + J(\infty),\end{aligned}$$

where $J(I)$, $J(P)$, $J(R)$ and $J(\infty)$ denote respectively the identity component, the hyperbolic component, the elliptic component and the parabolic component of the traces. Then the components $J(I)$, $J(P)$ and $J(R)$ are as in Section 2 and in the following we shall calculate the component $J(\infty)$ (cf. [9]).

Let $\tilde{D}_i$ be a fundamental domain of the stabilizer Γ_i of cusp κ_i in Γ. Then we have

$$\begin{aligned}J(\infty) = \lim_{Y \to \infty} \Big\{ &\sum_{i=1}^{2} \int_{\tilde{D}_i^Y} \sum_{\substack{M \in \Gamma_i \\ M \neq I}} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) \\ &- \int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi) d(z, \phi) \Big\},\end{aligned}$$

where $\tilde{D}_i^Y$ denotes the domain consisting of all points (z, ϕ) in $\tilde{D}_i$ such that $\mathrm{Im}(\sigma_i^{-1}z) < Y$, and $\tilde{D}_Y$ the domain consisting of all $(z, \phi) \in \tilde{D}$ such that $\mathrm{Im}(\sigma_i^{-1}z) < Y$ for all $i = 1, 2$. Making use of a summation formula due to Euler-MacLaurin and the Maass-Selberg relation, we have the following (cf. [2], [9]):

$$\int_{\tilde{D}_i^Y} \sum_{\substack{M \in \Gamma_i \\ M \neq I}} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) = 2^2\pi \frac{\Gamma(1/2)\Gamma((\delta+1)/2)}{\Gamma(1+(\delta/2))} \log Y + \varepsilon(\delta) + o(1)$$

as $Y \to \infty$, where $\varepsilon(\delta)$ denotes a function of δ such that $\lim_{\delta\to 0}\delta\varepsilon(\delta)=0$;

$$\frac{1}{8\pi^2}\int_{\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)E_i\left(z,\phi;\frac{1}{2}+\sqrt{-1}r\right)\overline{E_i\left(z,\phi;\frac{1}{2}+\sqrt{-1}r\right)}dr\,d(z,\phi)$$
$$=2^2\pi\frac{\Gamma(1/2)\Gamma((\delta+1)/2)}{\Gamma(1+(\delta/2))}\log Y$$
$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\psi'_{ij}}{\psi_{ij}}\left(\frac{1}{2}+\sqrt{-1}r\right)dr+o(1)$$

as $Y\to\infty$ $(j\neq i)$. By the expression (10) of $h(r)$, we have

$$h(r)=O\left(\frac{|r|^\delta}{|r|e^{\pi|r|}}\right);$$

and the operator $\tilde{\kappa}_\delta$ is complete continuous on $L^2(\Gamma\backslash\tilde{S})$. Therefore we have

$$\lim_{\delta\to+0}\delta\int_{-\infty}^{\infty} h(r)\frac{\psi'_{ij}}{\psi_{ij}}\left(\frac{1}{2}+\sqrt{-1}r\right)dr=0.$$

It is now clear that the above result, combined with the formula (6), proves the following ([9]):

Theorem 1. *Let Γ be a fuchsian group of the first kind not containing the element $-I$ and suppose that the number of regular cusps of Γ is two. Let d_1 be the dimension for the space consisting of all cusp forms of weight 1 with respect to Γ. Then d_1 is given by*

$$d_1=\frac{1}{2}\operatorname*{Res}_{s=0}\zeta_1^*(s), \tag{11}$$

where $\zeta_1^*(s)$ denotes the Selberg type zeta-function defined by (5) in Section 2.

Remark 2. Let Γ be a general discontinuous group of finite type not containing the element $-I$. Then we can prove that in the same way as in the above case, the contribution from parabolic classes to d_1 vanishes.

§ 4. The finite case 2 ($:\Gamma \ni -I$)

Let Γ be a fuchsian group of the first kind and assume that Γ contains the element $-I$ and has a non-compact fundamental domain $\tilde{D}$ in the space $\tilde{S}$. Let χ be a unitary representation of Γ of degree 1 such that $\chi(-I)=-1$. We denote by $S_1(\Gamma,\chi)$ the linear space of cusp forms of weight 1 on the group Γ with the character χ and by d_1 the dimension of the space $S_1(\Gamma, \chi)$. In this section we shall give a similar formula of the number d_1 when the group Γ is of finite type reduced at infinity and $\chi^2 \neq 1$.

Since Γ is of finite type reduced at ∞, ∞ is a cusp of Γ and the stabilizer Γ_∞ of ∞ in Γ is equal to $\pm\Gamma_0$ with $\Gamma_0 = \left\{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in Z\right\}$. The Eisenstein series $E_\chi(z, \phi; s)$ attached to ∞ and χ is then defined by

$$E_\chi(z, \phi; s) = \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \binom{* \ *}{c \ d}}} \frac{\bar{\chi}(M) y^s}{|cz+d|^{2s}} e^{-\sqrt{-1}(\phi + \arg(cz+d))}, \tag{12}$$

where $s = \sigma + \sqrt{-1}r$ with $\sigma > 1$. The constant term in the Fourier expansion of (12) at ∞ is given by

$$a_0(y, \phi; s) = e^{-\sqrt{-1}\phi}(y^s + \psi_\chi(s) y^{1-s}),$$

$$\psi_\chi(s) = -\sqrt{-1}\sqrt{\pi}\, \frac{\Gamma(s)}{\Gamma\left(s + \dfrac{1}{2}\right)} \sum_{\substack{c>0 \\ d \bmod c \\ \binom{* \ *}{c \ d} \in \Gamma}} \frac{\bar{\chi}(c, d)}{|c|^{2s}}.$$

In the following we only consider the case $\chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1$. As shown in [3], the parabolic component $J(\infty)$ in the trace formula is given by

$$\begin{aligned} J(\infty) &= \lim_{Y \to \infty} \left\{ 2\int_0^Y \int_0^1 \int_0^\pi \sum_{\substack{M \in \Gamma \\ M \neq I}} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) - \int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi) d(z, \phi) \right\} \\ &= -\frac{1}{4\pi} \int_{-\infty}^\infty h(r) \frac{\psi_\chi'((1/2) + \sqrt{-1}r)}{\psi_\chi((1/2) + \sqrt{-1}r)} dr - \frac{1}{4} h(0) \psi_\chi\left(\frac{1}{2}\right) + \varepsilon(\delta) \end{aligned}$$

with $\lim_{\delta \to 0} \delta\varepsilon(\delta) = 0$. When we combine this with the formula (7), we are led to the following theorem which is our main purpose in this section.

Theorem 2. *Let Γ be a fuchsian group of the first kind containing the element $-I$ and suppose that Γ is reduced at infinity. Let χ be a one-dimensional unitary representation of Γ such that $\chi(-I) = -1$, $\chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1$ and $\chi^2 \neq 1$. We denote by d_1 the dimension of the linear space consisting of cusp forms of weight 1 with respect to Γ with χ. Then the dimension d_1 is given by*

$$d_1 = \frac{1}{2} \sum_{\{M\}} \frac{\chi(M)}{[\Gamma(M) : \pm I]} \frac{\bar{\zeta}}{1 - \bar{\zeta}^2} + \frac{1}{2} \operatorname*{Res}_{s=0} \zeta_2^*(s) - \frac{1}{4} \psi_\chi\left(\frac{1}{2}\right), \tag{13}$$

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of $\Gamma/\{\pm I\}$, $\Gamma(M)$ denotes the centralizer of M in Γ, ζ is one of the eigenvalues of M, and $\zeta_2^*(s)$ denotes the Selberg type zeta-function defined by (8) in Section 2.

We may call the formulas (11) and (13) a kind of Riemann-Roch type theorem for automorphic forms of weight 1.

Remark 3. For a general discontinuous group Γ of finite type containing the element $-I$, we obtain the contribution from parabolic classes to d_1 in the same way as in the case of reduced at ∞.

§ 5. The case of $\Gamma_0(p)$

Let p be a prime number such that $p\equiv 3 \bmod 4$, $p\neq 3$ and let $\Phi_0(p)$ be the group generated by the group $\Gamma_0(p)$ and the element $\kappa=\begin{pmatrix} 0 & -\sqrt{p}^{-1} \\ \sqrt{p} & 0 \end{pmatrix}$, namely, $\Phi_0(p)=\Gamma_0(p)+\kappa\Gamma_0(p)$. Let ε be the Legendre symbol on $\Gamma_0(p)$: $\varepsilon(L)=\left(\frac{d}{p}\right)$ for $L=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(p)$. Since $\varepsilon(\kappa^2)=\varepsilon(-I)=-1$, we can define the odd characters $\varepsilon^{\pm}$ on $\Phi_0(p)$ such that $\varepsilon^{\pm}(\kappa)=\pm\sqrt{-1}$. Then we have

$$S_1(\Gamma_0(p),\varepsilon)=S_1(\Phi_0(p),\varepsilon^+)\oplus S_1(\Phi_0(p),\varepsilon^-).$$

We put

$$\mu_1^{\pm}=\dim S_1(\Phi_0(p),\varepsilon^{\pm}).$$

Then

$$\dim S_1(\Gamma_0(p),\varepsilon)=d_1=\mu_1^+ + \mu_1^-.$$

We denote by $\bar{\Gamma}_0(p)$, $\bar{\Phi}_0(p)$ the inhomogeneous linear transformation group attached to $\Gamma(p)$, $\Phi_0(p)$ respectively. If $\sigma(p)$ is the parabolic class number of $\bar{\Gamma}_0(p)$, then $\sigma(p)=2$; and if $e_2(p)$, $e_3(p)$ are the number of elliptic classes of order 2, 3 respectively of $\bar{\Gamma}_0(p)$, then

$$e_2(p)=0,\quad e_3(p)=1+\left(\frac{p}{3}\right).$$

Let $\sigma^*(p)$, $e_2^*(p)$, $e_3^*(p)$ denote respectively the number of parabolic classes, the number of elliptic classes of order 2, the number of elliptic classes of order 3 for $\bar{\Phi}_0(p)$. Then we have

$$\sigma^*(p)=\frac{1}{2}\sigma(p)=1;$$

$$e_3^*(p)=\frac{1}{2}e_3(p)=\frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right);$$

$$e_2^*(p)=\frac{1}{2}e_2(p)+e_2'(p)=e_2'(p),$$

where $e_2'(p)$ denotes the number of classes of elliptic elements of order 2 of $\kappa\bar{\Gamma}_0(p)$. It is known that

$$e_2'(p)=\left(3-\left(\frac{2}{p}\right)\right)=\begin{cases}4h & \text{if } p\equiv 3 \bmod 8,\\ 2h & \text{if } p\equiv 7 \bmod 8,\end{cases}$$

where h denotes the class number of $Q(\sqrt{-p})$, which is an odd integer. Let ϑ_2 denote the number of the elements L in $\bar{\Gamma}_0(p)$ such that $\varepsilon^-(\kappa L)=+\sqrt{-1}$. Then, by [5], we have the following

$$\vartheta_2=\begin{cases}h & \text{if } p\equiv 3 \bmod 8,\\ 0 & \text{if } p\equiv 7 \bmod 8.\end{cases}$$

In the following, we shall calculate the contribution from elliptic elements to $\mu_1^{\pm}$. Let $\{M\}$ be a complete system of representatives of the elliptic conjugacy classes of order 2 in $\bar{\Phi}_0(p)$. Then $\{M\}$ is given by $\left\{\kappa\begin{pmatrix} a & b\\ pb & d\end{pmatrix}\right\}$, where $\left\{\begin{pmatrix} a & b\\ pb & d\end{pmatrix}\right\}$ denotes the representatives of positive definite integral quadratic forms $\begin{pmatrix} a & pb\\ pb & pd\end{pmatrix}$ such that $\det\begin{pmatrix} a & pb\\ pb & pd\end{pmatrix}=p$. Then the result of calculation is given in the following table:

p	$\varepsilon(L)$	The number of elliptic classes of order 2	ζ	$\frac{1}{[\Gamma(M):\pm I]}\frac{\bar{\zeta}}{1-\bar{\zeta}^2}\varepsilon^{\pm}(\kappa L)$
$p\equiv 3 \bmod 8$	$\varepsilon(L)=1$	$3h$	$\sqrt{-1}$	$\frac{1}{2}\frac{\sqrt{-1}}{2}(\pm\sqrt{-1})=\mp\frac{1}{4}$
$p\equiv 3 \bmod 8$	$\varepsilon(L)=-1$	h	$\sqrt{-1}$	$\frac{1}{2}\frac{\sqrt{-1}}{2}(\mp\sqrt{-1})=\pm\frac{1}{4}$
$p\equiv 7 \bmod 8$	$\varepsilon(L)=1$	$2h$	$\sqrt{-1}$	$\frac{1}{2}\frac{\sqrt{-1}}{2}(\pm\sqrt{-1})=\mp\frac{1}{4}$

It is clear that there is no contribution from elliptic classes of order 3 to $\mu_1^{\pm}$. Therefore the contribution from elliptic classes to $\mu_1^{\pm}$ is given by

$$\frac{1}{2}\sum_{\{M\}}\frac{1}{[\Gamma(M):\pm I]}\frac{\bar{\zeta}}{1-\bar{\zeta}^2}\varepsilon^{\pm}(M)=\mp\frac{1}{4}h.$$

We also have $\psi_{\varepsilon^{\pm}}(1/2)=\mp 1$. Let $\{P_\alpha\}$ be a complete system of representatives of the primitive hyperbolic conjugacy classes in $\bar{\Gamma}_0(p)$ and let $\lambda_{0,\alpha}$ be the eigenvalue ($\lambda_{0,\alpha}>1$) of representative P_α. We put

$$Z^*(\delta)=\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}\frac{\varepsilon(P_\alpha)^k\log\lambda_{0,\alpha}}{|\lambda_{0,\alpha}^k-\lambda_{0,\alpha}^{-k}|}|\lambda_{0,\alpha}^k+\lambda_{0,\alpha}^{-k}|^{-\delta}.$$

Then, we have consequently the following

$$d_1=\mu_1^+ +\mu_1^- =\frac{1}{2}\operatorname*{Res}_{\delta=0}Z^*(\delta). \tag{14}$$

Remark 4. Combining the above (14) with Serre's result[1], we have the following remarkable equality

$$\operatorname*{Res}_{\delta=0}Z^*(\delta)=(h-1)+4(s+2a).$$

References

[1] Hiramatsu, T., Eichler classes attached to automorphic forms of dimension −1, Osaka J. Math., **3** (1966), 39–48.
[2] ——, On some dimension formula for automorphic forms of weight one I, Nagoya Math. J., **85** (1982), 213–221.
[3] ——, On some dimension formula for automorphic forms of weight one II, Nagoya Math. J., **105** (1987), 169–186.
[4] Kubota, T., Elementary theory of Eisenstein series, Kodansha and Halsted, Tokyo-New York, 1973.
[5] Petersson, H., Über Eisensteinsche Reihen und automorphe Formen von der Demension −1, Comment. Math. Helv., **31** (1956), 111–144.
[6] Selberg, A., Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20** (1956), 47–87.
[7] ——, Discontinuous groups and harmonic analysis, in Proc. International Math. Congr., Stockholm, 177–189, 1962.
[8] Serre, J.-P., Modular forms of weight one and Galois representations, in Proc. Symposium on Algebraic Number Fields, 193–268, Academic Press, London, 1977.
[9] Hiramatsu, T. and Akiyama, S., On some dimension formula for automorphic forms of weight one III, Nagoya Math. J., **111** (1988), 157–163.

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[1] cf. Serre [8], p. 253.