Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 65–87

On the Graded Rings of Modular Forms in Several Variables

Shigeaki Tsuyumine

The theory of modular forms in one variable has a long history. Gauß already knew modular functions in one variable in connection with elliptic functions (Takagi [49]). The theory of modular forms in several variables is younger than it, but is now nearly a century old. It was noted by Hilbert at the end of the last century, and first treated by his student Blumenthal [2]. In 1935 Siegel introduced a new class of modular forms in several variables in a celebrated paper [47]. Nowadays the theory of automorphic forms on a bounded symmetric domain has been developed in several directions as well as the theory of automorphic forms in a rather wider class. However, in spite of this long history, there is still only a small number of examples of rings of automorphic forms in several variables for which much is known about the structure. Here we give a list of papers concerning the structure of the graded ring, the generators or the relations among them, which is arranged nearly chronologically in each case:

Siegel modular forms; Igusa [28, 29, 30], Hammond [15], Freitag [6], Tsuyumine [56], Satoh [41].

Hilbert modular forms; Gundlach [13, 14], Hammond [16], Fomenko [5], Resnikoff [36], Hirzebruch [21, 22, 23], Van der Geer [11], Van der Geer-Zagier [12], Hermann [17, 18], Nagaoka [35], Müller [32, 33].

Hermitian modular forms; Freitag [7].

Automorphic forms on the complex 2-ball; Resnikoff-Tai [37, 48]. There are some other papers concerning the structure of a ring of modular forms over $\boldsymbol{Z}$. Here we should mention that the dimension formulas for spaces of modular forms, or rather cusp forms, are known for wider class. They have helped, and will keep to help, the work of determining the structure of a graded ring. Shimizu's formula [45] has been continually employed to study a graded ring of Hilbert modular forms (mainly in two dimensional case yet), and the formula by Busam [3] in the symmetric case. The works by Resnikoff-Tai [37, 48] were done by

Received March 16, 1987.

relying on the formula by Cohn [4], and the work by Satoh [41] which is concerned with the module structure of the space of vector valued Siegel modular forms of degree two, was done by relying on Tsushima's formula [51, 52].

The works of this kind have been done after algebraic geometry was fully developed, by which we never mean that all the important methods and ingredients have come from algebraic geometry. The theory of compactification introduced by Satake [39, 40] (cf. [43]) makes us apply algebraic geometry effectively to arithmetic quotients of the Siegel spaces. The works by Igusa [28, 30] have, more or less, flavor of the moduli theory of hyperelliptic curves and of abelian varieties. In the Hilbert modular case, Hirzebruch [19–24] (see also Hirzebruch-Van de Ven [25], Hirzebruch-Zagier [26, 27]) has developed an elegant and powerful geometric method. However it can be said that although we are now equipped with several excellent tools particularly for low dimensional cases, there is still plenty of difficulty because graded rings of modular forms have a character of their own and each of them requires us to treat it in distinct way.

We do not try in the present survey paper such an awful thing as looking round all the tools and the technique to determine the graded rings which have ever been developed. We introduce the method first employed by Gundlach [13] which, or whose modification, is used repeatedly by several authors (Siegel modular case; Hammond [15], Freitag [6], Tsuyumine [56]: Hilbert modular case; Hammond [16], Hermann [17, 18]). After preliminaries in § 1, we introduce it in § 2. In the remaining sections, its applications are shown. In § 3, Gundlach's original argument is introduced from our point of view. In § 4, the structure theorem for the graded ring of Siegel modular forms of degree two is discussed. It is based on the papers by Freitag and by Hammond. In § 5, Siegel modular forms of degree three is discussed by sketching [56], and in the last section we give a small observation about Siegel modular forms of degree four. We expect that the method will be still useful in more cases, and wish that the present paper would be of benefit to people trying further investigation.

The present work was done while the author was staying at Göttingen. He wishes to express his heartfelt gratitude to Sonderforschungsbereich 170 "Géométrie und Analysis" in Göttingen and especially to Prof. U. Christian for his hospitality and support during the author's his visit.

§ 1. A graded ring

We start with a quasi-projective variety X over C and a quasi-coherent sheaf $\mathscr{L}$ on X on which the following assumption is made:

Assumption I. X has a projective compactification $\bar{X}$ such that $\operatorname{codim}(\bar{X}-X)\geq 2$. $\mathscr{L}$ is a torsion-free graded $\mathcal{O}_X$-module $\mathscr{L}=\oplus_k \mathscr{L}(k)$, $k \in \boldsymbol{Z}$, ≥ 0, with $\mathscr{L}(0)=\mathcal{O}_X$ where $\mathscr{L}(k)$ is coherent and is locally free of rank one except on proper subvarieties. There is a positive integer d such that if $k>0$ is divisible by d, then $\mathscr{L}(k)$ is an ample invertible sheaf on X which has an extension to an ample invertible sheaf on $\bar{X}$.

For k, $k'\geq 0$, $\mathscr{L}(k)\otimes\mathscr{L}(k')$ is contained in $\mathscr{L}(k+k')$ as an $\mathcal{O}_X$-submodule. The equality holds if k, $k'\equiv 0 \pmod d$. It is true also for extensions of $\mathscr{L}(k)$'s, $k\equiv 0 \pmod d$, for the extended invertible sheaves are unique from our assumption that $\operatorname{codim}(\bar{X}-X)\geq 2$. The ampleness of $\mathscr{L}(d)$ implies in particular that $\mathcal{O}_X\subset\mathscr{L}(k)$ if k is large enough. Let $d_k\equiv 0 \pmod d$ be such that $k\leq d_k$ and $\mathcal{O}_X\subset\mathscr{L}(d_k-k)$. Then for any non-negative integer m, we have $\mathscr{L}(k+md)\subset\mathscr{L}(d_k+md)=\mathscr{L}(d_k)\otimes\mathscr{L}(d)^m$. There is an ascending sequence of $\mathcal{O}_X$-submodules of $\mathscr{L}(d_k)$; $\mathscr{L}(k)\subset\mathscr{L}(k+d)\otimes\mathscr{L}(d)^{-1}\subset\mathscr{L}(k+2d)\otimes\mathscr{L}(d)^{-2}\subset\cdots\subset\mathscr{L}(d_k)$. The coherency shows that $\mathscr{L}(k+md)\otimes\mathscr{L}(d)^{-m}$ are all equal for large m. We have shown that $\mathscr{L}(k)\otimes\mathscr{L}(k')=\mathscr{L}(k+k')$ if $k'\geq 0$ is $\equiv 0 \pmod d$ and if k is large enough, or $\equiv 0 \pmod d$. Hence $\mathscr{L}$ is finite over the submodule $\oplus_k \mathscr{L}(k)$, $k\equiv 0 \pmod d$.

By assumption $H^0(X, \mathscr{L}(k))$ is finite dimensional for any $k\geq 0$. Since by a standard argument (see, for instance, Mumford [34]) $H^0(X, \mathscr{L}(k))\otimes H^0(X, \mathscr{L}(k'))\to H^0(X, \mathscr{L}(k+k'))$ is surjective for sufficiently large k, k' which are $\equiv 0 \pmod d$, a graded ring

$$A^{(d)}:=\bigoplus_k H^0(X, \mathscr{L}(k))$$

is noetherian. Let

$$A:=\bigoplus_k H^0(X, \mathscr{L}(k)),$$

which we call *the graded ring associated with X and $\mathscr{L}$*, or *the graded ring associated with X* when the quasi-coherent sheaf under consideration is obvious. Then by the above argument the graded ring A is finite over $A^{(d)}$ as a module, and hence noetherian. Let

$$X^*:=\operatorname{Proj}(A).$$

X^* is a compactification of X satisfying the condition in the Assumption I, namely it is a projective variety satisfying

$$\operatorname{codim}(X^*-X)\geq 2.$$

Let $i\colon X \hookrightarrow X^*$ be the natural injection. We write $\mathscr{L}(k)^* = i_*(\mathscr{L}(k))$, which is a coherent sheaf on X^* and which is an ample invertible sheaf if $k \equiv 0 \pmod d$. $H^0(X^*, \mathscr{L}(k)^*)$ is equal to $H^0(X, \mathscr{L}(k))$ for any k. There is a finite morphism of X^* onto $\bar{X}$ which is the identity on X. X^*, $\bar{X}$ are not necessarily equal, however they are when $\bar{X}$ is normal.

Let $A_k = H^0(X, \mathscr{L}(k))$, which is the graded part in A of degree k, so that $A = \oplus_k A$. A formal power series $P_A(t)$ in a variable t is called *the generating function of* A if

$$P_A(t) = \sum_k (\dim_C A_k) t^k.$$

Since A is noetherian, $P_A(t)$ is a rational function of t by the Hilbert theorem.

Let V be an irreducible subvariety in X, and let $\bar{V}$ be the closure of V in X^*. We note that if $\bar{V} - V$ is of codimension at least two in $\bar{V}$, then V and $\mathscr{L}|_V$ satisfy the Assumption I.

Example 1. Let $\mathscr{D}$ be a bounded symmetric domain. $\mathscr{D}$ is written as a quotient $G(\boldsymbol{R})^\circ/K$ where $G(\boldsymbol{R})^\circ$ is the (topological) identity component of the group of real points of a semi-simple algebraic group G over $\boldsymbol{Q}$, and K is a maximal compact subgroup of $G(\boldsymbol{R})^\circ$. Let Γ be an arithmetic subgroup of G, which acts properly discontinuously on $\mathscr{D}$. By Baily-Borel [1], there is a natural compactification $(\mathscr{D}/\Gamma)^*$ which is called a Baily-Borel-Satake compactification. It is a normal projective algebraic variety and has $\mathscr{D}/\Gamma$ as an open subvariety. Now let us suppose that G has no normal $\boldsymbol{Q}$-subgroup of dimension three with $\boldsymbol{Q}$-rank one. Then $\operatorname{codim}((\mathscr{D}/\Gamma)^* - \mathscr{D}/\Gamma)$ is at least two. So if we put $X = \mathscr{D}/\Gamma$, then X satisfies the first condition in the Assumption I. We take as $\mathscr{L}$, the quasi-coherent sheaf corresponding to automorphic forms for Γ on $\mathscr{D}$, whose details are given in the following.

Let ρ be a (holomorphic) automorphy factor, i.e., the function on $\Gamma \times \mathscr{D}$ with values in $\boldsymbol{C} - \{0\}$ such that (i) $\rho(\gamma, z)$ is holomorphic in z for any fixed $\gamma \in \Gamma$, (ii) $\rho(\gamma\gamma', z) = \rho(\gamma, \gamma' z) \cdot \rho(\gamma', z)$ for $\gamma, \gamma' \in \Gamma$, and (iii) $\rho(\gamma, z) = \rho(\gamma', z)$ if γ, γ' induce the same automorphism of $\mathscr{D}$. Let $j(\gamma, z)$ be the jacobian at a point $z \in \mathscr{D}$, of an automorphism of $\mathscr{D}$ induced by $\gamma \in \Gamma$. $j(\gamma, z)$ is an example of automorphy factors. Let us consider the automorphy factor ρ which is of the form

$$\rho(\gamma, z) = v(\gamma) j(\gamma, z)^{-r}$$

where r is a positive rational number and v is a multiplier whose value for $\gamma \in \Gamma$ is an m-th root of unity for some fixed $m \in \boldsymbol{Z}$. v is depending on the choice of the branches of $j(\gamma, z)^{-r}$ if $r \in \boldsymbol{Q} - \boldsymbol{Z}$. Let π denote the canonical projection of $\mathscr{D}$ onto $\mathscr{D}/\Gamma$. We define $\mathscr{L}(k)$ to be the coherent sheaf corresponding to ρ^k-automorhpic forms, i.e., the sheaf defined by

$$H^0(U, \mathscr{L}(k)) = \{f \in \mathscr{O}_{\pi^{-1}(U)} \mid f(\gamma z) = \rho(\gamma, z)^k f(z) \quad \text{for } z \in \pi^{-1}(U), \gamma \in \Gamma\},$$

where U is any analytic open subset of $\mathscr{D}/\Gamma$, and $\mathscr{O}_{\pi^{-1}(U)}$ denotes the structure sheaf of $\pi^{-1}(U)$ in analytic sense. $\mathscr{L}(k)$ extends to a coherent sheaf on $(\mathscr{D}/\Gamma)^*$ (Serre [44]), and in particular it is algebraic. By Baily-Borel [1], $\mathscr{L}(k)$ extends to an ample invertible sheaf on $(\mathscr{D}/\Gamma)^*$ if k is sufficiently divisible.

So $X = \mathscr{D}/\Gamma$ and $\mathscr{L} = \bigoplus_k \mathscr{L}(k)$ satisfy the conditions in the Assumption I, if G has no normal $\boldsymbol{Q}$-subgroup of dimension three with $\boldsymbol{Q}$-rank one. $A_k = H^0(X, \mathscr{L}(k))$ is the space of ρ^k-automorphic forms, i.e., holomorphic functions f on $\mathscr{D}$ satisfying $f(\gamma z) = \rho(\gamma, z)^k f(z)$ for $\gamma \in \Gamma$. $A = \bigoplus_k A_k$ is the graded ring of such automorphic forms.

Example 2. Let $\mathscr{D}$, Γ be as above. Let X be an irreducible subvariety of $\mathscr{D}/\Gamma$ such that the closure $\bar{X}$ of X in $(\mathscr{D}/\Gamma)^*$ satisfies that $\operatorname{codim}(\bar{X} - X) \geq 2$. Then X and the sheaf given by restricting to X, the quasi-coherent sheaf on $\mathscr{D}/\Gamma$ corresponding to automorphic forms, satisfy the conditions in the Assumption I. Such is the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 3$, which we discuss later.

Remark. Let the notation be as in the Example 1. To investigate the structure of A, usually we had better to take $r > 0$ as small at possible. If otherwise, it might make the structure of A unnecessarily complicated. For instance, if we compare a polynomial ring and its subring consisting of polynomials of degree $\equiv 0 \pmod d$ for $d > 1$, then the former is easier to handle than the latter.

§ 2. A graded ring and a subring

Let X, $\mathscr{L}$ be as in the preceding section. Let D be an irreducible subvariety in X of codimension one, and let D^* be the closure of D in X^*. Let

$$B := \bigoplus_k B_k, \qquad B_k := H^0(D^*, \mathscr{L}(k)^*|_{D^*}).$$

B is noetherian and is a homogeneous coordinate ring of D^*, hence $D^* = \operatorname{Proj}(B)$. A global section f of $\mathscr{L}(k)$ on X can be regarded as that of $\mathscr{L}(k)^*$ on X^*, and hence $f|_D$ determines the unique element of B_k whose restriction to D equals $f|_D$. We have a homomorphism of graded rings

$$\Psi: A \longrightarrow B$$
$$f \longrightarrow f|_D$$

which is associated with the inclusion map of D into X, or D^* into X^*. We write

$$\bar{A} := \operatorname{Im} \Psi$$

which is a graded subring of B. Now we make the following assumption on D.

Assumption II. There is a homogeneous element χ in A such that χ vanishes only at D. Any such element is equal to a power of χ up to a constant factor.

Under the first condition in the Assumption II, the existence of χ satisfying the second is equivalent to the assertion that if $\varphi: \tilde{X} \to X$ is the normalization, then $\tilde{D} = \varphi^{-1}(D)$ is irreducible. For a positive rational number r we call $D(r)$ the following condition on D; $D(r)$: χ defines $r\tilde{D}$ in $\tilde{X}$ where χ is regarded as an element of $\oplus_k H^0(\tilde{X}, \varphi^*(\mathscr{L}(k)))$. For simplicity, we shall deal only with the case r integral, because the similar argument is applicable to the general case. We note that $D(1)$ is the case that D is defined ideal-theoretically by χ in X. χ defines also a subscheme of X^*, which equals D^* set-theoretically. All irreducible subvarieties in X of codimension one satisfy $D(r)$, $r \leq r_X$ for some fixed $r_X \in Z$, if $\operatorname{Pic}(X) \simeq Z + \{\text{finite torsion}\}$, which is the case if $X = \mathscr{D}/\Gamma$ and $\mathscr{L}$ are as in the Example 1 where $\mathscr{D}$ is an irreducible bounded symmetric domain under a certain condition and Γ has a finite commutator factor group (cf. Tsuyumine [55]). All of them satisfy in $D(1)$ particular if $\operatorname{Pic}(X) \simeq Z$ and if an automorphy factor ρ is taken suitably (loc. cit., see also Freitag [8, 9]).

The Assumption II implies that the divisor nD, for some positive integer n, corresponding to an ample invertible sheaf, and hence that Ψ is surjective for graded parts of sufficiently large degree which is $\equiv 0 \pmod{d'}$, d' being sufficiently divisible.

Let us consider the primitive case $D(1)$. Suppose that the structure of $\bar{A}$ is known, e.g., B is known and Ψ is surjective which is often the case (but not always) in the study of rings of automorphic forms. Then we can deduce some information on A from $\bar{A}$. *In the case of* $D(1)$, the *generating function* $P_A(t)$ *of* A *is given by* $(1 - t^{\deg(\chi)}) P_{\bar{A}}(t)$, $\deg(\chi)$ *denoting the degree of* χ *in* A. *If* $\{f_i\}$ *is any system of homogeneous elements in* A *whose images by* Ψ *generate* $\bar{A}$, *then* A *is generated by* χ *and the* f_i*'s.*

In particular if the ring $\bar{A}$ is isomorphic to a polynomial ring, then A is too, and furthermore A and $\bar{A}[\chi]$ are isomorphic as graded rings. The first assertion follows from an exact sequence

$$0 \longrightarrow (\chi) \longrightarrow A \xrightarrow{\Psi} \bar{A} \longrightarrow 0.$$

Let f be any homogeneous element in A, and let $k=\deg(f)$. We are able to prove by induction on k, that f is contained in the ring generated by χ and the f_i's. If $k<0$, then the assertion is trivial. Suppose that it is true for an element of degree $<k$. $\Psi(f)$ is written as a polynomial $Q(\Psi(f_1), \cdots)$ of $\Psi(f_i)$'s. Then the image of $f-Q(f_1, \cdots)$ by Ψ vanishes, and hence it is divisible by χ. If $f-Q(f_1, \cdots)=\chi g$, then g is of degree $<k$, and by the induction hypothesis g is written by χ and the f_i's, and hence f is. The third assertion is immediate from the fact that in the above exact sequence Ψ has a section by assumption.

The general case is to be treated somewhat delicately. Let $\varphi'\colon \tilde{X}^* \to X^*$ be the normalization, which is an extension of $\varphi\colon \tilde{X} \to X$. Let $\tilde{D}^*$ be the closure of $\tilde{D}$ in $\tilde{X}^*$, and let $\bar{A}':=\oplus_k H^0(\tilde{D}^*, \varphi'^*(\mathscr{L}(k)^*)|_{\tilde{D}^*})$, which is a homogeneous coordinate ring of $\tilde{D}^*$. Since $\varphi'|_{\tilde{D}^*}\colon \tilde{D}^* \to D^*$ is a finite morphism, it is shown that $\bar{A}'$ is finite over $\bar{A}$ as a module. Now let us define a valuation on A in terms of $\tilde{D}$. Let $f \in A$ be a homogeneous element. Then for some integer $m>0$, f^m can be regarded as a global section of the invertible sheaf $\mathscr{L}(k)$, $k\equiv 0 \pmod d$, and hence that of the invertible sheaf $\varphi^*(\mathscr{L}(k))$ on $\tilde{X}$. Then $\nu(f^m)$ is defined to be the vanishing order of f^m at $\tilde{D}$, and $\nu(f)$ is defined to be $\nu(f^m)/m$ ($\nu(0)=+\infty$). It is easy to check that ν is well-defined. If $\nu(f)=k'r$ for an integer k', then $f/\chi^{k'}$ defines a global section of $\varphi^*(\mathscr{L}(k))$ for $k=\deg(f)-\deg(\chi^{k'})$ (not necessarily that of $\mathscr{L}(k)$), and so $(f/\chi^{k'})|_{\tilde{D}} \in \bar{A}'$. We denote again by Ψ the map $f/\chi^{k'} \to (f/\chi^{k'})|_{\tilde{D}}$, which is an extension of the previous Ψ. For an integer i, let $A(i)$ be the ideal of A generated by homogeneous elements f with $\nu(f)\geq i$. There is a filtration

$$A=A(0)\supset A(1)\supset A(2)\supset \cdots.$$

Let

$$\bar{A}(i):=A(i)/A(i+1).$$

$\bar{A}(0)$ equals $\bar{A}$, which is a noetherian graded ring, and $\bar{A}(i)$'s are noetherian graded modules over $\bar{A}$. Let us fix some i'. If k' is large enough, then there is a homogeneous element g in $A(k'r-i')-A(k'r-i'+1)$, where g is taken to be 1 if $i'\equiv 0 \pmod r$. Fixing such g, we define a map $\Psi(i)$ of $A(i)$ of $\bar{A}'$ for $i\equiv i' \pmod r$, by

$$\begin{array}{ccccc} A(i) & \longrightarrow & A(i+\nu(g)) & \longrightarrow & \bar{A}' \\ f & \longrightarrow & fg & \longrightarrow & (fg/\chi^{(i+\nu(g))/r})|_{\bar{D}}. \end{array}$$

$\Psi(0)$ equals Ψ by definition. The kernel of $\Psi(i)$ is just $A(i+1)$ and hence $\Psi(i)$ is regarded also as an injective map of $\bar{A}(i)$ into $\bar{A}'$ which is an $\bar{A}$-module homomorphism. By definition, the map $A(i)$ to $\bar{A}'$ given by $f \to \Psi(i+r)(\chi f)$ equals $\Psi(i)$, and in particular there is an inclusion $\Psi(i)(\bar{A}(i)) \subset \Psi(i+r)(\bar{A}(i+r))$. If we identify $\bar{A}(i)$ with its $\Psi(i)$-image, then we have r ascending sequences of $\bar{A}$-modules;

$$\begin{aligned} &\bar{A}(0) \subset \bar{A}(r) \subset \bar{A}(2r) \subset \cdots \\ &\bar{A}(1) \subset \bar{A}(r+1) \subset \bar{A}(2r+1) \subset \cdots \\ &\vdots \\ &\bar{A}(r-1) \subset \bar{A}(2r-1) \subset \bar{A}(3r-1) \subset \cdots. \end{aligned}$$

Since all modules are submodules of $\bar{A}'$ and since $\bar{A}'$ is finite over $\bar{A}$, there is an integer i_0 such that

$$\bar{A}(i) = \bar{A}(i+r) \qquad \text{for any } i \geq i_0,$$

in other words, $\chi\bar{A}(i) = \bar{A}(i+r)$. Then if S is a graded subring of $\bar{A}$ isomorphic to a polynomial ring, then A is isomorphic as graded S-modules, to $\bar{A}(0) \oplus \bar{A}(1) \oplus \cdots \oplus \bar{A}(i_0-1) \oplus \bigoplus_{\mu=0}^{\infty} (\bar{A}(i_0) \oplus \cdots \oplus \bar{A}(i_0+r-1))\chi^{\mu}$, where A is regarded as an S-module by taking some homomorphism of S into A for which $S \to A \xrightarrow{\Psi} \bar{A}$ is the identity on S. Now suppose that the structures of $\bar{A}$ and of a finite number of $\bar{A}$-submodules $\bar{A}(i)$, $1 \leq i \leq i_0 + r-1$, in $\bar{A}'$ are determined. Then *the generating function of A is given by*

$$\sum_{i=0}^{i_0-1} P_{\bar{A}(i)}(t) + (1-t^{\deg(\chi)})\left(\sum_{i=i_0}^{i_0+r-1} P_{\bar{A}(i)}(t)\right).$$

If $\{f_{j,0}\}$ (resp. $\{f_{j,i}\}$, $i \geq 1$) is any system of homogeneous elements in A (resp. $A(i)$) whose images by Ψ (resp. $\Psi(i)$) generate $\bar{A}$ (resp. $\bar{A}(i)$ over $\bar{A}$), then A is generated by χ and $f_{j,i}$'s with $i \leq i_0+r-1$. To this assertion, a proof similar to the case $D(1)$ can be given, but we omit it.

Suppose that X, $\mathscr{L}$ satisfy the Assumption I. Let

$$X_0 \subset X_1 \subset \cdots \subset X_m = X$$

be a sequence of inclusions of irreducible subvarieties of X such that the codimensions of X_j in X_{j+1} is one for every $j=0, \cdots, m-1$. Furthermore suppose that the closure $\bar{X}_j$ of X_j in X^* satisfies a condition that the codimension of $\bar{X}_j - X_j$ in $\bar{X}_j$ is at least two for $j \geq 1$. Then X_j and $\mathscr{L}|_{X_j}$, $j \geq 1$, satisfy the Assumption I. If a divisor X_j in X_{j+1} satisfies the Assumption II for each $j=0, \cdots, m-1$ and if the above trick is successful for each j, then we may have some information about the graded ring associated with X, $\mathscr{L}$ from the graded ring $\oplus_k H^0(X_0^*, \mathscr{L}(k)^*|_{X_0^*})$, X_0^* denoting the closure of X_0 in X^*.

§ 3. Hilbert modular forms

Let K be a totally real algebraic number filed of degree $n>1$, and let O_K be the maximal order of K. $SL_2(K)$ acts on the product H^n on n copies of the upper half plane $H=\{z_1 \in \boldsymbol{C} \mid \operatorname{Im} z_1 > 0\}$ by the modular substitution;

$$z=(z_1, \cdots, z_n) \longrightarrow Mz = \left(\frac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}}, \cdots, \frac{\alpha^{(n)}z_n+\beta^{(n)}}{\gamma^{(n)}z_n+\delta^{(n)}}\right)$$

where $M=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$ and $\alpha^{(1)}, \cdots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$. The jacobian $j(M, z)$ at z, of the automorphism of H^n induced by M, is $N(\gamma z+\delta)^{-2} = \prod_{i=1}^n (\gamma^{(i)}z+\delta^{(i)})^{-2}$. Let Γ_K denote the Hilbert modular group $SL_2(O_K)$. Γ_K acts properly discontinuously on H^n. Let $X_K := H^n/\Gamma_K$. X_K has a natural compactification X_K^*, which is given by adding h points called cusps to X_K, h denoting the class number of K, and X_K^* is normal and projective. A holomorphic function f on H^n is called a *Hilbert modular form* for Γ_K of weight $k \in \boldsymbol{Z}$, ≥ 0 if it satisfies

$$f(Mz) = N(\gamma z+\delta)^k f(z), \qquad M=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K.$$

We denote by $A(\Gamma_K) = \oplus_k A(\Gamma_K)_k$ the graded ring of Hilbert modular forms. X_K^* is canonically isomorphic to Proj $(A(\Gamma_K))$.

For later use, we fix some notations of elliptic modular functions. Let $\Gamma_1 = SL_2(\boldsymbol{Z})$. $A(\Gamma_1)$ denotes the graded ring of elliptic modular forms for Γ_1. $g_4(\tau)$, $g_6(\tau)$, $\tau \in H$, denote the Eisenstein series of weight 4, 6 respectively, and $\Delta(\tau)$ denotes the cusp form of weight 12 which is written as a polynomial of g_4, g_6. $A(\Gamma_1)$ is generated by g_4, g_6, and $A(\Gamma_1)^{(4)}$ is generated by g_4 and g_6^2.

We embed H diagonally into H^n;

$$\begin{aligned} H &\longrightarrow H^n \\ \tau &\longrightarrow (\tau, \cdots, \tau). \end{aligned}$$

Then the stabilizer subgroup of Γ_K at H is equal to $\Gamma_1 \subset \Gamma_K$. Hence we have an embedding $H/\Gamma_1 \to X_K$. If $f(z)$ is a Hilbert modular form of weight k, then $f(\tau, \cdots, \tau)$ is an elliptic modular form of weight nk.

Let $K = \boldsymbol{Q}(\sqrt{5})$. We introduce Gundlach [13] from our point of view. Let D be the modular curve in X_K given as above. We have a homomorphism

$$\begin{aligned} \Psi: A(\Gamma_K) &\longrightarrow A(\Gamma_1). \\ f &\longrightarrow f|_H \end{aligned}$$

Let us denote by $G_2(z)$, $G_6(z)$ the Eisenstein series for Γ_K of weight 2, 6 respectively. Then it can be shown that $\Psi(G_2(z))$ and $\Psi(G_6(z))$ are algebraically independent. Hence $\Psi|_{A(\Gamma_K)^{(2)}}$ gives rise to a surjective map of $A(\Gamma_K)^{(2)}$ onto $A(\Gamma_1)^{(4)}$. Let $f(z)$ be of odd weight. Then the weight of $\Psi(f)$ is $\not\equiv 0 \pmod 4$, and so it is a multiple of g_6. If f is a unit with negative norm, then we have $f(\varepsilon z) = -f(z)$. The usual Fourier expansion argument shows that $\Psi(f)$ vanishes at the cusp of order at least two, and hence $\Psi(f)$ is divisible by Δ^2. So $\Psi(f)$ is a multiple of $g_6\Delta^2$ if f is of odd weight. Gundlach [13, p. 246] found out a Hilbert modular form of weight 15, which we denote by $h(z)$, such that $\Psi(h)$ equals $g_6\Delta^2$ up to a constant factor. Now the image of $A(\Gamma_K)\Psi$ by is determined as

$$\Psi(A(\Gamma_K)) = A(\Gamma_1)^{(4)}[g_6\Delta^2].$$

We define a theta constant with characteristic $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ $\alpha, \beta \in O_K$, by setting

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) = \sum_\nu \boldsymbol{e}\left(\mathrm{tr}\left(\left(\frac{1}{2\sqrt{5}}\left(\nu + \frac{\alpha}{2}\right)^2 z + \frac{1}{2\sqrt{5}}\left(\nu + \frac{\alpha}{2}\right)\beta\right)\right)\right)$$

where $\boldsymbol{e}(\ ) := \exp(2\pi\sqrt{-1}(\ ))$ and where ν runs over O_K, and $\mathrm{tr}(\nu z) := \nu^{(1)}z_1 + \cdots + \nu^{(n)}z_n$. A theta characteristic $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is said to be *even* or *odd* according as $\boldsymbol{e}(\mathrm{tr}(\alpha\beta)/4) = 1$ or -1, and $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is not identically zero if and only if $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is even. There are ten even theta characteristics mod 2, and the corresponding product

$$\Theta_0(z) := \prod_{\text{even}} \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)$$

is a moldular form of weight 5. With the aid of Götsky's observation, Gundlach obtained after some calculation, that

$$\operatorname{div}(\Theta_0)=D,$$

which is obtained also by the modular embedding argument as well as the result of Siegel modular forms of degree two (Hammond [16]). So D satisfies the Assumption II and is in the case $D(1)$. By the argument of the preceding section we have the following:

Theorem (Gundlach). *Let* $K=\boldsymbol{Q}(\sqrt{5})$. *Then* $A(\Gamma_K)=\boldsymbol{C}[G_2, G_6, \Theta_0, h]$. *The generating function* $P_{A(\Gamma_K)}(t)$ *equals* $(1+t^{15})/(1-t^2)(1-t^5)(1-t^6)$.

Let K be an arbitrary real quadratic field. Let σ be the automorphism of H^2 given by

$$z=(z_1, z_2)\longrightarrow \sigma z=(z_2, z_1).$$

A Hilbert modular form f is said to be *symmetric* if $f(z_1, z_2)=f(z_2, z_1)$. Let $\hat{\Gamma}_K$ be the composite of Γ_K and $\langle\sigma\rangle$ as groups acting on H^2, and let $\hat{X}_K=H^2/\hat{\Gamma}_K\simeq X_K/\langle\sigma\rangle$. We denote by $A(\hat{\Gamma}_K)$ the graded ring of symmetric Hilbert modular forms. $\hat{X}_K$ has a natural compactification $\hat{X}_K^*:=X_K^*/\langle\sigma\rangle$ which equals $\operatorname{Proj}(A(\hat{\Gamma}_K))$.

We return to the case $K=\boldsymbol{Q}(\sqrt{5})$. Let D' be the irreducible divisor of $\hat{X}_K$ defined by $z_1=z_2$, in other words, the image of D by the natural surjective map of X_K onto $\hat{X}_K$. In the above-mentioned system of generators of $A(\Gamma_K)$, G_2, G_6, h are symmetric and Θ_0 is anti-symmetric, i.e., $\Theta_0(z_2, z_1)=-\Theta(z_1, z_2)$. Then Θ_0^2 is symmetric and

$$\operatorname{div}(\Theta_0^2)=D'.$$

So D' is in the case $D(1)$, and we have the following:

Theorem (Gundlach). *Let* $K=\boldsymbol{Q}(\sqrt{5})$. *Then* $A(\hat{\Gamma}_K)=\boldsymbol{C}[G_2, G_6, \Theta_0^2, h]$. *The generating function* $P_{A(\hat{\Gamma})}(t)$ *equals* $(1+t^{15})/(1-t^2)(1-t^6)(1-t^{10})$.

The image of $A(\hat{\Gamma}_K)^{(2)}$ by Ψ equals $A(\Gamma_1)^{(4)}=\boldsymbol{C}[g_4, g_6^2]$, and the kernel is the ideal generated by Θ_0^2. By the argument in § 2, $A(\Gamma_K)^{(2)}$ is shown to be equal to $\boldsymbol{C}[G_2, G_6, \Theta_0^2]$ which is isomorphic to a polynomial ring. This shows in particular that the symmetric Hilbert modular function field for $\boldsymbol{Q}(\sqrt{5})$ is rational. h^2 is written as a polynomial of G_2, G_6, Θ_0^2, which has been explicitly done in Resnikoff [36], Hirzebruch [21].

Success in this line depends on finding out a "good" divisor of X_K or $\hat{X}_K$. We refer the reader to Hammond [16], Hermann [17, 18] for the case of real quadratic field K other than $\boldsymbol{Q}(\sqrt{5})$. Although Hermann [18] might look like the case when D is not irreducible, it is actually the case $D(8)$. It gives one good example which shows how we should treat the case $D(r)$, $r\geq 2$.

§ 4. Siegel modular forms of degree two

Let H_n be the Siegel space of degree n; $\{Z\in M_n(C)\,|\,{}^tZ=Z, \operatorname{Im} Z>0\}$. Let Γ_n be the modular group $\{M\in M_{2n}(Z)\,|\,{}^tMJM=J\}$ where J denotes $\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$, 1_n being the identity matrix of size n. Γ_n acts on H_n by the usual modular transformation;

$$Z \longrightarrow MZ=(AZ+B)(CZ+D)^{-1}, \qquad M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in \Gamma_n.$$

The jacobian $j(M, Z)$ at Z, of the automorphism of H_n induced by M, is $|CZ+D|^{-n-1}$. Let $\mathscr{A}_n:=H_n/\Gamma_n$, and let $\mathscr{A}_n^*$ be the Satake compactification which is normal and projective. $\mathscr{A}_n$ is the moduli space of principally polarized abelian varieties over C of dimension n. $\operatorname{codim}(\mathscr{A}_n^*-\mathscr{A}_n)$ equals n, and hence the variety $\mathscr{A}_n$ satisfies the first condition in the Assumption I if $n>1$. Let Γ be a congruence subgroup of Γ_n. A holomorphic function f on H_n is called a *Siegel modular form* for Γ of weight k if it satisfies

$$f(MZ)=|CZ+D|^k f(Z) \qquad \text{for } M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in \Gamma$$

where for $n=1$ we need the additional condition that f is holomorphic also at the cusps, which is automatic in the case $n>1$. We denote by $A(\Gamma)$ the graded ring of Siegel modular froms for Γ. $\mathscr{A}_n^*$ equals $\operatorname{Proj}(A(\Gamma_n))$. We denote by $E_k(Z)$ the Eisenstein series of even weight k, which is an example of Siegel modular forms. We define a theta constant by setting

$$\theta\begin{bmatrix} u \\ v \end{bmatrix}(Z)=\sum_{g\in \boldsymbol{Z}^n} \boldsymbol{e}\left(\frac{1}{2}\left(g+\frac{u}{2}\right)Z^t\left(g+\frac{u}{2}\right)+\frac{1}{2}\left(g+\frac{u}{2}\right)^t v\right)$$

where u, v as well as g, are integral row vectors of size n. $\begin{pmatrix} u \\ v \end{pmatrix}$ is called a *theta characteristic of degree n*, and is said to be *even* or *odd* according as $\boldsymbol{e}(u^tv/4)=1$ or -1. $\theta\begin{bmatrix} u \\ v \end{bmatrix}$ is not identically zero if and only if $\begin{pmatrix} u \\ v \end{pmatrix}$ is even. A point Z of H_n is called *reducible* if it is equivalent to a matrix of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ under Γ_n. A *reducible locus* in $\mathscr{A}_n$ is the subset consisting of all points corresponding to reducible points, which is closed in $\mathscr{A}_n$.

Now let us assume $n=2$. We introduce Freitag [6] (cf. [10]) and Hammond [15] from our point of view, in which Igusa's structure theorem for $A(\Gamma_2)^{(2)}$ was reproved. We take as D the reducible locus of $\mathscr{A}_2$, or equivalently, the divisor corresponding to the image of the embedding;

$$\begin{aligned} H^2 &\longrightarrow H_2. \\ (z_1, z_2) &\longrightarrow \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \end{aligned}$$

The stabilizer subgroup of Γ_2 at the image, is generated by the image of $\Gamma_1^{\times 2}$ by

$$\begin{aligned} \Gamma_1^{\times 2} &\longrightarrow \Gamma_2 \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right\} &\longrightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{bmatrix} \end{aligned}$$

and by a matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The latter matrix induces the transposition σ of z_1 and z_2. So D is isomorphic to $(H_1/\Gamma_1)^{\times 2}/\langle\sigma\rangle$. Since $(H/\Gamma_1)^* \simeq \mathrm{Proj}\,(\boldsymbol{C}[g_4, g_6])$, D is associated with a graded ring $B:=\boldsymbol{C}[g_4\otimes g_4, g_6\otimes g_6, g_4^3\otimes g_6^2, g_6^2\otimes g_4^3]^{\langle\sigma\rangle}$ where σ changes the first and second components of a tensor. It is easy to see that

$$B=\boldsymbol{C}[g_4\otimes g_4, g_6\otimes g_6, g_4^3\otimes g_6^2+g_6^2\otimes g_4^3].$$

Let

$$\begin{aligned} \Psi: A(\Gamma_2)^{(2)} &\longrightarrow B \\ f &\longrightarrow f|_{H^2} \end{aligned}$$

It is shown that $\Psi(E_4)$, $\Psi(E_6)$, $\Psi(E_{12})$ are algebraically independent. Thus Ψ is surjective. There are ten even theta characteristics of degree two. Let Θ be the square of the corresponding product of theta constants;

$$\Theta(Z):=\prod_{\text{even}} \theta\begin{bmatrix} u \\ v \end{bmatrix}(Z)^2.$$

$\Theta(Z)$ is a Siegel modular form for Γ_2 of weight ten, and

$$D=\operatorname{div}(\Theta).$$

The key to prove this is to show that $Z \in H_2$ is reducible if and only if one theta constant $\theta\begin{bmatrix} u \\ v \end{bmatrix}$ vanishes at Z. Hammond [15] has proved this by using moduli theory, that is, if $\theta\begin{bmatrix} u \\ v \end{bmatrix}(Z)=0$, then the principally polarized abelian variety corresponding to $Z \in H_2$ is decomposable. Freitag [6, 10] has done in a way similar to what Gundlach [13] did, namely by the function-theoretic argument of theta constants.

D is in the case $D(1)$, and hence $A(\Gamma_2)^{(2)}$ equals $C[E_4, E_6, E_{12}, \Theta]$ by the argument of § 2. Θ is equal to $E_4E_6-E_{10}$ up to a constant factor. So we have the following:

Theorem (Igusa). *$A(\Gamma_2)^{(2)}=C[E_4, E_6, E_{10}, E_{12}]$, and $E_4, \cdots, E_{12}$ are algebraically independent. The generating function is given by* $1/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})$.

By this, the Siegel modular function field of degree two is shown to be rational. $A(\Gamma_2)$ is generated by $A(\Gamma_2)^{(2)}$ and a cusp form χ_{35} of weight 35. For this fact and for the definition of χ_{35} we refer the reader to Igusa [30].

Remark. So far, a graded ring of modular forms is written as a finite *free* module over its graded subring isomorphic to a polynomial ring. Such a ring is called Cohen-Macaulay. In the case of real quadratic field K, $A(\Gamma_K)^{(r)}$ is shown to be Cohen-Macualay for any integer $r>1$ ([53]), and moreover it is very likely that $A(\Gamma_K)$ itself is Cohen-Macaulay, which is not generally proved yet. However, unfortunately this nice property does not hold for a general graded ring of modular forms. For instance, neither the graded ring of Hilbert modular forms for K of degree ≥ 3 nor the ring $A(\Gamma_n)$ of Siegel modular forms of degree $n \geq 3$ is Cohen-Macaulay ([53, 54, 58]).

§ 5. Siegel modular forms of degree three

Let $\mathfrak{M}_g$ be the moduli space of smooth curves of genus g. The Torelli map

$$\mathfrak{M}_g \longrightarrow \mathscr{A}_g$$

gives an embedding, and we identify $\mathfrak{M}_g$ with its image. $\mathfrak{M}_g$ is open in

$\mathscr{A}_g$ if $g \leq 3$, and is of codimension one if $g=4$. Let $\mathscr{H}_g$ be the locus in $\mathfrak{M}_g$ consisting of hyperelliptic points, and let $\bar{\mathscr{H}}_g$ be its closure in $\mathscr{A}_g$.

A hyperelliptic curve of genus g is given, as an affine curve, by an equation

$$y^2 = \prod_{i=1}^{2g+2} (x - \xi_i)$$

where $\xi_1, \cdots, \xi_{2g+2}$ are mutually distinct complex numbers. Let W_{2g+2} be an open subvariety of C^{2g+2} consisting of points with distinct coordinates, which parametrizes "all" hyperelliptic curves of genus g. There is a surjective morphism of W_{2g+2} onto $\mathscr{H}_g$ by sending $(\xi_1, \cdots, \xi_{2g+2})$ to the isomorphism class of the corresponding hyperelliptic curve. $\mathscr{H}_g$ is a quotient of W_{2g+2} by the composite of $SL_2(C)$ and of the symmetric group $\mathfrak{S}_{2g+2}$ of degree $2g+2$. Let $S(2, 2g+2)$ be the graded ring of invariants of a binary $(2g+2)$-form, which comes out as the invariant subring of $C[\xi_1, \cdots, \xi_{2g+2}]$ under the above-mentioned group where we refer the reader to Schur [42], Tsuyumine [56, Chap. I, II] for detail. $\mathscr{H}_g \to \operatorname{Proj}(S(2, 2g+2))$ is a compactification of $\mathscr{H}_g$. Let $S(2g+2)$ denote the invariant subring of $C[\xi_1, \cdots, \xi_{2g+2}]$ under $SL_2(C)$, so that the invariant subring of $S(2g+2)$ under $\mathfrak{S}_{2g+2}$ equals $S(2, 2g+2)$. If $\Gamma_g(2)$ denotes the principal congruence subgroup of level two, then Igusa [30] has obtained the homomorphism

$$\rho_g \colon A(\Gamma_g(2)) \longrightarrow S(2g+2)$$

of graded rings which is described explicitly in terms of theta constants. ρ_g maps the graded part of degree k to that of degree $gk/2$. He showed also that the restriction of ρ_g to $A(\Gamma_g)$, which we denote also by ρ_g, is a homomorphism of $A(\Gamma_g)$ to $S(2, 2g+2)$;

$$\rho_g \colon A(\Gamma_g) \longrightarrow S(2, 2g+2)$$

which is associated with an embedding $\mathscr{H}_g \longrightarrow \mathscr{A}_g$. The kernel of ρ_g is the ideal generated by modular forms vanishing identically on $\mathscr{H}_g \subset \mathscr{A}_g$. ρ_g is injective if $g \leq 2$ because all curves of genus $g \leq 2$ are hyperelliptic.

Now let us consider the case $g=3$. There are 36 even theta characteristics of degree three. The corresponding product of theta constants

$$\chi_{18} := \prod_{\text{even}} \theta\begin{bmatrix} u \\ v \end{bmatrix}$$

is a cusp form for Γ_3 of weight 18. It was shown by Igusa [30] that

$$\overline{\mathscr{H}}_3=\operatorname{div}(\chi_{18}).$$

We have an exact sequence

$$0\longrightarrow(\chi_{18})\longrightarrow A(\Gamma_3)\xrightarrow{\rho_3}S(2,8)$$

where for the structure of $S(2, 8)$, we refer the reader to Shioda [46].

Let R denote the reducible locus of $\mathscr{A}_3$. Then we have a sequence of inclusions

$$R\subset\overline{\mathscr{H}}_3\subset\mathscr{A}_3$$

where $\overline{\mathscr{H}}_3$ (resp. R) is irreducible of codimension one in $\mathscr{A}_3$ (resp. $\overline{\mathscr{H}}_3$). The sequence satisfies the conditions which are mentioned at the end of § 2. R corresponds to the subset

$$R_0:=\left\{\begin{bmatrix} Z_1 & 0\\ 0 & z_3\end{bmatrix}\in H_3\,\middle|\, Z_1\in H_2,\ z_3\in H\right\}$$

of H_3. The stabilizer subgroup of Γ_3 at R_0 is the image of

$$\begin{array}{ccc}\Gamma_2\times\Gamma_1 & \longrightarrow & \Gamma_3\\ \left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix},\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right\} & \longrightarrow & \begin{bmatrix} A & 0 & B & 0\\ 0 & a & 0 & b\\ C & 0 & D & 0\\ 0 & c & 0 & d\end{bmatrix}.\end{array}$$

The subset of R_0 consisting of diagonal matrices, is stable under the matrix $\begin{pmatrix} U & 0\\ 0 & U\end{pmatrix}\in\Gamma_3$ with

$$U=\begin{bmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

The graded ring associated with R, which we denote by $\bar{A}(0)$, is given by

$$\left\{\sum\psi\otimes j\in A(\Gamma_2)\otimes A(\Gamma_1)\,\middle|\,\sum\psi\begin{pmatrix} z_1 & 0\\ 0 & z_2\end{pmatrix}j(z_3)\ \text{is symmetric in } z_1, z_2, z_3\right\}$$

where for two graded modules $M=\oplus_k M_k$, $N=\oplus_k N_k$, $M\otimes N$ always denotes a graded module $\oplus_k M_k\otimes N_k$. $\bar{A}(0)$ is easily understood because the structures of $A(\Gamma_1)$, $A(\Gamma_2)$ are known. So if the trick mentioned in § 2 is applicable to $R\subset\overline{\mathscr{H}}_3$ and to $\overline{\mathscr{H}}_3\subset\mathscr{A}_3$, then we get some information about the structure of $A(\Gamma_3)$. It has been done in [56], whose sketch is given in the following (see also [57]).

$\overline{\mathscr{H}}_3 \subset \mathscr{A}_3$ is in the case $D(1)$, indeed $\overline{\mathscr{H}}_3$ is defined by χ_{18} as stated above. So the problem is reduced to determine the graded ring $A(\Gamma_3)/(x_{18})$, whose projective spectrum equals $\overline{\mathscr{H}}_3$. Let us denote by $\nu(f), f \in A(\Gamma_3)$, the vanishing order of $f|_{\overline{\mathscr{H}}_3}$ at R. There is a cusp form χ_{28} of weight 28 whose restriction to $\overline{\mathscr{H}}_3$ vanishes only at R, where for the definition of χ_{28} we refer the reader to [56]. χ_{28} is the modular form of lowest weight satisfying such a property, and $\nu(\chi_{28})=4$. Hence $R \subset \overline{\mathscr{H}}_3$ is in the case $D(4)$.

Remark. In [56], we have defined the order of vanishing to be twice as much as in the present paper in order to unify notations. So the vanishing order $\nu(\chi_{28})$ has been written as 8 there.

$\overline{\mathscr{H}}_3 \subset \mathscr{A}_3$ is corresponding to the union of loci in H_3 of theta constants $\theta\begin{bmatrix} u \\ v \end{bmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix}$ being even. There are just six theta constants which vanish identically on $R_0 \subset H_3$, e.g., $\theta\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(Z)$. Let V be the irreducible component of the analytic subvariety in H_3 defined by $\theta\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(Z)=0$ which contains R_0. Let f be a modular form for Γ_3. Suppose that $\nu(f)\equiv 0 \pmod 4$. Then $(f/\chi_{28}^{\nu(f)/4})|_{V-\Gamma_3 R_0}$ is holomorphic since χ_{28} vanishes nowhere on $V-\Gamma_3 R_0$, and it extends holomorphically to V by Riemann's removable singularity theorem. Thus

$$\Psi(f/\chi_{28}^{\nu(f)/4})(Z_1, z_3)=\lim_{\substack{Z\to Z_0\\ Z\in V}} (f/\chi_{28}^{\nu(f)/4})(Z), \qquad Z_0=\begin{bmatrix} Z_1 & 0 \\ 0 & z_3 \end{bmatrix} \in R_0,$$

is well defined. Let Γ_2' be the subgroup in Γ_2 of index six which leaves a theta characteristic $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ stable mod 2. Then $\Psi(f/\chi_{28}^{\nu(f)/4})$ is shown to be contained in $\bar{A}' := \left\{\sum \psi\otimes j \in A(\Gamma_2')\otimes A(\Gamma_1) \mid \sum \psi\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} j(z_3) \text{ is symmetric in } z_1, z_3\right\}$. The homomorphism Ψ and the graded ring $\bar{A}'$ can be constructed as the one which we introduced in § 2 where $X=\overline{\mathscr{H}}_3$, and $D=R$.

Let F be a meromorphic modular form for Γ_3 whose restriction to V is, however, holomorphic such as $f/\chi_{28}^{\nu(f)/4}$ above. It is not easy to calculate $\Psi(F)$ directly from definition, particularly if F is not holomorphic globally on H_3. $\rho_3(F)$ is obviously well-defined and is contained in $S(2, 8) \subset C[\xi_1, \cdots, \xi_8]$. Let $A'(\Gamma_3)$ be the graded ring of such modular forms. A hyperelliptic point Z in V moves toward $Z_0 \in R_0$ if the corresponding curve degenerates to the union of a hyperelliptic curve of genus two and of an elliptic curve, which is, for instance, written in terms of $\xi_1, \cdots, \xi_8$, as

$$(\xi_1, \cdots, \xi_5, t+\xi_6, t+\xi_7, t+\xi_8), \qquad t \longrightarrow \infty.$$

Let $S(2, 8)_0$ be the subring of $S(2, 8)$ which is generated by homogeneous elements $I(\xi_1, \cdots, \xi_8)$ of degree s, $s \geq 0$ such that

$$\lim_{t\to\infty} t^{-7s/3} I(\xi_1, \cdots, \xi_5, t+\xi_6, t+\xi_7, t+\xi_8)$$

makes sense. Then the image of $A'(\Gamma_3)$ by ρ_3 is contained in $S(2, 8)_0$. We obtain a homomorphism of graded rings

$$\Psi: S(2, 8)_0 \longrightarrow S(6) \otimes S(2, 4)$$

by using the above limit. There is a commutative diagram

$$\begin{array}{ccc} A'(\Gamma_3) & \xrightarrow{\quad\rho_3\quad} & S(2, 8) \\ \Psi \downarrow & & \Psi \downarrow \\ A(\Gamma_2) \otimes A(\Gamma_1) & \xrightarrow{\rho_2 \otimes \rho_1} & S(6) \otimes S(2, 4) \end{array}$$

The Ψ of right hand side is a map given by polynomial calculation, and is, although sometimes not very easy, surely computable. Thus since $\rho_2 \otimes \rho_1$ is injective, the Ψ of left hand side is computable (at least in principle).

Let $A(i)$ be the ideal of $A(\Gamma_3)/(\chi_{18})$ generated by the modular forms f with $\nu(f) \geq i$, and let $\bar{A}(i) = A(i)/A(i+1)$. Here we note that $A(i)$ is defined differently from [56] (the present $A(i)$ equals $A(i)/(\chi_{18})$ in [56]), but $\bar{A}(i)$'s are the same except for what we mentioned in the above remark. $\bar{A}(i)$'s are $\bar{A}(0)$-modules. We fix three modular forms β, γ, δ with $\nu(\beta) = 1$, $\nu(\gamma) = 2$, $\nu(\delta) = 3$ respectively. If $f \in A(\Gamma_3)$ is of order $i \equiv 0 \pmod 4$ (resp. $1, 2, 3 \pmod 4$), then $f/\chi_{28}^{i/4}$ (resp. $f\delta/\chi_{28}^{(i+3)/4}$, $f\gamma/\chi_{28}^{(i+2)/4}$, $f\beta/\chi_{28}^{(i+1)/4}$) is contained in $A'(\Gamma_3)$ and hence the Ψ-image is defined. For $i \equiv 0 \pmod 4$ (resp. $1, 2, 3 \pmod 4$) we denote by $\Psi(i)$ the map given by

$$f \longrightarrow \Psi(f/\chi_{28}^{i/4}) \text{ (resp. } \Psi(f\delta/\chi_{28}^{(i+3)/4}),\ \Psi(f\gamma/\chi_{28}^{(i+2)/4}),\ \Psi(f\beta/\chi_{28}^{(i+1)/4})),$$

where in [56], some specific modular forms are taken as β, γ, δ. $\Psi(i)$ induces an injective homomorphism

$$\bar{A}(i) \longrightarrow \bar{A}',$$

which we denote also by $\Psi(i)$. As mentioned above, this map is computable. Let us identify $\bar{A}(i)$ with its $\Psi(i)$-image. Then it is shown that

$$\bar{A}(i)=\bar{A}(i+4) \qquad \text{for } i\geq 3,$$

and so that

$$A(\Gamma_3)/(\chi_{18})\simeq \bar{A}(0)\oplus\bar{A}(1)\oplus\bar{A}(2)\oplus \bigoplus_{\mu=0}^{\infty}(\bar{A}(3)\oplus\bar{A}(4)\oplus\bar{A}(5)\oplus\bar{A}(6))\chi_{28}^{\mu}$$

as modules over a graded subring of $\bar{A}(0)$ isomorphic to a polynomial ring. The structure of $\bar{A}(i)$, $i\leq 6$, could be determined, and hence some of the structure of $A(\Gamma_3)$ too. We refer the reader to [56, pp. 831–2] for the theorem about the structure of $A(\Gamma_3)$. Here we note that *the problems about a minimal system of generators, relations among them, are untouched yet as well as the rationality problem of the variety H_3/Γ_3 which seems one of the outstanding problems.*

We shall close this section with Taniyama's words ([50, Letter to M. Sugiura]), which appears in context, to talk about the theory of complex multiplication. Perhaps the author takes them in a wider sense than Taniyama's original view, which, however, seems still true: When Siegel modular functions can be handled as "easily" as elliptic functions, number theory will have developed in many directions.

§ 6. Siegel modular forms of degree four

Let C be a non-hyperelliptic curve of genus four. We identify C with its canonical curve in $\boldsymbol{P}^3$. Then C is a complete intersection of a quadric and a cubic. There are two types of quadrics in $\boldsymbol{P}^3$, namely, smooth quadrics and quadric cones. A curve C exhibited as a complete intersection of a quadric cone and a cubic is said to *have a vanishing theta constant* because one of the theta constants with even characteristics vanishes at the jacobian point corresponding to C if and only if C is such a curve. Let $\mathfrak{M}_4'$ be the moduli of such curves.

Let R denote the irreducible subvariety of the reducible locus of $\mathscr{A}_4$ corresponding to the direct products of 3-dimensional abelian varieties and elliptic curves, and let $\overline{\mathfrak{M}}_4$, $\overline{\mathfrak{M}}_4'$ be the closures in $\mathscr{A}_4$ of $\mathfrak{M}_4$, $\mathfrak{M}_4'$. There is a sequence of inclusions;

$$R\subset\overline{\mathfrak{M}}_4'\subset\overline{\mathfrak{M}}_4\subset\mathscr{A}_4.$$

$\overline{\mathfrak{M}}_4$ is an irreducible divisor of $\mathscr{A}_4$, and is in the case $D(1)$, indeed it is defined by the single modular form J_8 of weight eight which is called *the Schottky invariant*. It is Schottky's theorem, however whose rigorous proof has been first given by Igusa [31]. Let χ_{68} be the modular form for Γ_4 defined to be the product of all theta constants with even characteristics (mod 2). Then $\overline{\mathfrak{M}}_4'$ is defined in $\overline{\mathfrak{M}}_4$, by $\sqrt{\chi_{68}|_{\overline{\mathfrak{M}}_4}}$ (Tsuyumine [55]). It is not known if there is Siegel modular form of weight 34 whose restriction to $\overline{\mathfrak{M}}_4$ equals $\sqrt{\chi_{68}|_{\overline{\mathfrak{M}}_4}}$. If this is the case, $\overline{\mathfrak{M}}_4' \subset \overline{\mathfrak{M}}_4$ is in the case $D(1)$. However even if we are not lucky, we have only the case $D(2)$. Here we pose the following question:

Question. Is there a Siegel modular form f for Γ_4 such that the ideal (J_8, χ_{68}, f) defines R set-theoretically?

Suppose that the answer is affirmative. Then it can be shown that $R \subset \overline{\mathfrak{M}}_4'$ satisfies the Assumption II, and hence that the above sequence if inclusions satisfies the conditions mentioned at the end of § 2. The result in the preceding section will help to determine the graded ring associated with R which corresponds to the image of the map;

$$\begin{array}{ccc} H_3 \times H & \longrightarrow & H_4 \\ (Z_1, z_4) & \longrightarrow & \begin{bmatrix} Z_1 & 0 \\ 0 & z_4 \end{bmatrix}. \end{array}$$

So there will be a chance we get some information about the structure of $A(\Gamma_4)$.

We introduce two facts concerning the above question. In his paper [31], Igusa constructed the modular form of weight 540 which defines $R \cup \mathscr{H}_4$ in $\overline{\mathfrak{M}}_4'$ set-theoretically, $\mathscr{H}_4$ denoting the hyperelliptic locus. Sasaki [38] gave for an arbitrary degree, a system of modular forms which defines set-theoretically the reducible locus. However its number is not small in general.

Though without evidence, the author believes that there should exist such a modular form as in the question. However it is hard to manage the step "$R \subset \overline{\mathfrak{M}}_4'$" if one follows the method in § 5. We shall need to use it jointly with some other method, e.g., an asymptotic dimension formula for spaces of cusp forms for Γ_4 (up to the first or the second degree terms? $\cdots$, I don't know), of course, the perfect one is better. Or is it possible to attack directly the graded ring associated with $\overline{\mathfrak{M}}_4'$ by geometric method? In this section we have shown only that the best card is still in a pile.

References

[1] Baily, W. L. and Borel, A., Compactification of arithmetic quotients of bounded symmetric domains, Ann. Math. (2), **84** (1966), 442–528.

[2] Blumenthal, O., Über Modulfunktionen von mehreren Veränderlichen, (Erste Hälfte), Math. Ann., **56** (1903), 509–548; (Zweite Hälfte), *ibid.*, **58** (1904), 497–527.

[3] Busam, R., Eine Verallgemeinerung gewisser Dimensionsformeln von Shimizu, Invent. Math., **11** (1970), 110–149.
[4] Cohn, L., The dimension of spaces of automorphic forms on a certain two dimensional complex domain, Mem. Amer. Math. Soc., **158** (1975).
[5] Fomenko, O. M., Modular forms and Hilbert functions for the field $Q(\sqrt{2})$ (Russian), Math. Zametki, **4** (1968), 129–136; (English transl.) Math. Notes, **4** (1968), 567–571.
[6] Freitag, E., Zur Theorie der Modulformen zweiten Grades, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (1965), 151–157.
[7] ——, Modulformen zweiten Grades zur rationalen und Gaußschen Zahlkörper, Sitz. der Heidelberg Akad. Wiss. No. 1, **49** (1967).
[8] ——, Stabile Modulformen, Math. Ann., **230** (1977), 197–211.
[9] ——, Die Irreduzibilität der Schottkyrelation (Bemerkung zu einem Satz von J. Igusa), Arch. Math., **40** (1983), 255–259.
[10] ——, Siegelsche Modulfunktionen, Grundlehren, **254**, Springer-Verlag, 1983.
[11] Van der Geer, G., Hilbert modular forms for the field $Q(\sqrt{6})$, Math. Ann., **233** (1978), 163–179.
[12] Van der Geer, G. and Zagier, D., The Hilbert modular group for the field $Q(\sqrt{13})$, Invent. Math., **42** (1977), 93–133.
[13] Gundlach, K.-B., Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers $Q(\sqrt{5})$, Math. Ann., **152** (1963), 226–256.
[14] ——, Die Bestimmung der Funktionen zu einigen Hilbertschen Modulgruppen, J. Reine Angew. Math., **220** (1965) 109–153.
[15] Hammond, W. F., On the graded ring of Siegel modular forms of genus two, Amer. J. Math., **87** (1965), 502–506.
[16] ——, The modular groups of Hilbert and Siegel, Amer. J. Math., **85** (1966), 497–516.
[17] Hermann, C. F., Symmetrische Hilbertsche Modulformen und Modulfunktionen zu $Q(\sqrt{17})$, Math. Ann., **256** (1981), 191–197.
[18] ——, Thetafunktionen und symmetrische Hilbertsche Modulformen zu $Q(\sqrt{65})$, J. Reine Angew. Math., **339** (1983), 147–162.
[19] Hirzebruch, F., Hilbert modular surfaces, L'Enseigment Math. (2), **21** (1973).
[20] ——, Kurven auf den Hilbertschen Modulflächen und Klassenzahlrelationen, Lect. Notes in Math., **412**, 75–93, Springer-Verlag, 1974.
[21] ——, Hilbert's modular group for the field $Q(\sqrt{5})$ and the cubic diagonal surface of Clebsch and Klein (Russian), Uspekhi Mat. Nauk, **31:5** (1976), 153–166; (English transl.) Russian Math. Surveys, **31:5** (1976), 96–110.
[22] ——, The ring of Hilbert modular forms for real quadratic fields in small discriminant, In: Modular functions of one variable, IV, Lect. Notes in Math., **627**, 287–323, Springer-Verlag, 1977.
[23] ——, Überlagerungen der projektiven Ebene und Hilbertsche Modulflächen, L'Enseigment Math., **24** (1978), 63–78.
[24] ——, Modulflächen und Modulkurven zur symmetrischen Hilbertschen Modulgruppe, Ann. Sci. École Nor. Sup. (4), **11** (1978), 101–165.
[25] Hirzebruch, F. and Van de Ven, A., Hilbert modular surfaces and the classification of algebraic surfaces, Invent. Math., **23** (1974), 1–29.
[26] Hirzebruch, F. and Zagier, D., Intersection number of curves on Hilbert modular surfaces and modular forms of Neben type, Invent. Math., **36** (1976), 57–113.
[27] ——, Classification of Hilbert modular surfaces, In: Complex analysis and algebraic geometry, 43–77, Iwanami Shoten, Tokyo, 1977.
[28] Igusa, J., On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175–200.

[29] ——, On Siegel modular forms of genus two (II), Amer. J. Math., **86** (1964), 817–855.
[30] ——, Modular forms and projective invariants, Amer. J. Math., **89** (1967), 817–855.
[31] ——, On the irreducibility of Schottky's divisor, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., **28** (1981), 531–545.
[32] Müller, R., Hilbertsche Modulformen und Modulfunktionen zu $Q(\sqrt{8})$, Math. Ann., **266** (1983), 83–103.
[33] ——, Hilbertsche Modulformen und Modulfunktionen zu $Q(\sqrt{5})$, Arch. Math., **45** (1985), 239–251.
[34] Mumford, D., Varieties defined by quadratic equations, In: Questions on algebraic varieties, (C.I.M.E., III Ciclo, Verenna, 1969), 29–100, Edizioni Cremonese, Rome, 1970.
[35] Nagaoka, S., On Hilbert modular forms, III, Proc. Japan. Acad., **59** Ser. A (1983), 346–348.
[36] Resnikoff, H. L., On the graded ring of Hilbert modular forms associated with $Q(\sqrt{5})$, Math. Ann., **208** (1974), 161–170.
[37] Resnikoff, H. L. and Tai, Y.-S., On the structure of a graded ring of automorphic forms on the 2-dimensional complex ball, Math. Ann., **238** (1978), 97–117.
[38] Sasaki, R., Modular forms vanishing at the reducible points of the Siegel upper-half space, J. Reine Angew. Math., **345** (1983), 111–121.
[39] Satake, I., On Siegel's modular functions, In: International symposium on algebraic number theory, 107–129, Tokyo and Nikko, 1955.
[40] ——, On the compactification of the Siegel space, J. Ind. Math. Soc. (NS), **20** (1956), 259–281.
[41] Satoh, T., On certain vector valued Siegel modular forms of degree two, Math. Ann., **274** (1986), 335–352.
[42] Schur, I., Vorlesungen über Invariantentheorie, Grundlehren 143, Springer-Verlag, 1968.
[43] Séminaire Henri Cartan, Fonctions automorphes, École Norm. Sup. 1957/58.
[44] Serre, J.-P., Prolongement de faiceaux analytique cohérents, Ann. Inst. Fourier (Grenoble), **16** (1966), 363–374.
[45] Shimizu, H., On discontinuous groups operating on the product of the upper half planes, Ann. Math., **77** (1963), 33–71.
[46] Shioda, T., On the graded ring of invariants of binary octavics, Amer. J. Math., **89** (1967), 1022–1046.
[47] Siegel, C. L., Einführung in die Theorie der Modulfunktionen n-ten Grades, Math. Ann., **116** (1939), 617–657.
[48] Tai, Y.-S. and Resnikoff, H. L., On the structure of a graded ring of automorphic forms on the 2-dimensional complex ball II, Math. Ann., **258** (1982), 367–382.
[49] Takagi, T., Kinsei Sûgaku Shidan (Japanese), Kyôritsu-shuppan, Tokyo, 1970.
[50] Taniyama, Y., The complete works of Yutaka Taniyama.
[51] Tsushima, R., An explicit dimension formula for the space of generalized automorphic forms with respect to $Sp(2, Z)$, Proc. Japan Acad., **59**A, (1983), 139–142.
[52] ——, An explicit dimension formula for the space of generalized automorphic forms with respect to $Sp(2, Z)$, preprint, 1983.
[53] Tsuyumine, S., Rings of modular forms—On Eichler's problem, Nagoya Math. J., **99** (1985), 31–44.
[54] ——, Rings of automorphic forms which are not Cohen-Macaulay, I, J. Math. Soc. Japan, **38** (1986), 147–162.
[55] ——, Factorial property of a ring of automorphic forms, Trans. Amer.

Math. Soc., **296** (1986), 111–123.
[56] ——, On Siegel modular forms of degree three, Amer. J. Math., **108** (1986), 755–862; Addendum, *ibid.*, 1001–1003.
[57] ——, On the modular function field of degree three, Comp. Math., **63** (1987), 83–98.
[58] ——, Rings of automorphic forms which are not Cohen-Macaulay, II, J. Math. Soc. Japan, **40** (1988), 369–381.

Department of Mathematics
Mie University
Tsu 514
Japan