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Special Values of L-functions Associated with the Space of Quadratic Forms and the Representation of $Sp(2n, F_p)$ in the Space of Siegel Cusp Forms

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Dedicated to Professor I. Satake and Professor F. Hirzebruch for their sixtieth birthdays

Contents

Introduction

0.1. To evaluate special values of various kinds of zeta functions and L-functions and to interpret the meaning of them have been providing fruitful problems to number theory.

Siegel [21], as an initiative work, established an ingenious method of

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evaluating the special values at non-positive integers of partial zeta functions for totally real fields and Klingen-Siegel proved that they are rational numbers (cf. [8]). Shintani [16] presented a more direct method of evaluating them, giving remarkable expressions of partial zeta functions by integrals taken over complex contour paths. Following Shintani's method, Satake [14], introducing zeta functions of self-dual homogeneous cones, studied a general method of obtaining nice expressions of the zeta functions by integrals over contour paths. In some cases, he succeeded in representing the special values at non-positive integers of the zeta functions of cones as a finite sum of certain integrals over some compact Lie group. Kurihara [9], also following Shintani, evaluated the special values at non-positive integers of Siegel zeta functions of $\boldsymbol{Q}$-anisotropic quadratic forms (non-zero forms) with signature $(1, n-1)$ $(n=3, 4)$. However, their methods are not applicable to the zeta functions of cones such that some of edge vectors of cones are contained in the boundary of the self-dual homogeneous cone Ω (see Introduction of [14]).

On the other hand, Hecke [6], [7], more than fifty years before, studied the decomposition into irreducible components of the representation μ of $SL_2(\boldsymbol{F}_p)$ in a certain space of elliptic cusp forms and obtained remarkable relations among multiplicities of irreducible representations in μ. Recently, Yamazaki [24], Tsushima [22], Lee-Weintraub [11], and Hashimoto [5] studied similar problems in the case of the representation μ_k of $Sp(4, \boldsymbol{F}_p)$ in a certain space of Siegel cusp forms of degree two and weight k. The former four authors employed algebro-geometric methods including the Hirzebruch-Riemann-Roch theorem, the holomorphic Lefschetz theorem, and Hashimoto used the Selberg trace formula. To attack the problems, they calculated the traces of μ_k for various elements of $Sp(4, \boldsymbol{F}_p)$. From the viewpoint of the Selberg trace formula, there appear special values of various kinds of zeta functions and L-functions in calculating the dimensions of the spaces of cusp forms and the traces of $\mu_k(\bar{\alpha})$ $(\bar{\alpha} \in Sp(4, \boldsymbol{F}_p))$, as is observed in [16], [1], [13], [4], [5]. In his lecture at Kyoto in 1985, Hashimoto introduced an interesting L-function attached to the ternary zero form $x_1x_2-x_{12}^2$ and expressed the traces of $\mu_k(\bar{\alpha})$ for certain unipotent elements $\bar{\alpha} \in Sp(4, \boldsymbol{F}_p)$ using the special value at $s=3/2$ of that L-function (see the identity (0.5) in 0.2).

The Main purpose of the present paper is to evaluate special values at non-positive integers of two kinds of L-functions, one of which is the one introduced by Hashimoto, associated with the ternary zero form $x_1x_2-x_{12}^2$. We shall follow the method of Satake-Kurihara basically. However, since the quadratic form $x_1x_2-x_{12}^2$ is a zero form (which represents zero non-trivially), we have to deal with partial zeta functions of cones whose edge vectors are not necessarily in the interior of $\mathscr{P}_2$, $\mathscr{P}_2$ being the self-dual homogeneous cone of positive definite symmetric matrices of size two. Because of this reason, Satake-Kurihara's method cannot be applied directly to our situation. We need some original ideas to obtain useful integral representations of partial zeta functions (see Chap. II). The special values of our L-functions are expressed explicitly by (generalized) Bernoulli numbers and special values of Bernoulli polynomials.

Moreover, we shall introduce certain zeta functions with a kind of Gauss sums attached to the space of quadratic forms and express the values of a certain class of integrals appearing in the Selberg trace formula for the trace of $\mu_k(\bar{\alpha})$ ($\bar{\alpha} \in Sp(2n, F_p)$) by using special values at non-positive integers of such zeta functions, where μ_k is the representation of $Sp(2n, F_p)$ in the space of cusp forms of degree n, weight k.

As an application of our results, in the case of $n=2$, we can obtain explicit formulae expressing the traces of $\mu_k(\bar{\alpha})$ by the special values at $s=0$ of our L-functions, which are explicitly evaluated as we stated above.

0.2. We fix the notation and explain our results more precisely. Take an odd prime p and fix it. Let L_n (resp. L_n^*) denote the lattice formed by integral symmetric (resp. half-integral symmetric) matrices of size n, and let $L_{n,+}$, $L_{n,+}^*$ be the subsets consisting of all positive definite matrices of L_n, L_n^*, respectively. Denote by $L_{n,+}^*/SL_n(\boldsymbol{Z})$ (resp. $L_{n,+}/SL_n(\boldsymbol{Z})$) the set of $SL_n(\boldsymbol{Z})$-equivalence classes in $L_{n,+}^*$ (resp. $L_{n,+}$). For an integral symmetric matrix S of size ν ($1 \leqq \nu \leqq n$) with $\det(S) \not\equiv 0 \bmod p$, let $\mathscr{L}_n(S)$ denote the subset of L_n consisting of all $x \in L_n$ such that $x \equiv U\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}{}^t U \bmod p$ with some $U \in GL_n(\boldsymbol{Z}/p\boldsymbol{Z})$. Define a kind of Gauss sum on L_n^* by $\tau_S^{(n)}(T) = \sum_x \exp(2\pi i \operatorname{tr}(Tx/p))$ ($T \in L_n^*$), x running over all residue classes of elements in $\mathscr{L}_n(S) \bmod pL_n$. Set

$$\xi_n^*(s, \tau_S^{(n)}) = \sum_{T \in L_{n,+}^*/SL_n(\boldsymbol{Z})} \tau_S^{(n)}(T) \varepsilon(T)^{-1} \det(T)^{-s},$$

$$L_n^*(s, \chi_{\det}) = \sum_{T \in L_{n,+}^*/SL_n(\boldsymbol{Z})} \chi(\det(T)) \varepsilon(T)^{-1} \det(T)^{-s},$$

where $\varepsilon(T)$ is the order of the unit group $\{U \in SL_n(\boldsymbol{Z}) \mid UT{}^tU = T\}$, and χ is a Dirichlet character $\bmod p$. These Dirichlet series, which are typical examples of Hurwitz-type zeta functions and L-functions of prehomogeneous vector spaces, are absolutely convergent for $\operatorname{Re}(s) > (n+1)/2$. As for a general theory of L-functions of prehomogeneous vector spaces, we refer to Sato [15], in a part of which the functional equations of L-functions (in a general situation) are derived from the works of Gyoja-Kawanaka [3] on prehomogeneous vector spaces over finite fields. In the case of $n=2$, Hashimoto introduced the following L-functions $L_2^*(s, \psi_{H,p})$, $L_2(s, \psi_{H,p})$.

Let ψ be the unique non-trivial quadratic character mod p, and let $\psi_{H,p}$ be a mapping from L_2^* to $\boldsymbol{R}$ given as follows; put $\psi_{H,p}(T)=\psi(t)$ if $T\equiv U\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}{}^tU \bmod p$ with some $U\in GL_2(\boldsymbol{Z}/p\boldsymbol{Z})$, $t\in\boldsymbol{Z}$, and otherwise, put $\psi_{H,q}(T)=0$. Set

$$L_2^*(s, \psi_{H,p})=\sum_{T\in L_{2,+}^*/SL_2(\boldsymbol{Z})}\psi_{H,p}(T)\varepsilon(T)^{-1}\det(T)^{-s},$$
$$L_2(s, \psi_{H,p})=\sum_{T\in L_{2,+}/SL_2(\boldsymbol{Z})}\psi_{H,p}(T)\varepsilon(T)^{-1}\det(T)^{-s}.$$

The L-functions $L_2^*(s, \psi_{H,p})$, $L_2(s, \psi_{H,p})$ are absolutely convergent for $\mathrm{Re}(s)>3/2$. The zeta functions and L-functions given above can be continued analytically to meromorphic functions of s in the whole complex plane which are holomorphic at non-positive integers (see Prop. 1.10, Theorem 1.11 in this paper). The L-functions $L_2^*(s, \chi_{\det})$, $L_2^*(s, \psi_{H,p})$, $L_2(s, \psi_{H,p})$ will be regarded as Siegel L-functions associated with the ternary zero form $x_1x_2-x_{12}^2$ (cf. [20]). Let B_n (resp. $B_n(x)$) be the n-th Bernoulli number (resp. n-th Bernoulli polynomial). Denote by $B_{n,\chi}$ the n-th generalized Bernoulli number attached to a Dirichlet character χ. For a real number x, $\langle x\rangle$ denotes the number satisfying $0<\langle x\rangle\leqq 1$ and $x-\langle x\rangle\in\boldsymbol{Z}$. Set

$$(0.1)\qquad \mathscr{A}=-\sum_{\alpha,\gamma}{}' B_1(\langle(\alpha^2-2\alpha\gamma)/p\rangle)B_1(\langle 2\alpha\gamma/p\rangle)B_1(\langle(\gamma^2-\alpha^2)/p\rangle)+\frac{1}{12}(3+\delta_{p,3})B_{1,\psi},$$

where α, γ run over all residue classes mod p satisfying $\alpha^2\not\equiv 2\alpha\gamma \bmod p$, $\alpha\gamma\not\equiv 0 \bmod p$, $\alpha^2\not\equiv\gamma^2 \bmod p$, and $\delta_{p,3}$ is the Kronecker symbol ($\delta_{p,3}=1$ if $p=3$, and otherwise, $\delta_{p,3}=0$). Moreover, set

$$(0.2)\qquad \mathscr{B}=-\frac{1}{3}\sum_{\alpha,\gamma}{}'' B_2(\langle(\alpha^2-2\alpha\gamma)/p\rangle)B_1(\langle(\gamma^2-\alpha^2)/p\rangle),$$

where α, γ run over all residue classes mod p satisfying $\alpha^2\not\equiv\gamma^2 \bmod p$.

Theorem 1. *Let p be an odd prime.*

(i) *The special values $L_2^*(1-m, \psi_{H,p})$ $(m=1, 2, \cdots)$ are rational numbers.*

(ii) *If $p\equiv 1$ mod 4, then $L_2^*(0, \psi_{H,p})=0$.*

(iii) *If $p\equiv 3$ mod 4, then,*

$$L_2^*(0, \psi_{H,p})=\mathscr{A}+\mathscr{B}+\frac{11}{36p}B_{3,\psi}-\frac{1}{24p}B_{1,\psi}.$$

We present a conjecture on the value $L_2^*(0, \psi_{H,p})$.

Conjecture I. *If* $p\equiv 3 \bmod 4$, $L_2^*(0, \psi_{H,p})=\dfrac{1}{24}B_{1,\psi}$.

(Note that $h(-p)=-B_{1,\psi}$ for $p>3$ and $p\equiv 3 \bmod 4$, where $h(-p)$ is the class number of the quadratic field $\boldsymbol{Q}(\sqrt{-p})$.)

Conjecture I is true for $p<500$ by the numerical calculation using a computer.

Theorem 2. *Let* χ *be a primitive character* mod p ($p>2$). *Then, for* $m=1, 2, \cdots$,

$$L_2^*(1-m, \chi_{\det})=\begin{cases} \dfrac{\chi(-4)^{-1}(-1)^m}{2^{2m+1}m}B_{2m,\chi^2}\cdots & \text{if } \chi\neq\psi, \\ \dfrac{\chi(-1)(-1)^{m-1}}{2^{2m+1}m}(p^{2m-1}-1)B_{2m}\cdots & \text{if } \chi=\psi. \end{cases}$$

Theorems 1, 2 will be regarded as a kind of generalization of the well-known formula $L(1-m, \chi)=-B_{m,\chi}/m$ for Dirichlet L-function $L(s, \chi)$.

Let $\Gamma_{2n}(p)$ be the principal congruence subgroup of $\Gamma_{2n}=Sp(2n, \boldsymbol{Z})$ with level p. The quotient group $\Gamma_{2n}/\Gamma_{2n}(p)$ is isomorphic to the finite symplectic group $Sp(2n, \boldsymbol{F}_p)$ of degree $2n$, and the surjection $\Gamma_{2n}\longrightarrow Sp(2n, \boldsymbol{F}_p)$ is denoted by $\alpha\longrightarrow\bar{\alpha}$ ($\alpha\in\Gamma_{2n}$). Let $\mathfrak{H}_n$ be the Siegel upper half plane of degree n, on which the real symplectic group $\mathfrak{G}_{2n}=Sp(2n, \boldsymbol{R})$ acts in a usual manner; for $Z=X+iY\in\mathfrak{H}_n$ and for $\gamma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\mathfrak{G}_{2n}$, $\gamma\langle Z\rangle =(AZ+B)(CZ+D)^{-1}$. Set $J(\gamma, Z)=\det(CZ+D)$, and

$$H(\gamma; Z)=J(\gamma, Z)^{-k}\det\left(\frac{\gamma\langle Z\rangle-\bar{Z}}{2i}\right)^{-k}\det(Y)^k,$$

$$dZ=\det(Y)^{-n-1}\prod_{1\leq i\leq j\leq n} dX_{ij}dY_{ij}.$$

For any subset Π of Γ_{2n} which is invariant by the conjugation of any elements in $\Gamma_{2n}(p)$, set

$$I_n(\Pi; k)=a(k)\int_{\Gamma_{2n}(p)\backslash\mathfrak{H}_n}\sum_{\gamma\in\Pi} H(\gamma; Z)dZ,$$

where $a(k)$ is a constant given by (3.1.3) in Chap. III. Denote by $\mathfrak{S}_k(\Gamma_{2n}(p))$ the space of Siegel cusp forms of degree n, weight k with respect to $\Gamma_{2n}(p)$. The representation μ_k of the group $Sp(2n, \boldsymbol{F}_p)$ in the space $\mathfrak{S}_k(\Gamma_{2n}(p))$ is given by $\mu_k(\bar{\alpha}^{-1})f(Z)=J(\alpha, Z)^{-k}f(\alpha\langle Z\rangle)(\alpha\in\Gamma_{2n}, f\in$

$\mathfrak{S}_k(\Gamma_{2n}(p))$, and see (3.1.2)). It is known by [2] that $\mathrm{tr}(\mu_k(\bar{\alpha}^{-1}))= I_n(\Gamma_{2n}(p)\alpha; k)$ for $k>2n$. For a symmetric matrix x of size ν, denote by $t_{n,\nu}(x)=\begin{pmatrix}1_n & \tilde{x}\\ 0 & 1_n\end{pmatrix}$, where $\tilde{x}=\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}$ (see (3.1.4)). For $\alpha\in\Gamma_{2n}$, let $\Pi_r(\alpha)$ $(1\leqq r\leqq n)$ be the set consisting of all elements in $\Gamma_{2n}(p)\alpha$ that are conjugate to some elements $t_{n,r}(x)$ with $x\in L_r$, $\det(x)\neq 0$ ($\Pi_r(\alpha)$ depends only on $\bar{\alpha}$). Following Shintani [16, Chap. 2], we will show that, if $k\geqq 2n+3$, then, for each $\alpha=t_{n,\nu}(S)$ with $S\in L_\nu$, $\det(S)\not\equiv 0 \bmod p$,

$$I_n(\Pi_r(\alpha); k)=[\Gamma_{\alpha,p}: \Gamma_{2n}(p)]p^{-r(n-(r-1)/2)}b(n, k, r)\Omega_{n,r}\xi_r^*(r-n, \tau_S^{(r)}) \qquad (\nu\leqq r\leqq n),$$

where $b(n, k, r)$, $\Omega_{n,r}$ are rational numbers given by (3.2.7), (3.2.8), respectively, and $\Gamma_{\alpha,p}=\{\gamma\in\Gamma_{2n}\mid \gamma^{-1}\alpha\gamma\equiv\alpha \bmod p\}$ (a subgroup of Γ_{2n}), of which $\Gamma_{2n}(p)$ is a normal subgroup with index finite. If $\alpha=1_{2n}$, the results above coincide with those of [16]. In the case of $n=2$, $\nu=1$, put, for any integer μ prime to p, $\alpha_\mu=t_{2,1}(\mu)$ ($\in\Gamma_4$). It is essentially known by [13], [1], [4], [5], and can be proved in a similar manner that, if $k\geqq 7$, $\mathrm{tr}(\mu_k(\bar{\alpha}_\mu))=\sum_{r=1}^{2} I_2(\Pi_r(\alpha_{-\mu}); k)$. It will be shown that, if $k\geqq 7$,

$$\begin{aligned}(0.3)\quad \mathrm{tr}(\mu_k(\bar{\alpha}_\mu))= &-2^{-5}3^{-1}p^2(p^2-1)\{\psi(-\mu)\tau_\psi B_{2,\psi}+(p^2-1)/6\}(2k-3)\\ &+2^{-1}p(p^2-1)\{\psi(-1)pL_2^*(0, \psi_{\det})+\psi(-\mu)\tau_\psi pL_2^*(0, \psi_{H,p})\\ &+2^{-4}3^{-1}(p^2-1)\},\end{aligned}$$

where τ_ψ is the Gauss sum associated with ψ.

Theorem 3. *Let p be an odd prime and let κ be a non-quadratic residue* mod p. *Let $k\geqq 7$. The difference of the two traces* $\mathrm{tr}(\mu_k(\bar{\alpha}_\mu))$ $(\mu=1, \kappa)$ *is given by*

$$(0.4)\quad \mathrm{tr}(\mu_k(\bar{\alpha}_1))-\mathrm{tr}(\mu_k(\bar{\alpha}_\kappa))=\begin{cases}-2^{-4}3^{-1}p^2(p^2-1)\sqrt{p}\,B_{2,\psi}(2k-3) & \cdots p\equiv 1 \bmod 4,\\ -p^2(p^2-1)\sqrt{-p}L_2^*(0, \psi_{H,p}) & \cdots p\equiv 3 \bmod 4.\end{cases}$$

We present another conjecture, which is based on Conjecture I.

Conjecture II. *If $p>3$, $p\equiv 3$ mod 4, and $k\geqq 7$, then,*

$$\mathrm{tr}(\mu_k(\bar{\alpha}_1))-\mathrm{tr}(\mu_k(\bar{\alpha}_\kappa))=\frac{1}{24}p^2(p^2-1)\sqrt{-p}h(-p).$$

Tsushima calculated the traces of $\mu_k(\bar{\alpha}_\mu)$ in his private notes, which are based on the results of [22]. Lee-Weintraub [11] announced the explicit values of the imaginary parts of $\mathrm{tr}(\mu_k(\bar{\alpha}_\mu))$ and presented a conjecture analogous to our Conjecture II. Their methods are purely geometric and the results do not involve special values of L-functions. We are much concerned how the traces can be expressed by special values of L-functions and how they can be explicitly evaluated.

In his lecture at Kyoto in 1985, Hashimoto announced that the difference of the traces of $\mu_k(\bar{\alpha}_\mu)$ $(\mu=1, \kappa)$ is given as follows:

$$\text{(0.5)} \quad \operatorname{tr}(\mu_k(\bar{\alpha}_1)) - \operatorname{tr}(\mu_k(\bar{\alpha}_\kappa)) = \frac{-p^4(p^2-1)i}{4\pi^2} L_2(3/2, \psi_{H,p}) \cdots p \equiv 3 \bmod 4.$$

It follows from (0.4), (0.5) that

$$\text{(0.6)} \qquad L_2(3/2, \psi_{H,p}) = 4\pi^2 p^{-3/2} L_2^*(0, \psi_{H,p}) \cdots \text{if } p \equiv 3 \bmod 4,$$

which, together with (iii) of Theorem 1, will be regarded as a Kronecker limit formula for $L_2(s, \psi_{H,p})$. It is guessed that $L_2^*(s, \psi_{H,p})$, $L_2(s, \psi_{H,p})$ are related with each other by a functional equation under $s \to 3/2 - s$. The relation (0.6) will be derived from the functional equation.

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Notation

Let $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the set of natural numbers, the ring of rational integers, the rational number field, the real number field, and the complex number field, respectively. For any commutative ring S, $M(m, n; S)$, $M_n(S)$, $GL_n(S)$, and $SL_n(S)$ denote the module of $m \times n$ matrices with entries in S, the ring of matrices of size n with entries in S, the group of invertible elements in $M_n(S)$, and the group of elements in $M_n(S)$ whose determinants are one, respectively. For any element A of $M_n(S)$, let ${}^tA$, $\operatorname{tr}(A)$, and $\det(A)$ denote the transposed matrix of A, the trace of A, and the determinant of A, respectively. We denote by 1_n the unit matrix of $M_n(S)$. Moreover, we put $S^\times = GL_1(S)$.

For any element Z of $M_n(\boldsymbol{C})$, we denote by $\operatorname{Re}(Z)$, $\operatorname{Im}(Z)$, and $\bar{Z}$, the real part of Z, the imaginary part of Z, and the complex conjugate of Z, respectively. For real symmetric matrices A, B of the same size, $A > B$ means that $A - B$ is positive definite. For any $x \in \boldsymbol{R}$, $\langle x \rangle$ denotes the real number with $x - \langle x \rangle \in \boldsymbol{Z}$, $0 < \langle x \rangle \leq 1$. Let $\Gamma(s)$ and $\zeta(s)$ be the gamma function and the Riemann zeta function, respectively. Finally, the symbol $e[w]$ ($w \in \boldsymbol{C}$) is used as an abbreviation for $\exp(2\pi i w)$.

Chapter I. *L*-functions of quadratic forms

1.1. Definition of zeta functions and *L*-functions

Following Shintani [16], we shall define certain zeta functions with Gauss sums and certain *L*-functions which are associated with the vector space of symmetric matrices. We shall not discuss a general theory of those functions but only some properties that will be needed in later chapters.

Let $G_{\boldsymbol{R}}^{(n)}$ be $GL_n(\boldsymbol{R})$, and let $V_{\boldsymbol{R}}^{(n)}$ be the $\boldsymbol{R}$-vector sapce of real symmetric matrices of size n. Then, the group $G_{\boldsymbol{R}}^{(n)}$ acts on $V_{\boldsymbol{R}}^{(n)}$ in a usual manner; for $g \in G_{\boldsymbol{R}}^{(n)}$, $x \in V_{\boldsymbol{R}}^{(n)}$, we put $\rho(g)x = gx^t g$. Let L_n be the lattice of $V_{\boldsymbol{R}}^{(n)}$ consisting of all integral symmetric matrices of size n, and let L_n^* be its dual with respect to the bilinear form $(x, y) = \mathrm{tr}(xy)$ ($x, y \in V_{\boldsymbol{R}}^{(n)}$). Namely, L_n^* is the lattice consisting of half-integral symmetric matrices of size n.

Let p be an odd prime and fix it once and for all. Denote by $\boldsymbol{Z}_{(2)}$ the $\boldsymbol{Z}$-module consisting of all rational numbers of the form $2^{-m}x$ with $m, x \in \boldsymbol{Z}$. For $a, b \in \boldsymbol{Z}_{(2)}$, $a \equiv b \bmod p$ means that $(a-b)/p$ is a p-adic integer. Let $1 \leqq \nu \leqq n$ and let $S \in L_\nu$ with $\det(S) \not\equiv 0 \bmod p$. We set

$$\mathscr{L}_n(S) = \left\{ x \in L_n \,|\, x \equiv U \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} {}^t U \bmod p \text{ with some } U \in GL_n(\boldsymbol{Z}/p\boldsymbol{Z}) \right\},$$

$\boldsymbol{Z}/p\boldsymbol{Z}$ being the ring of residue classes mod p. Then, $\mathscr{L}_n(S)$ is invariant under the action of $SL_n(\boldsymbol{Z})$; namely, $U\mathscr{L}_n(S){}^t U = \mathscr{L}_n(S)$ for any $U \in SL_n(\boldsymbol{Z})$. Moreover, $\mathscr{L}_n(VS{}^t V) = \mathscr{L}_n(S)$ for any $V \in GL_\nu(\boldsymbol{Z}/p\boldsymbol{Z})$. If $\nu < n$, this definition of $\mathscr{L}_n(S)$ amounts to saying that

$$\mathscr{L}_n(S) = \left\{ x \in L_n \,|\, x \equiv U \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} {}^t U \bmod p \text{ with some } U \in SL_n(\boldsymbol{Z}) \right\}.$$

We denote by $\mathscr{L}_n(S)/pL_n$ all residue classes of elements in $\mathscr{L}_n(S) \bmod pL_n$. Define a subgroup $\tilde{\Gamma}_{S,\infty}$ of $GL_n(\boldsymbol{Z}/p\boldsymbol{Z})$ by

$$\tilde{\Gamma}_{S,\infty} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_n(\boldsymbol{Z}/p\boldsymbol{Z}) \,|\, AS\,{}^t A \equiv S \bmod p,\ C \equiv 0 \bmod p \right\}.$$

A complete set of representatives of $\mathscr{L}_n(S)/pL_n$ is given by the set

$$\left\{ U \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} {}^t U \,|\, U \in GL_n(\boldsymbol{Z}/p\boldsymbol{Z})/\tilde{\Gamma}_{S,\infty} \right\}. \tag{1.1.1}$$

A kind of Gauss sum $\tau_S^{(n)}(T)$ $(T \in L_n^*)$ is defined as follows:

$$(1.1.2) \qquad \tau_S^{(n)}(T) = \sum_{x \in \mathscr{L}_n(S)/pL_n} e[\mathrm{tr}\,(Tx)/p],$$

which depends only on $T \bmod p$. Then we have $\tau_S^{(n)}(UT{}^tU) = \tau_S^{(n)}(T)$ for any $U \in GL_n(\boldsymbol{Z}/p\boldsymbol{Z})$. Denote by $L_{n,+}^*$ the subset of L_n^* consisting of all positive definite symmetric matrices. For each $T \in L_{n,+}^*$, let $\varepsilon(T)$ be the order of the unit group $\{U \in SL_n(\boldsymbol{Z}) \mid UT{}^tU = T\}$. Two matrices T_1, T_2 of L_n^* are called $SL_n(\boldsymbol{Z})$-equivalent, if there exists some $U \in SL_n(\boldsymbol{Z})$ with $T_2 = UT_1^tU$. Denote by $L_{n,+}^*/SL_n(\boldsymbol{Z})$ the set of $SL_n(\boldsymbol{Z})$-equivalence classes in $L_{n,+}^*$. We define a zeta function $\xi_n^*(s, \tau_S^{(n)})$ with the Gauss sum $\tau_S^{(n)}$ as follows:

$$\xi_n^*(s, \tau_S^{(n)}) = \sum_{T \in L_{n,+}^*/SL_n(\boldsymbol{Z})} \tau_S^{(n)}(T)\varepsilon(T)^{-1} \det(T)^{-s}.$$

For a primitive character $\chi \bmod p$, we also define an L-function $L_n^*(s, \chi_{\det})$ by putting

$$L_n^*(s, \chi_{\det}) = \sum_{T \in L_{n,+}^*/SL_n(\boldsymbol{Z})} \chi(\det(T))\varepsilon(T)^{-1} \det(T)^{-s},$$

where χ is naturally extended over $\boldsymbol{Z}_{(2)}$. Moreover, set

$$\xi_n^*(s) = \sum_{T \in L_{n,+}^*/SL_n(\boldsymbol{Z})} \varepsilon(T)^{-1} \det(T)^{-s}.$$

It is well-known that $\xi_n^*(s)$ is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$, Therefore, $\xi_n^*(s, \tau_S^{(n)})$, $L_n^*(s, \chi_{\det})$ also converge absolutely if $\mathrm{Re}(s) > (n+1)/2$. The zeta function $\xi_n^*(s)$ is one of the zeta functions intensively studied by Shintani in [16].

Denote by ψ the unique non-trivial quadratic character mod p, which is characterized by $\psi(a) = (a/p)$ for any integer a prime to p, (a/p) being the Legendre symbol. Let $\psi_{H,p}$ be a mapping from L_2^* to $\boldsymbol{R}$ given by

$$\psi_{H,p}(T) = \begin{cases} \psi(t) \cdots \text{if } T \equiv U\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} {}^tU \bmod p \text{ with some } U \in GL_2(\boldsymbol{Z}/p\boldsymbol{Z}) \text{ and} \\ \qquad\qquad t \in \boldsymbol{Z} \text{ prime to } p, \\ 0 \cdots\cdots \text{otherwise.} \end{cases}$$

In the case of $n=2$, Hashimoto introduced the following L-function:

$$L_2^*(s, \psi_{H,p}) = \sum_{T \in L_{2,+}^*/SL_2(\boldsymbol{Z})} \psi_{H,p}(T)\varepsilon(T)^{-1} \det(T)^{-s},$$

which is absolutely convergent for $\mathrm{Re}(s) > 3/2$. We have a relation among $\xi_2^*(s, \tau_S^{(2)})$, $L_2^*(s, \psi_{H,p})$, $L_2^*(s, \psi_{\det})$, and $\xi_2^*(s)$, which is given in Proposition 1.2.

Lemma 1.1. *Let μ be any integer prime to p and take μ as S in* (1.1.2) *in the case of $n=2$, $\nu=1$. Then, for $T\in L_2^*$, we have*

(i) $\tau_\mu^{(2)}(T)=\{\psi(-1)\psi(\det(T))p-1\}/2\cdots$ *if* $\det(T)\not\equiv 0 \bmod p$,

(ii) $\tau_\mu^{(2)}(T)=\{\psi(\mu)\psi_{H,p}(T)p\tau_\psi-1\}/2\cdots$ *if* $\det(T)\equiv 0 \bmod p$, $T\not\equiv 0 \bmod p$,

(iii) $\tau_\mu^{(2)}(T)=(p^2-1)/2\cdots$ *if* $T\equiv 0 \bmod p$,

where τ_ψ is the ordinary Gauss sum associated with ψ: $\tau_\psi=\sum_{a\not\equiv 0 \bmod p}\psi(a)\cdot e[a/p]$.

Proof. Writing $U=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL_2(Z/pZ)$, we have $U\begin{pmatrix}\mu & 0\\ 0 & 0\end{pmatrix}{}^tU=\mu\begin{pmatrix}\alpha^2 & \alpha\gamma\\ \alpha\gamma & \gamma^2\end{pmatrix}$. We set

(1.1.3) $\mathcal{M}(p)=\{(\alpha,\gamma)\in Z/pZ\times Z/pZ\,|\,(\alpha,\gamma)\not\equiv(0,0) \bmod p\}$.

It is easy to see from (1.1.1) that, for each $T\in L_2^*$,

$$\tau_\mu^{(2)}(T)=\frac{1}{2}\sum_{(\alpha,\gamma)\in\mathcal{M}(p)}e\left[\mu\,\mathrm{tr}\left(\begin{pmatrix}\alpha^2 & \alpha\gamma\\ \alpha\gamma & \gamma^2\end{pmatrix}T\right)/p\right].$$

If $\det(T)\not\equiv 0 \bmod p$, we may assume that $T\equiv\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix} \bmod p$ with $(t_j,p)=1$ $(j=1,2)$. Then we get

$$\tau_\mu^{(2)}(T)=\frac{1}{2}\left\{\sum_{(\alpha,\gamma)\in\mathcal{M}(p)}e[\mu(t_1\alpha^2+t_2\gamma^2)/p]\right\}.$$

As is well-known, if $(a,p)=1$, then, $\sum_{u \bmod p}e[au^2/p]=\psi(a)\tau_\psi$. Thus we have, with the help of the property $\tau_\psi^2=\psi(-1)p$,

$$\tau_\mu^{(2)}(T)=\frac{1}{2}\{\psi(-1)\psi(t_1t_2)p-1\}.$$

Next let $\det(T)\equiv 0 \bmod p$ and $T\not\equiv 0 \bmod p$. We may assume that $T\equiv\begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix} \bmod p$ with $(t,p)=1$. Then we get

$$\tau_\mu^{(2)}(T)=\frac{1}{2}\{\psi(\mu t)p\tau_\psi-1\},$$

from which the assertion (ii) follows. The assertion (iii) is clear.

Proposition 1.2. *Let μ be any integer prime to p. Then,*

$$\xi_2^*(s, \tau_\mu^{(2)})=\frac{1}{2}\{\psi(-1)pL_2^*(s, \psi_{\det})+\psi(\mu)p\tau_\psi L_2^*(s, \psi_{H,p})-(1-p^{2-2s})\xi_2^*(s)\}.$$

We omit the proof of Proposition 1.2, which is immediate from Lemma 1.1.

Let κ be a non-quadratic residue mod p. Taking 1, κ as μ in Proposition 1.2, we get the following corollary.

Corollary to Proposition 1.2. *We have*

$$p\tau_\psi L_2^*(s, \psi_{H,p})=\xi_2^*(s, \tau_1^{(2)})-\xi_2^*(s, \tau_\kappa^{(2)}),$$
$$\psi(-1)pL_2^*(s, \psi_{\det})=\xi_2^*(s, \tau_1^{(2)})+\xi_2^*(s, \tau_\kappa^{(2)})+(1-p^{2-2s})\xi_2^*(s).$$

1.2. Some properties of $\xi_n^*(s, \tau_S^{(n)})$, $L_2^*(s, \psi_{\det})$, and $L_2^*(s, \psi_{H,p})$ (analytic continuations, poles, residues)

We follow Section 2 of Chapter 2 in [16]. Let $\mathscr{P}_n$ be the symmetric space formed by positive definite real symmetric matrices of size n. For each $\lambda \in N$, the functions $f_n(x, \lambda), f_n^*(x, \lambda)$ $(x \in V_{\boldsymbol{R}}^{(n)})$ on $V_{\boldsymbol{R}}^{(n)}$ are defined as follows:

$$f_n(x, \lambda)=\begin{cases}\det(x)^{\lambda-(n+1)/2}\exp(-2\pi \operatorname{tr}(x))\cdots \text{if } x \in \mathscr{P}_n,\\ 0\cdots \text{if } x \in V_{\boldsymbol{R}}^{(n)},\ x \notin \mathscr{P}_n,\end{cases}$$
$$f_n^*(x, \lambda)=\det(1_n-ix)^{-\lambda}.$$

An Euclidean measure dx on $V_{\boldsymbol{R}}^{(n)}$ is normalized by $dx=\prod_{1\leq i\leq j\leq n} dx_{ij}$. Put $\chi(g)=\det(g)^2$. Moreover, we set

$$\gamma_n(s)=\prod_{i=0}^{n-1}\Gamma(s+1+i/2), \qquad C_n=\prod_{k=1}^{n}\frac{2\pi^{k/2}}{\Gamma(k/2)},$$

and

$$v_n=\begin{cases}\dfrac{1}{2}\zeta(2)\zeta(3)\cdots\zeta(n) & (n\geqq 2)\\[2ex] \dfrac{1}{2} & (n=1).\end{cases}$$

Let p be an odd prime and fix it.

Lemma 1.3. *Let* $1\leqq \nu\leqq n$, *and let* $S \in L_\nu$ *with* $\det(S)\not\equiv 0 \bmod p$. *If* $\lambda>n+1$, *then,*

$$\sum_{T\in L^*}\tau_S^{(n)}(T)f_n(\rho({}^tg^{-1})(T/p), \lambda)=\mu_{n,\lambda}\chi(g)^{(n+1)/2}\sum_{x\in\mathscr{L}_n(S)}f_n^*(\rho(g)x, \lambda),$$

where $g \in G_{\boldsymbol{R}}^{(n)}$, *and* $\mu_{n,\lambda}=(4\pi)^{n(n-1)/4}p^{n(n+1)/2}(2\pi)^{-\lambda n}\gamma_n(\lambda-(n+1)/2)$ (*both sides are absolutely convergent*).

Proof. The Fourier transform of $f_n(\rho({}^tg^{-1})x, \lambda)$ on $V_{\boldsymbol{R}}^{(n)}$ is given by

$$\int_{V_{\boldsymbol{R}}^{(n)}} f_n(\rho({}^tg^{-1})x, \lambda)\, e[\operatorname{tr}(xy)]dx$$
$$=\pi^{n(n-1)/4}(2\pi)^{-\lambda n}\gamma_n(\lambda-(n+1)/2)\chi(g)^{(n+1)/2}f_n^*(\rho(g)y, \lambda)$$

($g \in G_{\boldsymbol{R}}^{(n)}$, $y \in V_{\boldsymbol{R}}^{(n)}$, and the integral converges absolutely).

This identity is nothing but Hilfssatz 37 of [18]. Replacing y with $u+py$ ($u \in V_{\boldsymbol{R}}^{(n)}$) and changing the variables by $x \to x/p$, we get

$$\int_{V_{\boldsymbol{R}}^{(n)}} f_n(\rho({}^tg^{-1})(x/p), \lambda)\, e[\operatorname{tr}(ux/p)]\, e[\operatorname{tr}(xy)]dx$$
$$=2^{-n(n-1)/2}\mu_{n,\lambda}\chi(g)^{(n+1)/2}f_n^*(\rho(g)(u+py), \lambda).$$

By virtue of the Poisson summation formula (see (iii) of Lemma 19 in [16]), we have

$$\sum_{x \in L_n^*} f_n(\rho({}^tg^{-1})(x/p), \lambda)\, e[\operatorname{tr}(ux/p)]=\mu_{n,\lambda}(g)^{(n+1)/2}\sum_{y \in L_n} f_n^*(\rho(g)(u+py), \lambda)$$

(the both sides are absolutely convergent for $\lambda>n+1$).

If we let u run over $\mathscr{L}_n(S)/pL_n$, we obtain the formula in Lemma 1.3.

Let d_ng be a Haar measure on $G_{\boldsymbol{R}}^{(n)}$ normalized by

$$d_ng=\det(g)^{-n}\prod_{1\leq i, j\leq n} dg_{ij}.$$

Set $G_{\boldsymbol{R},+}^{(n)}=\{g \in G_{\boldsymbol{R}}^{(n)} \mid \det(g)>0\}$. We put

$$(1.2.1)\quad Z^*(s, \tau_S^{(n)})=\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z})} \chi(g)^s \sum_{T \in L_n^*} \tau_S^{(n)}(T) f_n(\rho(g)(T/p), \lambda)d_ng.$$

We denote by $Z_+^*(s, \tau_S^{(n)})$ the integral obtained by replacing the region of integration with the set $\{g \in G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z}) \mid \chi(g)\geq 1\}$ in the right side of (1.2.1). As is shown in Lemma 21, (i) in [16], the integral $Z^*(s, \tau_S^{(n)})$ is absolutely convergent if $\operatorname{Re}(s)>(n+1)/2$, $\operatorname{Re}(\lambda+s)>n$, and then,

$$(1.2.2)\quad Z^*(s, \tau_S^{(n)})$$
$$=p^{ns}\pi^{n(n-1)/4}(2\pi)^{-(\lambda+s-(n+1)/2)n}2^{-n-1}C_n\gamma_n(\lambda+s-n-1)\xi_n^*(s, \tau_S^{(n)}).$$

We keep the conditions: $S \in L_\nu$, $\det(S)\not\equiv 0 \bmod p$. Set, for $r \in N$ ($r\leq n$),

$$(1.2.3)\qquad \mathscr{L}_n^{(r)}(S)=\{x \in \mathscr{L}_n(S) \mid \operatorname{rank}(x)=r\}.$$

We see immediately that, if $\mathscr{L}_n^{(r)}(S)\neq\emptyset$, then, of necessity, $\nu\leqq r\leqq n$.

Lemma 1.4. *Let $\nu\leqq r\leqq n$. Each $x\in\mathscr{L}_n^{(r)}(S)$ can be written as $x=U\begin{pmatrix}x_1&0\\0&0\end{pmatrix}{}^tU$ with some $x_1\in\mathscr{L}_r^{(r)}(S)$, $U\in SL_n(\boldsymbol{Z})$.*

Proof. We assume that $r<n$, otherwise we have nothing to do. Take $x\in\mathscr{L}_n^{(r)}(S)$. Then there exist $U\in SL_n(\boldsymbol{Z})$ and $x_1\in L_r$ with $\det(x_1)\neq 0$ such that $x=U\begin{pmatrix}x_1&0\\0&0\end{pmatrix}{}^tU$. Since $x\in\mathscr{L}_n(S)$, the rank of x_1 mod p as a matrix of $M_r(\boldsymbol{Z}/p\boldsymbol{Z})$ is ν. Thus there exist some $W_1\in GL_r(\boldsymbol{Z}/p\boldsymbol{Z})$ and $x_2\in L_\nu$, $\det(x_2)\not\equiv 0 \bmod p$ with the condition

$$(1.2.4)\qquad x_1\equiv W_1\begin{pmatrix}x_2&0\\0&0\end{pmatrix}{}^tW_1 \bmod p.$$

Since $x\in\mathscr{L}_n(S)$, so is $\begin{pmatrix}x_1&0\\0&0\end{pmatrix}$. Hence we see easily from (1.2.4) that there exists some $V\in GL_n(\boldsymbol{Z}/p\boldsymbol{Z})$ such that $\begin{pmatrix}x_2&0\\0&0\end{pmatrix}\equiv V\begin{pmatrix}S&0\\0&0\end{pmatrix}{}^tV \bmod p$. Writing $V=\begin{pmatrix}V_1&V_2\\V_3&V_4\end{pmatrix}$ with $V_1\in M_\nu(\boldsymbol{Z}/p\boldsymbol{Z})$, we have $x_2\equiv V_1S{}^tV_1 \bmod p$, which implies that $V_1\in GL_\nu(\boldsymbol{Z}/p\boldsymbol{Z})$. We get, again by (1.2.4), $x_1\equiv W\begin{pmatrix}S&0\\0&0\end{pmatrix}{}^tW \bmod p$ with some $W\in GL_r(\boldsymbol{Z}/p\boldsymbol{Z})$. Thus, $x_1\in\mathscr{L}_r^{(r)}(S)$. q.e.d.

Let P_n^r denote the subgroup of $G_{\boldsymbol{R}}^{(n)}$ formed by all matrices whose left lower $(n-r)\times r$ blocks are zero. Denote by $P_{n,+}^r$ the connected component of 1_n in P_n^r.

Lemma 1.5. *The following decompositions hold.*

(i) $\mathscr{L}_n(S)=\bigcup_{r=\nu}^n\mathscr{L}_n^{(r)}(S)$ *(disjoint union).*

(ii) *For each r ($\nu\leqq r\leqq n$),*

$$\mathscr{L}_n^{(r)}(S)=\bigcup_{U\in SL_n(\boldsymbol{Z})/SL_n(\boldsymbol{Z})\cap P_n^r}\left\{U\begin{pmatrix}x&0\\0&0\end{pmatrix}{}^tU \,\middle|\, x\in\mathscr{L}_r^{(r)}(S)\right\}$$

(disjoint union).

The proof is due to Lemma 1.4 and is immediate.

We set

$$(1.2.5)\quad Z(f_n^*(x,\lambda),\mathscr{L}_n(S),s)=\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z})}\chi(g)^s\sum_{x\in\mathscr{L}_n^{(n)}(S)}f_n^*(\rho(g)x,\lambda)d_ng.$$

By virtue in Lemma 21, (ii) in [16] and Lemma 1.5 above, one has the following proposition (or one can prove it in the same manner).

Proposition 1.6. *Assume that λ, s satisfy the following inequalities*:

$$(1.2.6)\qquad \begin{cases} \lambda>1,\ \mathrm{Re}(s)<\lambda & \textit{for } n=1, \\ \lambda>\mathrm{Max}\,(13/2,\ 2\,\mathrm{Re}(s)+7/2) & \textit{for } n=2, \\ \lambda>n+7/2,\ \mathrm{Re}(s)<\lambda-(n-1)/2 & \textit{for } n\geqq 3. \end{cases}$$

In addition to (1.2.6), *if s satisfies* $\mathrm{Re}(s)>(n-1)/2$, *then the integral* $Z(f_n^*(x, \lambda), \mathscr{L}_n(S), s)$ *is absolutely convergent.*

We impose the following assumption on λ:

(1.2.7) $\lambda>1$ for $n=1$, $\lambda>13/2$ for $n=2$, and $\lambda>n+7/2$ for $n\geqq 3$.

For $\lambda\in \boldsymbol{N}$ satisfying (1.2.7), we put

$$(1.2.8)\qquad a_\lambda=\begin{cases} \lambda-1 & \text{for } n=1 \\ (\lambda-13/2)/2 & \text{for } n=2 \\ \lambda-n & \textit{for } n\geqq 3. \end{cases}$$

We define the integral $Z_+(f_n^*(x, \lambda), \mathscr{L}_n(S), s)$ by restricting the region of integration to the set $\{gG\in \mathscr{P}_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z})\,|\,\chi(g)\geqq 1\}$ in the definition (1.2.5) of $Z(f_n^*(x, \lambda), \mathscr{L}_n(S), s)$. We see easily from Proposition 1.6 that $Z(f_n^*(x, \lambda), \mathscr{L}_n(S), (n+1)/2-s)$ is absolutely convergent if $-a_\lambda<\mathrm{Re}(s)<1$, and hence that $Z_+(f_n^*(x, \lambda), \mathscr{L}(S), (n+1)/2-s)$ is absolutely convergent for $-a_\lambda<\mathrm{Re}(s)$.

Proposition 1.7. *Assume that λ satisfies the condition* (1.2.7). *If* $\mathrm{Re}(s)>(n+1)/2$, *then the following identity holds*:

$$(1.2.9)\qquad Z^*(s, \tau_S^{(n)})=Z_+^*(s, \tau_S^{(n)})+\mu_{n,\lambda}\Big\{Z_+(f_n^*(x, \lambda), \mathscr{L}_n(S), (n+1)/2-s) + \sum_{r=\nu}^{n-1}\frac{C_n v_{n-r}}{C_r C_{n-r}(s-(n+1-r)/2)} Z(f_r^*(x, \lambda), \mathscr{L}_r(S), n/2)\Big\}.$$

Proposition 1.7 can be proved quite in a similar manner as in the proof of Lemma 21, (iii) of [16]. For the convenience of the reader, we give a proof, which is based on Proposition 1.6.

Proof of Proposition 1.7. First we notice that the integral $Z(f_r^*(x, \lambda), \mathscr{L}_r(S), n/2)$ is absolutely convergent by Proposition 1.6. Using Lemma 1.3, we have, if $\mathrm{Re}(s)>(n+1)/2$,

$$(1.2.10)\quad Z^*(s,\tau_S^{(n)})=Z_+^*(s,\tau_S^{(n)}) + \mu_{n,\lambda}\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z}),\,\chi(g)\geqq 1}\chi(g)^{(n+1)/2-s}\sum_{x\in\mathscr{L}_n(S)} f_n^*(\rho(g)x,\lambda)d_n g.$$

We need the next lemma. For the proof, see [19] and also Lemma 17 in [16].

Lemma 1.8. *If* $\mathrm{Re}(s)>0$ *and* $t>0$, *then*

$$\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z}),\,\chi(g)\geqq 1/t}\chi(g)^{-s}d_n g=\frac{t^s}{s}v_n.$$

Let dp be the right invariant measure on $P_{n,+}^r$ normalized by

$$dp=\det(p_1)^{n-r}\det(p_2)^{-r}d_r p_1 d_{n-r}p_2 dq,$$

where $p=\begin{pmatrix}p_1 & 0\\ 0 & p_2\end{pmatrix}\begin{pmatrix}1_r & q\\ 0 & 1_{n-r}\end{pmatrix}$ with $p_1\in G_{\boldsymbol{R},+}^{(r)}$, $p_2\in G_{\boldsymbol{R},+}^{(n-r)}$, $q\in M(r,n-r;\boldsymbol{R})$, and $dq=\prod_{1\leqq i\leqq r,\,1\leqq j\leqq n-r}dq_{ij}$. Let $\nu\leqq r\leqq n-1$. With the help of Lemma 1.5, Lemma 18 in [16], and Lemma 1.8, we get, if $\mathrm{Re}(s)>(n+1)/2$,

$$\begin{aligned}
&\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z}),\,\chi(g)\geqq 1}\chi(g)^{(n+1)/2-s}\sum_{x\in\mathscr{L}_n^{(r)}(s)} f_n^*(\rho(g)x,\lambda)d_n g\\
&=2^{-1}\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z})\cap P_{n,+}^r,\,\chi(g)\geqq 1}\chi(g)^{(n+1)/2-s}\sum_{x\in\mathscr{L}_r^{(r)}(S)} f_n^*\left(\rho(g)\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix},\lambda\right)d_n g\\
&=\frac{C_n}{C_r C_{n-r}}\int_{\substack{P_{n,+}^r/SL_n(\boldsymbol{Z})\cap P_{n,+}^r\\ \chi(p_1)\chi(p_2)\geqq 1}}\chi(g)^{(n+1)/2-s}\sum_{x\in\mathscr{L}_r^{(r)}(S)} f_r^*(\rho(p_1)x,\lambda)dp\\
&=\frac{C_n v_{n-r}}{C_r C_{n-r}(s-(n+1-r)/2)}Z(f_r^*(x,\lambda),\mathscr{L}_r(S),n/2).
\end{aligned}$$

Thus we see from (1.2.10) that the expression (1.2.9) holds for $\mathrm{Re}(s)>(n+1)/2$. We have completed the proof of Proposition 1.7.

We notice that $Z_+^*(s,\tau_S^{(n)})$ is a holomorphic function of s in the whole complex plane and that $Z_+(f_n^*(x,\lambda),\mathscr{L}_n(S),(n+1)/2-s)$ is holomorphic for $\mathrm{Re}(s)>-a_\lambda$. Taking the identities (1.2.2), (1.2.9) into account of, we see that $\xi_n^*(s,\tau_S^{(n)})$ can be continued to a meromorphic function in the region $\mathrm{Re}(s)>-a_\lambda$. Since λ can be taken sufficiently large, $\xi_n^*(s,\tau_S^{(n)})$ is extended to a meromorphic function in the whole complex plane. Moreover, we see that the poles of $\xi_n^*(s,\tau_S^{(n)})$ are located only at $s=(n+1-r)/2$ $(\nu\leqq r\leqq n)$ and that they are simple poles.

The next Proposition 1.9 corresponds to Corollary to Lemma 21 in [16].

Proposition 1.9. *Let λ satisfy the condition* (1.2.7). *The following functional equation holds*:

$$Z(f_n^*(x, \lambda), \mathscr{L}_n(S), (n+1)/2-s) = \frac{\pi^{n(n+1)/2} p^{n(s-(n+1)/2)} C_n \gamma_n(\lambda+s-n-1)}{2(2\pi)^{ns}\gamma_n(\lambda-(n+1)/2)} \xi_n^*(s, \tau_S^{(n)})$$

(*the left side is defined at least if* $-a_\lambda < \mathrm{Re}(s) < 1$).

Proof. We start from the identity (1.2.9). If $-a_\lambda < \mathrm{Re}((s) < 1$, then we can get, similarly as in the proof of Proposition 1.7,

$$Z_+^*(s, \tau_S^{(n)}) = \mu_{n,\lambda}\int_{G_{\boldsymbol{R},+}^{(n)}/SL_n(\boldsymbol{Z}),\chi(g)\leqq 1} \chi(g)^{(n+1)/2-s} \sum_{x\in\mathscr{L}_n^{(n)}(S)} f_n^*(\rho(g)x, \lambda) d_n g$$
$$-\mu_{n,\lambda}\sum_{r=\nu}^{n-1} \frac{C_n v_{n-r}}{C_r C_{n-r}(s-(n+1-r)/2)} Z(f_r^*(x, \lambda), \mathscr{L}_r(S), n/2),$$

from which we obtain

$$Z(s, \tau_S^{(n)}) = \mu_{n,\lambda} Z(f_n^*(x, \lambda), \mathscr{L}_n(S), (n+1)/2-s).$$

Thus, by (1.2.2), we get the functional equation in Proposition 1.9,

Proposition 1.10. *The zeta function* $\xi_n^*(s, \tau_S^{(n)})$ *can be continued analytically to a meromorphic function in the whole complex plane which has simple poles only at* $s=(n+1-r)/2$ $(\nu\leqq r\leqq n-1)$. *The residue of the pole at* $s=(n+1-r)/2$ *is given by*

$$\frac{2^{(n-r)(n+r+1)/2} v_{n-r}}{C_{n-r}} \xi_r^*((r+1-n)/2, \tau_S^{(r)}) \qquad (\nu\leqq r\leqq n-1).$$

Proof. The former part of Proposition 1.10 has been verified. We have only to calculate the residues of the poles. We see easily from Proposition 1.7, Proposition 1.9 that the residue of the pole at $s=(n+1-r)/2$ $(\nu\leqq r\leqq n)$ is given by

$$\frac{(2\pi)^{rn/2} C_n v_{n-r} \gamma_r(\lambda-(r+1+n)/2)}{2^{1+r(r+1)/2} p^{nr/2} \gamma_r(\lambda-(r+1)/2)} \mu_{n,\lambda} \xi_r^*((r+1-n)/2, \tau_S^{(r)}).$$

Thus, using the identity (1.2.2), we get the explicit residue at $s=(n+1-r)/2$ of $\xi_n^*(s, \tau_S^{(n)})$ as in Proposition 1.10.

For a primitive character χ mod p, $B_{k,\chi}$ denotes the k-th generalized Bernoulli number given by

(1.2.11) $$B_{k,\chi}=p^{k-1}\sum_{a=1}^{p-1}\chi(a)B_k(a/p).$$

In the case of $n=2$, one can derive some information of $L_2^*(s, \psi_{H,p})$, $L_2^*(s, \psi_{\det})$ from Proposition 1.10.

Theorem 1.11. *Let p be an odd prime and let ψ be the unique non-trivial quadracic character mod p. The L-functions $L_2^*(s, \psi_{H,p})$, $L_2^*(s, \psi_{\det})$ are continued analytically to meromorphic functions in the whole complex plane which are holomorphic except at $s=3/2$, 1. Then, $L_2^*(s, \psi_{H,p})$ has the unique simple poles at $s=1$ with residue $-B_{1,\psi}/p$, and $L_2^*(s, \psi_{\det})$ has simple poles at $s=3/2$, 1. The residue of $L_2^*(s, \psi_{\det})$ at $s=3/2$ (resp. $s=1$) is given by $\psi(-1)3^{-1}p^{-2}(p-1)\pi$ (resp. $-\psi(-1)2^{-1}p^{-1}(p-1)$).*

Proof. The former part is clear from Corollary to Proposition 1.2 and Proposition 1.10. Let $\mu\in \boldsymbol{Z}$ with $(\mu, p)=1$. An elementary computation shows that, in the case of $n=1$,

(1.2.12) $$\xi_1^*(s, \tau_\mu^{(1)})=\frac{1}{2}\{\psi(\mu)\tau_\psi L(s, \psi)-(1-p^{1-s})\zeta(s)\},$$

where $L(s, \psi)$ is the Dirichlet L-function associated with ψ. Since $L(0, \psi)=-B_{1,\psi}$, $\zeta(0)=-1/2$. We have

$$\xi_1^*(0, \tau_\mu^{(1)})=-\frac{1}{2}\psi(\mu)\tau_\psi B_{1,\psi}+\frac{1}{4}(1-p).$$

We see immediately from Proposition 1.10 that the residue of $\xi_2^*(s, \tau_\mu^{(2)})$ at the pole $s=1$ is given by $\xi_1^*(0, \tau_\mu^{(1)})$. It is known by [16, Theorem 2 or Corollary to Lemma 21] that $\xi_2^*(s)$ has simple poles only at $s=3/2$, 1 with residues $\pi/3$, -1 respectively. Since $\xi_2^*(s, \tau_\mu^{(2)})$ is holomorphic at $s=3/2$, so is $L_2^*(s, \psi_{H,p})$. Thus we get, by Corollary to Proposition 1.2, the assertion of Theorem 1.11.

Chapter II. Evaluation of special values of L-functions (the cases of degree two)

2.1. L-functions, and partial zeta functions

Let $\partial\mathscr{P}_2$ denote the boundary of the domain $\mathscr{P}_2$ in $V_{\boldsymbol{R}}^{(2)}$, that is, $\partial\mathscr{P}_2$ is the set of positive semi-definite symmetric matrices of size two. Let $\{W_1, W_2, \cdots, W_r\}$ be an r-tuple of elements in $\mathscr{P}_2\cup\partial\mathscr{P}_2$ such that $W_1, W_2, \cdots, W_r$ are linearly independent over $\boldsymbol{R}$. Then, necessarily, $r\leqq 3$. For any r-tuple $\xi=(\xi_1, \cdots, \xi_r)$ of positive numbers, we define a partial zeta function $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ as follows (see (2.2) of [14] and (1.4) of [9]):

$$(2.1.1) \qquad \zeta(s; \{W_1, \cdots, W_r\}, \xi) = \sum_{m_1, \cdots, m_r = 0}^{\infty} \det\left(\sum_{j=1}^{r} (\xi_j + m_j) W_j\right)^{-s}.$$

Let $C = C(W_1, \cdots, W_r)$ be a simplicial cone spanned by $W_1, \cdots, W_r$:

$$C = C(W_1, \cdots, W_r) = \left\{\sum_{j=1}^{r} \lambda_j W_j \,|\, \lambda_j > 0 \ (1 \leqq j \leqq r)\right\}.$$

We assume that the cone $C(W_1, \cdots, W_r)$ is contained in $\mathscr{P}_2$. Then it is easily shown that the zeta functions $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ is absolutely convergent for $\mathrm{Re}(s) > r/2$. For any subset M of $V_{\boldsymbol{R}}^{(2)}$, the zeta function $\zeta(s; C, M)$, if it converges absolutely, is defined by

$$(2.1.2) \qquad \zeta(s; C, M) = \sum_{T \in C \cap M} \det(T)^{-s}.$$

It is well-known that, as a fundamental domain of $\mathscr{P}_n$ under the usual action of the group $GL_n(\boldsymbol{Z})$, one can take the so-called Minkowski domain $\mathscr{R}_n$ of reduced matrices (see, for instance, § 9 of [12]). In the case of $n=2$, the domain $\mathscr{R}_2$ has a simple form:

$$\mathscr{R}_2 = \left\{\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} \,|\, 0 \leqq 2y_{12} \leqq y_1 \leqq y_2,\ 0 < y_1\right\}.$$

We fix three special elements V_1, V_2, V_3 in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ throughout Chapter II; put

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad \text{and} \quad V_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We set, for simplicity,

$$(2.1.3) \qquad \begin{aligned} &C_{123} = C(V_1, V_2, V_3), \qquad C_{ij} = C(V_i, V_j) \quad (1 \leqq i < j \leqq 3), \\ &C_j = C(V_j) \quad (j = 1, 2), \end{aligned}$$

which are simplicial cones contained in $\mathscr{P}_2$. Then the domain $\mathscr{R}_2$ has the decomposition

$$(2.1.4) \qquad \mathscr{R}_2 = C_{123} \cup C_{12} \cup C_{13} \cup C_{23} \cup C_1 \cup C_2 \qquad \text{(disjoint union)}.$$

For each cone C in (2.1.3) and any $Y \in C$, the order $\varepsilon^*(Y)$ of the group $\{U \in GL_2(\boldsymbol{Z}) \,|\, UY\,{}^tU = Y\}$ takes the same value independent of Y belonging to C, and one can put

$$\varepsilon^*(C) = \varepsilon^*(Y) \qquad (Y \in C).$$

It is easily verified that

$$(2.1.5)\quad \varepsilon^*(C_{123})=2,\quad \varepsilon^*(C_{ij})=4\ (1\leqq i<j\leqq 3),\quad \varepsilon^*(C_1)=8,\quad \varepsilon^*(C_2)=12.$$

For a real number x, we denote by $\langle x\rangle$ the unique real number which satisfies $0<\langle x\rangle\leqq 1$ and $x-\langle x\rangle\in \boldsymbol{Z}$. Let p be an odd prime.

The aim of this section is to represent the L-functions $L_2^*(s, \psi_{H,p})$, $L_2^*(s, \chi_{\det})$ and the zeta function $\xi_2^*(s)$ as a finite linear combination of partial zeta functions (2.1.1).

First we shall discuss the L-function $L_2^*(s, \psi_{H,p})$. Let $\mathcal{M}(p)$ be the set given by (1.1.3). For each integer μ prime to p, let $L^*(\mu)$ be the set consisting of all elements $T\in L_2^*$ satisfying $\psi_{H,p}(T)=\psi(\mu)$. Then it immediately follows that

$$L^*(\mu)=\left\{T\in L_2^*\mid T\equiv\mu\begin{pmatrix}\alpha^2 & \alpha\gamma\\ \alpha\gamma & \gamma^2\end{pmatrix} \bmod p \text{ for some } (\alpha,\gamma)\in\mathcal{M}(p)\right\}$$

and that $L^*(\mu l^2)=L^*(\mu)$ for any integer l prime to p. For each $(\alpha,\lambda)\in\mathcal{M}(p)$ and for each integer μ prime to p, we put

$$(2.1.6)\qquad \xi_{\alpha,\gamma,\mu}=(\langle\mu(\alpha^2-2\alpha\gamma)/p\rangle,\ \langle 2\mu\alpha\gamma/p\rangle,\ \langle\mu(\gamma^2-\alpha^2)/p\rangle).$$

Let $\Xi_{H,\mu}$ be the set of all triples $\xi_{\alpha\gamma\mu}$: $\Xi_{H,\mu}=\{\xi_{\alpha,\gamma,\mu}\mid(\alpha,\gamma)\in\mathcal{M}(p)\}$. Then, $\mathcal{M}(p)/\{\pm 1\}$ corresponds to $\Xi_{H,\mu}$ bijectively by $\pm(\alpha,\gamma)\rightarrow\xi_{\alpha,\gamma,\mu}(=\xi_{-\alpha,-\gamma,\mu})$. For any integers i,j with $1\leqq i<j\leqq 3$, we set

$$\Xi_{H,\mu}^{(i,j)}=\{\xi=(\xi_1,\xi_2,\xi_3)\in\Xi_{H,\mu}\mid \xi_k=1\},$$

where k is the unique integer of 1, 2, 3 satisfying $\{i,j,k\}=\{1,2,3\}$. We notice that

$$(2.1.7)\qquad \begin{cases}\Xi_{H,\mu}^{(1,2)}=\{\xi_{\alpha,\gamma,\mu}\mid(\alpha,\gamma)\in\mathcal{M}(p),\ \alpha^2\equiv\gamma^2 \bmod p\},\\ \Xi_{H,\mu}^{(1,3)}=\{\xi_{\alpha,\gamma,\mu}\mid(\alpha,\gamma)\in\mathcal{M}(p),\ \alpha\gamma\equiv 0 \bmod p\},\\ \Xi_{H,\mu}^{(2,3)}=\{\xi_{\alpha,\gamma,\mu}\mid(\alpha,\gamma)\in\mathcal{M}(p),\ \alpha^2\equiv 2\alpha\gamma \bmod p\}.\end{cases}$$

For each cone C of the form (2.1.3), the zeta function $\zeta(s;\,C,\,L^*(\mu))$ given by (2.1.2) is absolutely convergent at least for $\operatorname{Re}(s)>3/2$.

Proposition 2.1. *The following expressions for the zeta functions* $\zeta(s;\,C,\,L^*(\mu))$ *hold*:

$$\zeta(s;\,C_{123},\,L^*(\mu))=p^{-2s}\sum_{\xi\in\Xi_{H,\mu}}\zeta(s;\,\{V_1,V_2,V_3\},\,\xi),$$

$$\zeta(s;\,C_{ij},\,L^*(\mu))=p^{-2s}\sum_{(\xi_1,\xi_2,\xi_3)\in\Xi_{H,\mu}^{(i,j)}}\zeta(s;\,\{V_i,V_j\},\,(\xi_i,\xi_j))\qquad(1\leqq i<j\leqq 3),$$

$\zeta(s;\, C_1,\, L^*(\mu))=0,$

$$\zeta(s;\, C_2,\, L^*(\mu))=\begin{cases}0 & \cdots if\ p>3,\\ p^{-2s}\zeta(s;\, \{V_2\},\, \langle\mu/p\rangle)\cdots & if\ p=3.\end{cases}$$

Proof. Take $T\in C_{123}\cap L^*(\mu)$ and write $T=\sum_{j=1}^{3} m_j V_j$ with all $m_j\in N$. If we take a pair $(\alpha, \gamma)\in \mathscr{M}(p)$ such that

$$T\equiv\mu\begin{pmatrix}\alpha^2 & \alpha\gamma\\ \alpha\gamma & \gamma^2\end{pmatrix} \bmod p, \tag{2.1.8}$$

then, m_j's satisfy the congruences:

$$\begin{cases}m_1\equiv\mu(\alpha^2-2\alpha\gamma) \bmod p, & m_2\equiv 2\mu\alpha\gamma \bmod p,\\ m_3\equiv\mu(\gamma^2-\alpha^2) \bmod p.\end{cases}$$

Therefore, there exists a triple $l=(l_1, l_2, l_3)$, l_j being nonnegative integers, such that $(m_1, m_2, m_3)=p(\xi_{\alpha,\gamma,\mu}+l)$. Each $T\in C_{123}\cap L^*(\mu)$ determines a triple l uniquely and also $(\alpha, \gamma)\in\mathscr{M}(p)$ uniquely up to (± 1)-multiplication. Thus the first identity of Proposition 2.1 follows. Next, for instance, let $T\in C_{12}\cap L^*(\mu)$ and write $T=\sum_{j=1}^{2} m_j V_j$ $(m_j\in N)$. A pair (α, γ) can be so taken as in (2.1.8). Then the congruences $m_1\equiv\mu(\alpha^2-2\alpha\gamma) \bmod p$, $m_2\equiv 2\mu\alpha\gamma \bmod p$ follow, and necessarily, the relation $\alpha^2\equiv\gamma^2 \bmod p$ has to hold. Therefore, the identity for $\zeta(s;\, C_{12},\, L^*(\mu))$ immediately follows. Other identities left are quite similarly verified. So the proof is omitted.

q.e.d.

Let κ be a non-quadratic residue mod p as in Chapter I.

Proposition 2.2. *Let ψ be the unique non-trivial quadratic character mod p. Then we have*

$$L_2^*(s, \psi_{H,p})=\sum_{\mu}\psi(\mu)\Big\{\zeta(s;\, C_{123},\, L^*(\mu))+\frac{1}{2}\sum_{i<j}\zeta(s;\, C_{ij},\, L^*(\mu))$$
$$+\frac{1}{6}\delta_{p,3}\zeta(s;\, C_2,\, L^*(\mu))\Big\},$$

where μ is taken over 1 and κ, and the summation $\sum_{i<j}$ indicates that i, j run over all integers with $1\leqq i<j\leqq 3$. Moreover, $\delta_{p,3}=0$ if $p\neq 3$, and $\delta_{p,3}=1$ if $p=3$.

Proof. Only in this proof, we introduce the L-function $M_2^*(s, \psi_{H,p})$ which is quite similar to $L_2^*(s, \psi_{H,p})$. We set

$$M_2^*(s, \psi_{H,p})=\sum_{T\in L_{2,+}^*/GL_2(Z)}\psi_{H,p}(T)\varepsilon^*(T)^{-1}\det(T)^{-s}$$

where T is taken over $GL_2(\mathbf{Z})$-equivalence classes of positive definite half-integral symmetric matrices of size two, and $\varepsilon^*(T)$ is the order of the unit group $\{U \in GL_2(\mathbf{Z}) \mid UT{}^tU=T\}$ of T. Then an elementary observation shows that $L_2^*(s, \psi_{H,p})=2M_2^*(s, \psi_{H,p})$. In view of the decomposition (2.1.4) of $\mathscr{R}_2$, we may take a disjoint union $\cup_C (C\cap L_2^*)$, C varying all simplicial cones in (2.1.3), as a complete set of $GL_2(\mathbf{Z})$-equivalence classes of all elements in $L_{2,+}^*$. Thus we get, with the help of the decomposition $L_2^*=L^*(1)\cup L^*(\kappa)$ (disjoint union),

$$L_2^*(s, \psi_{H,p})=2\sum_C \varepsilon^*(C)^{-1}\sum_\mu \psi(\mu)\zeta(s; C, L^*(\mu)),$$

which, together with (2.1.5) and Proposition 2.1, completes the proof of Proposition 2.2.

Let χ be a primitive character mod p. Secondly, we treat the L-function $L_2^*(s, \chi_{\det})$. For each integer δ prime to p, we set

$$M^*(\delta)=\{T\in L_2^* \mid \det(T)\equiv\delta \bmod p\}.$$

For each $T=\begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix}\in L_2^*$, a triple ξ_T is defined by

$$\xi_T=(\langle(t_1-2t_{12})/p\rangle, \langle 2t_{12}/p\rangle, \langle(t_2-t_1)/p\rangle),$$

which depends only on T mod p. Let Ξ_δ be the set of all triples ξ_T, T varying all elements of L_2^*/pL_2^* with $\det(T)\equiv\delta \bmod p$: $\Xi_\delta=\{\xi_T \mid T\in L_2^* \bmod pL_2^*, \det(T)\equiv\delta \bmod p\}$. For integers i, j $(1\leqq i<j\leqq 3)$, we set

$$\Xi_\delta^{(i,j)}=\{\xi=(\xi_1, \xi_2, \xi_3)\in\Xi_\delta \mid \xi_k=1\},$$

k being the the unique integer of 1, 2, 3 with $\{i, j, k\}=\{1, 2, 3\}$.

For any cone C of the form (2.1.3), the zeta function $\zeta(s; C, M^*(\delta))$, which is absolutely convergent for $\mathrm{Re}(s)>3/2$, has the following expression.

Proposition 2.3. *Let δ be any integer prime to p. Then,*

$$\zeta(s; C_{123}, M^*(\delta))=p^{-2s}\sum_{\xi\in\Xi_\delta}\zeta(s; \{V_1, V_2, V_3\}, \xi),$$

$$\zeta(s; C_{ij}, M^*(\delta))=p^{-2s}\sum_{(\xi_1,\xi_2,\xi_3)\in\Xi_\delta^{(i,j)}}\zeta(s; \{V_i, V_j\}, (\xi_i, \xi_j))$$

$$(1\leqq i<j\leqq 3).$$

For the cone C_j $(j=1, 2)$, we have

$$\sum_{\delta\not\equiv 0 \bmod p}\chi(\delta)\zeta(s; C_1, M^*(\delta))=\sum_{t_1\not\equiv 0 \bmod p}\chi(t_1^2)\zeta(s; \{V_1\}, \langle t_1/p\rangle),$$

$$\sum_{\delta\not\equiv 0 \bmod p} \chi(\delta)\zeta(s;\, C_2,\, M^*(\delta)) = \sum_{t_{12}\not\equiv 0 \bmod p} \chi(3t_{12}^2)\zeta(s;\, \{V_2\},\, \langle 2t_{12}/p\rangle)$$

(*note that, if* $p=3$, *then, the right side of the last equality coincides identically with zero*).

Proposition 2.4. *Let* χ *be a primitive character* mod p. *Then,*

$$L_2^*(s, \chi_{\det}) = \sum_{\delta\not\equiv 0 \bmod p} \chi(\delta)\Big\{\zeta(s:\, C_{123},\, M^*(\delta)) + \frac{1}{2}\sum_{i<j}\zeta(s;\, C_{ij},\, M^*(\delta))$$
$$+\frac{1}{4}\zeta(s;\, C_1,\, M^*(\delta)) + \frac{1}{6}\zeta(s;\, C_2,\, M^*(\delta))\Big\}.$$

We omit the proofs of Proposition 2.3 and Proposition 2.4, which are quite similar to those of Proposition 2.1 and Proposition 2.2.

Finally, we obtain Proposition 2.5, which asserts that the zeta function $\xi_2^*(s)$ can be represented as a finite linear combination of partial zeta functions.

Proposition 2.5. *We have*

$$\xi_2^*(s) = \zeta(s;\{V_1, V_2, V_3\}, (1, 1, 1)) + \frac{1}{2}\sum_{i<j}\zeta(s;\{V_i, V_j\}, (1, 1))$$
$$+\frac{1}{4}\zeta(s;\{V_1\}, 1) + \frac{1}{6}\zeta(s;\{V_2\}, 1).$$

The proof of Proposition 2.5 is omitted as well.

2.2. Integral representations of partial zeta functions I

The aim of subsequent two sections is to obtain convenient expressions of partial zeta functions as integrals over contour paths, and then to evaluate special values of them at non-positive integers.

Let $\zeta(s;\{W_1, \cdots, W_r\}, \xi)$ be a partial zeta function as is defined in (2.1.1). We assume that the cone $C(W_1, \cdots, W_r)$ is contained in $\mathcal{P}_2$. The following formula is well-known (see for instance [12] and also Lemma 1 of [14]):

$$(2.2.1)\quad \det(T)^{-s} = \frac{1}{\Gamma_2(s)}\int_{\mathcal{P}_2} \det(Y)^s e^{-\operatorname{tr}(TY)} dv(Y) \quad (T\in\mathcal{P}_2,\ \operatorname{Re}(s)>1/2),$$

where we put

$$\Gamma_2(s) = \pi^{1/2}\Gamma(s)\Gamma(s-1/2) \quad\text{and}\quad dv(Y) = \det(Y)^{-3/2}\prod_{1\leq i\leq j\leq 2} dY_{ij}.$$

We set, for $t \in \boldsymbol{C}$, $x \in \boldsymbol{R}$,

$$\phi(t; x)=\frac{e^{tx}}{e^{t}-1},$$

which is the generating function of Bernoulli polynomials $B_k(x)$. Namely, the Laurent expansion at $t=0$ of $\phi(t; x)$ is given by

$$\phi(t; x)=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}t^{k-1} \qquad (|t|<2\pi). \tag{2.2.2}$$

By a usual argument which uses the formula (2.2.1), we get an expression of $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ for $\mathrm{Re}(s)>r/2$ by the integral taken over $\mathscr{P}_2$:

$$\zeta(s; \{W_1, \cdots, W_r\}, \xi)=\frac{1}{\Gamma_2(s)}\int_{\mathscr{P}_2}\det(Y)^s\prod_{j=1}^{r}\phi(\mathrm{tr}(W_jY); 1-\xi_j)dv(Y).$$

We set, for $\theta \in \boldsymbol{R}$, $k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.

Following Satake [14, 2.2], we make a change of variables $Y \to (t, u, \theta)$ with $Y=tk_\theta\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}{}^tk_\theta$ $(0<t, 0<u\leqq 1, 0\leqq\theta\leqq\pi)$. We thus obtain, using the relation $dv(Y)=t^{-1}u^{-3/2}(1-u)dtdud\theta$,

$$\zeta(s; \{W_1, \cdots, W_r\}, \xi)=\frac{1}{\Gamma_2(s)} \times\int_0^\infty dt\int_0^1 du\int_0^\pi d\theta\cdot t^{2s-1}u^{s-3/2}(1-u)\Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi), \tag{2.2.3}$$

where we put

$$\Phi((t, u, \theta), \{W_1, \cdots, W_r\}, \xi)=\prod_{j=1}^{r}\phi(t\lambda((u, \theta), W_j); 1-\xi_j)$$

and

$$\lambda((u, \theta), W)=\mathrm{tr}\left(Wk_\theta\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}{}^tk_\theta\right) \qquad \text{for any } W \in \mathscr{P}_2\cup\partial\mathscr{P}_2.$$

The following condition (2.2.4) on vectors W_j;

(2.2.4) all W_j $(1\leqq j\leqq r)$ are contained in $\mathscr{P}_2$,

being imposed, then, the integral (2.2.3) has been studied in a full generality by Satake [14] and by Kurihra [9] in a special but significant case.

However, for our aim to evaluate special values of L-functions discussed in 2.1, it is indispensable to get rid of the condition (2.2.4). In view of Proposition 2.1, Proposition 2.3, and Proposition 2.5, we have only to consider the cases in which, with respect to an r-tuple $\{W_1, \cdots, W_r\}$, the vectors $W_1, \cdots, W_{r-1}$ are all in $\mathscr{P}_2$, and W_r coincides with the special vector V_3 in $\partial\mathscr{P}_2$.

Now we set

$$\psi(t; x) = \phi(t; x) - \frac{1}{t},$$

which is a holomorphic function of t in the region $|t| < 2\pi$. Let $\{W_1, \cdots, W_{r-1}, V_3\}$ ($r=2$ or 3) be an r-tuple of vectors in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, \cdots, W_{r-1}$ are all in $\mathscr{P}_2$. We set, for an r-tuple $\xi = (\xi_1, \cdots, \xi_r)$ of positive numbers,

$$\begin{aligned}\Phi_P((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi) \\ = \prod_{j=1}^{r-1} \phi(t\lambda((u, \theta), W_j); 1-\xi_j)\psi(t\lambda((u, \theta), V_3); 1-\xi_3),\end{aligned}$$

and, for an $(r-1)$-tuple $\xi' = (\xi_1, \cdots, \xi_{r-1})$,

$$\begin{aligned}\Phi_S((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi') \\ = \frac{1}{\lambda((u, \theta), V_3)} \prod_{j=1}^{r-1} \phi(t\lambda((u, \theta), W_j); 1-\xi_j).\end{aligned}$$

Moreover, we set

$$\begin{aligned}(2.2.5) \qquad \zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi) \\ = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) \\ \times \Phi_P((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi),\end{aligned}$$

$$\begin{aligned}(2.2.6) \qquad \zeta_S(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi') \\ = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) \\ \times \Phi_S((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi')\end{aligned}$$

(the letter P (resp. S) is used to intend that the function given in (2.2.5) (resp. in (2.2.6)) is a principal (resp. singular) part of $\zeta(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)$).

The integrals in (2.2.5), (2.2.6) are absolutely convergent at least for $\mathrm{Re}(s) > 3/2$. Obviously,

$$\zeta(s;\{W_1,\cdots,V_3\},\xi)=\zeta_P(s;\{W_1,\cdots,V_3\},\xi)+\zeta_S(s;\{W_1,\cdots,V_3\},\xi').$$

For our later use, we prepare some symbols. For a positive number ε, let $I_\varepsilon(\infty)$ (resp. $I_\varepsilon(1)$) be the contour path consisting of the oriented half line $(+\infty,\varepsilon)$ (resp. $(1,\varepsilon)$), a counterclockwise circle of radius ε around the origin, and the oriented half line $(\varepsilon,+\infty)$ (resp. $(\varepsilon,1)$). We would like to modify the integral in (2.2.3) directly into the integral taken over contour paths $I_\varepsilon(\infty)$ and $I_\varepsilon(1)$ (for a small ε) with respect to t and u, respectively. However, the function $\phi(t\lambda((u,\theta),V_3);1-\xi_3)$ has serious singularities as a function of t and u on the paths $I_\varepsilon(\infty)$, $I_\varepsilon(1)$, because of the form of $\lambda((u,\theta),V_3)=u\sin^2\theta+\cos^2\theta$, and therefore such a modification cannot be done easily. To avoid the difficulties derived from such singularities, we divide zeta function $\zeta(s;W_1,\cdots,W_{r-1},V_3\},\xi)$ into two parts as above. In the rest of this section, we shall mainly discuss the function $\zeta_P(s:\{W_1,\cdots,W_{r-1},V_3\},\xi)$ and its expression by an integral over contour paths. The singular part $\zeta_S(s;\{W_1,\cdots,W_{r-1},V_3\},\xi)$ will be dealed with in the next section.

For a positive number δ, we denote by $D_\delta(\infty)$ and $D_\delta(1)$ the regions given as follows:

$$D_\delta(\infty)=\{z\in\boldsymbol{C}\mid|z|<\delta\}\cup\{z\in\boldsymbol{C}\mid\mathrm{Re}(z)>0\text{ and }|\mathrm{Im}(z)|<\delta\},$$
$$D_\delta(1)=D_\delta(\infty)\cap\{z\in\boldsymbol{C}\mid|z|\leqq1\}.$$

If W is in $\mathscr{P}_2$, we can take a positive constant a, b satisfying

$$a1_2<W<b1_2. \tag{2.2.7}$$

We may write $\lambda((u,\theta),W)=\alpha_1u+\alpha_2$ with $a<\alpha_1$, $\alpha_2<b$. It then follows that

$$\begin{aligned}&|\lambda((u,\theta),W)|<b(1+|u|),\\&\mathrm{Re}(\lambda((u,\theta),W))>a-b\delta\quad\text{if }\mathrm{Re}(u)>-\delta\quad(\delta>0).\end{aligned} \tag{2.2.8}$$

We need some analytic properties of the functions $\phi(t\lambda((u,\theta),W);1-\xi)$ and $\psi(t\lambda((u,\theta),V_3);1-\xi)$ $(\xi>0)$.

Lemma 2.6. *Suppose that $W\in\mathscr{P}_2$ satisfies the condition* (2.2.7). *Let $\xi>0$ and $0<\delta<a/b$.*

(i) *If $|t|<\pi/2b$, and $u\in D_\delta(1)$, then, $t\phi(t\lambda((u,\theta),W);1-\xi)$, which is a holomorphic function of (t,u) for each θ in that region of (t,u), has the power series expansion with respect to t:*

$$t\phi(t\lambda((u,\theta),W);1-\xi)=\frac{1}{\lambda((u,\theta),W)}+\sum_{k=1}^{\infty}\frac{B_k(1-\xi)}{k!}\{\lambda((u,\theta),W)\}^{k-1}t^k,$$

(ii) *If* $t>0$ *and* $0\leqq u\leqq 1$, *then,*

$$t\phi(t\lambda((u,\theta),W);1-\xi)<\frac{te^{-t\xi a}}{1-e^{-ta}}.$$

Proof. If $|t|<\pi/2b$, $u\in D_\delta(1)$, then, we get, by (2.2.8),

$$|t\lambda((u,\theta),W)|<\pi(1+|u|)/2<2\pi.$$

Therefore, the Laurent expansion (2.2.2) implies the assertion (i). The assertion (ii) is immediately derived from the inequality $\lambda((u,\theta),W)>a$ $(0\leqq u\leqq 1)$.

Lemma 2.7. *Let* $\xi>0$ *and* $0<\delta<1$. *Then* $\psi(t\lambda((u,\theta),V_3);1-\xi)$ *is a holomorphic function of* t, u *for each* θ *in the region* $\{(t,u)\,|\,|t|<2\pi,\ u\in D_\delta(1)\}$, *and has a Taylor expansion with respect to* t:

$$\psi(t\lambda((u,\theta),V_3);1-\xi)=\sum_{k=1}^{\infty}\frac{B_k(1-\xi)}{k!}(u\sin^2\theta+\cos^2\theta)^{k-1}t^{k-1}.$$

Proof. Recalling that $\lambda((u,\theta),V_3)=u\sin^2\theta+\cos^2\theta$, and moreover that $|t\lambda((u,\theta),V_3)|<2\pi$ if $|t|<2\pi$, $u\in D_\delta(1)$, we immediately get the assertion of Lemma 2.7.

The function $\psi(t;1-\xi)$ has a preferable property which will be used in the proof of Proposition 2.9.

Lemma 2.8. *Let* $\xi>0$. *There exist positive constants* M_k $(k=1,2,\cdots)$ *independent of* t *such that, if* $0\leqq t<+\infty$,

$$|\psi^{(k)}(t;1-\xi)|<M_k,$$

where $\psi^{(k)}(t;1-\xi)$ *denotes the k-th derivative of* $\psi(t;1-\xi)$ *as a function of* t.

We omit the proof of Lemma 2.8, which is an easy exercise of differential calculus.

It is easy to see from (2.2.8) and Lemma 2.6 that, if δ is taken sufficiently small, then, $t\phi(t\lambda((u,\theta),W);1-\xi)$ $(W\in\mathscr{P}_2,\ \xi>0)$ is holomorphic as a function of t, u for each $\theta\in\boldsymbol{R}$ in the region $D_\delta(\infty)\times D_\delta(1)$. Moreover, taking (2.2.8), Lemma 2.6, and Lemma 2.7 into account, we see without difficulty that the integral

$$t^2\int_0^{\pi}\Phi_P((t,u,\theta);\{W_1,\cdots,W_{r-1},V_3\},\xi)d\theta$$

indicates a holomorphic function of t, u in the region $D_\delta(1)\times D_\delta(1)$ for a sufficiently small δ. We notice here that the range of t is the region $D_\delta(1)$ (not $D_\delta(\infty)$).

To define the function $t^s=e^{s\log t}$, we take the branch of $\log t$ with $0<\arg t<2\pi$.

Proposition 2.9. *The function $\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)$ is analytically continued to a meromorphic function in the whole complex plane which is holomorphic at $s=1-m$ $(m=1, 2, \cdots)$. Moreover, the special value at $s=1-m$ is given by*

$$\begin{aligned}&\zeta_P(1-m; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)\\&\quad=C(m)\int_{\Gamma_\varepsilon} dt\int_{I_\varepsilon(\infty)} du\int_0^\pi d\theta t^{1-2m}u^{-m-1/2}(1-u)\\&\quad\quad\times\phi_P((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi),\end{aligned}$$

where $C(m)=(2m-1)!/2^{2m+2}\pi^2 i$ and Γ_ε denotes a circle of radius ε around the origin oriented counterclockwise, ε being taken sufficiently small.

Proof. We set, only in the proof of this proposition,

$$f(t, u, \theta)=\Phi_P((t, u, \theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi).$$

We divide the integral in (2.2.5) into two parts by the range of the variable t. We set

$$\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)=\frac{1}{\Gamma_2(s)}(I_1(s)+I_2(s)),$$

where

$$I_1(s)=\int_0^1 dt\int_0^1 du\int_0^\pi d\theta\cdot t^{2s-1}u^{s-3/2}(1-u)f(t, u, \theta),$$

$$I_2(s)=\int_1^\infty dt\int_0^1 du\int_0^\pi d\theta\cdot t^{2s-1}u^{s-3/2}(1-u)f(t, u, \theta).$$

It is easy to see from the remark just before the statement of Proposition 2.9 that $I_1(s)$ has the following expression by an integral over contour paths:

$$(2.2.9)\qquad \begin{aligned}I_1(s)=&\frac{1}{(e[2s]-1)(e[s-3/2]-1)}\int_{I_\varepsilon(1)} dt\int_{I_\varepsilon(1)} du\int_0^\pi d\theta\\&\times t^{2s-1}u^{s-3/2}(1-u)f(t, u, \theta),\end{aligned}$$

where ε is taken sufficiently small. Since the integral in (2.2.9) indicates an entire function of s, the function $I_1(s)$ can be continued analytically to a meromorphic function in the whole complex plane. Thus we easily obtain

$$(2.2.10)\quad \lim_{s\to 1-m}\frac{I_1(s)}{\Gamma_2(s)}=C(m)\int_{\Gamma_\varepsilon}dt\int_{I_\varepsilon(1)}du\int_0^\pi d\theta\cdot t^{1-2m}u^{-m-1/2}(1-u)f(t,u,\theta)$$
$$(m=1,2,\cdots).$$

On the one hand, we see easily from Lemma 2.8 and so on that the function $f(t, u, \theta)$ is a C^∞-function of (t, u, θ) in the region $(0, +\infty)\times[0, 1]\times[0, \pi]$, and especially that the partial derivatives $(\partial^k f/\partial u^k)(t, u, \theta)$ $(k=0, 1, 2, \cdots)$ are bounded in the region $[1, +\infty]\times[0, 1]\times[0, \pi]$ of (t, u, θ). We set, for $\mathrm{Re}(s)>0$,

$$F(s;(t,\theta))=\int_0^1 u^{s-1}f(t,u,\theta)du.$$

Then we have

$$(2.2.11)\quad I_2(s)=\int_1^\infty dt\int_0^\pi d\theta\cdot t^{2s-1}\{F(s-1/2;(t,\theta))-F(s+1/2;(t,\theta))\}.$$

Using the integration by parts recurrently, we obtain

$$(2.2.12)\quad F(s;(t,\theta))=\sum_{j=0}^{m-1}\frac{(-1)^j}{s(s+1)\cdots(s+j)}\frac{\partial^j f}{\partial u^j}(t,1,\theta)$$
$$+\frac{(-1)^m}{s(s+1)\cdots(s+m-1)}\int_0^1 u^{s+m-1}\frac{\partial^m f}{\partial u^m}(t,u,\theta)du$$
$$(\mathrm{Re}(s)>-m).$$

Since any $m\in\boldsymbol{N}$ can be taken, it follows from (2.2.11), (2.2.12) that $I_2(s)$ can be continued to an entire function of s. Thus we get

$$(2.2.13)\quad \left[\frac{I_2(s)}{\Gamma_2(s)}\right]_{s=1-m}=0.$$

The analytic continuation of $\zeta_P(s;\{W_1,\cdots,W_{r-1},V_3\},\xi)$ immediately follows from those of $I_1(s)$, $I_2(s)$. The last assertion of Proposition 2.9 is derived from (2.2.10), (2.2.13). q.e.d.

The following proposition which deals with partial zeta functions whose edge vectors are all in $\mathscr{P}_2$ is only a small part of the results obtained by Satake in [14].

Proposition 2.10. *Let all vectors W_j $(1 \leqq j \leqq r)$, which are linearly independent over $\mathbf{R}$, be in $\mathcal{P}_2$. Then the zeta function $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ has the analytic continuation to a meromorphic function of s in the whole complex plane, which is holomorphic at $s=1-m$ $(m=1, 2, \cdots)$. The special value at $s=1-m$ is then given by*

$$\zeta(1-m; \{W_1, \cdots, W_r\}, \xi) = C(m)\int_{\Gamma_\varepsilon} dt \int_{I_\varepsilon(1)} du \int_0^\pi d\theta \cdot t^{1-2m} u^{-m-1/2}(1-u) \times \Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi).$$

Proof. For the completeness, we give a proof. Lemma 2.6 implies that the integral

$$t^2 \int_0^\pi \Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi) d\theta$$

indicates a holomorphic function of (t, u) in some region $D_\delta(\infty) \times D_\delta(1)$. Thus one obtains, for a sufficiently small ε,

$$\zeta(s; \{W_1, \cdots, W_r\}, \xi) = \frac{1}{\Gamma_2(s)(e[2s]-1)(e[s-3/2]-1)} \int_{I_\varepsilon(\infty)} dt \int_{I_\varepsilon(1)} du \int_0^\pi d\theta \times t^{2s-1} u^{s-3/2}(1-u) \Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi).$$

Since the integral in the right side of the equality is absolutely convergent, this identity gives the analytic continuation of $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ to a meromorphic function of s in the whole complex plane. Substituting $s = 1-m$, we get the identity in Proposition 2.10.

In view of Proposition 2.1, Proposition 2.3, and Proposition 2.5, we need only partial zeta functions of the form

$$\zeta(s; \{V_1, V_2, V_3\}, (\xi_1, \xi_2, \xi_3)), \quad \zeta(s; \{V_i, V_j\}, (\xi_i, \xi_j)) \quad (1 \leqq i < j \leqq 3)$$
$$\zeta(s; \{V_j\}, \xi) \quad (j=1, 2, \text{ and } \xi > 0).$$

Now we discuss the evaluation of $\zeta_P(1-m; \{V_1, V_2, V_3\}, (\xi_1, \xi_2, \xi_3))$ $(m \in N)$ and so on as a continuation of Proposition 2.9, Proposition 2.10. For each triple (k_1, k_2, k_3) of integers such that $k_1, k_2 \geqq -1$, $k_3 \geqq 0$, and $k_1 + k_2 + k_3 = 2(m-1)$, we define a number $\Lambda_{(k_1, k_2, k_3)}$ by putting

$$(2.2.14) \quad \Lambda_{(k_1, k_2, k_3)} = \int_{I_\varepsilon(1)} du \int_0^\pi d\theta \cdot u^{-m-1/2}(1-u) \prod_{j=1}^{3} \lambda((u, \theta), V_j)^{k_j},$$

where the integral in the right side is independent of the choice of a small positive number ε.

Remark. The numbers $\Lambda_{(k_1,k_2,k_3)}$ essentially coincide with $N(1-m; k_1+1, k_2+1, k_3+1; v_1, v_2, v_3)$ in Kurihara [9], though his definition is different from ours.

Proposition 2.11. *Let $\xi=(\xi_1, \xi_2, \xi_3)$ be a triple of positive numbers, and $m \in \mathbf{N}$. Then we have*

$$\zeta_P(1-m; \{V_1, V_2, V_3\}, \xi)=-2\pi i C(m) \sum_{k_1,k_2,k_3}{}' \left\{\prod_{j=1}^{3} \frac{B_{k_j+1}(\xi_j)}{(k_j+1)!}\right\} \Lambda_{(k_1,k_2,k_3)},$$

where k_1, k_2, k_3 run over all integers satisfying the conditions $k_1, k_2 \geqq -1$, $k_3 \geqq 0$, and $k_1+k_2+k_3=2(m-1)$.

Proof. We take δ sufficiently small so that Lemma 2.6 for V_j $(j=1, 2)$ and Lemma 2.7 hold. Then we get the following power series expansion, if $|t|<\delta$, $u \in D_\delta(1)$,

$$\Phi_P((t, u, \theta); \{V_1, V_2, V_3\}, \xi)=\sum_{k_j} \prod_{j=1}^{3} \left\{\frac{B_{k_j+1}(1-\xi_j)}{(k_j+1)!} \lambda((u, \theta), V_j)^{k_j}\right\} \cdot t^{k_1+k_2+k_3},$$

where k_1, k_2, k_3 run over all integers satisfying $k_1, k_2 \geqq -1$ and $k_3 \geqq 0$. Applying Proposition 2.9, we obtain the expression for $\zeta_P(1-m; \{V_1, V_2, V_3\}, \xi)$ in Proposition 2.11.

Quite in the same manner as in the proof of Proposition 2.11, one can evaluate special values at $s=1-m$ of the functions $\zeta_P(s; \{V_j, V_3\}, (\xi_j, \xi_3))$ $(j=1, 2)$, $\zeta(s; \{V_1, V_2\}, (\xi_1, \xi_2))$ and $\zeta(s; \{V_j\}, \xi)$ $(j=1, 2, \xi>0)$. So we omit the proof of the following proposition.

Proposition 2.12. *Let ξ, ξ_j $(j=1, 2, 3)$ be positive numbers and $m \in \mathbf{N}$. Then the following expressions hold.*

(a) $$\zeta_P(1-m; \{V_1, V_3\}, (\xi_1, \xi_3))=2\pi i C(m) \sum_{k_1,k_3}{}' \frac{B_{k_1+1}(\xi_1) B_{k_3+1}(\xi_3)}{(k_1+1)!(k_3+1)!} \Lambda_{(k_1,0,k_3)},$$

where k_1, k_3 run over all integers with $k_1 \geqq -1$, $k_3 \geqq 0$, $k_1+k_3=2(m-1)$.

(b) $$\zeta_P(1-m; \{V_2, V_3\}, (\xi_2, \xi_3))=2\pi i C(m) \sum_{k_2,k_3}{}' \frac{B_{k_2+1}(\xi_2) B_{k_3+1}(\xi_3)}{(k_2+1)!(k_3+1)!} \Lambda_{(0,k_2,k_3)},$$

where k_2, k_3 run over all integers with $k_2 \geqq -1$, $k_3 \geqq 0$, $k_2+k_3=2(m-1)$.

(c) $$\zeta(1-m; \{V_1, V_2\}, (\xi_1, \xi_2))=2\pi i C(m) \sum_{k_1,k_2}{}' \frac{B_{k_1+1}(\xi_1) B_{k_2+1}(\xi_2)}{(k_1+1)!(k_2+1)!} \Lambda_{(k_1,k_2,0)},$$

where k_1, k_2 run over all integers with $k_1, k_2 \geqq -1$, $k_1+k_2=2(m-1)$.

(d) $$\zeta(1-m;\{V_1\},\xi)=-2\pi i C(m)\frac{B_{2m-1}(\xi)}{(2m-1)!}\Lambda_{(2m-2,0,0)}.$$

(e) $$\zeta(1-m;\{V_2\},\xi)=-2\pi i C(m)\frac{B_{2m-1}(\xi)}{(2m-1)!}\Lambda_{(0,2m-2,0)}.$$

To complete the evaluation of special values of zeta functions above, we have to study some properties of the numbers $\Lambda_{(k_1,k_2,k_3)}$.

Proposition 2.13. *Let k_1, k_2, k_3 be integers with $k_1, k_2\geqq -1$, $k_3\geqq 0$, and $k_1+k_2+k_3=2(m-1)$ ($m\in\boldsymbol{N}$). If k_1, k_2, k_3 satisfy one of the following three conditions, then, $(1/\pi)\Lambda_{(k_1,k_2,k_3)}$ is a rational number.*

(i) *k_1, k_2, k_3 are non-negative integers,*
(ii) *$k_1=-1$ and k_2 is a positive odd integer,*
(iii) *$k_2=-1$ and k_1, k_3 are non-negative integers.*

Proof. A straightforward computation shows that

$$(2.2.15)\qquad \begin{cases}\lambda((u,\theta),V_1)=1+u, & \lambda((u,\theta),V_2)=1+u+(1-u)\sin\theta\cos\theta,\\ \lambda((u,\theta),V_3)=u\sin^2\theta+\cos^2\theta.\end{cases}$$

Changing the variable by $\cot\theta=x$ in (2.2.14), we obtain

$$(2.2.16)\qquad \frac{1}{2\pi}\Lambda_{(k_1,k_2,k_3)}=\int_{I_\varepsilon(1)}u^{-m-1/2}(1-u)P_{k_1,k_2,k_3}(u)du,$$

where ε is a sufficiently small number and

$$(2.2.17)\quad P_{k_1,k_2,k_3}(u)=\frac{1}{2\pi}(1+u)^{k_1}\int_{\boldsymbol{R}}\left\{1+u+(1-u)\frac{x}{1+x^2}\right\}^{k_2}\left(\frac{x^2+u}{1+x^2}\right)^{k_3}\frac{dx}{1+x^2}.$$

The computation of the integral in (2.2.17) easily shows that, if k_1, k_2, k_3 satisfy either of the conditions (i), (ii), then, $P_{k_1,k_2,k_3}(u)$ is a polynomial of u with rational coefficients. On the one hand,

$$(2.2.18)\qquad \int_{I_\varepsilon(1)}u^{k-1/2}du=-\frac{4}{2k+1}\qquad \text{for any } k\in\boldsymbol{Z}.$$

Therefore, according to (2.2.16), the value $(1/2\pi)\Lambda_{(k_1,k_2,k_3)}$ is a rational number, if each triple (k_1,k_2,k_3) satisfies either of the conditions (i), (ii).

Suppose that $k_2=-1$ and $k_1, k_3\geqq 0$. Set

$$Q(u)=3u^2+10u+3,$$

and

$$\omega(u)=\frac{-(1-u)+i\sqrt{Q(u)}}{2(1+u)} \qquad (0\leqq u\leqq 1).$$

Then, $\omega(u)$, $\overline{\omega(u)}$ (the complex conjugate of $\omega(u)$) are the distinct roots of the quadratic equation: $(1+u)x^2+(1-u)x+1+u=0$. We write simply ω for $\omega(u)$. Applying the residue theorem in computing the integral in (2.2.17), we obtain

$$\text{(2.2.19)} \quad P_{k_1,-1,k_3}(u)=i\frac{(1+u)^{k_1}}{(1+u)(\omega-\bar{\omega})}\left(1-\frac{1-u}{1+\omega^2}\right)^{k_3}+R_{k_1,k_3}(u),$$

where we put

$$R_{k_1,k_3}(u)=i(1+u)^{k_1}\operatorname*{Res}_{x=i}\{(1+u)x^2+(1-u)x+(1+u)\}^{-1}\left(1-\frac{1-u}{1+x^2}\right)^{k_3}.$$

Since the residue at $x=i$ of the function

$$\{(1+u)x^2+(1-u)x+1+u\}^{-1}\left(\frac{1-u}{1+x^2}\right)^{l} \qquad (l\in \boldsymbol{Z},\ l>0),$$

is a polynomial of u with coefficients in the Gaussian field $\boldsymbol{Q}(i)$, so is $R_{k_1,k_3}(u)$. Therefore, the real part of $R_{k_1,k_3}(u)$ is a polynomial of u with rational coefficients. An elementary calculation shows that

$$(1+u)(\omega-\bar{\omega})=i\sqrt{Q(u)}, \quad 1+\omega^2=-\frac{1-u}{1+u}\omega, \quad \omega\bar{\omega}=1, \quad \text{and}$$

$$1-\frac{1-u}{1+\omega^2}=\frac{1+u-i\sqrt{Q(u)}}{2}.$$

Then, we get, by (2.2.19),

$$\text{(2.2.20)} \quad P_{k_1,-1,k_3}(u)=2^{-k_3}(1+u)^{k_1}\{1+u-i\sqrt{Q(u)}\}^{k_3}Q(u)^{-1/2}+R_{k_1,k_3}(u).$$

The function $u^{-m+1}(1+u)^{2m-2-2j}Q(u)^j$ $(0\leqq j\leqq m-1)$ is invariant under the transformation $u\to 1/u$, and consequently, is a polynomial of $(u+1/u)$ with degree $m-1$. Thus one can write

$$\text{(2.2.21)} \quad u^{-m+1}(1+u)^{2m-2-2j}Q(u)^j=\sum_{k=0}^{m-1}b_{j,m,k}(u^k+u^{-k}) \qquad (0\leqq j\leqq m-1)$$

with some $b_{j,m,k}\in\boldsymbol{Q}$. We set, for a sufficiently small $\varepsilon>0$,

$$g(s)=\int_{I_\varepsilon(1)} u^{s-1/2}Q(u)^{-1/2}du,$$

where the branch of $Q(u)^{1/2}$ is so taken that $Q(u)^{1/2}>0$, if $u\in\boldsymbol{R}$. The integral in the right side is independent of the choice of ε and converges for arbitrary $s\in\boldsymbol{C}$. Accordingly, $g(s)$ stands for an entire function of s. We define a sequence $\{\alpha_n\}$ by putting

$$\alpha_n=g(n)-g(-n) \qquad (n=0, 1, 2, \cdots).$$

Lemma 2.14. *The sequence* $\{\alpha_n\}$ *satisfies the recursive formula*

$$3(n-1/2)\alpha_n+10(n-1)\alpha_{n-1}+3(n-3/2)\alpha_{n-2}=-16 \qquad (n\geqq 2)$$

with $\alpha_0=0$, $\alpha_1=-16/3$. *Consequently, all* α_n *are rational numbers.*

Proof. We begin with, if $\mathrm{Re}\,(s)>-1/2$.

$$\begin{aligned}3g(s+1)+5g(s)&=\int_{I_\varepsilon(1)} u^{s-1/2}(3u+5)Q(u)^{-1/2}du\\&=(e[s-1/2]-1)\int_0^1 u^{s-1/2}\frac{d}{du}Q(u)^{1/2}du.\end{aligned}$$

The integration by parts then implies that, if $\mathrm{Re}\,(s)>1/2$,

$$3g(s+1)+5g(s)=-4(1+e[s])-(s-1/2)\int_{I_\varepsilon(1)} u^{s-3/2}Q(u)^{1/2}du.$$

Writing $Q(u)^{1/2}=Q(u)\cdot Q(u)^{-1/2}$, we get the following functional equation:

(2.2.22) $\quad 3(s+1/2)g(s+1)+10sg(s)+3(s-1/2)g(s-1)=-4(1+e[s]),$

which is valid for arbitrary $s\in\boldsymbol{C}$ by the analytic continuation. Putting $s=0$, one gets $\alpha_1=g(1)-g(-1)=-16/3$. Moreover, if we substitute $s=m-1$ and $s=1-m$, respectively in (2.2.22), and add the both equalities so obtained, then we have the recursive formula in Lemma 2.14.

We continue the proof of Proposition 2.13. We see from (2.2.17) that $P_{k_1,-1,k_3}(u)$ is real valued, if $0\leqq u\leqq 1$, and hence from (2.2.20) that $P_{k_1,-1,k_3}(u)$ is a polynomial of u with rational coefficients plus a $\boldsymbol{Q}$-linear sum of the functions $(1+u)^{2m-1-2j}Q(u)^{j-1/2}$ $(0\leqq j\leqq m-1)$. Moreover, we find from (2.2.21) that

$$\begin{aligned}&\int_{I_\varepsilon(1)} u^{-m-1/2}(1-u)(1+u)^{2m-1-2j}Q(u)^{j-1/2}du\\&\quad=-2b_{j,m,0}\alpha_1+\sum_{k=1}^{m-1} b_{j,m,k}(\alpha_{k-1}-\alpha_{k+1}) \qquad (0\leqq j\leqq m-1),\end{aligned}$$

which is a rational number owing to Lemma 2.14. Thus, taking (2.2.16), (2.2.18) into account of, we can conclude that $(1/2\pi)\Lambda_{(k_1,-1,k_3)}(k_1, k_3 \geqq 0)$ is a rational number. q.e.d.

Remark For triples (k_1, k_2, k_3) not satisfying the conditions of Proposition 2.13, it will be hard to compute $\Lambda_{(k_1,k_2,k_3)}$ in an elementary manner. However we do not need the explicit values of them (see 2.4, 2.5 of this paper).

To evaluate the special values at $s=0$ of L-functions, we need the following explicit values of $\Lambda_{(k_1,k_2,k_3)}$.

Proposition 2.15. *We have*

$$\Lambda_{(0,0,0)}=\Lambda_{(-1,1,0)}=8\pi, \quad \Lambda_{(1,-1,0)}=32\pi/3, \quad \textit{and} \quad \Lambda_{(0,-1,1)}=16\pi/3.$$

Proof. The first two identities are straightforward. Then, the identity (2.2.20) implies that $P_{1,-1,0}(u)=2P_{0,-1,1}(u)=(1+u)Q(u)^{-1/2}$. Therefore, we get

$$\Lambda_{(1,-1,0)}=2\Lambda_{(0,-1,1)}=2\pi(g(-1)-g(1))=-2\pi\alpha_1=32\pi/3.$$

2.3. Integral representations of partial zeta functions II

We keep the notation used in 2.2. We shall study the analytic continuations of the functions $\zeta_S(s, \{V_1, V_2, V_3\}, (\xi_1, \xi_2))$, $\zeta_S(s: \{V_j, V_3\}, \xi)$ $(j=1, 2)$, and determine the first and, if possible, the second term of the Laurent expansions at $s=1-m$ $(m \in N)$ of them.

For simplicity we write λ_j for $\lambda((u, \theta), V_j)$ $(j=1, 2, 3)$, if there is no fear of confusion. We see easily from (2.2.6) that, for positive numbers ξ_1, ξ_2, ξ, and for $\mathrm{Re}(s)>3/2$,

$$\text{(2.3.1)} \quad \begin{aligned} &\zeta_S(s; \{V_1, V_2, V_3\}, (\xi_1, \xi_2))=\frac{1}{\Gamma_2(s)(e[2s]-1)}I(s; (\xi_1, \xi_2)) \\ &\zeta_S(s; \{V_j, V_3\}, \xi)=\frac{1}{\Gamma_2(s)(e[2s]-1)}I_j(s; \xi) \qquad (j=1, 2), \end{aligned}$$

where we put

$$I(s; (\xi_1, \xi_2))=\int_{I_\varepsilon(\infty)} dt\int_0^1 du\int_0^\pi d\theta\cdot t^{2s-2}u^{s-3/2}(1-u)\,\frac{1}{\lambda_3}\prod_{j=1}^{2}\phi(t\lambda_j; 1-\xi_j),$$

$$I_j(s; \xi)=\int_{I_\varepsilon(\infty)} dt\int_0^1 du\int_0^\pi d\theta\cdot t^{2s-2}u^{s-3/2}(1-u)\,\frac{1}{\lambda_3}\phi(t\lambda_j; 1-\xi) \quad (j=1, 2),$$

ε being taken sufficiently small. The absolutely convergence for $\mathrm{Re}(s) > 3/2$ of the integrals above is easily verified by Lemma 2.6. We shall first integrate with respect to θ. Changing the variable by $\cot\theta = x$ $(-\infty < x < +\infty)$, we get, by (2.2.15),

$$\lambda_1 = 1+u, \quad \lambda_2 = 1+u+(1-u)\frac{x}{1+x^2}, \quad \lambda_3 = \frac{u+x^2}{1+x^2}. \tag{2.3.2}$$

As is easily seen, for each positive number $\beta < 1$, there exists a small positive number $\delta = \delta(\beta)$ such that $\phi(t(1+z);\, 1-\xi)$ $(\xi > 0)$, as a function of t, z, is holomorphic in the region $\{(t, z) \in C^2 \,|\, \{t \in D_\delta(\infty),\ t \neq 0,\ |z| \leqq \beta\}$. Then, $\phi(t(1+z);\, 1-\xi)$, as a function of z, has the power series expansion

$$\phi(t(1+z);\, 1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(t;\, 1-\xi)t^k}{k!} \cdot z^k \tag{2.3.3}$$
$$(t \in D_\delta(\infty),\ t \neq 0,\ |z| \leqq \beta).$$

It follows from (2.3.3) that, if $t \in D_\delta(\infty)$, $t \neq 0$, and $0 \leqq u \leqq 1$,

$$\phi(t\lambda_2;\, 1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(t(1+u);\, 1-\xi)t^k(1-u)^k}{k!} \cdot \left(\frac{x}{1+x^2}\right)^k, \tag{2.3.4}$$

where δ is taken sufficiently small. We then define a function $\mathscr{H}_k(u)$ for each non-negative integer k:

$$\mathscr{H}_k(u) = (1-u)^k \int_R \frac{1}{x^2+u}\left(\frac{x}{1+x^2}\right)^k dx \qquad (u > 0). \tag{2.3.5}$$

Obviously, we have $\mathscr{H}_k(u) = 0$ for any odd k. Applying the residue theorem in calculating the integral in (2.3.5), one can divide $\mathscr{H}_{2k}(u)$ $(k \in Z,\ k \geqq 0)$ into two parts as follows:

$$\mathscr{H}_{2k}(u) = \pi u^{-1/2}(-u)^k + \pi \mathscr{A}_{2k}(u), \tag{2.3.6}$$

where

$$\mathscr{A}_{2k}(u) = 2i(1-u)^{2k} \operatorname*{Res}_{x=i}\left(\frac{1}{x^2+u}\left(\frac{x}{1+x^2}\right)^{2k}\right).$$

An elementary computation shows that $\mathscr{A}_0(u) = 0$, $\mathscr{A}_2(u) = (1+u)/2$. Moreover, we observe that each $\mathscr{A}_{2k}(u)$ is a polynomial of u with rational coefficients. We set, for $\xi > 0$,

$$F_1(t, u;\, 1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(t(1+u);\, 1-\xi)}{(2k)!} \cdot t^{2k}(-u)^k,$$

$$F_2(t, u; 1-\xi)=\sum_{k=0}^{\infty}\frac{\phi^{(2k)}(t(1+u); 1-\xi)}{(2k)!}\cdot t^{2k}\mathscr{A}_{2k}(u).$$

We shall discuss the convergence and the regularity of $F_j(t, u; 1-\xi)$ $(j=1, 2)$ as functions of t, u. For that purpose, some preparations will be needed. We put, for $a \in \boldsymbol{R}$, and $j \in \boldsymbol{N}$,

$$\phi_j(t; a)=\frac{e^{ta}}{(e^t-1)^j}.$$

Lemma 2.16. *Let $n \in \boldsymbol{N}$ and $a<1$. If we write, in a unique way,*

$$\phi^{(n)}(t; a)=\sum_{j=0}^{n}\lambda_{j,n}(a)\phi_{j+1}(t; a) \qquad \text{with some } \lambda_{j,n}(a) \in \boldsymbol{R}, \tag{2.3.7}$$

then, we have $(-1)^n\lambda_{j,n}(a)>0$ for each j $(0\leqq j\leqq n)$.

Proof. Differentiating the both sides of (2.3.7) with respect to t, and using the identity $\phi'_{j+1}(t; a)=(a-j-1)\phi_{j+1}(t; a)-(j+1)\phi_{j+2}(t; a)$, we get recursive relations:

$$\begin{cases}\lambda_{0,n+1}(a)=(a-1)\lambda_{0,n}(a),\\ \lambda_{j,n+1}(a)=(a-j-1)\lambda_{j,n}(a)-j\lambda_{j-1,n}(a) \qquad (1\leqq j\leqq n),\\ \lambda_{n+1,n+1}(a)=-(n+1)\lambda_{n,n}(a).\end{cases} \tag{2.3.8}$$

In the case of $n=1$, we have, trivially, $(-1)\lambda_{j,1}(a)>0$ $(j=0, 1)$. Thus the assertion follows by induction on n from (2.3.8).

Taking the k-th derivative of (2.2.2), one gets, if $|t|<2\pi$,

$$\frac{\phi^{(k)}(t; a)}{k!}=(-1)^k t^{-k-1}+\sum_{n=k+1}^{\infty}\frac{B_n(a)}{n!}\binom{n-1}{k}t^{n-1-k}. \tag{2.3.9}$$

For $x \in \boldsymbol{R}$, $[x]$ denotes the largest integer less than or equal to x.

Lemma 2.17. *Let $0<\beta<1$ and $a \in \boldsymbol{R}$. If $|t|\leqq\pi/2$, $|w|\leqq\beta$, then,*

$$t\sum_{k=0}^{\infty}\frac{\phi^{(2k)}(t; a)t^{2k}}{(2k)!}\cdot w^k=\frac{1}{1-w}+\sum_{n=1}^{\infty}\frac{B_n(a)}{n!}\left(\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}w^k\right)t^n, \tag{2.3.10}$$

where we put $\kappa_n=[(n-1)/2]$, and the infinite series of both sides are absolutely convergent. Moreover, the function defined by the infinite series (2.3.10) *indicates a holomorphic function of t, w in the region $\{(t, w)|\ |t|\leqq \pi/2, |w|\leqq\beta\}$.*

Proof. By virtue of the fact that $t\phi(t;a)=\sum_{k=0}^{\infty}(B_k(a)/k!)t^k$ is absolutely convergent for $|t|<2\pi$, there exists a positive constant C_1 independent of k which satisfies

$$|B_k(a)/k!|<C_1(3\pi/2)^{-k} \qquad (k=1,2,\cdots).$$

Using (2.3.9), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\frac{\phi^{(2k)}(t;a)t^{2k+1}}{(2k)!}\cdot w^k\right| &\leqq \sum_{k=0}^{\infty}|w|^k+C_1\sum_{k=0}^{\infty}\sum_{n=2k+1}^{\infty}\binom{n-1}{2k}|w|^k\left(\frac{2|t|}{3\pi}\right)^n\\ &\leqq \frac{1}{1-|w|}+C_1\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}\beta^k\right)\left(\frac{2|t|}{3\pi}\right)^n\\ &\leqq \frac{1}{1-|w|}+C_1\sum_{n=1}^{\infty}\left(\frac{4|t|}{3\pi}\right)^n,\end{aligned}$$

where the last infinite series is convergent for $|t|\leqq\pi/2$. Thus the infinite series in the both sides of (2.3.10) are absolutely and uniformly convergent for $|t|\leqq\pi/2$, $|w|\leqq\beta$. In a similar manner, the identity (2.3.10) is easily shown to hold.

Proposition 2.18. *If we take δ sufficiently small, then, the infinite series $tF_j(t,u;1-\xi)$ ($\xi>0, j=1,2$) are absolutely and uniformly convergent in the region $D_\delta(\infty)\times D_\delta(1)$. Consequently, $tF_j(t,u;1-\xi)$ ($j=1,2$) are holomorphic in the same region. Moreover, $tF_j(t,u;1-\xi)$, as functions of t, have the following power series expansions; if $|t|<\delta$, $u\in D_\delta(1)$, then, we have*

$$tF_1(t,u;1-\xi)=\frac{1+u}{1+3u+u^2}+\sum_{n=1}^{\infty}\frac{B_n(1-\xi)}{n!}\cdot\mu_n(u)t^n, \tag{2.3.11}$$

$$tF_2(t,u;1-\xi)=\sum_{n=0}^{\infty}\frac{\mathscr{A}_{2n}(u)}{(1+u)^{2n+1}}+\sum_{n=1}^{\infty}\frac{B_n(1-\xi)}{n!}\cdot\nu_n(u)t^n, \tag{2.3.12}$$

where we put

$$\left\{\begin{aligned}\mu_n(u)&=\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}(1+u)^{n-1-2k}(-u)^k,\\ \nu_n(u)&=\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}(1+u)^{n-1-2k}\mathscr{A}_{2k}(u) \quad \left(\kappa_n=\left[\frac{n-1}{2}\right], n\geqq 1\right).\end{aligned}\right. \tag{2.3.13}$$

Proof. First we consider the infinite series $tF_1(t,u;1-\xi)$. We put $t'=t(1+u)$, $w=-u/(1+u)^2$. Obviously,

$$tF_1(t, u; 1-\xi) = \frac{t'}{1+u} \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(t'; 1-\xi)t'^{2k}}{(2k)!} \cdot w^k.$$

By the inequality $|-u/(1+u)^2| \leqq 1/4$ for $0 \leqq u \leqq 1$, there exists a small positive constant δ_1 such that $|-u/(1+u)^2| \leqq 1/2$ for $u \in D_{\delta_1}(1)$. If $|t| < \pi/4$, $u \in D_{\delta_1}(1)$, then we get $|t'| < \pi/2, |w| \leqq 1/2$. It follows easily from Lemma 2.17 that $tF_1(t, u; 1-\xi)$ converges absolutely and is holomorphic in the region $\{(t, u) |\ |t| < \pi/4, u \in D_{\delta_1}(1)\}$, and moreover that the power series expansion (2.3.11) holds in the same region. We take δ sufficiently small, comparing with δ_1. Let $|t| \geqq \pi/4$, $t \in D_\delta(\infty)$, and moreover, $u \in D_\delta(1)$. Set $\tau' = \mathrm{Re}(t')$. Then we may have $\tau' > |t'|/\sqrt{2}$, δ being taken sufficiently small. An elementary observation shows that

$$|\phi_j(t'; a)| \leqq \phi_j(\tau'; a) \qquad (a \in \boldsymbol{R}, j=0, 1, 2, \cdots),$$

from which, in addition to Lemma 2.16, we get

$$|\phi^{(2k)}(t', 1-\xi)| \leqq \phi^{(2k)}(\tau', 1-\xi) \qquad (k=0, 1, 2, \cdots).$$

Hence we see from Lemma 2.16 and the expansion (2.3.4) that

$$\begin{aligned}\sum_{k=0}^{\infty} \left| \frac{\phi^{(2k)}(t'; 1-\xi)t'^{2k}}{(2k)!} \cdot w^k \right| &\leqq \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(\tau'; 1-\xi)|t'|^{2k}}{(2k)!} \cdot (1/2)^k \\ &\leqq \sum_{k=0}^{\infty} \frac{\phi^{(k)}(\tau'; 1-\xi)\tau'^k}{k!} \cdot \left(\frac{-|t'|}{\sqrt{2}\tau'} \right)^k \\ &= \phi\left(\tau' - \frac{|t'|}{\sqrt{2}}; 1-\xi \right).\end{aligned}$$

Thus, $F_1(t, u; 1-\xi)$ is absolutely convergent in the region $\{(t, u) |\ |t| \geqq \pi/4, t \in D_\delta(\infty), u \in D_\delta(1)\}$. Moreover, we see from the observation above that $F_1(t, u; 1-\xi)$ is uniformly convergent in some small neighbourhood of (t, u) contained in the region above. Consequently, $F_1(t, u; 1-\xi)$ is holomorphic in that region.

Next we consider the series $tF_2(t, u; 1-\xi)$. We have to estimate $\mathscr{A}_{2k}(u)$ from the above. The definition of $\mathscr{A}_{2k}(u)$ implies that

$$\pi \mathscr{A}_{2k}(u) = (1-u)^{2k} \int_{\boldsymbol{R}} \frac{1}{x^2+u} \left\{ \left(\frac{x}{1+x^2} \right)^{2k} - \left(\frac{-u}{(1-u)^2} \right)^k \right\} dx.$$

Putting $s(x) = (x/1+x^2)^2$ for simplicity, we get the expression

$$\pi \mathscr{A}_{2k}(u) = \int_{\boldsymbol{R}} \frac{ux^2+1}{(1+x^2)^2} \sum_{j=0}^{k-1} \{s(x)(1-u)^2\}^{k-1-j}(-u)^j dx, \tag{2.3.14}$$

which holds for any $u \in \boldsymbol{C}$. We take a positive number δ_2 in such a manner that, if $|u|<\delta_2$, then, $|-u/(1-u)^2|<1/16$. Accordingly, $\delta_2 \leqq 9-4\sqrt{5}=0.0557\cdots$. Using the inequalities $s(x)\leqq 1/4$, we see easily from (2.3.14) that, if $|u|<\delta_2$,

$$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right|<(1/4)^{k-1}\frac{|1-u|^{2(k-1)}}{|1+u|^{2k}}\sum_{j=0}^{k-1}\left|\frac{-4u}{(1-u)^2}\right|^j$$
$$<\frac{1}{(1-\delta_2)^2}(1/4)^{k-1}\left(\frac{1+\delta_2}{1-\delta_2}\right)^{2(k-1)}\cdot\frac{4}{3}.$$

Since we have the inequality $(1+\delta_2)/2(1-\delta_2)<3/5$, there exists a positive constant C_2 independent of k such that

(2.3.15) $$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right|<C_2(3/5)^{2k} \quad \text{if } |u|<\delta_2 \quad (k=0, 1, 2, \cdots).$$

We put $\delta_3=\delta_2/2$. On the other hand, if $|u|\geqq\delta_2$, and $u \in D_{\delta_3}(1)$, then,

$$|1+u|\geqq 1+\mathrm{Re}\,(u)>1+\delta_3, \quad |1-u|\leqq 1, \quad \text{and} \quad |u|\leqq 1.$$

Thus the inequalities just above and (2.3.14) imply that, for any non-negative integer k,

(2.3.16) $$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right|<\frac{4}{3}\left(\frac{1}{1+\delta_3}\right)^{2k} \qquad \text{if } |u|\geqq\delta_2 \text{ and } u \in D_{\delta_3}(1).$$

Putting $t(1+u)=t'$, we get, formally,

(2.3.17) $$tF_2(t, u; 1-\xi)=\frac{t'}{1+u}\sum_{k=0}^{\infty}\frac{\Phi^{(2k)}(t'; 1-\xi)t'^{2k}}{(2k)!}\cdot\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}.$$

With the help of (2.3.15), (2.3.16), we can prove that the right side of (2.3.17) is absolutely convergent if $|t|<\pi/4$, $u \in D_{\delta_3}(1)$, and we obtain, similarly as in the proof of (2.3.11), the identity (2.3.12). Since $\mathscr{A}_{2k}(u)$ is a polynomial of u, we see easily from the expression (2.3.12) that $tF_2(t, u; 1-\xi)$ is holomorphic in the region $\{t \in \boldsymbol{C} \mid |t|<\pi/4\}\times D_{\delta_3}(1)$. The rest of assertions for $tF_2(t, u; 1-\xi)$ can be verified in the same manner as in the case of $tF_1(t, u; 1-\xi)$ by using the inequlities (2.3.15), (2.3.16). q.e.d.

We take δ sufficiently small so that the identity (2.3.4) and Proposition 2.18 simultaneously hold. Then, taking the identities (2.3.4), (2.3.5), (2.3.6) and the inequality $|x/(1+x^2)|\leqq 1/2$ into account of, we obtain

$$\int_0^\pi \frac{1}{\lambda_3}\phi(t\lambda_2;\,1-\xi)d\theta=\pi u^{-1/2}F_1(t,u;\,1-\xi)+\pi F_2(t,u;\,1-\xi) \qquad (t\in D_\delta(\infty),\ 0<u\leqq 1). \tag{2.3.18}$$

We set, for positive numbers ξ_1, ξ_2, ξ,

$$\begin{cases}\Phi^{(0)}(t,u;\,\xi_1,\xi_2)=\phi(t(1+u);\,1-\xi_1)F_1(t,u;\,1-\xi_2),\\ \Phi^{(1)}(t,u;\,\xi)=\phi(t(1+u);\,1-\xi),\\ \Phi^{(2)}(t,u;\,\xi)=F_1(t,u;\,1-\xi),\end{cases} \tag{2.3.19}$$

$$\begin{cases}\Psi^{(0)}(t,u;\,\xi_1,\xi_2)=\phi(t(1+u);\,1-\xi_1)F_2(t,u;\,1-\xi_2),\\ \Psi^{(1)}(t,u;\,\xi)=0,\\ \Psi^{(2)}(t,u;\,\xi)=F_2(t,u;\,1-\xi).\end{cases} \tag{2.3.20}$$

Let $\Phi(t,u)$ (resp. $\Psi(t,u)$) be one of the three functions given in (2.3.19) (resp. in (2.3.20)). We set

$$\begin{aligned} J(s;\Phi)&=\int_{I_\varepsilon(\infty)}dt\int_{I_\varepsilon(1)}du\cdot t^{2s-2}u^{s-2}(1-u)\Phi(t,u),\\ K(s;\Psi)&=\int_{I_\varepsilon(\infty)}dt\int_{I_\varepsilon(1)}du\cdot t^{2s-2}u^{s-3/2}(1-u)\Psi(t,u),\end{aligned} \tag{2.3.21}$$

where ε is taken sufficiently small with $\varepsilon<\delta$, δ being the same as in (2.3.18). Then, by virtue of Proposition 2.18 and its proof, the integrals $J(s;\Phi)$, $K(s;\Psi)$ are independent of the choice of ε and absolutely convergent for arbitrary $s\in \boldsymbol{C}$. Moreover, they indicate entire functions of s. We write, for convenience,

$$\begin{cases}J(s;(\xi_1,\xi_2))=J(s;\Phi^{(0)}(t,u;\,\xi_1,\xi_2)),\\ J_j(s;\xi)=J(s;\Phi^{(j)}(t,u;\,\xi)) \qquad (j=1,2),\\ K(s;(\xi_1,\xi_2))=K(s;\Psi^{(0)}(t,u;\,\xi_1,\xi_2)),\\ K_j(s;\xi)=K(s;\Psi^{(j)}(t,u;\,\xi)) \qquad (j=1,2).\end{cases} \tag{2.3.22}$$

Trivially, $K_1(s;\xi)=0$. Thus, using (2.3.18), we obtain convenient expressions for $I(s;(\xi_1,\xi_2))$, $I_j(s;\xi)$ $(j=1,2)$ by the integrals (2.3.22) over contour paths $I_\varepsilon(\infty)$, $I_\varepsilon(1)$.

Proposition 2.19. *Let ξ_1, ξ_2, ξ be positive numbers. We have*

$$I(s;(\xi_1,\xi_2))=\frac{\pi}{(e[s]-1)}J(s;(\xi_1,\xi_2))+\frac{\pi}{(e[s-3/2]-1)}K(s;(\xi_1,\xi_2)),$$

$$I_j(s;\xi)=\frac{\pi}{(e[s]-1)}J_j(s;\xi)+\frac{\pi}{(e[s-3/2]-1)}K_j(s;\xi)\qquad (j=1,2),$$

which give the analytic continuation to meromorphic functions of s in the whole complex plane.

Corollary to Proposition 2.19. *The functions* $\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2))$, $\zeta_S(s;\{V_j,V_3\},\xi)$ $(j=1,2)$ *can be continued analytically to meromorphic functions of s in the whole complex plane.*

The corollary is an immediate consequence of Proposition 2.19 and (2.3.1).

Proposition 2.19 shows us that the Laurent expansions at $s=1-m$ $(m\in \boldsymbol{N})$ of $I(s;(\xi_1,\xi_2))$, $I_j(s;\xi)$ $(j=1,2)$ are given as follows:

(2.3.23) $$I(s;(\xi_1,\xi_2))=\frac{J(1-m;(\xi_1,\xi_2))}{2i(s+m-1)}+\frac{1}{2i}\{J'(1-m;(\xi_1,\xi_2)) -\pi iJ(1-m;(\xi_1,\xi_2))-\pi iK(1-m;(\xi_1,\xi_2))\} +\text{higher terms of }(s+m-1),$$

(2.3.24) $$I_j(s;\xi)=\frac{J_j(1-m;\xi)}{2i(s+m-1)}+\frac{1}{2i}\{J_j'(1-m;\xi)-\pi iJ_j(1-m;\xi) -\pi iK_j(1-m;\xi)\}+\text{higher terms of }(s+m-1) \quad (j=1,2).$$

Let $C(m)$ $(m\in \boldsymbol{N})$ be the constant given in Proposition 2.9. Then, as a Taylor expansion at $s=1-m$ $(m\in \boldsymbol{N})$, we have

$$\frac{1}{\Gamma_2(s)(e[2s]-1)}=-2C(m)+\beta_m(s+m-1)+\text{higher terms of }(s+m-1)$$

with some constant $\beta_m\in \boldsymbol{C}$. Thus, by (2.3.1), (2,3.23), we get the Laurent expansion at $s=1-m$ of $\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2))$:

(2.3.25) $$\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2)) =\frac{iC(m)J(1-m;(\xi_1,\xi_2))}{(s+m-1)}+\frac{1}{2i}\{(\beta_m+2\pi iC(m))J(1-m;(\xi_1,\xi_2)) -2C(m)J'(1-m;(\xi_1,\xi_2))+2\pi iC(m)K(1-m;(\xi_1,\xi_2))\} +\text{higher terms of }(s+m-1).$$

At this stage we have to evaluate $J(1-m;(\xi_1,\xi_2))$, $J'(1-m;(\xi_1,\xi_2))$, $K(1-m;(\xi_1,\xi_2))$ and so on.

We consider the integral $J(s, \Phi)$ in (2.3.21). Putting $s=1-m$, we get

$$J(1-m; \Phi)=\int_{\Gamma_\varepsilon} dt\int_{\Gamma_\varepsilon} du\cdot t^{-2m}u^{-m-1}(1-u)\Phi(t, u) \tag{2.3.26}$$

(for the path Γ_ε, see Proposition 2.9).

Furthermore, derivating the integrand of $J(s; \Phi)$ with respect to s, we obtain

$$\begin{aligned} J'(1-m; \Phi) &\\ =\int_{I\varepsilon(\infty)} dt\int_{I\varepsilon(1)} &du\cdot t^{-2m}u^{-m-1}(1-u)(2\log t+\log u)\Phi(t, u), \end{aligned} \tag{2.3.27}$$

where the integral is absolutely convergent again by Proposition 2.18. For non-negative integers n, the functions $\Phi_n(t)$, according to the choice of $\Phi(t, u)$, are defined as follows; We set

$$\begin{cases} \Phi_n(t)=\dfrac{\phi(t; 1-\xi_1)\phi^{(2n)}(t; 1-\xi_2)(-1)^n}{(2n)!} & \text{if } \Phi(t, u)=\Phi^{(0)}(t, u; \xi_1, \xi_2), \\ \Phi_0(t)=\phi(t; 1-\xi), \quad \Phi_n(t)=0 \quad (n\geqq 1) & \text{if } \Phi(t, u)=\Phi^{(1)}(t, u; \xi), \\ \Phi_n(t)=\dfrac{\phi^{(2n)}(t; 1-\xi)(-1)^n}{(2n)!} & \text{if } \Phi(t, u)=\Phi^{(2)}(t, u; \xi). \end{cases}$$

Moreover, we see from Proposition 2.18 that $\Phi(t, u)$ has a Laurent expansion with respect to t:

$$\Phi(t, u)=\sum_{n=-2}^{\infty} b_n(u; \Phi)t^n \qquad \text{if } |t|<\delta,\ u\in D_\delta(1). \tag{2.3.28}$$

Proposition 2.20. *Let $m\in N$. We have, for a sufficiently small ε,*

(i) $J(1-m; \Phi)=2\pi i\int_{\Gamma_\varepsilon} \Phi_m(t)dt,$

(ii) $$J'(1-m; \Phi)=4\pi i\int_{I\varepsilon(\infty)} \log t\cdot\Phi_m(t)dt-4\pi i\sum_{j=0}^{m-1}\frac{(2m-2j-1)!}{\{(m-j)!\}^2}$$
$$\times\int_{\Gamma_\varepsilon} t^{2(j-m)}\Phi_j(t)dt+2\pi i\int_{I\varepsilon(1)} u^{-m-1}(1-u)\log u\cdot b_{2m-1}(u; \Phi)du.$$

Proof. The function $\Phi(t, u)$, which is holomorphic if $t\in D_\delta(\infty)$, $t\neq 0$, and $|u|<\delta$, can be expanded in a power series of u as follows:

$$\Phi(t, u)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n\Phi}{\partial u^n}(t, 0)u^n. \tag{2.3.29}$$

On the other hand, it is easy to see from the definition of $\Phi_n(t)$ that

$$(2.3.30)\quad t^2\Phi(t, u)=t^2\sum_{n=0}^{\infty}\Phi_n(t(1+u))t^{2n}u^n \qquad (t\in D_\delta(\infty),\ u\in D_\delta(1)).$$

Since the infinite series in the right side of (2.3.30) is uniformly convergent in $D_\delta(\infty)\times D_\delta(1)$ by Proposition 2.18, and each term $t^2\Phi_n(t(1+u))t^{2n}u^n$ is a holomorphic function of t, u, we can differentiate it termwise. Thus, taking the k-th derivative of (2.3.30) with respect to u, we get

$$\frac{\partial^k\Phi}{\partial u^k}(t, u)=\sum_{j=0}^{k}\sum_{n=j}^{\infty}\binom{k}{j}n(n-1)\cdots(n-j+1)\Phi_n^{(k-j)}(t(1+u))t^{k-j+2n}u^{n-j}.$$

Therefore,

$$(2.3.31)\qquad \frac{\partial^k\Phi}{\partial u^k}(t, 0)=\sum_{j=0}^{k}\binom{k}{j}j!\,\Phi_j^{(k-j)}(t)t^{k+j}.$$

It follows from (2.3.29), (2.3.31) that

$$\begin{aligned}(2.3.32)\quad &\int_{I_\varepsilon(1)}u^{-m-1}(1-u)\Phi(t, u)du\\ &=2\pi i\left\{\frac{1}{m!}\cdot\frac{\partial^m\Phi}{\partial u^m}(t, 0)-\frac{1}{(m-1)!}\cdot\frac{\partial^{m-1}\Phi}{\partial u^{m-1}}(t, 0)\right\}\\ &=2\pi i\left(\Phi_m(t)t^{2m}+\sum_{j=0}^{m-1}\left\{\frac{t^{m+j}}{(m-j)!}\Phi_j^{(m-j)}(t)-\frac{t^{m-1+j}}{(m-1-j)!}\Phi_j^{(m-1-j)}(t)\right\}\right).\end{aligned}$$

With the use of the integration by parts, the identity

$$(2.3.33)\qquad -s\int_{I_\varepsilon(\infty)}t^{s-1}\Phi_j^{(m-1-j)}(t)dt=\int_{I_\varepsilon(\infty)}t^s\Phi_j^{(m-j)}(t)dt$$

holds for each j $(0\leqq j\leqq m-1)$. Putting $s=j-m$, we get

$$(2.3.34)\quad (m-j)\int_{I_\varepsilon(\infty)}t^{j-m-1}\Phi_j^{(m-1-j)}(t)dt=\int_{I_\varepsilon(\infty)}t^{j-m}\Phi_j^{(m-j)}(t)dt.$$

Differentiating the both sides of (2.3.33) with respect to s and then, putting $s=j-m$, we have

$$\begin{aligned}(2.3.35)\quad &(m-j)\int_{I_\varepsilon(\infty)}\log t\cdot t^{j-m-1}\Phi_j^{(m-1-j)}(t)dt-\int_{I_\varepsilon(\infty)}t^{j-m-1}\Phi_j^{(m-1-j)}(t)dt\\ &=\int_{I_\varepsilon(\infty)}\log t\cdot t^{j-m}\Phi_j^{(m-j)}(t)dt \qquad (0\leqq j\leqq m-1).\end{aligned}$$

Therefore, the identities (2.3.26), (2.3.32), and (2.3.34) imply the assertion (i). The recurrent use of the integration by parts yields

$$(2.3.36)\quad \int_{I_\varepsilon(\infty)} t^{j-m-1}\Phi_j^{(m-1-j)}(t)dt=\frac{(2m-2j-1)!}{(m-j)!}\int_{\Gamma_\varepsilon} t^{2(j-m)}\Phi_j(t)dt.$$

It is easy to see from (2.3.32), (2.3.35), and (2.3.36) that

$$(2.3.37)\quad 2\int_{I_\varepsilon(\infty)} dt\int_{I_\varepsilon(1)} du\cdot t^{-2m}u^{-m-1}(1-u)\log t\cdot\Phi(t,u)$$
$$=4\pi i\int_{I_\varepsilon(\infty)}\log t\cdot\Phi_m(t)dt-4\pi i\sum_{j=0}^{m-1}\frac{(2m-2j-1)!}{\{(m-j)!\}^2}\int_{\Gamma_\varepsilon} t^{2(j-m)}\Phi_j(t)dt.$$

Moreover, we get, using the expansion (2.3.28),

$$\int_{I_\varepsilon(\infty)} dt\int_{I_\varepsilon(1)} du\cdot t^{-2m}u^{-m-1}(1-u)\log u\cdot\Phi(t,u)$$
$$=2\pi i\int_{I_\varepsilon(1)} u^{-m-1}(1-u)\log u\cdot b_{2m-1}(u;\Phi)du,$$

which, in addition to (2.3.27), (2.3.37), completes the proof.

Let $m\in\boldsymbol{N}$. For integers k, n with $k, n\geqq 0$, $k+n=2m+1$, we define the numbers $\mathscr{M}_{(k-1,n-1)}$ by putting

$$(2.3.38)\qquad \mathscr{M}_{(k-1,n-1)}=\int_{I_\varepsilon(1)}\log u\cdot u^{-m-1}(1-u)(1+u)^{k-1}\mu_n(u)du,$$

where $\mu_n(u)$ $(n\geqq 1)$ is a polynomial of u given by (2.3.13), and

$$\mu_0(u)=\frac{1+u}{1+3u+u^2}.$$

The numbers $\mathscr{M}_{(k-1,n-1)}$ are independent of the choice of small ε.

Lemma 2.21. *If $k, n\geqq 1$ with $k+n=2m+1$, then $(1/2\pi i)\mathscr{M}_{(k-1,n-1)}$ are rational numbers.*

Proof. It follows from (2.3.13) that

$$(1-u)(1+u)^{k-1}\mu_n(u)=\sum_{j=0}^{\kappa_n}\binom{n-1}{2j}(1-u)^{2(m-j)-1}(-u)^j.$$

By the conditions $k\geqq 1$ and $k+n=2m+1$, we have $m>j$ for each j $(0\leqq j\leqq\kappa_n=[(n-1)/2])$. As is easily shown, the coefficient of the term u^m

of the polynomial $(1-u)(1+u)^{2(m-j)-1}(-u)^j$ vanishes. Therefore, the assertion of Lemma 2.21 is reduced to the formula

$$\text{(2.3.39)} \qquad \int_{I_\varepsilon(1)} \log u \cdot u^p du = \frac{2\pi i}{p+1} \qquad \text{for } p \in Z,\ p \neq -1,$$

In the case of $\Phi(t, u)=\Phi^{(0)}(t, u; \xi_1, \xi_2)$, Proposition 2.20 yields

Proposition 2.22. *Let ξ_1, ξ_2 be positive numbers and $m \in N$. Then,*

$$\text{(i)} \qquad J(1-m; (\xi_1, \xi_2)) = \frac{4\pi^2(-1)^m}{(2m+1)!}\{B_{2m+1}(\xi_1)+B_{2m+1}(\xi_2)\},$$

$$\begin{aligned}\text{(ii)} \quad J'(1-m; (\xi_1, \xi_2)) &= \frac{4\pi i(-1)^m}{(2m)!}\int_{I_\varepsilon(\infty)} \log t \cdot \phi(t; 1-\xi_1)\phi^{(2m)}(t; 1-\xi_2)dt \\ &\quad -8\pi^2\sum_{j=0}^{m-1}\frac{(2m-2j-1)!(-1)^j}{\{(m-j)!\}^2} \\ &\quad \times\left\{\frac{B_{2m+1}(\xi_1)}{(2m+1)!}+\sum_{n=2j+1}^{2m+1}\frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!n!}\cdot\binom{n-1}{2j}\right\} \\ &\quad -2\pi i\sum_{n=0}^{2m+1}\frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!n!}\cdot\mathcal{M}_{(2m-n,\, n-1)}.\end{aligned}$$

Proof. In the proof we have $\Phi(t, u)=\Phi^{(0)}(t, u; \xi_1, \xi_2)$, and

$$\Phi_n(t) = \frac{\phi(t; 1-\xi_1)\phi^{(2n)}(t; 1-\xi_2)(-1)^n}{(2n)!} \qquad (n=0, 1, 2, \cdots).$$

The expansions (2.2.2), (2.3.9) show that the coefficient of the term t^{-1} in the Laurent expansion at $t=0$ of $\Phi_m(t)$ is given by

$$\frac{(-1)^m}{(2m+1)!}\{B_{2m+1}(1-\xi_1)+B_{2m+1}(1-\xi_2)\}.$$

Thus, by (i) of Proposition 2.18, the assertion (i) follows. In view of the expansions (2.2.2), (2.3.11) of Proposition 2.18, the coefficient $b_{2m-1}(u; \Phi)$ in (2.3.28) is given as follows:

$$\begin{aligned} b_{2m-1}(u; \Phi) &= \frac{B_{2m+1}(1-\xi_1)(1+u)^{2m+1}}{(2m+1)!(1+3u+u^2)} + \sum_{n=1}^{2m+1}\frac{B_{2m+1-n}(1-\xi_1)B_n(1-\xi_2)}{(2m+1-n)!n!} \\ &\quad \times (1+u)^{2m-n}\mu_n(u).\end{aligned}$$

Therefore, we see from (2.3.38) that

$$\int_{I_\varepsilon(1)} u^{-m-1}(1-u)\log u\cdot b_{2m-1}(u;\Phi)du$$
$$=-\sum_{n=0}^{2m+1}\frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!n!}\cdot\mathcal{M}_{(2m-n,n-1)}.$$

Computing the coefficient of the term $t^{2(m-j)-1}$ $(0\leqq j\leqq m-1)$ in the Laurent expansion at $t=0$ of $\Phi_j(t)$, we have

$$\int_{\Gamma_\varepsilon} t^{2(j-m)}\Phi_j(t)dt$$
$$=2\pi i(-1)^j\left\{\frac{B_{2m+1}(1-\xi_1)}{(2m+1)!}+\sum_{n=2j+1}^{2m+1}\frac{B_{2m+1-n}(1-\xi_1)B_n(1-\xi_2)}{(2m+1-n)!n!}\cdot\binom{\mathrm{u}-1}{2j}\right\}.$$

Summing up the results above, we obtain the assertion (ii).

In another two cases of $\Phi(t,u)$, we obtain the following

Proposition 2.23. *Let* $m\in\boldsymbol{N}$ *and* $\xi>0$. *Then,*

(i) $J_j(1-m;\xi)=0 \qquad (j=1,2)$,

(ii) $J_1'(1-m;\xi)=\dfrac{4\pi^2B_{2m}(\xi)}{m(m!)^2}+2\pi i\dfrac{B_{2m}(\xi)}{(2m)!}\mathcal{M}_{(2m-1,0)}$,

$$J_2'(1-m;\xi)=\frac{4\pi^2(-1)^mB_{2m}(\xi)}{m(2m!)}+\frac{4\pi^2B_{2m}(\xi)}{m}\sum_{j=0}^{m-1}\frac{(-1)^j}{\{(m-j)!\}^2(2j)!}$$
$$+2\pi i\frac{B_{2m}(\xi)}{(2m)!}\mathcal{M}_{(0,2m-1)}.$$

(iii) *In particular,* $J_j'(1-m;\xi)\in\pi^2\boldsymbol{Q}$ $(j=1,2)$.

Proof. If $\Phi(t,u)=\phi(t(1+u);1-\xi)$, then, we have $\Phi_0(t)=\phi(t;1-\xi)$, $\Phi_n(t)=0$ $(n\geqq 1)$, and $b_{2m-1}(u;\Phi)=B_{2m}(1-\xi)(1+u)^{2m-1}/(2m)!$. Hence, we see immediately from Proposition 2.20 that $J_1(1-m;\xi)=0$, and that

$$J_1'(1-m;\xi)=-\frac{4\pi i(2m-1)!}{(m!)^2}\int_{\Gamma_\varepsilon} t^{-2m}\phi(t;1-\xi)dt$$
$$+\frac{2\pi iB_{2m}(1-\xi)}{(2m)!}\int_{I_\varepsilon(\infty)}\log u\cdot u^{-m-1}(1-u)(1+u)^{2m-1}du,$$

from which we get the expression for $J_1'(1-m;\xi)$ in the assertion (ii) (note that $\mu_1(u)=1$). In the case of $\Phi(t,u)=F_1(t,u;1-\xi)$, we have

$$\Phi_n(t)=\frac{\phi^{(2n)}(t;1-\xi)(-1)^n}{(2n)!}\quad\text{and}\quad b_{2m-1}(u;\Phi)=\frac{B_{2m}(1-\xi)}{(2m)!}\cdot\mu_{2m}(u).$$

Since the integration by parts implies that

$$-s\int_{I_\varepsilon(\infty)} t^{s-1}\phi^{(2m-1)}(t;\,1-\xi)dt=\int_{I_\varepsilon(\infty)} t^{s}\phi^{(2m)}(t;\,1-\xi)dt,$$

we get

$$\text{(2.3.40)}\qquad \int_{I_\varepsilon(\infty)} \log t\cdot\phi^{(2m)}(t;\,1-\xi)dt=-\int_{I_\varepsilon(\infty)} t^{-1}\phi^{(2m-1)}(t;\,1-\xi)dt = -\frac{\pi i B_{2m}(1-\xi)}{m}.$$

Thus,

$$4\pi i\int_{I_\varepsilon(\infty)} \log t\cdot\Phi_m(t)dt=\frac{4\pi^2(-1)^m B_{2m}(\xi)}{m(2m)!}.$$

Moreover, we have, by a usual argument,

$$\int_{\Gamma_\varepsilon} t^{2(j-m)}\Phi_j(t)dt=\frac{2\pi i(-1)^j B_{2m}(1-\xi)}{(2m)!}\cdot\binom{2m-1}{2j}.$$

Hence, the assertion (ii) of Proposition 2.20 implies the expression for $J_2'(1-m;\,\xi)$ in (ii). The equality $J_2(1-m;\,\xi)=0$ is clear.

Finally, we evaluate the special values at $s=1-m$ of $K(s;\,(\xi_1,\,\xi_2))$ $K_2(s;\,\xi)$ (for the definition, see (2.3.22)).

Let $\nu_n(u)$ $(n\geqq 1)$ be the polynomials of u defined by (2.3.13). For instance, $\nu_1(u)=\nu_2(u)=0$, $\nu_3(u)=(1+u)/2$. We put, for convenience,

$$\nu_0(u)=\sum_{j=0}^{\infty}\frac{\mathscr{A}_{2j}(u)}{(1+u)^{2j+1}}\qquad \text{for } |u| \text{ sufficiently small.}$$

For any pairs $(k,\,n)$ of non-negative integers with $k+n=2m+1$ $(m\in \boldsymbol{N})$, we define the numbers $\mathscr{N}_{(k-1,\,n-1)}$ by putting

$$\text{(2.3.41)}\qquad \mathscr{N}_{(k-1,\,n-1)}=\int_{I_\varepsilon(1)} u^{-1/2-m}(1-u)(1+u)^{k-1}\nu_n(u)du,$$

where ε is taken sufficiently small. The integral in the right side of (2.3.41) is independent of the choice of small ε. Then the identity (2.2.18) implies that

$$\text{(2.3.42)}\qquad \mathscr{N}_{(k-1,\,n-1)}\in \boldsymbol{Q}\qquad \text{for } k,\,n\geqq 1.$$

Proposition 2.2.4. *Let* $\xi_1,\,\xi_2,\,\xi>0$ *and* $m\in \boldsymbol{N}$. *Then,*

$$\text{(i)}\qquad K(1-m;\,(\xi_1,\,\xi_2))=-2\pi i\sum_{n=0}^{2m+1}\frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!\,n!}\cdot\mathscr{N}_{(2m-n,\,n-1)},$$

(ii) $$K_2(1-m;\ \xi)=\frac{2\pi i B_{2m}(\xi)}{(2m)!}\cdot\mathcal{N}_{(0,2m-1)}.$$

Proof. Let $\Psi(t, u)$ be one of the functions given in (2.3.20). Recalling the definition (2.3.21) of $K(s;\ \Psi)$, we have by the Fubini theorem,

$$K(1-m;\ \Psi)=\int_{I\varepsilon(1)} u^{-m-1/2}(1-u)du\int_{\Gamma_\varepsilon} dt\cdot t^{-2m}\Psi(t, u).$$

If $\psi(t, u)=\psi^{(0)}(t, u;\ \xi_1, \xi_2)$, then, observing the expansions (2.2.2), (2.3.13), we get

$$\int_{\Gamma_\varepsilon} t^{-2m}\psi(t, u)dt=2\pi i\sum_{n=0}^{\infty}\frac{B_{2m+1-n}(1-\xi_1)B_n(1-\xi_2)}{(2m+1-n)!n!}\cdot(1+u)^{2m-n}\nu_n(u).$$

Thus we obtain the assertion (i). The assertion (ii) is similarly verified.

As is observed in (2.3.25), the function $\zeta_S(s;\ \{V_1, V_2, V_3\}, (\xi_1, \xi_2))$ has a possible simple pole at $s=1-m\ (m\in\boldsymbol{N})$. On the other hand, it is easy to see from (2.3.1), Proposition 2.19, and (i) of Proposition 2.23 that the functions $\zeta_S(s;\ \{V_j, V_3\}, \xi)\ (j=1, 2)$ are holomorphic at $s=1-m$. The special values at $s=1-m$ of them can be given explicitly by applying Proposition 2.23, Proposition 2.24.

Proposition 2.25. *Let $m\in\boldsymbol{N}$ and $\xi>0$. Then,*
(i) $\zeta_S(1-m;\ \{V_j, V_3\}, \xi)\in\boldsymbol{Q}\ (j=1, 2)$.
(ii) *In particular,*

$$\zeta_S(0;\ \{V_1, V_3\}, \xi)=\frac{1}{2}B_2(\xi),$$

$$\zeta_S(0;\ \{V_2, V_3\}, \xi)=\frac{3}{8}B_2(\xi).$$

Proof. Proposition 2.19 and (i) of Proposition 2.23 show that

$$I_j(1-m;\ \xi)=\frac{1}{2i}J_j'(1-m;\ \xi)-\frac{\pi}{2}K_j(1-m;\ \xi)\qquad (j=1, 2),$$

We observe from (2.3.1) that

$$\zeta_S(1-m;\ \{V_j, V_3\}, \xi)=-2C(m)I_j(1-m;\ \xi)\qquad (j=1, 2).$$

Thus the assertion (i) follows from Proposition 2.23, Lemma 2.21, Proposition 2.24, and (2.3.42) (for the explicit value of $C(m)$, see Proposition 2.9). Let $m=1$. We have, by (2.3.39),

$$(2.3.43)\qquad \mathscr{M}_{(1,0)}=\mathscr{M}_{(0,1)}=\int_{I_\varepsilon(1)} \log u\cdot u^{-2}(1-u^2)du=-4\pi i.$$

Therefore, we find from (ii) of Proposition 2.23 that $J_1'(0;\xi)=8\pi^2B_2(\xi)$ and $J_2'(0;\xi)=6\pi^2B_2(\xi)$. Since $\mathscr{N}_{(0,1)}=0$ and accordingly, $K_2(0;\xi)=0$, we get the assertion (ii) (notice that $K_1(s;\xi)$ is identically zero).

2.4. Evaluation of special values of $L_2^*(s, \psi_{H,p})$

The aim of the subsequent two sections is to prove the rationality of special values of L-functions at non-positive integers and in particular to obtain the explicit special values at $s=0$ of them. We keep the notation used in the previous sections.

Suppose that p is an odd prime. For any integers μ, δ prime to p, let $L^*(\mu)$, $M^*(\delta)$ be the same as in 2.1. Corresponding to $M=L^*(\mu)$, $M^*(\delta)$, we shall define the principal part $\zeta_P(s; C, M)$ and the singular part $\zeta_S(s; C, M)$ of the zeta function $\zeta(s; C, M)$, C being the simplicial cones C_{123}, C_{j3} $(j=1, 2)$. In view of Proposition 2.1, Proposition 2.3, we set, for $M=L^*(\mu)$ or $M^*(\delta)$,

$$(2.4.1)\qquad \begin{aligned}
&\zeta_P(s; C_{123}, M)=p^{-2s}\sum_{\xi\in\Xi_M}\zeta_P(s;\{V_1, V_2, V_3\}, \xi),\\
&\zeta_S(s; C_{123}, M)=p^{-2s}\sum_{\xi\in\Xi_M}\zeta_S(s;\{V_1, V_2, V_3\}, (\xi_1, \xi_2)),\\
&\zeta_P(s; C_{j3}, M)=p^{-2s}\sum_{\xi\in\Xi_M^{(j,3)}}\zeta_P(s;\{V_j\, V_3\}, (\xi_j, \xi_3)),\\
&\zeta_S(s; C_{j3}, M)=p^{-2s}\sum_{\xi\in\Xi_M^{(j,3)}}\zeta_S(s;\{V_j, V_3\}, \xi_j)\qquad (j=1, 2),
\end{aligned}$$

(ξ being denoted by (ξ_1, ξ_2, ξ_3)),

where we set $\Xi_M=\Xi_{H,\mu}$, $\Xi_M^{(j,3)}=\Xi_{H,\mu}^{(j,3)}$ (resp. $\Xi_M=\Xi_\delta$, $\Xi_M^{(j,3)}=\Xi_\delta^{(j,3)}$), if $M=L^*(\mu)$ (resp. if $M=M^*(\delta)$). Proposition 2.2 then makes it possible to define the principal and singular parts of the L-function $L_2^*(s, \psi_{H,p})$. We set

$$(2.4.2)\qquad \begin{aligned}
L_{2,P}^*(s, \psi_{H,p})=\sum_\mu\psi(\mu)\Big\{&\{\zeta_P(s; C_{123}, L^*(\mu))+\frac{1}{2}\sum_{j=1}^{2}\zeta_P(s; C_{j3}, L^*(\mu))\\
&+\frac{1}{2}\zeta(s; C_{12}, L^*(\mu))+\frac{\delta_{p,3}}{6}\,\zeta(s; C_2, L^*(\mu))\Big\},
\end{aligned}$$

$$L_{2,S}^*(s, \psi_{H,p})=\sum_\mu\psi(\mu)\Big\{\zeta_S(s; C_{123}, L^*(\mu))+\frac{1}{2}\sum_{j=1}^{2}\zeta_S(s; C_{i3}, L^*(\mu))\Big\}.$$

where μ runs over 1 and κ, (κ being a non-quadratic residue mod p). Let

χ be a primitive character mod p. Viewing Proposition 2.4, Proposition 2.5, we define P- nad S-parts of $L_2^*(s, \chi_{\det})$, $\xi_2^*(s)$ as follows; we set

(2.4.3) $L_{2,P}^*(s, \chi_{\det})$

$$= \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta)\Big\{\zeta_P(s; C_{123}, M^*(\delta)) + \frac{1}{2}\sum_{j=1}^{2}\zeta_P(s; C_{j3}, M^*(\delta)) + \frac{1}{2}\zeta(s; C_{12}, M^*(\delta)) + \frac{1}{4}\zeta(s; C_1, M^*(\delta)) + \frac{1}{6}\zeta(s; C_2, M^*(\delta))\Big\},$$

$L_{2,S}^*(s, \chi_{\det})$

$$= \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta)\Big\{\zeta_S(s; C_{123}, M^*(\delta)) + \frac{1}{2}\sum_{j=1}^{2}\zeta_S(s; C_{j3}, M^*(\delta))\Big\},$$

$$\xi_{2,P}^*(s) = \zeta_P(s; \{V_1, V_2, V_3\}, (1, 1, 1)) + \frac{1}{2}\sum_{j=1}^{2}\zeta_P(s; \{V_j, V_3\}, (1, 1)) + \frac{1}{2}\zeta(s; \{V_1, V_2\}, (1, 1)) + \frac{1}{4}\zeta(s; \{V_1\}, 1) + \frac{1}{6}\zeta(s; \{V_2\}, 1),$$

$$\xi_{2,S}^*(s) = \zeta_S(s; \{V_1, V_2, V_3\}, (1, 1)) + \frac{1}{2}\sum_{j=1}^{2}\zeta_S(s; \{V_j, V_3\}, 1\}.$$

The following obvious identities hold:

$$(2.4.4) \qquad \begin{cases} L_2^*(s, \psi_{H,p}) = L_{2,P}^*(s, \psi_{H,p}) + L_{2,S}^*(s; \psi_{H,p}) \\ L_2^*(s, \chi_{\det}) = L_{2,P}^*(s, \chi_{\det}) + L_{2,S}^*(s, \chi_{\det}) \\ \xi_2^*(s) = \xi_{2,P}^*(s) + \xi_{2,S}^*(s). \end{cases}$$

It is easy to see from Proposition 2.9, Proposition 2.10 and Corollary to Proposition 2.19 that $L_{2,P}^*(s, \psi_{H,p})$, $L_{2,S}^*(s, \psi_{H,p})$, $L_{2,P}^*(s, \chi_{\det})$, $L_{2,S}^*(s, \chi_{\det})$, $\xi_{2,P}^*(s)$, and $\xi_{2,S}^*(s)$ can be continued analytically to meromorphic functions of s in the whole complex plane.

In the rest of this section we shall discuss the evaluation of special values at $s=1-m$ ($m \in N$) of $L_2^*(s, \psi_{H,p})$. First of all, we need two lemmas related to the Bernoulli polynomials.

Lemma 2.26. *Let $m \in N$. Then,*

$$\sum_{\xi \in \Xi_{H,\mu}} B_{2m+1}(\xi_j) = 0 \qquad (j=1, 2, 3),$$

where ξ_j is the j-component of $\xi \in \Xi_{H,\mu}$.

Proof. The proof is based on the distribution property of $B_k(x)$:

$$(2.4.5) \qquad B_k(x) = p^{k-1}\sum_{r=0}^{p-1} B_k\left(\frac{x+r}{p}\right) \qquad \text{(see for instance [10, p. 35]).}$$

Moreover, if we note that $B_{2m+1}(0)=B_{2m+1}(1)=0$, then, the assertion immediately follows.

Lemma 2.27. *Let $m \in \boldsymbol{N}$ and let k_2, k_3 be non-negative integers with $k_2+k_3=2m-1$.*

(i) *If k_2 is an even positive integer, then,*

$$\sum_{\xi \in \Xi_{H,\mu}} B_{k_2+1}(\xi_2)B_{k_3+1}(\xi_3)=0.$$

(ii) *If $k_2=0$, then,*

$$\sum_{\xi \in \Xi_{H,\mu}, \xi_2 \neq 1} B_1(\xi_2)B_{2m}(\xi_3)=0$$

where $\xi=(\xi_1, \xi_2, \xi_3)$ runs over all triples of $\Xi_{H,\mu}$ with $\xi_2 \neq 1$.

Proof. Let k_2 be an even positive integer. We note that

$$\langle x\rangle+\langle -x\rangle=1 \qquad \text{if } x \in \boldsymbol{R}-\boldsymbol{Z}. \tag{2.4.6}$$

The identity $B_{k_2+1}(\langle x\rangle)=-B_{k_2+1}(\langle -x\rangle)$ $(x \in \boldsymbol{R}-\boldsymbol{Z})$ shows that

$$\begin{aligned}&\sum_{(\alpha,\gamma) \in \mathcal{M}(p)} B_{k_2+1}(\langle 2\mu\alpha\gamma/p\rangle)B_{k_3+1}(\langle \mu(\gamma^2-\alpha^2)/p\rangle)\\ &\qquad =-\sum_{(\alpha,\gamma) \in \mathcal{M}(p)} B_{k_2+1}(\langle -2\mu\alpha\gamma/p\rangle)B_{k_3+1}(\langle \mu(\gamma^2-\alpha^2)/p\rangle).\end{aligned}$$

Then, (α, γ) being replaced with $(-\alpha, \gamma)$, the assertion (i) follows. The proof of the assertion (ii) is quite similar to that of (i).

Let $m \in \boldsymbol{N}$. In the below, let k_1, k_2, k_3 be integers satisfying $k_1, k_2 \geqq -1$, $k_3 \geqq 0$ and $k_1+k_2+k_3=2(m-1)$. For any triple $\xi=(\xi_1, \xi_2, \xi_3)$ of positive numbers, we write, for convenience,

$$B(k_1, k_2, k_3; \xi)=\prod_{j=1}^{3} \frac{B_{k_j+1}(\xi_j)}{(k_j+1)!}. \tag{2.4.7}$$

Let μ be any integer prime to p. Viewing Proposition 2.11, Proposition 2.12, we define the numbers $\mathcal{A}_{(k_1,k_2,k_3)}(\mu)$ as follows;

(i) If $k_1k_2k_3 \neq 0$, we set

$$\mathcal{A}_{(k_1,k_2,k_3)}(\mu)=-2\pi i C(m)\Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_{H,\mu}} B(k_1, k_2, k_3; \xi).$$

(ii) Let r be an integer with $1 \leqq r \leqq 3$. If $k_r=0$ and other k_j's $(j \neq r)$ are non-zero, we set

$$\mathcal{A}_{(k_1,k_2,k_3)}(\mu)=-2\pi i C(m)\Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_{H,\mu}, \xi_r \neq 1} B(k_1, k_2, k_3; \xi).$$

(iii) Let r, n be integers with $1 \leqq r < n \leqq 3$. If $k_r = k_n = 0$, and the rest of k_j's is non-zero (then, necessarily, $m>1$), we set

$$\mathscr{A}_{(k_1,k_2,k_3)}(\mu) = -2\pi i C(m) \Lambda_{(k_1,k_2,k_3)} \{ \sum_{\xi \in \Xi_{H,\mu}, \xi_r, \xi_n \neq 1} B(k_1, k_2, k_3; \xi) - s(k_1, k_2, k_3; \mu) \},$$

where

$$s(k_1, k_2, k_3; \mu) = \begin{cases} \frac{1}{8} g_{2m-1}(\mu) & (k_1, k_2, k_3) = (0, 0, 2m-2), \\ \frac{1}{24} \delta_{p,3} g_{2m-1}(\mu) & (k_1, k_2, k_3) = (0, 2m-2, 0), \\ 0 & (k_1, k_2, k_3) = (2m-2, 0, 0), \end{cases}$$

$g_d(\mu)$ ($d \in \boldsymbol{Z}$, $d \geqq 0$) being $\sum_{\alpha \not\equiv 0 \bmod p} B_d(\langle \mu\alpha^2/p \rangle)/d!$.

(iv) In the case of $(k_1, k_2, k_3) = (0, 0, 0)$, we set

$$\mathscr{A}_{(0,0,0)}(\mu) = -2\pi i C(1) \Lambda_{(0,0,0)} \left\{ \sum_{\xi \in \Xi_{H,\mu}, \xi_j \neq 1\ (j=1,2,3)} B(0, 0, 0; \xi) - \frac{1}{8} g_1(\mu) - \frac{1}{24} \delta_{p,3} g_1(\mu) \right\}.$$

Note that, if d is prime to p, then, $\mathscr{A}_{(k_1,k_2,k_3)}(\mu d^2) = \mathscr{A}_{(k_1,k_2,k_3)}(\mu)$. The special values at $s = 1-m$ ($m \in \boldsymbol{N}$) of $L^*_{2,P}(s, \psi_{H,p})$ can be evaluated with the use of the numbers defined above.

Proposition 2.28. *Let* $m \in \boldsymbol{N}$. *Then,*

(i) $L^*_{2,P}(1-m, \psi_{H,p}) = p^{2(m-1)} \sum_{\mu} \sum'_{k_1,k_2,k_3} \psi(\mu) \mathscr{A}_{(k_1,k_2,k_3)}(\mu)$,

where μ *is over* 1 *and* κ, *and* (k_1, k_2, k_3) *runs over all triples of integers with* $k_1, k_2 \geqq -1$, $k_3 \geqq 0$, *and* $k_1 + k_2 + k_3 = 2(m-1)$.

(ii) *Accordingly,* $L^*_{2,P}(1-m, \psi_{H,p}) \in \boldsymbol{Q}$.

(iii) *If* $p \equiv 1 \bmod 4$, *then,* $L^*_{2,P}(1-m, \psi_{H,p}) = 0$,
If $p \equiv 3 \bmod 4$, *then, in particular,*

$$L^*_{2,P}(0, \psi_{H,p}) = \mathscr{A} + \mathscr{B},$$

where $\mathscr{A}$, $\mathscr{B}$ *are the constants given by* (0.1), (0.2) *in the introduction.*

Proof. Let $\xi_{\alpha,\gamma,\mu}$ be the triple of $\Xi_{H,\mu}$ given by (2.1.6). We notice that $B_1(1) = 1/2$, and moreover that

$$(2.4.8)\quad \begin{cases} \Xi_{H,\mu}^{(2,3)} \cap \Xi_{H,\mu}^{(1,3)} = \{\xi_{\alpha,\gamma,\mu} \mid \alpha \equiv 0 \bmod p,\ \gamma \not\equiv 0 \bmod p\}, \\ \Xi_{H,\mu}^{(2,3)} \cap \Xi_{H,\mu}^{(1,2)} = \begin{cases} \phi & (p>3), \\ \{\xi_{\alpha,\gamma,\mu} \mid \gamma \equiv -\alpha \bmod p,\ \alpha \not\equiv 0 \bmod p\} & (p=3), \end{cases} \\ \Xi_{H,\mu}^{(1,3)} \cap \Xi_{H,\mu}^{(1,2)} = \phi. \end{cases}$$

Taking very carefully (2.4.1), (2.4.2), Proposition 2.11, Proposition 2.12, and (2.4.8) into account of, we obtain the expression (i) for $L_{2,P}^{*}(1-m, \psi_{H,p})$.

If a triple (k_1, k_2, k_3) satisfies any of the conditions (i), (ii), (iii) in Proposition 2.13, then, $(1/\pi)\Lambda_{(k_1,k_2,k_3)}$ is a rational number. Therefore, $\mathscr{A}_{(k_1,k_2,k_3)}(\mu)$ is also a rational number. If $k_1=k_2=-1$ (resp. If $k_1=-1$ and k_2 is even), then, Lemma 2.26 (resp. Lemma 2.27) shows that $\mathscr{A}_{(-1,-1,2m)}(\mu)=0$ (resp. $\mathscr{A}_{(-1,k_2,k_3)}(\mu)=0$). Therefore, the assertion (ii) follows from (i).

For convenience, we write $\xi_{\alpha,\gamma,\mu}=(\xi_{\alpha,\gamma,\mu}^{(1)}, \xi_{\alpha,\gamma,\mu}^{(2)}, \xi_{\alpha,\gamma,\mu}^{(3)})$. Exchanging μ for $-\mu$, we observe from (2.1.6), (2.4.6) that

$$(2.4.9)\quad \begin{cases} \xi_{\alpha,\gamma,-\mu}^{(j)} = 1-\xi_{\alpha,\gamma,\mu}^{(j)} & \text{if } \xi_{\alpha,\gamma,\mu}^{(j)} \neq 1, \\ \xi_{\alpha,\gamma,-\mu}^{(j)} = 1 & \text{if } \xi_{\alpha,\gamma,\mu}^{(j)} = 1 \quad (j=1,2,3) \end{cases}$$

Further, we note that $B_k(1)=(-1)^k B_k(1)$ for $k\neq 1$. Then we see easily from (2.4.9) that, for any triple (k_1, k_2, k_3),

$$(2.4.10)\qquad \mathscr{A}_{(k_1,k_2,k_3)}(-\mu) = -\mathscr{A}_{(k_1,k_2,k_3)}(\mu).$$

If $p\equiv 1 \bmod 4$, then, -1 is a quadratic residue mod p. Hence, $\mathscr{A}_{(k_1,k_2,k_3)}(-\mu)=\mathscr{A}_{(k_1,k_2,k_3)}(\mu)$, which implies that $\mathscr{A}_{(k_1,k_2,k_3)}(\mu)=0$. Thus the first assertion of (iii) follows.

Suppose that $p\equiv 3 \bmod 4$. Then we may take -1 as κ. The assertion (i), together with (2.4.10) and the property $\psi(-1)=-1$, show, that

$$L_{2,P}^{*}(0, \psi_{H,p}) = 2\{\mathscr{A}_{(-1,1,0)}(1)+\mathscr{A}_{(1,-1,0)}(1)+\mathscr{A}_{(0,-1,1)}(1)+\mathscr{A}_{(0,0,0)}(1)\}.$$

We have, by Proposition 2.15,

$$(2.4.11)\quad \mathscr{A}_{(-1,1,0)}(1) = \frac{-1}{4} \sum_{\substack{(\alpha,\gamma)\in\mathscr{M}(p), \\ \alpha^2 \not\equiv \gamma^2 \bmod p}} B_2(\langle 2\alpha\gamma/p\rangle) B_1(\langle(\gamma^2-\alpha^2)/p\rangle),$$

$$\mathscr{A}_{(1,-1,0)}(1)=\mathscr{B},$$

$$\mathscr{A}_{(0,-1,1)}(1) = \frac{-1}{6} \sum_{\substack{(\alpha,\gamma)\in\mathscr{M}(p), \\ \alpha^2 \not\equiv 2\alpha\gamma \bmod p}} B_1(\langle(\alpha^2-2\alpha\gamma)/p\rangle) B_2(\langle\gamma^2-\alpha^2)/p\rangle),$$

$$\mathscr{A}_{(0,0,0)}(1)=\mathscr{A}/2,$$

where, to show the last equality, we used the fact that $B_{1,\psi}=g_1(1)$. In the first equality of (2.4.11), if we replace (α, γ) with (γ, α), then, we get, with the help of (2.4.6), $\mathscr{A}_{(-1,1,0)}(1)=0$. In the third equality of (2.4.11), replacing (α, γ) with $(\alpha-\gamma, -\gamma)$, we have

$$\mathscr{A}_{(0,-1,1)}(1)=\frac{-1}{6}\sum_{\substack{(\alpha,\gamma)\in\mathscr{M}(p)\\ \alpha^2\not\equiv\gamma^2 \bmod p}} B_2(\langle(2\alpha\gamma-\alpha^2)/p\rangle)B_1(\langle(\alpha^2-\gamma^2)/p\rangle)=-\mathscr{B}/2.$$

Thus the second assertion of (iii) follows. q.e.d.

Now we study the singular part $L^*_{2,S}(s, \psi_{H,p})$.

Proposition 2.29. *Let $m\in N$ and let μ be an integer prime to p. The function $\zeta_S(s;\ C_{123},\ L^*(\mu))$ is holomorphic at $s=1-m$, and the special value at $s=1-m$ is given by*

$$\zeta_S(1-m;\ C_{123},\ L^*(\mu))$$
$$=iC(m)p^{2(m-1)}\sum_{\xi\in\Xi_{H,\mu}}\{J'(1-m;\ (\xi_1,\ \xi_2))-\pi iK(1-m;\ (\xi_1,\ \xi_2))\}.$$

Proof. Proposition 2.22 and Lemma 2.26 show that

$$\sum_{\xi\in\Xi_{H,\mu}} J(1-m;\ (\xi_1,\ \xi_2))=0.$$

Thus we see immediately from (2.4.1) and the expansion (2.3.25) that $\zeta_S(s;\ C_{123},\ L^*(\mu))$ is holomorphic at $s=1-m$, and that the special value at $s=1-m$ is expressed as in the proposition.

If follows from Proposition 2.25, Proposition 2.29, and (2.4.2) that $L^*_{2,S}(s,\ \psi_{H,p})$ is holomorphic at $s=1-m$ $(m\in N)$. The following proposition plays a key role to evaluate its special value at $s=1-m$.

Proposition 2.30. *Let $m\in N$. Then,*

$$\frac{4\pi i(-1)^m}{(2m)!}\sum_{\mu}\psi(\mu)\sum_{\xi\in\Xi_{H,\mu}}\int_{I_\varepsilon(\infty)}\log t\cdot\phi(t;\ 1-\xi_1)\phi^{(2m)}(t;\ 1-\xi_2)dt$$
$$=\frac{8\pi^2(-1)^m}{p^{2m-1}}\sum_{j=1}^{2m+1}\frac{B_{2m+1-j}B_{j,\psi}}{j!(2m+1-j)!\,j},$$

where μ runs over 1 and κ, and ε is taken sufficiently small.

Proof. As in the proof of Proposition 2.28, let $\xi^{(j)}_{\alpha,\gamma,\mu}$ $(j=1, 2, 3)$ be the j-component of $\xi_{\alpha,\gamma,\mu}\in\Xi_{H,\mu}$ (see (2.1.6)). We set $\mu\alpha^2=x$, $2\mu\alpha\gamma=u$. If (α, γ) runs over all elements of $\mathscr{M}(p)$ with $\alpha\not\equiv 0 \bmod p$, and μ is over 1 and κ, then, $(x, u)=(\mu\alpha^2, 2\mu\alpha\gamma)$ just doubly covers all elements of $\mathscr{M}(p)$ with $x\not\equiv 0 \bmod p$. If $\alpha\equiv 0 \bmod p$, then, $\xi^{(1)}_{\alpha,\gamma,\mu}=\xi^{(2)}_{\alpha,\gamma,\mu}=1$. Thus,

$$(2.4.12)\quad \frac{1}{2}\sum_{\mu}\psi(\mu)\sum_{(\alpha,\gamma)\in\mathcal{M}(p)}\int_{I_\varepsilon(\infty)}\log t\cdot\phi(t;1-\xi^{(1)}_{\alpha,\gamma,\mu})\phi^{(2m)}(t;1-\xi^{(2)}_{\alpha,\gamma,\mu})dt$$
$$=\sum_{x\not\equiv 0 \bmod p}\sum_{u \bmod p}\psi(x)\int_{I_\varepsilon(\infty)}\log t\cdot\phi(t;1-\langle(x-u)/p\rangle)\phi^{(2m)}(t;1-\langle u/p\rangle)dt.$$

We need the following lemma.

Lemma 2.31. *Let x be any integer prime to p. Then,*
(i)

$$\sum_{u \bmod p}\phi(t;1-\langle(x-u)/p\rangle)\phi^{(2m)}(t;1-\langle u/p\rangle)=\sum_{j=1}^{2m+1}\gamma_{j,2m}\phi^{(j)}(t;1-\langle x/p\rangle)$$

with

$$\gamma_{j,2m}=\frac{(-1)^j(2m)!}{j!(2m+1-j)!p^{2m-j}}\cdot B_{2m+1-j}\qquad(1\leqq j\leqq 2m+1),$$

(ii) $$\int_{I_\varepsilon(\infty)}\log t\cdot\sum_{u \bmod p}\phi(t;1-\langle(x-u)/p\rangle)\phi^{(2m)}(t;1-\langle u/p\rangle)dt$$
$$=-2\pi i\sum_{j=1}^{2m+1}\gamma_{j,2m}\frac{(-1)^jB_j(\langle x/p\rangle)}{j}.$$

Proof. We may take x so that $1\leqq x\leqq p-1$. We get, with the help of Lemma 2.16,

$$\sum_{u \bmod p}\phi(t;1-\langle(x-u)/p\rangle)\phi^{(2m)}(t;1-\langle u/p\rangle)$$
$$=\sum_{j=0}^{2m}\sum_{u \bmod p}\lambda_{j,2m}(1-\langle u/p\rangle)\frac{\exp t(2-\langle u/p\rangle-\langle(x-u)/p\rangle)}{(e^t-1)^{j+2}}$$
$$=\sum_{j=0}^{2m}\frac{1}{(e^t-1)^{j+2}}\left\{\sum_{u=1}^{x-1}\lambda_{j,2m}(1-u/p)e^{t(2-x/p)}+\sum_{u=x}^{p}\lambda_{j,2m}(1-u/p)e^{t(1-x/p)}\right\}$$
$$=\sum_{j=0}^{2m}\left\{\sum_{u=1}^{x-1}\lambda_{j,2m}(1-u/p)\phi_{j+1}(t;1-x/p)+\sum_{u=1}^{p}\lambda_{j,2m}(1-u/p)\phi_{j+2}(t;1-x/p)\right\}.$$

Then, using the formula $\phi_{j+1}(t;a)=-(1/j)\phi_j'(t;a)+((a-j)/j)\phi_j(t;a)$ $(j\geqq 1)$ recurrently, we obtain the expression

$$(2.4.13)\qquad \sum_{u \bmod p}\phi(t;1-\langle(x-u)/p\rangle)\phi^{(2m)}(t;1-\langle u/p\rangle)$$
$$=\sum_{j=0}^{2m+1}\gamma_{j,2m}(x)\phi^{(j)}(t;1-x/p),$$

where $\gamma_{j,2m}(x)$ $(0\leqq j\leqq 2m+1)$ are certain rational numbers (they may depend on x). Recalling the Laurent expansion (2.3.9) of $\phi^{(k)}(t;a)$, and comparing the coefficients of the term t^{-j-1} in the Laurent expansions at $t=0$ of the both sides of (2.4.13), we get the explicit values of $\gamma_{j,2m}(x)$:

$$(-1)^j j!\gamma_{j,2m}(x)=\sum_{u \bmod p}\frac{(2m)!B_{2m+1-j}(1-\langle(x-u)/p\rangle)}{(2m+1-j)!}\qquad(1\leqq j\leqq 2m+1),$$

$$\gamma_{0,2m}(x)=\frac{1}{2m+1}\sum_{u \bmod p}\{B_{2m+1}(1-\langle(x-u)/p\rangle)+B_{2m+1}(1-\langle u/p\rangle)\}.$$

Then, the property (2.4.5) of $B_k(x)$ shows that

$$\gamma_{0,2m}(x)=0,\qquad \gamma_{j,2m}(x)=\frac{(-1)^j(2m)!B_{2m+1-j}}{j!(2m+1-j)!p^{2m-j}}\qquad(1\leqq j\leqq 2m+1).$$

Thus the assertion (i) follows. Similarly as in (2.3.40), we have

$$\int_{I\varepsilon(\infty)}\log t\cdot\phi^{(j)}(t;1-x/p)dt=-2\pi i\frac{B_j(1-x/p)}{j},$$

which, together with (i), completes the proof of the assertion (ii).

Thus, (2.4.6) and Lemma 2.31 with the definition (1.2.11) of the generalized Bernoulli numbers imply Proposition 2.30.

Proposition 2.32. *Let $m\in N$. Then,*
(i) $\sum_\mu \psi(\mu)\zeta_S(1-m;C_{123},L^*(\mu))\in \boldsymbol{Q}$.
(ii) *In particular, if $m=1$,*

$$\sum_\mu \psi(\mu)\zeta_S(0;C_{123},L^*(\mu))=\frac{1}{p}\left\{\frac{11}{36}B_{3,\psi}-\frac{7}{16}B_{2,\psi}-\frac{1}{24}B_{1,\psi}\right\}.$$

Proof. Taking (ii) of Proposition 2.22, Lemma 2.21, Lemma 2.26, and Proposition 2.30 into account of, we observe that

$$\sum_\mu \psi(\mu)\sum_{\xi\in\Xi_{H,\mu}}J'(1-m;(\xi_1,\xi_2))\in\pi^2\boldsymbol{Q}.$$

We get, immediately by (i) of Proposition 2.24, (2.3.42), and Lemma 2.26,

$$\sum_{\xi\in\Xi_{H,\mu}}K(1-m;(\xi_1,\xi_2))\in\pi i\boldsymbol{Q}.$$

Thus the assertion (i) follows from Proposition 2.29. Next suppose that $m=1$. Since obviously, $\mathcal{N}_{(0,1)}=\mathcal{N}_{(1,0)}=0$, we see easily again from Proposition 2.24, Lemma 2.26 that

$$\sum_{\xi\in\Xi_{H,\mu}}K(0;(\xi_1,\xi_2))=0.$$

Therefore we find from Proposition 2.29, Proposition 2.22, (2.3.43), Lemma 2.26, and Proposition 2.30 that

$$\sum_{\mu} \psi(\mu)\zeta_S(0;\, C_{123},\, L^*(\mu)) = \frac{1}{16\pi^2} \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}} J'(1-m;\, (\xi_1,\, \xi_2))$$
$$= \frac{1}{p}\left\{-\frac{1}{24} B_{1,\psi} + \frac{1}{16} B_{2,\psi} - \frac{1}{36} B_{3,\psi}\right\}$$
$$-\frac{1}{2} \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}} \{B_1(\xi_1)B_2(\xi_2) + B_2(\xi_1)B_1(\xi_2)\}.$$

Hence the following lemma completes the proof of Proposition 2.32.

Lemma 2.33. *We have*

$$\sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}} \{B_1(\xi_1)B_2(\xi_2) + B_2(\xi_1)B_1(\xi_2)\} = \frac{1}{p} B_{2,\psi} - \frac{2}{3p} B_{3,\psi}.$$

Proof. We notice that

$$\int_{\Gamma_\varepsilon} t^{-2}\phi(t;\, 1-\xi_1)\phi(t;\, 1-\xi_2)dt = -2\pi i\left\{\frac{1}{6}\sum_{j=1}^{2} B_3(\xi_j) + \frac{1}{2}\sum_{j=1}^{2} B_1(\xi_j)B_2(\xi_{3-j})\right\}.$$

Then it follows from Lemma 2.26 that the left side of the equality in the lemma is equal to

$$\frac{-1}{\pi i}\int_{\Gamma_\varepsilon} t^{-2} \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}} \phi(t;\, 1-\xi_1)\phi(t;\, 1-\xi_2)dt$$
$$= \frac{-1}{\pi i}\int_{\Gamma_\varepsilon} t^{-2} \sum_{x \not\equiv 0 \bmod p} \psi(x) \sum_{u \bmod p} \phi(t;\, 1-\langle (x-u)/p\rangle)\phi(t;\, 1-\langle u/p\rangle)dt$$
$$= \frac{1}{\pi i}\int_{\Gamma_\varepsilon} t^{-2} \sum_{x \not\equiv 0 \bmod p} \psi(x)\{\phi(t;\, 1-\langle x/p\rangle) + p\phi'(t;\, 1-\langle x/p\rangle)\}dt$$
$$= 2 \sum_{x \not\equiv 0 \bmod p} \psi(x)\{B_2(1-\langle x/p\rangle)/2 + pB_3(1-\langle x/p\rangle)/3\},$$

which, in addition to (1.2.11), completes the proof of Lemma 2.33.

Finally, we evaluate the special values at $s=0$ of the zeta functions attached to the cones C_{j3} $(j=1,\, 2)$.

Proposition 2.34. *The following identities hold;*

$$\sum \psi(\mu)\zeta_S(0;\, C_{23},\, L^*(\mu)) = \frac{3}{8p} B_{2,\psi},$$

$$\sum \psi(\mu)\zeta_S(0;\, C_{13},\, L^*(\mu)) = \frac{1}{2p} B_{2,\psi}.$$

Proof. Only the first identity will be proved. Proposition 2.25 and (2.4.1) show that

$$\sum \psi(\mu)\zeta_S(0;\, C_{23},\, L^*(\mu)) = \frac{3}{16} \sum \psi(\mu) \sum_{\substack{(\alpha,\gamma)\in\mathcal{M}(p) \\ \alpha^2\equiv 2\alpha\gamma \bmod p}} B_2(\langle 2\mu\alpha\gamma\rangle/p)$$
$$= \frac{3}{8p} B_{2,\psi}. \qquad \text{q.e.d.}$$

Gathering (2.4.2), Proposition 2.25, Proposition 2.32, and Proposition 2.34 together, we obtain the following proposition.

Proposition 2.35. *We have*
(i) $L^*_{2,S}(1-m,\, \psi_{H,p}) \in \boldsymbol{Q}$ $(m \in N)$.
(ii) $L^*_{2,S}(0,\, \psi_{H,p}) = (11/36p)B_{3,\psi} - (1/24p)B_{1,\psi}$.

Now Theorem 1 (the main theorem) in the introduction follows immediately from Proposition 2.28, Proposition 2.35, (2.4.4), and the fact that if $p \equiv 1 \bmod 4$, $B_{k,\psi} = 0$ for any odd k.

2.5. Evaluation of special values of $L^*_{2,P}(s, \chi_{\det})$, $\xi^*_2(s)$.

Let χ be a primitive character mod p, p being an odd prime. We have defined the functions $L^*_{2,P}(s, \chi_{\det})$, $L^*_{2,S}(s, \chi_{\det})$, $\xi^*_{2,P}(s)$, $\xi^*_{2,S}(s)$ in (2.4.3). As we see from Proposition 2.9, Proposition 2.10, the principal parts $L^*_{2,P}(s, \chi_{\det})$, $\xi^*_{2,P}(s)$ are holomorphic at $s = 1-m$ $(m \in N)$.

We need a lemma concerning the Bernoulli polynomials.

Lemma 2.36. *Let* $m \in N$. *Then,*
(i) $\sum_{a\not\equiv 0 \bmod p} \chi(a^2)B_{2m-1}(\langle a/p\rangle) = 0$.
(ii) *Let* k *be any integer with* $0 \leqq k \leqq 2m+1$. *Then,*

$$\sum_{a\not\equiv 0 \bmod p} \chi(a^2)B_{2m+1-k}(\langle -a/p\rangle)B_k(\langle a/p\rangle) = 0.$$

Proof. Replacing a with $-a$, and then, using the property $B_k(1-x) = (-1)^k B_k(x)$ and (2.4.6), we have the assertions (i), (ii).

Then, Proposition 2.3, (d), (e) of Proposition 2.12, and Lemma 2.36 show that

$$(2.5.1) \qquad \sum_{\delta\not\equiv 0 \bmod p} \chi(\delta)\zeta(1-m;\, C_j,\, M^*(\delta)) = 0 \qquad (j=1, 2).$$

For each triple (k_1, k_2, k_3) of integers satisfying $k_1, k_2 \geqq -1$, $k_3 \geqq 0$, and $k_1+k_2+k_3=2(m-1)$ $(m \in N)$, we define the numbers $\mathscr{B}_{(k_1,k_2,k_3)}$ as follows;

(i) If $k_1k_2k_3 \neq 0$, we set

$$\mathscr{B}_{(k_1,k_2,k_3)} = -2\pi i C(m) A_{(k_1,k_2,k_3)} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\xi \in \Xi_\delta} B(k_1, k_2, k_3; \xi).$$

(ii) Let r be an integer with $1 \leqq r \leqq 3$. If $k_r=0$ and other k_j's $(j \neq r)$ are non-zero, we set

$$\mathscr{B}_{(k_1,k_2,k_3)} = -2\pi i C(m) A_{(k_1,k_2,k_3)} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\xi \in \Xi_\delta, \xi_r \neq 1} B(k_1, k_2, k_3; \xi).$$

(iii) Let r, n be integers with $1 \leqq r < n \leqq 3$. If $k_r = k_n = 0$ and the rest of k_j's is non-zero $(m>1)$, we set

$$\mathscr{B}_{((k_1,k_2,k_3)} = -2\pi i C(m) A_{(k_1,k_2,k_3)} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\substack{\xi \in \Xi_\delta \\ \xi_r, \xi_n \neq 1}} B(k_1, k_2, k_3, \xi).$$

(iv) If $(k_1, k_2, k_3)=(0, 0, 0)$, we set

$$\mathscr{B}_{(0,0,0)} = -2\pi i C(1) A_{(0,0,0)} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\substack{\xi \in \Xi_\delta \\ \xi_j \neq 1\ (j=1,2,3)}} B(0, 0, 0; \xi).$$

As the following Lemma 2.37 shows, all the numbers $\mathscr{B}_{(k_1,k_2,k_3)}$ are proved to be zero. However, these numbers are useful to evaluate the special values $L^*_{2,P}(1-m, \chi_{\det})$.

Lemma 2.37. *For any triple (k_1, k_2, k_3) as above, $\mathscr{B}_{(k_1,k_2,k_3)}=0$.*

Proof. For each $T=\begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix} \in L_2^*$, we write

$$\xi_T = (\langle (t_1-2t_{12})/p \rangle, \langle 2t_{12}/p \rangle, \langle (t_2-t_1)/p \rangle) = (\xi_T^{(1)}, \xi_T^{(2)}, \xi_T^{(3)}).$$

Replacing T with $-T$, we observe that, if $\xi_T^{(j)} \neq 1$, then, $\xi_{-T}^{(j)} = 1-\xi_T^{(j)}$, and that, if $\xi_T^{(j)}=1$, then, $\xi_{-T}^{(j)}=1 (1 \leqq j \leqq 3)$. Thus it follows from the definition of $\mathscr{B}_{(k_1,k_2,k_3)}$ and some properties of $B_k(x)$ that $\mathscr{B}_{(k_1,k_2,k_3)} = -\mathscr{B}_{(k_1,k_2,k_3)}$.

q.e.d.

The special values at $s=1-m$ $(m \in N)$ of $L^*_{2,P}(s, \chi_{\det})$ are obtained in a similar manner as in the proof of Proposition 2.28. The result is very simple.

Proposition 2.38. *We have $L^*_{2,P}(1-m, \chi_{\det})=0$ $(m=1, 2, \cdots)$.*

Proof. We notice that, for each integer δ prime to p,

$$\Xi_\delta^{(1,2)} \cap \Xi_\delta^{(1,3)} = \left\{ \xi_T \mid T = \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix},\ t_1 \in Z/pZ,\ t_1^2 \equiv \delta \bmod p \right\},$$

$$\Xi_\delta^{(1,3)} \cap \Xi_\delta^{(2,3)} = \phi,$$

$$\Xi_\delta^{(2,3)} \cap \Xi_\delta^{(1,2)} = \left\{ \xi_T \mid T = \begin{pmatrix} 2t_{12} & t_{12} \\ t_{12} & 2t_{12} \end{pmatrix},\ t_{12} \in Z/pZ,\ 3t_{12}^2 \equiv \delta \bmod p \right\}.$$

Then we have, by (i) of Lemma 2.36,

$$\text{(2.5.2)} \qquad \begin{cases} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\xi \in \Xi_\delta^{(1,2)} \cap \Xi_\delta^{(1,3)}} B_{2m-1}(\xi_1) = 0, \\ \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\xi \in \Xi_\delta^{(2,3)} \cap \Xi_\delta^{(1,2)}} B_{2m-1}(\xi_2) = 0. \end{cases}$$

Thus, making use of the definition (2.4.3) of $L_{2,P}^*(s, \chi_{\det})$, Proposition 2.11, Proposition 2.12, and the identities (2.5.1), (2.5.2), we obtain

$$L_{2,P}^*(1-m, \chi_{\det}) = p^{2(m-1)} \sum_{(k_1,k_2,k_3)}' \mathscr{B}_{(k_1,k_2,k_3)},$$

where (k_1, k_2, k_3) is taken over all triples of integers satisfying $k_1, k_2 \geqq -1$, $k_3 \geqq 0$, and $k_1 + k_2 + k_3 = 2(m-1)$. Therefore, Lemma 2.37 completes the proof of Proposition 2.38.

Now we consider the function $L_{2,S}^*(s, \chi_{\det})$. We have

$$\text{(2.5.3)} \qquad \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \zeta_S(s; C_{123}, M^*(\delta))$$
$$= p^{-2s} \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \sum_{\substack{T \in L_2^*/pL_2^* \\ \det(T) \equiv \delta \bmod p}} \zeta_S(s; \{V_1, V_2, V_3\}, (\xi_T^{(1)}, \xi_T^{(2)})).$$

Dividing the summation into two parts according to $t_1 \not\equiv 0 \bmod p$ or not in $T = \begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix}$, we see that the left side of (2.5.3) is equal to

$$p^{-2s} \sum_{\substack{\delta, t_1 \not\equiv 0 \bmod p \\ t_{12} \bmod p}} \chi(\delta) \zeta_S(s; \{V_1, V_2, V_3\}, (\langle (t_1 - 2t_{12})/p \rangle, \langle 2t_{12}/p \rangle))$$
$$+ p^{-2s} \sum_{\substack{t_{12} \not\equiv 0 \bmod p \\ t_2 \bmod p}} \chi(-t_{12}^2) \zeta_S(s; \{V_1, V_2, V_3\}, (\langle -2t_{12}/p \rangle, \langle 2t_{12}/p \rangle)).$$

Using the well-known fact that $\sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) = 0$, we get

$$\text{(2.5.4)} \qquad \sum_{\delta \not\equiv 0 \bmod p} \chi(\delta) \zeta_S(s; C_{123}, M^*(\delta))$$
$$= p^{1-2s} \chi(-4)^{-1} \sum_{a \not\equiv 0 \bmod p} \chi(a^2) \zeta_S(s; \{V_1, V_2, V_3\}, (\langle -a/p \rangle, \langle a/p \rangle)).$$

Since Proposition 2.22, Lemma 2.36 show that

$$\sum_{a\not\equiv 0 \bmod p} \chi(a^2)J(1-m; (\langle -a/p\rangle, \langle a/p\rangle))=0 \qquad (m \in N),$$

it is not difficult to see from (2.5.4), (2.3.25), Proposition 2.22, Proposition 2.24, and Lemma 2.36 that the function in the left side of the equality (2.5.4) is holomorphic at $s=1-m$, and moreover that its special value at $s=1-m$ is given by

$$(2.5.5) \qquad \sum_{\delta\not\equiv 0 \bmod p} \chi(\delta)\zeta_S(s; C_{123}, M^*(\delta))\Big|_{s=1-m} = \frac{\chi(-4)^{-1}4\pi(-1)^{m+1}p^{2m-1}C(m)}{(2m)!}$$

$$\times \int_{I_\varepsilon(\infty)} \log t \cdot \sum_{a\not\equiv 0 \bmod p} \chi(a^2)\phi(t; \langle a/p\rangle)\phi^{(2m)}(t; 1-\langle a/p\rangle)dt,$$

where ε is taken sufficiently small. In a similar manner as in Lemma 2.31, we can write

$$(2.5.6) \qquad \sum_{a\not\equiv 0 \bmod p} \chi(a^2)\phi(t; \langle a/p\rangle)\phi^{(2m)}(t; 1-\langle a/p\rangle)=\sum_{j=1}^{2m+1} \lambda_j\phi^{(j)}(t),$$

where $\phi(t)=\phi(t; 0)=1/(e^t-1)$, and λ_j $(1\leqq j\leqq 2m+1)$ are given by

$$(2.5.7) \qquad (-1)^j j!\, \lambda_j = \frac{(2m)!}{(2m+1-j)!} \sum_{a\not\equiv 0 \bmod p} \chi^2(a)B_{2m+1-j}(\langle a/p\rangle).$$

We notice that $\lambda_j=0$ if j is even. By a similar argument as in the proof of Lemma 2.31, we have

$$(2.5.8) \qquad \int_{I_\varepsilon(\infty)} \log t \cdot \sum_{j=1}^{2m+1} \lambda_j\phi^{(j)}(t)dt = -2\pi i \sum_{j=1}^{2m+1} \frac{\lambda_j B_j}{j} = \pi i\lambda_1$$

(note that $\lambda_j B_j=0$ if $j\neq 1$). Computing the number λ_1 from (2.5.7), we get, by (1.2.11), (2.4.5),

$$(2.5.9) \qquad \lambda_1 = \begin{cases} -p^{1-2m}B_{2m,\chi^2} & \text{if } \chi^2 \text{ is non-trivial,} \\ \left(1-\dfrac{1}{p^{2m-1}}\right)B_{2m} & \text{if } \chi \text{ is quadratic.} \end{cases}$$

Moreover, we find that

$$\begin{aligned} &\sum_{\delta\not\equiv 0 \bmod p} \chi(\delta)\zeta_S(s; C_{13}, M^*(\delta)) \\ &\qquad = p^{-2s} \sum_{t_1,t_2\not\equiv 0 \bmod p} \chi(t_1t_2)\zeta_S(s; \{V_1, V_2\}, \langle t_1/p\rangle) \\ &\qquad = 0. \end{aligned}$$

The case of the cone C_{23} is similar. Thus we have, identically,

$$(2.5.10) \qquad \sum_{\delta \not\equiv \bmod p} \chi(\delta)\zeta_S(s; C_{j3}, M^*(\delta))=0 \qquad (j=1, 2).$$

In view of the identities from (2.5.4) to (2.5.10), we obtain the following proposition.

Proposition 2.39. *The sepcial values at* $s=1-m$ $(m=1, 2, \cdots)$ *of* $L_{2,S}^*(s, \chi_{\det})$ *are given as follows:*

$$L_{2,S}^*(1-m, \chi_{\det})=\begin{cases} \dfrac{\chi(-4)^{-1}(-1)^m}{2^{2m+1}m} B_{2m,\chi^2} & \text{if } \chi \neq \psi, \\ \dfrac{\chi(-1)(-1)^{m-1}}{2^{2m+1}m}(p^{2m-1}-1)B_{2m} & \text{if } \chi=\psi. \end{cases}$$

ψ *being the unique non-trivial quadratic character mod p.*

By virtue of Proposition 2.38 and Proposition 2.39, we can evaluate the special values $L_2^*(1-m, \chi_{\det})$ explicitly. The result is given in Theorem 2 in the introduction.

Finally, we shall evaluate the value $\xi_2^*(0)$. Using (2.4.3), Proposition 2.11, Proposition 2.12, and Proposition 2.15, we get

$$\xi_{2,P}^*(0)=1/24.$$

Further, we see from Proposition 2.22 that $J(1-m; (1, 1))=0$ $(m \in N)$ and that

$$J'(0; (1, 1))=-2\pi i\int_{I_\varepsilon(\infty)} \log t\cdot\phi(t)\phi^{(2)}(t)dt+16\pi^2 B_1 B_2.$$

Since we can write $\phi(t)\phi^{(2)}(t)=\mu_1\phi'(t)+\mu_2\phi^{(2)}(t)+\mu_3\phi^{(3)}(t)$ with $\mu_1=-1/6$, $\mu_2=-1/2$, $\mu_3=-1/3$, we have

$$\int_{I_\varepsilon(\infty)} \log t\cdot\phi(t)\phi^{(2)}(t)dt=-2\pi i\sum_{j=1}^{3}\frac{\mu_j B_j}{j}=-\pi i/12,$$

which shows that $J'(0; (1, 1))=-3\pi^2/2$. We see from Proposition 2.24 that $K(0; (1, 1))=0$ (note that $\mathcal{N}_{(0,1)}=\mathcal{N}_{(1,0)}=0$). Therefore, the function $\zeta_S(s; \{V_1, V_2, V_3\}, (1, 1))$ is holomorphic at $s=1-m$ $(m \in N)$, and its special value at $s=0$ is given by

$$(2.5.11) \qquad \zeta_S(0; \{V_1, V_2, V_3\}, (1, 1))=iC(1)J'(0; (1, 1))=-3/32.$$

Moreover, we have by Proposition 2.25,

$$\zeta_S(0; \{V_1, V_3\}, 1)=1/12, \qquad \zeta_S(0; \{V_2, V_3\}, 1)=1/16,$$

which, in addition to (2.5.11), imply that $\xi^*_{2,S}(0)=-1/48$. Accordingly,

$$\xi^*_2(0)=\xi^*_{2,P}(0)+\xi^*_{2,S}(0)=1/48.$$

Thus we are successful in giving another proof of the following theorem due to Siegel [20, Satz 3], Shintani [16, Theorem 2].

Theorem 2.40 (Siegel-Shintani). *The special value at $s=0$ of the zeta function $\xi^*_2(s)$ is given by*

$$\xi^*_2(0)=1/48.$$

Chapter III. Some applications to the representation of $Sp(2n, \boldsymbol{F}_p)$ in the space of Siegel cusp forms

3.1. The representatinn μ_k of $Sp(2n, \boldsymbol{F}_p)$ in the space of cusp forms

Let $\mathfrak{H}_n$ be the Siegel upper half plane of degree n: $\mathfrak{H}_n=\{Z \in M_n(\boldsymbol{C}) \mid Z={}^tZ \operatorname{Im}(Z)>0\}$. The real symplectic group $\mathfrak{G}_{2n}=Sp(2n, \boldsymbol{R})$ of degree 2n acts on $\mathfrak{H}_n$ in a usual manner:

$$Z \longrightarrow g\langle Z\rangle=(AZ+B)(CZ+D)^{-1} \qquad \left(Z \in \mathfrak{H}_n,\ g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{G}_{2n}\right).$$

By this action, $\mathfrak{G}_{2n}/\{\pm 1\}$ gives the group of biholomorphic automorphisms of $\mathfrak{H}_n$. We put

$$J(g, Z)=\det(CZ+D) \qquad \text{for } g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{G}_{2n}.$$

Denote by $\Gamma_{2n}(l)$ the principal congruence subgroup with level $l\,(l \in \boldsymbol{N})$ of the Siegel modular group $Sp(2n, \boldsymbol{Z})$:

$$\Gamma_{2n}(l)=\{\gamma \in Sp(2n, \boldsymbol{Z}) \mid \gamma \equiv 1_{2n} \bmod l\}.$$

For a positive integer k, let $\mathfrak{S}_k(\Gamma_{2n}(l))$ be the space of Siegel cusp forms of degree n, weight k with respect to $\Gamma_{2n}(l)$; namely, it consists of all holomorphic functions $f(Z)$ on $\mathfrak{H}_n$ which satisfy the following two conditions:

$$\begin{cases} \text{(i)} \quad f(\gamma\langle Z\rangle)=J(\gamma, Z)^k f(Z) & \text{for any } \gamma \in \Gamma_{2n}(l), \\ \text{(ii)} \quad \det(\operatorname{Im}(Z))^{k/2}\,|f(Z)| & \text{is bounded on } \mathfrak{H}_n. \end{cases}$$

We write, simply, $\Gamma_{2n}=\Gamma_{2n}(1)\ (=Sp(2n, \boldsymbol{Z}))$. The group $\Gamma_{2n}(l)$ is a normal subgroup of Γ_{2n}. For $\alpha \in \Gamma_{2n}$ and $f \in \mathfrak{S}_k(\Gamma_{2n}(l))$, we put

(3.1.1) $$(f|[\alpha]_k)(Z)=J(\alpha, Z)^{-k}f(\alpha\langle Z\rangle).$$

Obviously, $f|[\alpha]_k \in \mathfrak{S}_k(\Gamma_{2n}(l))$. In the following we suppose that p is an odd prime. It is well-known that the quotient group $\Gamma_{2n}/\Gamma_{2n}(p)$ is isomorphic to the finite symplectic group $Sp(2n, \boldsymbol{F}_p)$ of degree $2n$ over the finite field $\boldsymbol{F}_p$ of p-elements. For $\alpha \in \Gamma_{2n}$, $\bar{\alpha}=\alpha \bmod p$ is regarded as an element of $Sp(2n, \boldsymbol{F}_p)$ via the isomorphism $\Gamma_{2n}/\Gamma_{2n}(p)\cong Sp(2n, \boldsymbol{F}_p)$. By means of (3.1.1), one can construct a representation μ_k of $Sp(2n, \boldsymbol{F}_p)$ in the space $\mathfrak{S}_k(\Gamma_{2n}(p))$; for $\bar{\alpha} \in Sp(2n, \boldsymbol{F}_p)$ $(\alpha \in \Gamma_{2n})$ and $f \in \mathfrak{S}_k(\Gamma_{2n}(p))$, we set

(3.1.2) $$\mu_k(\bar{\alpha})f=f\,|\,[\alpha^{-1}]_k.$$

It is easily checked that (3.1.2) is well-defined. We put, for $\gamma \in \mathfrak{G}_{2n}$, $Z \in \mathfrak{H}_n$,

$$H(\gamma; Z)=\det(\mathrm{Im}(Z))^k \det\left(\frac{\gamma\langle Z\rangle-\bar{Z}}{2i}\right)^{-k} J(\gamma, Z)^{-k}.$$

We have $H(\gamma; g\langle Z\rangle)=H(g^{-1}\gamma g; Z)$ for $g, \gamma \in \mathfrak{G}_{2n}$. Denote by dZ the invariant measure on $\mathfrak{H}_n$ given by

$$dZ=\det(Y)^{-n-1}\prod_{1\leqq i\leqq j\leqq n} dX_{ij}dY_{ij} \qquad \text{for } Z=X+iY.$$

The following theorem is due to Godement [2] (see also Lemma 1 of [5]).

Theorem 3.1 (Godement). *Let $k>2n$. The trace of $\mu_k(\bar{\alpha}^{-1})$ $(\alpha \in \Gamma_{2n})$ is given by the following formula*:

$$\mathrm{tr}\,(\mu_k(\bar{\alpha}^{-1}))=a(k)\int_{\Gamma_{2n}(p)\backslash\mathfrak{H}_n}\sum_{\gamma\in\Gamma_{2n}(p)\alpha} H(\gamma; Z)dZ,$$

where we put

(3.1.3) $$a(k)=\frac{\gamma_n(k-(n+1)/2)}{2^n(2\pi)^{n(n+1)/2}\gamma_n(k-n-1)} \qquad (\textit{for } \gamma_n(s), \textit{ see Chap. I}).$$

For a symmetric matirx x of size ν $(1\leqq\nu\leqq n)$, we write

(3.1.4) $$t_{n,\nu}(x)=\begin{pmatrix} 1_n & \begin{matrix} x & 0 \\ 0 & 0 \end{matrix} \\ 0 & 1_n \end{pmatrix}.$$

Suppose that $\bar{\alpha}$ $(\alpha \in \Gamma_{2n})$ is $Sp(2n, \boldsymbol{F}_p)$-conjugate to some element $t_{n,\nu}(S)$ with $S \in L_\nu$, $\det(S)\not\equiv 0 \bmod p$. For each integer r $(1\leqq r\leqq n)$, let $\Pi_r(\alpha)$ be the set consisting of all elements $\gamma \in \Gamma_{2n}(p)\alpha$ that are Γ_{2n}-conjugate to some elements $t_{n,r}(x)$ with $x \in L_r$, $\det(x) \neq 0$. Set, following [16, §3 of Chap. 2],

$$(3.1.5) \qquad I_n(\Pi_r(\alpha); k) = a(k) \int_{\Gamma_{2n}(p) \backslash \mathfrak{H}_n} \sum_{\gamma \in \Pi_r(\alpha)} H(\gamma; Z) dZ.$$

Put
$$U_n = \prod_{k=1}^{n} (2\pi^k / \Gamma(k)).$$

We denote by $\delta_n g$ the invariant measure on $\mathfrak{G}_{2n}$ which satisfies

$$\int_{\mathfrak{G}_{2n}} f(g\langle i1_n \rangle) \delta_n g = 2^{-n} U_n \int_{\mathfrak{H}_n} f(Z) dZ$$

for any integrable function f on $\mathfrak{H}_n$. Then we have

$$(3.1.6) \qquad I_n(\Pi_r(\alpha); k) = a'(k) \int_{\Gamma_{2n}(p) \backslash \mathfrak{G}_{2n}} \sum_{\gamma \in \Pi_r(\alpha)} H(g^{-1}\gamma g; i1_n) \delta_n g$$

where $a'(k) = 2^n a(k)/U_n$.

3.2. On the integrals $I_n(\Pi_r(\alpha); k)$

In the case of $\alpha = 1_{2n}$, Shintani [16] proved the absolutely convergence of the integrals $I_n(\Pi_r(\alpha); k)$ under a certain condition for k and expressed the values of them as elementary constant multiples of the special values at non-positive integers of the zeta function $\xi_r^*(s)$. In the following, as quite an analogy of Shintani's results, we evaluate the values of the integrals $I_n(\Pi_r(\alpha); k)$ for a general α by using the special values of $\xi_r^*(s; \tau_S^{(r)})$. We keep the notation used in Chap. I.

Let $1 \leqq \nu \leqq n$ and let $S \in L_\nu$ with $\det(S) \not\equiv 0 \bmod p$. We may assume that $\alpha = t_{n,\nu}(S)$. Then we observe that $\Pi_r(\alpha) \neq \phi$ only if $\nu \leqq r \leqq n$. Set

$$\Gamma_{\alpha,p} = \{\sigma \in \Gamma_{2n} \mid \sigma^{-1}\alpha\sigma \equiv \alpha \bmod p\}.$$

Then, $\Gamma_{\alpha,p}$ is a subgroup of Γ_{2n}, and $\Gamma_{2n}(p)$ is a normal subgroup of $\Gamma_{\alpha,p}$. In a formal manner, we get, by (3.1.6),

$$(3.2.1) \qquad I_n(\Pi_r(\alpha); k) = a'(k)[\Gamma_{\alpha,p} : \Gamma_{2n}(p)] \times \int_{\Gamma_{2n} \backslash \mathfrak{G}_{2n}} \sum_{\sigma \in \Gamma_{\alpha,p} \backslash \Gamma_{2n}} \sum_{\gamma \in \sigma^{-1}\Pi_r(\alpha)\sigma} H(g^{-1}\gamma g; i1_n) \delta_n g,$$

where $[\Gamma_{\alpha,p} : \Gamma_{2n}(p)]$ denotes the group index of $\Gamma_{\alpha,p}$ to $\Gamma_{2n}(p)$. Let $\mathscr{L}_n^{(r)}(S)$ be the same as in (1.2.3). Denote by $\mathfrak{G}_{2n,r}$ the subgroup of $\mathfrak{G}_{2n}$ consisting of all matrices whose left lower $(2n-r) \times r$ blocks are zero. Every element q of $\mathfrak{G}_{2n,r}$ has the following block decomposition:

$$(3.2.2)\qquad q=\begin{pmatrix}1_n & \begin{matrix} y_1 & y_{12} \\ {}^t y_{12} & 0\end{matrix} \\ & 1_n\end{pmatrix}\begin{pmatrix}\begin{matrix}1_r & x_{12}\\ 0 & 1_{n-r}\end{matrix} & \\ & \begin{matrix}1_r & 0\\ -{}^t x_{12} & 1_{n-r}\end{matrix}\end{pmatrix}\begin{pmatrix}a & & & \\ & \alpha & & \beta \\ & & {}^t a^{-1} & \\ & \gamma & & \delta\end{pmatrix},$$

where $a\in GL_r(\boldsymbol{R})$, $h=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in \mathfrak{G}_{2(n-r)}$, $x_{12}, y_{12}\in M(r, n-r; \boldsymbol{R})$, $y_1\in M_r(\boldsymbol{R})$ (${}^t y_1=y_1$). Let $\delta_{n,r}q$ be the left invariant measure of $\mathfrak{G}_{2n,r}$ given by

$$\delta_{n,r}q=|\det(a)|^{-(2n-r+1)}d_r a\delta_{n-r}h dx_{12}dy_1 dy_{12}.$$

Lemma 3.2. *Let* $\alpha=t_{n,\nu}(S)$ *with* $S\in L_\nu$, $\det(S)\not\equiv 0 \bmod p$. *For each integer* r *with* $\nu\leqq r\leqq n$, *we have*

$$\bigcup_{\sigma\in\Gamma_{\alpha,p}\backslash\Gamma_{2n}}\sigma^{-1}\Pi_r(\alpha)\sigma=\bigcup_{\gamma\in\Gamma_{2n}\cap\mathfrak{G}_{2n,r}\backslash\Gamma_{2n}}\gamma^{-1}\{t_{n,r}(x)\,|\,x\in\mathscr{L}_r^{(r)}(S)\}\gamma$$

(*the both sides are disjoint unions*).

Proof. In the proof we use, implicitly, the fact that p is an odd prime. The disjointness of the unions is clear. First, let $x\in\mathscr{L}_r^{(r)}(S)$, $\gamma\in\Gamma_{2n}$. There exists $U\in GL_r(\boldsymbol{Z}/p\boldsymbol{Z})$ with $x\equiv U\begin{pmatrix}S&0\\0&0\end{pmatrix}{}^tU \bmod p$. Since the map $\beta\to\bar\beta$ of Γ_{2n} to $Sp(2n, \boldsymbol{F}_p)$ is surjective, there exists $\sigma_1\in\Gamma_{2n}$ such that $t_{n,r}(x)\equiv\sigma_1^{-1}\alpha\sigma_1 \bmod p$. Thus, $\gamma^{-1}t_{n,r}(x)\gamma=(\sigma_1\gamma)^{-1}\tau_1\alpha(\sigma_1\gamma)$ with some $\tau_1\in\Gamma_{2n}(p)$. Write $\sigma_1\gamma=\rho\sigma$ with $\rho\in\Gamma_{\alpha,p}$ and $\sigma\in\Gamma_{\alpha,p}\backslash\Gamma_{2n}$. Then, $\gamma^{-1}t_{n,r}(x)\gamma=\sigma^{-1}\rho^{-1}\tau_1\alpha\rho\sigma$. Since $\rho^{-1}\alpha\rho\equiv\alpha \bmod p$, we have $\rho^{-1}\alpha\rho=\tau_2\alpha$ with some $\tau_2\in\Gamma_{2n}(p)$. Hence, $\rho^{-1}\tau_1\alpha\rho\in\Pi_r(\alpha)$.

Conversely, let $\sigma\in\Gamma_{\alpha,p}\backslash\Gamma_{2n}$, and let $\tau\in\Gamma_{2n}(p)$ with $\tau\alpha\in\Pi_r(\alpha)$. Then we have $\sigma^{-1}\tau\alpha\sigma=\gamma^{-1}t_{n,r}(x)\gamma$ with some $\gamma\in\Gamma_{2n}\cap\mathfrak{G}_{2n,r}\backslash\Gamma_{2n}$ and $x\in L_r$, $\operatorname{rank}(x)=r$. The task we have to do is to prove that $x\in\mathscr{L}_r^{(r)}(S)$. If we put $\gamma\sigma^{-1}=\gamma_1$, then, $\gamma_1^{-1}\alpha\gamma_1\equiv t_{n,r}(x) \bmod p$. Writing $\gamma_1=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_{2n}$ we have

$$(3.2.3)\qquad \begin{aligned}&\begin{pmatrix}S&0\\0&0\end{pmatrix}C\equiv 0 \bmod p,\qquad \begin{pmatrix}S&0\\0&0\end{pmatrix}D\equiv A\begin{pmatrix}x&0\\0&0\end{pmatrix} \bmod p,\\ &C\begin{pmatrix}x&0\\0&0\end{pmatrix}\equiv 0 \bmod p.\end{aligned}$$

Since ${}^tDA-{}^tBC=1_n$, we get, by (3.2.3),

$$(3.2.4)\qquad {}^tD\begin{pmatrix}S&0\\0&0\end{pmatrix}D\equiv{}^tDA\begin{pmatrix}x&0\\0&0\end{pmatrix}\equiv(1_n+{}^tBC)\begin{pmatrix}x&0\\0&0\end{pmatrix}\equiv\begin{pmatrix}x&0\\0&0\end{pmatrix} \bmod p.$$

We decompose A, C, D as follows:

$$A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad C=\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad D=\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \qquad \text{with } a_1, c_1, d_1 \in M_\nu(\boldsymbol{Z}).$$

Since $\det(S)\not\equiv 0 \bmod p$, we have, again by (3.2.3), $c_1\equiv 0 \bmod p$, $c_2\equiv 0 \bmod p$. Thus we see easily from the relation $A^tD-B^tC=1_n$ that $a_1^t d_1 + a_2^t d_2 \equiv 1_\nu \bmod p$. Accordingly, there exists some $V\in GL_n(\boldsymbol{Z}/p\boldsymbol{Z})$ of the form $V=\begin{pmatrix} d_1 & d_2 \\ * & * \end{pmatrix}$. We have, by (3.2.4), ${}^tV\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}V\equiv \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \bmod p$. Thus, $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\in \mathscr{L}_n^{(r)}(S)$. In the same manner as in the proof of Lemma 1.4, we can conclude that $x\in \mathscr{L}_r^{(r)}(S)$. q.e.d.

We set

$$\omega_n=\zeta(2)\zeta(4)\cdots\zeta(2n).$$

Let $\nu\leqq r\leqq n$. Using the decomposition in Lemma 3.2, we get, by the equality (3.2.1),

$$\text{(3.2.5)}\quad I_n(\Pi_r(\alpha);k) = a'(k)[\Gamma_{\alpha,p}:\Gamma_{2n}(p)]\int_{\Gamma_{2n}\cap\mathfrak{G}_{2n,r}\backslash\mathfrak{G}_{2n}}\sum_{x\in\mathscr{L}_r^{(r)}(S)} H(g^{-1}t_{n,r}(x)g;\, i1_n)\delta_n g.$$

Making use of Lemma 22 of [16] and the decomposition (3.2.2) of $q\in\mathfrak{G}_{2n,r}$, we have

$$\begin{aligned}\text{(3.2.6)}\quad I_n(\Pi_r(\alpha);k) &= a'(k)[\Gamma_{\alpha,p}:\Gamma_{2n}(p)]\frac{U_n}{U_{n-r}C_r} \\ &\quad\times\int_{\Gamma_{2n}\cap\mathfrak{G}_{2n,r}\backslash\mathfrak{G}_{2n,r}}\sum_{x\in\mathscr{L}_r^{(r)}(S)} H(t_{n,r}(a^{-1}x^ta^{-1});\, i1_n)\delta_{n,r}q \\ &= a'(k)[\Gamma_{\alpha,p}:\Gamma_{2n}(p)]2^{r(2n-r+1)/2}\frac{U_n\omega_{n-r}}{U_{n-r}C_r} \\ &\quad\times\int_{G_{\boldsymbol{R},+}^{(r)}/SL_r(\boldsymbol{Z})}\chi(g)^{n-(r-1)/2}\sum_{x\in\mathscr{L}_r^{(r)}(S)}\det(1_r-igx^tg)^{-k}d_rg \\ &= a'(k)[\Gamma_{\alpha,p}:\Gamma_{2n}(p)]2^{r(2n-r+1)/2}\frac{U_n\omega_{n-r}}{U_{n-r}C_r} \\ &\quad\times Z(f_r^*(x,k),\mathscr{L}_r(S),n-(r-1)/2).\end{aligned}$$

By virtue of Proposition 1.6, the integral $Z(f_r^*(x,k),\mathscr{L}_r(S),n-(r-1)/2)$ is absolutely convergent for $k\geqq 2n+3$, and hence the equalities (3.1.6), (3.2.1), (3.2.5), (3.2.6) can be justified definitely. Thus the absolutely convergence of the integral $I_n(\Pi_r(\alpha);k)$ follows for $k\geqq 2n+3$. We obtain

Proposition 3.3.

Proposition 3.3. *Let* $1 \leqq \nu \leqq n$ *and take* $S \in L_\nu$ *with* $\det(S) \not\equiv 0 \bmod p$. *Put* $\alpha = t_{n,\nu}(S)$. *For each* r $(\nu \leqq r \leqq n)$, *the integral* $I_n(\Pi_r(\alpha); k)$ *given by* (3.1.5) *is absolutely convergent for* $k \geqq 2n+3$ *and is equal to*

$$[\Gamma_{\alpha,p}: \Gamma_{2n}(p)] p^{-r(n-(r-1)/2)} b(n, k, r) \Omega_{n,r} \xi_r^*(r-n, \tau_S^{(r)}),$$

where we put

$$b(n, k, r) = \prod_{j=1}^{n-r} (2k-n-j)(2k-n-j+2) \cdots (2k-n+j-2), \tag{3.2.7}$$

$$\Omega_{n,r} = \frac{\omega_{n-r} 2^{r(n-r)-1}}{U_{n-r}(4\pi)^{(n-r)(n-r+1)/2}} \tag{3.2.8}$$

(*we understand* $b(n, k, r) = 1$, $\Omega_{n,r} = 1$ *for* $r = n$).

Proof. The absolutely convergence of the integral $I_n(\Pi_r(\alpha); k)$ has already been verified. The functional equation in Proposition 1.9 shows that

$$\begin{aligned} &Z(f_r^*(x, k), \mathscr{L}_r(S), n-(r-1)/2) \\ &= \frac{\pi^{r(r+1)/2} p^{-r(n-(r-1)/2)} C_r \gamma_r(k-n-1)}{2(2\pi)^{r(r-n)} \gamma_r(k-(r+1)/2)} \cdot \xi_r^*(r-n, \tau_S^{(r)}). \end{aligned} \tag{3.2.9}$$

The latter half of the assertion is a direct consequence of (3.2.6), (3.2.9).

3.3. Traces of $\mu_k(\bar{\alpha})$ in the case of degree 4 ($n=2$)

We consider the case of $n=2$, $\nu=1$. Take a non-quadratic residue κ mod p and fix it. For any integer μ prime to p, we put

$$\alpha_\mu = t_{2,1}(\mu) = \begin{pmatrix} 1_2 & \begin{matrix} \mu & 0 \\ 0 & 0 \end{matrix} \\ 0 & 1_2 \end{pmatrix} \in \Gamma_4.$$

It is easy to see that

$$[\Gamma_{\alpha_\mu,p}: \Gamma_4(p)] = 2p^4(p^2-1).$$

Since $L(-1, \psi) = -B_{2,\psi}/2$, $\zeta(-1) = -1/12$, the identity (1.2.12) shows that

$$\xi_1^*(-1, \tau_\mu^{(1)}) = -\frac{1}{4}\psi(\mu)\tau_\psi B_{2,\psi} + \frac{1}{24}(1-p^2).$$

It follows from Proposition 3.3 that, if $r=1$.

$$I_2(\Pi_1(\alpha_\mu);\ k)=-2^{-5}3^{-1}p^2(p^2-1)\{\psi(\mu)\tau_\psi B_{2,\psi}+(p^2-1)/6\}(2k-3)\quad (k\geqq 7).$$

It is immediate to see from Proposition 1.2, Proposition 3.3 that, if $r=2$,

$$I_2(\Pi_2(\alpha_\mu);\ k)=\frac{p(p^2-1)}{2}\{p\psi(-1)L_2^*(0,\ \psi_{\mathrm{det}})+p\psi(\mu)\tau_\psi L_2^*(0,\ \psi_{H,p})$$
$$+(p^2-1)\xi_2^*(0)\}\qquad (k\geqq 7).$$

Thus, by virtue of Theorem 2 in the introduction and Theorem 2.40, we obtain

$$I_2(\Pi_2(\alpha_\mu);\ k)=\frac{p(p^2-1)}{2}\{p\psi(\mu)\tau_\psi L_2^*(0,\ \psi_{H,p})+2^{-4}3^{-1}(2p^2-p-1)\}.$$

It is essentially known by [1], [13], [14], [5] and verified in a similar manner that, if $n=2$, in the trace formula for $\mathrm{tr}\,(\mu_k(\bar{\alpha}_\mu^{-1}))$ in Theorem 3.1, the contributions from any other conjugacy classes except from $\Pi_r(\alpha_\mu)$ $(r=1, 2)$ vanish identically. Thus one obtains, if $k\geqq 7$,

$$\mathrm{tr}\,(\mu_k(\bar{\alpha}_\mu))=\sum_{j=1}^{2} I_2(\Pi_j(\alpha_{-\mu});\ k).$$

Summing up the results above, we obtain the following theorem (and also (0.3) in the introduction).

Theorem 3.4. *Let μ be any integer prime to p. If $k\geqq 7$, then,*

$$\mathrm{tr}\,(\mu_k(\bar{\alpha}_\mu))=-2^{-5}3^{-1}p^2(p^2-1)\{\psi(-\mu)\tau_\psi B_{2,\psi}+(p^2-1)/6\}(2k-3)$$
$$+2^{-1}p(p^2-1)\{p\psi(-\mu)\tau_\psi L_2^*(0,\ \psi_{H,p})+2^{-4}3^{-1}(2p^2-p-1)\}.$$

It is well-known that

$$\tau_\psi=\begin{cases}\sqrt{p} & p\equiv 1 \bmod 4,\\ i\sqrt{p} & p\equiv 3 \bmod 4.\end{cases}$$

Substituting 1, κ for μ and subtracting $\mathrm{tr}\,(\mu_k(\bar{\alpha}_\kappa))$ from $\mathrm{tr}\,(\mu_k(\bar{\alpha}_1))$ in Theorem 3.4, we obtain Theorem 3 in the introduction. As a direct corollary of Theorem 3.4, the imaginary part of $\mathrm{tr}\,(\mu_k(\bar{\alpha}_\mu))$ is given as follows:

$$\mathrm{Im}\,(\mathrm{tr}\,(\mu_k(\bar{\alpha}_\mu))=\begin{cases}0 & p\equiv 1 \bmod 4,\\ 2^{-1}p^{5/2}(p^2-1)\psi(-\mu)L_2^*(0,\ \psi_{H,p}) & p\equiv 3 \bmod 4.\end{cases}$$

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