Advanced Studies in Pure Mathematics 14, 1988
Representations of Lie Groups, Kyoto, Hiroshima, 1986
pp. 651–660

Boundedness of Certain Unitarizable Harish-Chandra Modules

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§ 0. Introduction

Let G be a connected real semisimple Lie group, Z be the center of G, K be a maximal compact subgroup of G modulo Z, $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_c$ of the Lie algebra $\mathfrak{g}$ of G and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. An element X of $\mathfrak{g}$ defines vector fields $\pi(X)$ and $D_R(X)$ on G by

$$(\pi(X)\phi)(g)=\frac{d}{dt}\phi(e^{-tX}g)|_{t=0}$$

and

$$(D_R(X)\phi)(g)=\frac{d}{dt}\phi(ge^{tX})|_{t=0}$$

for $\phi\in C^\infty(G)$. Then π and D_R extend to algebra homomorphisms of $U(\mathfrak{g})$ to the algebra of differential operators on G. For an element x of G we also define an endomorphism $\pi(x)$ of $C^\infty(G)$ by $(\pi(x)\phi)(g)=\phi(x^{-1}g)$ for $\phi\in C^\infty(G)$.

Let f be an element of $C^\infty(G)$ or a column vector of elements of $C^\infty(G)$. Suppose f is left K-finite and $Z(\mathfrak{g})$-finite (i.e. $\dim\sum_{k\in K}C\pi(k)f<\infty$ and $\dim\pi(Z(\mathfrak{g}))f<\infty$). Put $V_f=\pi(U(\mathfrak{g}))f$. Then V_f is a $(\mathfrak{g},K)$-module under π. Moreover we say that V_f is a unitarizable Harish-Chandra module if there exists a unitary representation (τ, E) of G with finite length (i.e. (τ, E) is isomorphic to a finite direct sum of irreducible unitary representations) such that V_f is isomorphic to the Harish-Chandra module of (τ, E). In this paper we consider the following problem:

Suppose V_f is a unitarizable Harish-Chandra module. Then is the function $f(g)$ bounded when g tends to a certain infinite point?

Of course if we do not impose any other assumption on f, we have nothing to conclude. We have in mind that f satisfies some more conditions, such as, f corresponds to a section of the G-homogeneous vector bundle associated to a representation of a certain subgroup of G and/or f

Received November 9, 1987.

satisfies certain differential equations etc. For example, if f is a zonal spherical function, then we can conclude that f is bounded because f coincides with the matrix coefficient with respect to a normalized K-fixed vector of the corresponding irreducible unitary representation of G.

Let σ be an involutive automorphism of G and H be the fixed point group of σ. In the case when f is right H-fixed, that is, f is (identified with) a function on the semisimple symmetric space G/H, then we can conclude that f is also bounded (Corollary 2.2). This follows from our general theorem (Theorem 1.1). If we apply Theorem 1.1 to the case when f is a section of a representation space belonging to the principal series of G, then we have some restriction for the representation (Corollary 2.4). More generally, in this paper, we apply Theorem 1.1 to the case when f is a section of a vector bundle over G/Q induced from a certain representation of Q where Q is a fixed point group of an involution of G or Q is a nilpotent subgroup of G.

The result in this paper was obtained when the second author was visiting university of Copenhagen in the summer of 1984. The second author expresses his sincere gratitude to University of Copenhagen and Danish Mathematical Society for their hospitality during the visit.

The first author should like to thank the Taniguchi Foundation for the invitation to visit Japan in the fall of 1986. He also wants to thank Mathematical Society of Japan and the Universities of Hiroshima, Kyoto and Tokyo for their hospitality.

§ 1. Main Theorem

Retain the notation in Section 0. To state our result we prepare some more notation. Let $G=KA_{\mathfrak{p}}N$ be an Iwasawa decomposition of G, $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}$ be the corresponding decomposition of $\mathfrak{g}, \theta$ be the Cartan involution of both G and $\mathfrak{g}$ with respect to K, Σ be the root system defined for the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$, Σ^+ be the positive root system corresponding to $\mathfrak{n}$ and $\Psi=\{\alpha_1, \cdots, \alpha_l\}$ be the fundamental system. Let $\{H_1, \cdots, H_l\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_l\}$, that is, $H_j \in \mathfrak{a}_{\mathfrak{p}}$ and $\alpha_i(H_j)=\delta_{ij}$. The number l is called the rank of G/K. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}_c$ and $S(\mathfrak{g})_{(m)}$ be the totality of the homogeneous elements of $S(\mathfrak{g})$ with degree m. Put $S(\mathfrak{g})^{(m)}=\sum_{i\leq m} S(\mathfrak{g})_{(i)}$. By the symmetrization $\Lambda: S(\mathfrak{g})\to U(\mathfrak{g})$ we define $U(\mathfrak{g})^{(m)}=\Lambda(S(\mathfrak{g})^{(m)})$. Thus we have

$$U(\mathfrak{g})^{(m)}/U(\mathfrak{g})^{(m-1)}\simeq S(\mathfrak{g})^{(m)}/S(\mathfrak{g})^{(m-1)}\simeq S(\mathfrak{g})_{(m)}.$$

Using this isomorphism we can define $\bar{p}\in S(\mathfrak{g})_{(m)}$ for any $p\in U(\mathfrak{g})^{(m)}-U(\mathfrak{g})^{(m-1)}$. If $p=0$, then we put $\bar{p}=0$.

By the map

$$\begin{array}{ccc} \boldsymbol{R}^l & \longrightarrow & A_{\mathfrak{p}} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ t=(t_1, \cdots, t_l) & \longmapsto & a(t)=\exp\left(-\sum_{t_j\neq 0} H_j \log|t_j|\right) \end{array}$$

we can identify $(0, \infty)^l$ and $A_{\mathfrak{p}}$. For any $t=(t_1, \cdots, t_l) \in \boldsymbol{R}^l$, we put $\Sigma_t=\sum_{t_j\neq 0} \boldsymbol{R}\alpha_j \cap \Sigma$ and define a parabolic subalgebra $\mathfrak{p}_t$ with the Langlands decomposition $\mathfrak{p}_t=\mathfrak{m}_t+\mathfrak{a}_t+\mathfrak{n}_t$, where $\mathfrak{a}_t\subset\mathfrak{a}_{\mathfrak{p}}$, $\mathfrak{n}_t=\sum_{\alpha\in\Sigma^+-\Sigma_t}\mathfrak{g}^\alpha$ and $\mathfrak{g}^\alpha$ is the root space corresponding to the root $\alpha\in\Sigma$. Let $P_t=M_tA_tN_t$ be the corresponding parabolic subgroup and its Langlands decomposition. We will identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$ by the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$.

Theorem 1.1. *Let f be a non-zero left K-finite and $Z(\mathfrak{g})$-finite function (or column vector of functions) on G. Fix any $g_0\in G$ and $t\in[0, \infty)^l$. Suppose $V_f=\pi(U(\mathfrak{g}))f$ is a unitarizable Harish-Chandra module. Moreover suppose f satisfies the following condition.*

(A.1) *There exist a subset J of $U(\mathfrak{g})$ such that f is right J-finite (i.e. there exists a finite dimensional subspace F of $C^\infty(G)$ satisfying $F\ni$ (each component of) f and $D_R(p)F\subset F$ for any $p\in J$) and moreover*

$$N(J)\cap \mathrm{Ad}\,(g_o)\theta(\mathfrak{n}_t)=0$$

by denoting

$$N(J)=\{X\in\mathfrak{g};\ \bar{p}(X)=0 \text{ for any } p\in J\}.$$

Then for any $g_1\in G$ there exist neighborhoods $U(g_1)$ of g_1 in G, $U(g_o)$ of g_0 in G and a neighborhood $U(t)$ of t in $\boldsymbol{R}^l$ such that $f(g)$ is bounded on the set

$$\{xa(s)y^{-1};\ x\in U(g_1),\ y\in U(g_0) \text{ and } s\in U(t)\cap(0, \infty)^l\}.$$

Remark 1.2. i) It is clear that the following condition implies (A.1).

(A.2) There exists a $\boldsymbol{R}$-subalgebra $\mathfrak{b}$ of $\mathfrak{g}_c$ such that f is right $\mathfrak{b}$-finite and

$$\mathfrak{b}^\perp\cap \mathrm{Ad}\,(g_o)\theta(\mathfrak{n}_t)=0.$$

ii) In Section 3 we give a stronger result than the above theorem, which is valid without the assumption of the unitarizability of V_f (cf. Theorem 3.3).

Example 1.3 (cf. Lemma 3.1.). Suppose f is a left and right K-finite and $Z(\mathfrak{g})$-finite function on G and moreover suppose V_f is a unitarizable Harish-Chandra module. To show that f is bounded by using Theorem 1.2, we may assume V_f is irreducible. Then there exists a character $\chi: Z\to\{z\in\boldsymbol{C};|z|=1\}$ such that $\pi(z)f=\chi(z)f$ for $z\in Z$. Put $\mathfrak{b}=\mathfrak{k}$ in (A.2).

Then the elements of $\mathfrak{b}^{\perp} \cap \mathfrak{g}$ being semisimple, we have always (A.2) for any $g_o \in G$ and any $t \in [0, \infty)^l$. Since $G = K\{a(t); t \in (0, 1]^l\}K$ and $(K/Z) \times [0, 1]^l \times (K/Z)$ is compact, Theorem 1.1 and Remark 1.2 imply that f is bounded.

§ 2. Applications

In this section we will apply Theorem 1.1 to a section of the G-homogeneous vector bundle associated to a representation of a closed subgroup of G. Let Q be a closed subgroup of G and ξ be a matrix representation of Q. This means there exist a non-negative integer m such that the map $Q \ni x \mapsto \xi(x) \in \mathrm{GL}(m, C)$ is a Lie group homomorphism. A C^∞-section of the vector bundle over G/Q associated to the representation ξ of Q is identified with a column vector f of m components in $C^\infty(G)$ which satisfies

$$f(gx) = \xi(x)^{-1} f(g) \qquad \text{for } g \in G \text{ and } x \in Q.$$

Theorem 2.1. *Let σ be an involutive automorphism of $\mathfrak{g}$ which commutes with θ, $\mathfrak{h}$ be the fixed point subalgebra of σ and H be the analytic subgroup of G with the Lie algebra $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (resp. $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$) be the decompositions of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for θ (resp. σ). Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ and put $A = \exp \mathfrak{a}$. Let f be a C^∞-section of the vector bundle over G/H associated with a finite dimensional matrix representation ξ of H. Assume that f is left K-finite and $Z(\mathfrak{g})$-finite and moreover $V_f = \pi(U(\mathfrak{g}))f$ is a unitarizable Harish-Chandra module. Then each component of f is bounded on the subset KA of G.*

Corollary 2.2. *In Theorem* 2.1, *if ξ is a unitary representation, then f is bounded on G. Especially, if ξ is trivial (i.e. $f \in C^\infty(G/H)$) in Theorem* 2.1, *then f is bounded.*

Proof. Note that $G = KAH$. Then the corollary follows from Theorem 2.1 because each component of $\xi(x)$ $(x \in H)$ is bounded on H if ξ is unitary.

Proof of Theorem 2.1. Let $\Sigma(\mathfrak{a})$ be the root system corresponding to the pair $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}(\mathfrak{a}, \alpha)$ be the root space corresponding to $\alpha \in \Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a})^+$ be a positive system of $\Sigma(\mathfrak{a})$. We may assume $\mathfrak{a}_\mathfrak{p} \supset \mathfrak{a}$ and Σ^+ is compatible with $\Sigma(\mathfrak{a})^+$. Put $\bar{\mathfrak{a}}_+ = \{X \in \mathfrak{a}; \alpha(X) \geq 0 \text{ for any } \alpha \in \Sigma(\mathfrak{a})^+\}$, $A = \exp \mathfrak{a}$ and $\bar{A}_+ = \exp \bar{\mathfrak{a}}_+$. Let $Z_K(\mathfrak{a})$ and $N_K(\mathfrak{a})$ be the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. Then the quotient group $W(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is identified with the Weyl group of $\Sigma(\mathfrak{a})$. For any element $w \in W(\mathfrak{a})$ we fix a representative $\bar{w}$ of w in $N_K(\mathfrak{a})$. Then $KA =$

$\bigcup_{w \in W(\mathfrak{a})} K\bar{A}_+\bar{w}$. Let B be the closure of the set $\{t \in \boldsymbol{R}^l; a(t) \in \bar{A}_+\}$ in $\boldsymbol{R}^l$. Fix $w \in W(\mathfrak{a})$ and $t \in B$. We remark that $B \subset [0, 1]^l$, $\theta(\mathfrak{n}_t) \subset \sum_{\alpha \in \Sigma(\mathfrak{a})+} \mathfrak{g}(\mathfrak{a}, -\alpha)$ and $\sigma(\mathfrak{g}(\mathfrak{a}, \alpha)) = \mathfrak{g}(\mathfrak{a}, -\alpha)$. Hence $\mathfrak{h}^\perp \cap \mathrm{Ad}(\bar{w})\theta(\mathfrak{n}_t) \subset \mathfrak{q} \cap \sum_{\alpha \in \Sigma(\mathfrak{a})+} \mathfrak{g}(\mathfrak{a}, -w\alpha) = 0$. Applying Theorem 1.1 and Remark 1.2 with $\mathfrak{b} = \mathfrak{h}$ and $g_o = \bar{w}$, we have Theorem 2.1 because $(K/Z) \times B$ is compact. Q.E.D.

Theorem 2.3. *Retain the notation in Section* 0 *and Section* 1. *Fix* $t = (t_1, \cdots, t_l) \in \{0, 1\}^l$. *Let* f *be a* C^∞*-section of the vector bundle associated to a finite dimensional matrix representation of* N_t. *Assume that* f *is left* K*-finite,* $Z(\mathfrak{g})$*-finite and* V_f *is a unitarizable Harish-Chandra module. Then for any compact subset* V *of* G *and any real number* C, f *is bounded on the set*

$$B = \{ga \in G; g \in V, a \in A_t, \alpha(\log a) \geq C \text{ for any } \alpha \in \Sigma^+\}.$$

Proof. Suppose $t' = (t_1', \cdots, t_l') \in (0, \infty)^l$ satisfies $a(t') \in A_t$ and $\alpha(\log(a(t'))) \geq C$ for any $\alpha \in \Sigma^+$. Then for each j, if $t_j' \neq 1$, then $t_j = 0$ and $t_j' < e^{-C}$. Hence Theorem 2.3 is a direct consequence of Theorem 1.1 and Remark 1.2 because $\mathfrak{n}_t^\perp \cap \theta(\mathfrak{n}_{t'}) \subset \mathfrak{n}_t^\perp \cap \theta(\mathfrak{n}_t) = 0$. Q.E.D.

Corollary 2.4. *Use the notation* 2.3. *Let* v *be a column vector of finite elements of* $C^\infty(G)$. *Let* λ *and* μ *be elements of the dual space of* $\mathfrak{a}_\mathfrak{p}$. *Assume*

$$v(gan) = v(g)a^{\lambda + i\mu} \qquad \text{for } g \in G,\ a \in A_t \text{ and } n \in N_t.$$

Suppose moreover v *is left* K*-finite,* $Z(\mathfrak{g})$*-finite and* $\pi(U(\mathfrak{g}))v$ *is a unitarizable Harish-Chandra module. Then we have*

$$\lambda(H_j) \leq 0 \qquad \text{if } t_j = 0.$$

Proof. Fix j with $t_j = 0$. Since $H_j \in \mathfrak{a}_t$, Theorem 2.3 assures that the function $[0, \infty) \ni s \mapsto v(g \exp sH_j)$ is bounded. Then the corollary is clear. Q.E.D.

Remark 2.5. Use the notation in Theorem 2.1. Let $\{X_1, \cdots, X_k\}$ and $\{Y_1, \cdots, Y_n\}$ be the basis of $\mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h} \cap \mathfrak{p}$, respectively, such that $\langle X_i, X_j \rangle = -\delta_{ij}$, $\langle Y_i, Y_j \rangle = \delta_{ij}$. Here $\langle\ ,\ \rangle$ denotes the Killing form of $\mathfrak{g}$. Put $\Delta = -\sum X_i^2 + \sum Y_j^2$ and $\Delta' = -\sum X_i^2$. Let f' be a column vector of C^∞-functions on G. Suppose f' is left K-finite and $Z(\mathfrak{g})$-finite and that $V_{f'}$ is a unitarizable Harish-Chandra module. Moreover suppose that there exist non-trivial polynomials $P(\Delta)$ of Δ and $P'(\Delta')$ of Δ' such that $D_R(P(\Delta))f' = D_R(P'(\Delta'))f' = 0$. Then each component of f' is bounded on KA.

The proof of the above statement is similar to that of Theorem 2.1 by using (A.1) in place of (A.2). We note that the system of the equations $D_R(P(\Delta))u = D_R(P'(\Delta'))u = 0$ on H is elliptic. A similar generalization is possible for Theorem 2.3.

§ 3. Proof of Theorem 1.1.

In Theorem 1.1, the assumption implies that V_f is a finite direct sum of irreducible unitarizable Harish-Chandra modules and therefore to prove the theorem we may assume that V_f is irreducible and that f is a scalar valued function on G. Then Theorem 1.1 clearly follows from the following two lemmas. We remark that Lemma 3.2 does not require the unitarizability of V_f and Lemma 3.1 is known. For completeness we also give its proof.

Lemma 3.1. *Use the notation in Theorem* 1.1. *Suppose that a function f on G satisfies all the assumptions in the theorem and moreover suppose f is right K-finite, then f is bounded on G.*

Lemma 3.2. *Let χ be a unitary character of the center Z of G and f be a left K-finite element of $C^\infty(G)$ with $f(zg) = \chi(z)f(g)$ for $g \in G$ and $z \in Z$. Given $g_o \in G$ and $t \in [0, \infty)^l$ satisfying* (A.1). *Let $\hat{K}$ be the set of equivalence classes of the irreducible unitary representations of K and for $\delta \in \hat{K}$, let χ_δ be the character corresponding to δ. Suppose f is $Z(\mathfrak{g})$-finite and V_f defines* ($\mathfrak{g}$, K)*-module with finite length and moreover suppose that*

$$f_\delta(g) = \chi_\delta(e) \int_{K/Z} f(gk)\chi_\delta(k^{-1})\,dk$$

is a bounded function on G for any $\delta \in \hat{K}$ satisfying $\chi_\delta(z) = \chi(z)\chi_\delta(e)$ for $z \in Z$. Then we have the same conclusion for f as in Theorem 1.2.

Now we will prove the above lemmas. Put $\tilde{G} = G \times G$, $\sigma(g_1, g_2) = (g_2, g_1)$ for $(g_1, g_2) \in \tilde{G}$ and $\Delta G = \{(g, g) \in \tilde{G}; g \in G\}$. Then the group manifold G is identified with the semisimple symmetric space $X = \tilde{G}/\Delta G$ by the map induced from the map $\tilde{G} \ni (g_1, g_2) \mapsto g_1 g_2^{-1}$. The action of the element of $\tilde{G}$ on G is given by $\tilde{G} \times G \ni ((g_1, g_2), x) \mapsto g_1 x g_2^{-1} \in G$. To prove Lemma 3.2 we will use some results in [O2]. In [O2, § 1] we construct an equivariant open imbedding of a semisimple symmetric space in a manifold $\tilde{X}$. In our case, $\tilde{X}$ is defined as follows:

We define an equivalence relation that the elements (g, t) and (g', t') in $\tilde{G} \times \boldsymbol{R}^l$ are equivalent if and only if $\operatorname{sgn} t = \operatorname{sgn} t'$ and $g\tilde{a}(t)\tilde{Q}_t = g'\tilde{a}(t')\tilde{Q}_{t'}$. Here $\operatorname{sgn} t = (\operatorname{sgn} t_1, \cdots, \operatorname{sgn} t_l) \in \{-1, 0, 1\}^l$, $\tilde{a}(t) = (a(t), a(t)^{-1}) \in \tilde{G}$ and $\tilde{Q}_t = \{(man, ma'n') \in \tilde{G}; m \in M_t, a \in A_t, a' \in A_t, n \in N_t$ and $n' \in \theta(N_t)\}$. Then $\tilde{X}$ is the quotient space of $\tilde{G} \times \boldsymbol{R}^l$ by this equivalence relation.

Let ω be the projection of $\tilde{G}\times \boldsymbol{R}^l$ onto $\tilde{X}$. The action of an element of $\tilde{G}$ on $\tilde{X}$ is defined through the left translation on the first component of $\tilde{G}\times \boldsymbol{R}^l$. We can define a compatible real analytic structure on $\tilde{X}$. The number of $\tilde{G}$-orbits in $\tilde{X}$ is 3^l and every open orbit is isomorphic to X. We identify G with the open orbit $\tilde{G}\cdot\omega(e, (1, \cdots, 1))$.

First we want to prove Lemma 3.2. We may assume V_f is irreducible. The assumption in Lemma 3.2 implies that there exists a column vector v with components in $C^\infty(G)$ such that the first component of v equals f and moreover v satisfies a system

$$\mathcal{N}:\begin{cases}\pi(H)v=A_H v & (\forall H\in \mathfrak{k}),\\ D_R(p)v=B_p v & (\forall p\in J),\\ \pi(q)v=C_q v & (\forall q\in Z(\mathfrak{g})).\end{cases}$$

Here A_H, B_p are constant square matrices and C_q are constant scalar matrices. Put $z=w((g_1, g_0), t)$ with an element $g_1\in G$ and also put $Y=\tilde{G}\cdot z$. Here g_o and t are given in Lemma 3.2. We can choose a local coordinate system $(t_1, \cdots, t_r, x_1, \cdots, x_n)$ of $\tilde{X}$ in a neighborhood of z such that G corresponds to the region defined by $t_1>0, \cdots, t_r>0$ and Y corresponds to the submanifold defined by $t_1=\cdots=t_r=0$.

Let $SS\mathcal{N}$ be the characteristic variety of $\mathcal{N}$. The cotangent space $T^*_z\tilde{X}$ is identified with $\boldsymbol{R}^r\times T^*_z Y$. Moreover, since $Y\simeq\tilde{G}/\tilde{Q}_t$, we have

$$\begin{aligned}T^*_z Y&\simeq \mathrm{Lie}\,(\tilde{Q}_t)^\perp\cap(\mathfrak{g}\oplus\mathfrak{g})\\ &=\{(X+Y, -X+Z); X\in\mathfrak{m}_t, Y\in\mathfrak{n}_t \text{ and } Z\in\theta(\mathfrak{n}_t)\}.\end{aligned}$$

Let $A=(X+Y, -X+Z)$ be an element of $T^*_z Y$ with the above notation. Suppose $SS\mathcal{N}\cap T^*_z\tilde{X}\cap(\boldsymbol{R}^r\times\{A\})\neq\emptyset$. The equations $\pi(H)v=A_H v$ $(H\in\mathfrak{k})$ imply that $\langle \mathrm{Ad}\,(g_1)^{-1}\mathfrak{k}, X+Y\rangle=0$. Let Δ be the Casimir operator of $\mathfrak{g}$. By the imbedding $\mathfrak{g}\simeq\mathfrak{g}\oplus\{0\}\subset\mathfrak{g}\oplus\mathfrak{g}$, we extend Δ to an element of $Z(\mathfrak{g}\oplus\mathfrak{g})$. Then the equation $\pi(\Delta)v=c_\Delta v$ implies $\bar{\Delta}(X+Y)=0$. Since $\bar{\Delta}\,|\,(\mathrm{Ad}\,(g_1)^{-1}\mathfrak{k})^\perp$ is positive definite, we have $X+Y=0$ and therefore $X=Y=0$. Similarly the equations $D_R(p)v=B_p v$ $(p\in J)$ imply $\mathrm{Ad}\,(g_0)^{-1}N(J)\ni -X+Z$. Combining this with the assumption $N(J)\cap\mathrm{Ad}\,(g_0)\theta(\mathfrak{n}_t)=0$, we can conclude $X=Y=Z=0$ and we have

$$SS\mathcal{N}\cap T^*_z\tilde{X}\subset T^*_Y\tilde{X}.$$

Hence it follows from [O1, Theorem 5.2] that each component of v is ideally analytic at z and v has the following expression in the intersection of G and a neighborhood of z in $\tilde{X}$:

$$v(t, x)=\sum_{\nu=1}^{N} a_\nu(t, x)p_\nu(\log t_1, \cdots, \log t_r)t_1^{\lambda_{\nu,1}}\cdots t_r^{\lambda_{\nu,r}}. \tag{3.1}$$

Here N is a positive number, p_ν are non-zero homogeneous polynomials, $a_\nu(t, x)$ are (vectors of) real analytic functions and $\lambda_\nu=(\lambda_{\nu,1}, \cdots, \lambda_{\nu,r}) \in C^r$. Each λ_ν is called a characteristic exponent and the condition $a_\nu(0, x)= 0$ $(\forall\nu)$ means v is identically zero.

Put $S=\{u \in C^\infty(G); \pi(q)u=C_q u(\forall q \in Z(\mathfrak{g}))\}$. We use the boundary value maps $\beta_1, \cdots, \beta_M$ of S for the boundary Y which are defined in [O2, § 3]. The maps have the following properties (cf. [O1, Theorem 5.3] and [O2, Theorem 3.4]).

Each β_j corresponds to a characteristic exponent $\lambda_{\nu(j)}$. Put $S_j= \{u \in S; \beta_i(u)=0 \text{ for } 1\leq i\leq j\}$. Then β_j defines a $\tilde{G}$-equivariant map of S_{j-1} to the space of hyperfunction sections of a certain $\tilde{G}$-homogeneous line bundle over Y. The condition $\operatorname{Re}\lambda_{\nu(i),k}\leq\operatorname{Re}\lambda_{\nu(j),k}(\forall k)$ implies $i\leq j$. If the infinitesimal character of v is generic, then we can assume $p_\nu\equiv 1$, $M= N$, $\nu(j)=j$ and $\beta_j(v)(x)=a_j(0, x)$. In general, we can choose an integer L such that if $u \in S$ is ideally analytic at $y \in Y$, then the following two conditions are equivalent:

$$\text{(3.2)} \qquad \operatorname{supp}\beta_i(u) \not\ni y \qquad \text{for } 1\leq i\leq L.$$

(3.3) u is bounded in the intersection of G and a sufficiently small neighborhood of y in $\tilde{X}$.

We remark that a left and right K-finite element of S is ideally analytic at any point of Y. Hence $f_\delta \in S_L$ for any $\delta \in \hat{K}$. By the $\tilde{G}$-equivariance of $\beta_j|S_{j-1}$ we can conclude $f \in S_L$, which proves Lemma 3.2 also by the above equivalence.

Next we will prove Lemma 3.1. We may still assume V_f is an irreducible $(\mathfrak{g}, K)$-module and use the identification $\tilde{G}/\Delta G\simeq G$. Decomposing $D_R(U(\mathfrak{k}))f$ into a direct sum of irreducible $(\{e\}\times K)$-modules, we may assume $D_R(U(\mathfrak{k}))f$ is an irreducible $(\{e\}\times K)$-module. Put $U_f= \pi(U(\mathfrak{g}))D_R(U(\mathfrak{k}))f$. Then U_f is an irreducible $(\mathfrak{g}\oplus 0, K\times K)$-module. Note that for $\phi \in U_f$, ϕ is bounded if and only if $\phi \in S_L$. We choose $p \in U(\mathfrak{g})$ such that $(\pi(p)f)(e)=1$. Hence replacing f by $\pi(p)f$, we may assume $f(e) =1$ because $U_f=U_\phi$ with any non-zero $\phi \in U_f$ and $\beta_j|S_{j-1}$ define $\tilde{G}$-equivariant maps for any j. Moreover the non-zero function $G \ni g\mapsto \int_{K/Z} f(kgk^{-1})dk$ belongs to U_f, we may assume both $f(e)=1$ and $f(kgk^{-1}) =f(g)$ for $g \in G$ and $k \in K$.

Suppose V_f is isomorphic to the Harish-Chandra module of an irreducible unitary representation (τ, E) of G with an inner product $(\ ,\ )$. We identify V_f with a subset of E by the isomorphism. Let $U(\mathfrak{g})^K$ be the totality of K-invariant elements of $U(\mathfrak{g})$. Fix a orthonormal basis

$\{v_1, \cdots, v_n\}$ of $U(\mathfrak{g})^K f$ and put $v=\sum v_j(e)v_j$. For any $D \in U(\mathfrak{g})$ we also put $\bar{D}=\int_K \mathrm{Ad}\,(k)Ddk$ and $\pi(\bar{D})f=\sum C_D^i v_i$ with $C_D^i \in \boldsymbol{C}$. Then

$$\begin{aligned}(\pi(D)f)(e)&=(\pi(\bar{D})f)(e)=\sum C_D^i v_i(e)=(\sum C_D^i v_i, \sum v_j(e)v_j)\\ &=(\pi(\bar{D})f, v)=\int_{K/Z}(\pi(\mathrm{Ad}\,(k)D)f, v)dk=(\pi(D)f')(e)\end{aligned}$$

with

$$f'(g)=\int_{K/Z}(\pi(kg^{-1}k^{-1})f, v)dk.$$

This proves $f=f'$ because their Taylor expansions at the identity element of G are equal. Since f' is clearly bounded on G, we have obtained Lemma 3.1.

For λ and $\lambda' \in \boldsymbol{C}^r$, we define $\mathrm{Re}\,\lambda=(\mathrm{Re}\,\lambda_1, \cdots, \mathrm{Re}\,\lambda_r)$ and $\mathrm{Re}\,\lambda \leq \mathrm{Re}\,\lambda'$ if $\mathrm{Re}\,\lambda_j \leq \mathrm{Re}\,\lambda'_j$ $(\forall j=1, \cdots, r)$. In the expression (3.1) of v we put $\bar{\Lambda}=\{(\lambda_\nu, \deg p_\nu); a_\nu \neq 0\}$ and

$$\Lambda=\{(\lambda, m) \in \bar{\Lambda}; \{(\lambda', m') \in \bar{\Lambda}; \mathrm{Re}\,\lambda' < \mathrm{Re}\,\lambda \text{ or } (\lambda'=\lambda \text{ and } m'>m)\}=\emptyset\}.$$

We call Λ *the set of leading exponents* of v at z. Then the argument in the proof of Lemma 3.2 gives the following result.

Theorem 3.3. i) *Let f be a left K-finite and $Z(\mathfrak{g})$-finite function on G. Fix $g_o \in G$ and $t \in [0, \infty)^l$. If f satisfies the conditon* (A.1), *then f is ideally analytic at the point $\omega((g_1, g_0), t) \in \tilde{X}$ for any $g_1 \in G$.*

ii) *Let χ be a character of the center Z of G and f be a non-zero left K-finite and $Z(\mathfrak{g})$-finite function on G satisfying $f(zg)=\chi(z)f(g)$ for $g \in G$ and $z \in Z$. Suppose V_f is an irreducible* $(\mathfrak{g}, K)$*-module and f is ideally analytic at a boundary point y of G in $\tilde{X}$. Choose $\delta \in \hat{K}$ such that $\chi(z)=\chi_\delta(z)/\chi_\delta(e)$ $(\forall z \in Z)$ and moreover the function*

$$f_\delta(g)=\chi_\delta(e)\int_{K/Z} f(gk)\chi_\delta(k^{-1})dk$$

is non-trivial. Then the set of leading exponents of f at y coincides with that of f_δ at y. Especially, if V_f is a unitarizable Harish-Chandra module, then the set of leading exponents of f at y coincides with that of a matrix coefficient of the corresponding irreducible unitary representation of G.

Proof. We have only to prove Theorem 3.3. ii). Retain the notation in the proof of Lemma 3.2. Let $v \in S$ which is ideally analytic at y. Then we have the following (cf. [O1, § 5]):

For a given $(\lambda, m) \in C^r \times \{0, 1, 2, \cdots\}$, we can choose an integer L by changing the indices of boundary value maps β_i if necessary so that $\beta_j | S_{j-1}$ are still $\tilde{G}$-equivariant and moreover the condition that supp $\beta_i(u) \not\ni y$ for $1 \leq i < L$ and supp $\beta_L(u) \ni y$ is equivalent to the condition that (λ, m) is a leading exponent of v at y.

Thus Theorem 3.3. ii) follows from the argument in the proof of Lemma 3.2.

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