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# Algebraic Structures on Virtual Characters of a Semisimple Lie Group

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## Introduction

Let $G$ be a connected semisimple Lie group with finite centre. Take a continuous representation $\pi$ of $G$ on a Hilbert space $\mathfrak{H}$. For a maximal compact subgroup $K$ of $G$, put

$$\mathfrak{H}_K=\{v\in\mathfrak{H}\,|\,\dim\pi(K)v<\infty\}.$$

We call a vector $v$ in $\mathfrak{H}_K$ a $K$-finite vector. A representation $(\pi, \mathfrak{H})$ is called admissible if, for any irreducible representation $\delta$ of $K$, the multiplicity of $\delta$ in $\mathfrak{H}_K$ is finite. All the irreducible admissible representations of $G$ are classified under infinitesimal equivalence. There are at least three different methods to classify them. Namely, Langlands' classification ([30]), classification using the theory of $D$-modules ([2]) and Vogan's minimal $K$-type arguments ([41]). However, we still cannot understand the admissible representations well. For example, Langlands' parameter of a finite dimensional representation is very difficult to calculate out. So we need not only the classification theory but also more easier description or structures of admissible representations. There are many improvements in this direction, for example, theory of primitive ideals, $K_C$-orbits on flag varieties, Weyl group representations on virtual character modules and so on.

We treat here algebraic structures on virtual character modules. In [28, Appendix], G. J. Zuckerman defined a representation of Weyl groups on virtual character modules on $G$ with regular infinitesimal character. This Weyl group representation provides powerful methods to calculate invariants of admissible representations, such as Gelfand-Kirillov dimensions, $\tau$-invariants, primitive ideals and so on, and to classify them in large. In part I of this paper, we improve his definitions and define a representation of a Hecke algebra on a virtual character module with singular infinitesimal character. After this, we show how useful the representations of Weyl groups and their Hecke algebras are in invariant theory of repre-

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sentations of $G$, treating character polynomials, Gelfand-Kirillov dimensions and Goldie rank polynomials. Our definition of the representations of Weyl groups (or Hecke algebras) is slightly different from Zuckerman's. It requires only the informations of the structure of invariant eigendistributions.

Moreover, in Part II, we study the representations of the group $U(p, 1)$ ($p \geqq 2$) and provide many useful examples. The author could not find the classification of irreducible admissible representations of $U(p, 1)$ with detailed proofs in any available publication (c.f. [14, 26, 27], and see also [45]). Therefore we give the classification with proofs in Part II. We also give the irreducible components of principal series representations and explicit form of irreducible characters. However, many results are dut to T. Hirai ([14]) and the author could not achieve the results in Part II without his suggestions.

Let us explain each section of this article. In Part I, we study general theory for semisimple Lie groups. § 1 is for preliminaries and notations involving the structure of invariant eigendistributions. A survey of the author's works on representations of Weyl groups and their Hecke algebras on virtual character modules is given in the former part of § 2. In the latter part of § 2, we treat complex semisimple Lie groups and get fairly natural results. In § 3, we define character polynomials and Gelfand-Kirillov dimensions for virtual characters (or, equivalently, constant coefficient invariant eigendistributions). After that, we study the connection between Weyl group (or Hecke algebra) representations and character polynomials and Gelfand-Kirillov dimensions.

In Part II, we treat only the group $U(p ,1)$. In §§ 1–4, we classify the irreducible admissible representations of $U(p, 1)$ and get their characters. We also study the structure of principal series representations in § 3. In § 5, we write down explicitly the acton of Weyl groups and their Hecke algebras on irreducible characters. Using the results of § 5 and Part I, we calculate Gelfand-Kirillov dimensions, $\tau$-invariants and character polynomials for $U(p, 1)$ in § 6.

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## Part I

### §1. Preliminaries

Let $G$ be a connected semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. In this paper, we always denote the Lie algebra of a Lie group $H$ by corresponding German small letter $\mathfrak{h}$, and its complexification by $\mathfrak{h}_C$. We denote the enveloping algebra of $\mathfrak{h}_C$ by $U(\mathfrak{h}_C)$.

**Definition 1.1.** Let $G$ be as above. We call $G$ *acceptable* if there exists a complex Lie group $G_C$ with Lie algebra $\mathfrak{g}_C$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism $j$ of $G$ into $G_C$. (2) For a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$, let $\rho$ be half the sum of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\xi_\rho = \exp \rho$ is a well-defined character of $H_C = \exp \mathfrak{h}_C$ into $C^\times$.

We assume $G$ acceptable in Part I and fix a complex Lie group $G_C$ above.

**1.1. Cartan subgroups.** Let $\mathrm{Car}(G)$ be all the conjugacy classes of Cartan subgroups of $G$ under inner automorphisms of $G$. By $\mathrm{Car}^0(G)$, we denote all the conjugacy classes of connected components of Cartan subgroups of $G$. If $H$ is a Cartan subgroup (or its connected component), $[H]$ denotes its conjugacy class in $\mathrm{Car}(G)$ (respectively $\mathrm{Car}^0(G)$).

Choose a Cartan subgroup $H$ of $G$. Let $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ be the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$ and $W = W(\mathfrak{g}_C, \mathfrak{h}_C)$ be its Weyl group. We define real roots $\Delta_R$ and imaginary roots $\Delta_I$ as follows.

$$\Delta_R = \{\alpha \in \Delta \mid \alpha \text{ takes real values on } \mathfrak{h}\},$$
$$\Delta_I = \{\alpha \in \Delta \mid \alpha \text{ takes purely imaginary values on } \mathfrak{h}\}.$$

For $\alpha \in \Delta_R$, let $H_\alpha$ be the element of $\mathfrak{h}_C$ such that $\alpha(X) = B(H_\alpha, X)$ holds for any $X \in \mathfrak{h}$, where $B(\,,\,)$ denotes the Killing form on $\mathfrak{g}_C$. Take root vectors $X_{\pm\alpha}$ from $\mathfrak{g}_C$ in such a way that $[X_\alpha, X_{-\alpha}]=H_\alpha$, and we put $H'_\alpha=(2/|\alpha|^2)H_\alpha$, $X'_{\pm\alpha}=(\sqrt{2}/|\alpha|)X_{\pm\alpha}$. Let $\nu=\nu_\alpha$ be the automorphism of $\mathfrak{g}_C$ defined by

$$\nu=\nu_\alpha=\exp\{-\tfrac{1}{4}\sqrt{-1}\pi\ \mathrm{ad}\,(X'_\alpha+X'_{-\alpha})\},$$

so-called *Cayley transform* with respect to $\alpha$. Put $\mathfrak{b}=\nu(\mathfrak{h}_C)\cap\mathfrak{g}$. Then $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$ not conjugate to $\mathfrak{h}$ under any automorphism of $\mathfrak{g}$ and $\beta=\nu(\alpha)\in\Delta_I(\mathfrak{g}_C, \mathfrak{b}_C)$. This relation between $(\mathfrak{h}, \alpha)$ and $(\mathfrak{b}, \beta)$ is denoted by $(\mathfrak{h}, \alpha)\to(\mathfrak{b}, \beta)$ or simply by $\mathfrak{h}\to\mathfrak{b}$. We introduce an order $<$ in $\mathrm{Car}(G)$ by defining $[H]<[B]$ when $\mathfrak{h}\to\mathfrak{b}$ for an appropriate choice of a representative $B$ of the class $[B]$, and extend it transitively. In this order, maximally split Cartan subgroups form the smallest class and Cartan subgroups with maximal troidal part form the largest class. There are some examples of this ordering in [17, § 3.5].

We denote by $J$ a maximally split Cartan subgroup of $G$. Fix an order on $\Delta=\Delta(\mathfrak{g}_C, \mathfrak{j}_C)$ and write $\Delta^+$ for the set of positive roots with respect to this order. We choose positive systems $\{\Delta^+(\mathfrak{g}, \mathfrak{h}_C)\,|\,[H]\in\mathrm{Car}(G)\}$ consistent with $\Delta^+(\mathfrak{g}_C, \mathfrak{j}_C)$ by means of Cayley transforms. Similarly, $\{W=W(\mathfrak{g}_C, \mathfrak{h}_C)\,|\,[H]\in\mathrm{Car}(G)\}$ are all identified by Cayley transforms. Therefore we just write $W$ instead of $W(\mathfrak{g}_C, \mathfrak{h}_C)$ for any Cartan subalgebra $\mathfrak{h}_C$.

**1.2. Some functions on a Cartan subgroup.** Fix a Cartan subgroup $H$ of $G$. We call $W(G; H)=N_G(H)/Z_G(H)$ a Weyl group of $(G, H)$, where $N_G(H)$ is the normalizer of $H$ in $G$ and $Z_G(H)$ is the centralizer (for a subset $D$ in $G$ and a subgroup $B$ of $G$, we write $W(B; D)=N_B(D)/Z_B(D)$ in general).

Take $\alpha\in\Delta(\mathfrak{g}, \mathfrak{h}_C)$ and a non-zero root vector $X_\alpha$ for $\alpha$. We define an analytic function $\xi_\alpha$ on $H$ by $\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha$ for $h\in H$. We also define some functions which take values in $\{0, \pm1\}$. Put

$$\varepsilon_R(h)=\prod_{\alpha\in\Delta_R^+}\mathrm{sgn}(1-\xi_\alpha(h)^{-1})\qquad(h\in H),$$

where $\Delta_R^+=\Delta^+\cap\Delta_R$ and we put $\varepsilon_R(h)=0$ if $\xi_\alpha(h)=1$ for some $\alpha\in\Delta_R^+$. For $s\in W(G; H)$, we put

$$N(s)=\#\{\alpha\in\Delta_I^+=\Delta^+\cap\Delta_I\,|\,s^{-1}\alpha<0\},$$
$$R(s)=\{\alpha\in\Delta_R^+\,|\,s^{-1}\alpha<0\},$$

where $\#A$ is the order of the finite set $A$. Define a locally constant function $\varepsilon(h; s)$ on $H$ by

$$\varepsilon(h; s)=(-1)^{N(s)} \prod_{\alpha \in R(s)} \operatorname{sgn}(\xi_{s^{-1}\alpha}(h)) \qquad (h \in H,\ s \in W(G; H))$$

We write $\varepsilon(s)=\det(s)$. Then it holds that $\varepsilon(s)\varepsilon_R(sh)=\varepsilon_R(h)\varepsilon(h; s)$ (see Lemma 2.2 in [17, p. 36]).

For $\lambda \in \mathfrak{h}_C^*$, we define a family $\mathfrak{B}(H; \lambda)$ of analytic functions on $H$ by

$\mathfrak{B}(H; \lambda)=\{\zeta \mid \zeta$ is analytic on $H$, satisfying the following conditions (1) and (2)$\}$,

(1) $\zeta$ is an eigenfunction of $S(\mathfrak{h}_C)^W$ with eigenvalue $\lambda$, where $S(\mathfrak{h}_C)$ is the symmetric algebra of $\mathfrak{h}_C$ identified with differential operators of constant coefficients on $H$ and $S(\mathfrak{h}_C)^W$ consists of $W$-invariant elements of $S(\mathfrak{h}_C)$.

(2) $\zeta$ is $\varepsilon$-symmetric under $W(G; H)$, i.e.,

$$\zeta(wh)=\varepsilon(h; w)\zeta(h) \qquad (h \in H,\ w \in W(G; H)).$$

Each element $\zeta \in \mathfrak{B}(H; \lambda)$ can be written as

$$\zeta(h \exp x)=\sum_{s \in W} a_s(h; x) \exp s\lambda(x) \qquad (x \in \mathfrak{h},\ h \in H),$$

where $a_s(h; x)$ is a polynomial function in $x$ depending on $h$. If $a_s(h; x)$ can be taken as a constant with respect to $x$ for any $h \in H$ and $s \in W$, we say $\zeta$ has constant coefficients. Let $\mathfrak{B}'(H; \lambda)$ be the set of $\zeta \in \mathfrak{B}(H; \lambda)$ which has constant coefficients.

We want to describe the space $\mathfrak{B}'(H; \lambda)$ in the following. Let $\{H_i \mid 0 \leqq i \leqq l\}$ be a complete system of representatives of conjugacy classes of connected components of $H$ under the action of $G$. We denote this set by $\mathrm{Car}_H^0(G)$. Let $H_0$ be the neutral component of $H$ and choose $\{a_i \mid 0 \leqq i \leqq l\}$ so that $a_i \in H_i$ $(1 \leqq i \leqq l)$ and $a_0=e$. Let $L=\mathrm{Ker}(\exp\colon \mathfrak{h} \to H_0)$ and put $W_{\tilde{H}}(\lambda)=\{w \in W \mid \langle w\lambda, L\rangle \subset 2\pi\sqrt{-1}\,\boldsymbol{Z}\}$, where, $\langle\ ,\ \rangle$ is the natural pairing of $\mathfrak{h}_C^* \times \mathfrak{h}_C$. For $t \in W_{\tilde{H}}(\lambda)$ and $i$ $(0 \leqq i \leqq l)$, we define an analytic function $\zeta(a_i, t\lambda; h)$ on $H$ as follows. For $h \in H_i$, we put

$$\zeta(a_i, t\lambda; h)=\sum_{s \in W(G; H_i)} \varepsilon(a_i; s)\xi_{t\lambda}(a_i^{-1}(sh)),$$

where $\xi_\mu(\mu \in \mathfrak{h}_C^*)$ is an analytic function on $H_0$ defined by $\xi_\mu(\exp x)=\exp \mu(x)$ $(x \in \mathfrak{h})$. On $W(G; H)$-orbit of $H_i$, we put $\zeta(a_i, t\lambda; h)$ as

$$\zeta(a_i, t\lambda; wh)=\varepsilon(h; w)\zeta(a_i, t\lambda; h) \qquad (h \in H_i,\ w \in W(G; H)),$$

and for $h \in H$ outside of $W(G; H)$-orbit of $H_i$, we put $\zeta(a_i, t\lambda; h)=0$.

**Lemma 1.2.** (1) *The function* $\zeta(a_i, t\lambda; \cdot)$ *belongs to* $\mathfrak{B}'(H; \lambda)$.

*Moreover*, $\{\zeta(a_i, t\lambda; \cdot) \mid 0 \leq i \leq l,\ t \in W_{\tilde{H}}(\lambda)\}$ *generates* $\mathfrak{B}'(H; \lambda)$ *as a vector space.*

(2) *If* $\lambda$ *is regular, i.e.,* $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ *is trivial, then* $\mathfrak{B}'(H; \lambda) = \mathfrak{B}(H; \lambda)$.

*Proof.* (1) is a consequence of easy calculations. (2) is well-known. Q.E.D.

**1.3. Invariant eigendistributions.** Let $\mathfrak{Z}$ be the centre of $U(\mathfrak{g}_C)$ and fix a Cartan subgroup $H$. Then by Harish-Chandra homomorphism, $\mathfrak{Z}$ is isomorphic to $S(\mathfrak{h}_C)^W$ as an algebra. Since $\mathrm{Hom}_{\mathrm{alg}}(S(\mathfrak{h}_C)^W, \boldsymbol{C})$ is canonically isomorphic to the set $\mathfrak{h}_C^*/W$, there is a map of $\mathfrak{h}_C^*$ to $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{Z}, \boldsymbol{C})$:

$$\mathfrak{h}_C^* \xrightarrow{\text{projection}} \mathfrak{h}_C^*/W \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{Z}, \boldsymbol{C})$$

Take $\lambda \in \mathfrak{h}_C^*$ and we denote by $\chi_\lambda$ the corresponding algebra homomorphism of $\mathfrak{Z}$ to $\boldsymbol{C}$.

**Definition 1.3.** A distribution $\Theta$ on $G$ is called *invariant eigendistribution* (IED) with infinitesimal character $\lambda$ if it satisfies the following two conditions.

(1) $\Theta$ is invariant under the action of $G$.

(2) $\Theta$ is an eigendistribution of $\mathfrak{Z}$ with eigenvalue $\lambda$, i.e., $z\Theta = \chi_\lambda(z)\Theta$ for any $z \in \mathfrak{Z}$.

By Harish-Chandra's famous theorem [11, Th. 2], $\Theta$ is essentially equal to a locally summable function on $G$. Moreover it is analytic on $G'$, open dense subset of regular elements in $G$. Since $G' \subset \bigcup_{[H] \in \mathrm{Car}(G), g \in G} gHg^{-1}$ and $\Theta$ is invariant under $G$, $\Theta$ is completely determined by the values on a complete system of representatives $\{H \mid [H] \in \mathrm{Car}(G)\}$.

**Definition 1.4.** For an IED $\Theta$, we put

$$\mathrm{Supp}(\Theta) = \{[H] \in \mathrm{Car}(G) \mid \Theta \text{ is not identically zero on } H\},$$
$$\mathrm{Hght}(\Theta) = \{[H] \in \mathrm{Supp}(\Theta) \mid [H] \text{ is maximal in } \mathrm{Supp}(\Theta)\}.$$

We call an element $[H]$ in $\mathrm{Hght}(\Theta)$ a *height* of $\Theta$. Remark that height of $\Theta$ is not unique in general. If $\Theta$ has a unique height $[H]$, we call $\Theta$ *extremal* of height $[H]$.

**Theorem 1.5** (T. Hirai [17, 18]). (1) *If* $[H] \in \mathrm{Hght}(\Theta)$, *then* $\varepsilon_R D(\Theta|_H)$ *is contained in* $\mathfrak{B}(H; \lambda)$, *where* $\lambda$ *is an infinitesimal character of* $\Theta$ *and* $D$ *is an analytic function on* $H$ *given by*

$$D(h)=\xi_{\rho}(h)\prod_{\alpha\in\Delta^{+}}(1-\xi_{\alpha}(h)^{-1}) \qquad (h\in H).$$

(2) *For any* $\zeta\in\mathfrak{B}(H;\lambda)$, *we can construct an extremal* IED $\boldsymbol{T}\zeta$ *which has the following two properties.* (i) $\boldsymbol{T}\zeta$ *has the unique height* $[H]$. (ii) *On* $H$, $\varepsilon_{R}D(\boldsymbol{T}\zeta)|_{H}$ *is equal to* $\zeta$. *Moreover* $\boldsymbol{T}$ *is a linear map of* $\mathfrak{B}(H;\lambda)$ *into the space of* IEDs.

(3) *Any* IED $\Theta$ *with infinitesimal character* $\lambda$ *can be written as a linear combination of extremal* IEDs *of the form* $\{\boldsymbol{T}\zeta\,|\,\zeta\in\mathfrak{B}(H;\lambda),\,[H]\in\mathrm{Car}(G)\}$.

We call the map $\boldsymbol{T}$ Hirai's method $\boldsymbol{T}$. There is some confusion of terminology, but we think it is better to keep to the initial meaning of T. Hirai.

Denote by $\mathfrak{A}(\lambda)$ the space of IEDs with infinitesimal character $\lambda$ and put $\mathfrak{A}_{H}(\lambda)=\boldsymbol{T}(\mathfrak{B}(H;\lambda))$. Then from Theorem 1.5, it is clear that

$$\mathfrak{A}(\lambda)=\sum_{[H]\in\mathrm{Car}(G)}^{\oplus}\mathfrak{A}_{H}(\lambda).$$

Now let us investigate more detailed structure of $\mathfrak{A}(\lambda)$. Take $\Theta\in\mathfrak{A}(\lambda)$. Then by Harish-Chandra, restriction of $\Theta$ to $H$ ($[H]\in\mathrm{Supp}(\Theta)$) has the form

$$D\Theta(h\exp x)=\sum_{s\in W}a_{s}(h;x)\exp s\lambda(x) \qquad (x\in\mathfrak{h}),$$

where $h\in H$ is regular and $x$ is small. The coefficient $a_{s}(h;x)$ is a polynomial function in $x$ generally. We say $\Theta$ *has constant coefficients* (or is a *constant coefficient IED*) if $a_{s}(h;x)$'s can be taken as a constant in $x$ for any $s\in W$ and $h\in H$ regular. We denote by $\mathfrak{A}'(\lambda)$ the space of all the constant coefficient IEDs and put $\mathfrak{A}'_{H}(\lambda)=\mathfrak{A}'(\lambda)\cap\mathfrak{A}_{H}(\lambda)$. Then it holds that $\mathfrak{A}'_{H}(\lambda)=\boldsymbol{T}(\mathfrak{B}'(H;\lambda))$ and

$$\mathfrak{A}'(\lambda)=\sum_{[H]\in\mathrm{Car}(G)}^{\oplus}\mathfrak{A}'_{H}(\lambda).$$

**1.4. Virtual character modules.** Let $(\pi,\mathfrak{H})$ be an irreducible quasi-simple representation of $G$ on a Hilbert space $\mathfrak{H}$. We can define an IED $\Theta_{\pi}$ on $G$ called an (*irreducible*) *character* of $\pi$ as follows. Let $C_{0}^{\infty}(G)$ be the space of $C^{\infty}$-functions on $G$ with compact supports. For each $f\in C_{0}^{\infty}(G)$, we put

$$\pi(f)=\int_{G}f(g)\pi(g)dg,$$

where $dg$ is a Haar measure on $G$. Harish-Chandra proved that $\pi(f)$ is a trace class operator and that

$$\Theta_\pi(f)=\mathrm{Trace}\ \pi(f) \qquad (f \in C_0^\infty(G))$$

is a distribution on $G$. We denote all the irreducible quasi-simple representation of $G$ with infinitesimal character $\chi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{Z}, \boldsymbol{C})$ by $M(\chi)$. Here an infinitesimal character of an irreducible representation $(\pi, \mathfrak{H})$ means that of $\Theta_\pi$ or of the corresponding algebra homomorphism of $\mathfrak{Z}$ to $\boldsymbol{C}$. Let $V(\chi)$ be a vector space generated by $\{\Theta_\pi \mid \pi \in M(\chi)\}$. Then we call an element of $V(\chi)$ a virtual character. Of course $V(\chi_\lambda)$ is contained in $\mathfrak{A}(\lambda)$. Moreover we have

**Theorem 1.6** ([35]). *The space of virtual characters with infinitesimal character $\lambda$ is equal to the space of constant coefficient* IEDs *with infinitesimal character $\lambda$: $V(\chi_\lambda)=\mathfrak{A}'(\lambda)$.*

*Proof.* The fact that $V(\chi_\lambda)$ is contained in $\mathfrak{A}'(\lambda)$ is proved by Fomin and Shapovalov [8]. So it is sufficient to see that $\mathfrak{A}'_H(\lambda)$ is contained in $V(\chi_\lambda)$. Using explicit formulae of IEDs obtained by T. Hirai, one can prove that any element of $\mathfrak{A}'_H(\lambda)$ can be expressed as a linear combination of characters of principal series representations induced from cuspidal parabolic subgroups. For detailed proof, see [35]. Q.E.D.

## § 2. Representations of Weyl groups and their shrinked Hecke algebras

In subsections 2.1 and 2.2, we review the results of [34] and [37]. Fix a Cartan subgroup $H$ and define $L$ and $W_H^\sim(\lambda)$ for $\lambda \in \mathfrak{h}_C^*$ as in § 1.2. Put

$$L_\lambda=\sum_{w \in W_H^\sim(\lambda)} w^{-1}L,$$

and $W_H(\lambda)=\{w \in W \mid wL_\lambda=L_\lambda\}$. Then $W_H(\lambda)$ is the largest subgroup of $W$ which leaves $W_H^\sim(\lambda)$ invariant under the right multiplication ([34, Prop. 1.5]). We also define an integral Weyl group $W(\lambda)$ with respect to $\lambda$, dropping the subscript $H$, as follows:

$$W(\lambda)=\{w \in W \mid w\lambda-\lambda \text{ is a sum of roots}\}.$$

Then $W(\lambda)$ is a Weyl group of the root system $\Delta(\lambda)=\{\alpha \in \Delta \mid \langle\lambda, \alpha^\vee\rangle \in \boldsymbol{Z}\}$, where $\alpha^\vee=2\alpha/\langle\alpha, \alpha\rangle$.

**Lemma 2.1.** *For any Cartan subgroup $H$, $W(\lambda)$ is a subgroup of $W_H(\lambda)$.*

*Proof.* Take $s \in W(\lambda)$ and we show that $W(\lambda)$ leaves $W_{\tilde{H}}(\lambda)$ invariant under the right multiplication. Then we have $W(\lambda) \subset W_H(\lambda)$. Take $w \in W_{\tilde{H}}(\lambda)$. Then exp $w\lambda$ is a well-defined character on $H_0$. We write

$$ws\lambda = w((s\lambda - \lambda) + \lambda) = w(s\lambda - \lambda) + w\lambda.$$

Since $s\lambda - \lambda$ is a sum of roots, we know that exp $w(s\lambda - \lambda)$ is well-defined on $H_0$. So exp $ws\lambda =$ exp $w(s\lambda - \lambda) \cdot$ exp $w\lambda$ is well-defined character on $H_0$, which proves $ws \in W_{\tilde{H}}(\lambda)$. Q.E.D.

Consider the following two assumptions on $G$. For $H$, put $\mathrm{Car}^0_H(G) = \{H_i \mid 0 \leqq i \leqq l\}$ as in § 1.2. Take $a_i \in H_i$ for each $1 \leqq i \leqq l$ and put $a_0 = e \in H_0$.

**Assumption 2.2.** For each Cartan subgroup $H$ of $G$, there exists $\{a_i \mid 0 \leqq i \leqq l\}$ such that

$$\xi_{t\lambda}(a_i^{-1}(sa_i)) = 1 \qquad \text{for any } t \in W_{\tilde{H}}(\lambda) \text{ and } s \in W(G; H_i).$$

**Assumption 2.3.** For each Cartan subgroup $H$ of $G$, there exists $\{a_i \mid 0 \leqq i \leqq l\}$ such that $sa_i = a_i$ for any $s \in W(G; H_i)$.

Clearly Assumption 2.3 is stronger than Assumption 2.2. We think these assumptions are fairly natural ones (see Lemma 1.6 in [34] and Remark 2.2 in [37]).

In this section, we only assume Assumption 2.2, while in the part of the next § 3, we assume Assumption 2.3.

**2.1. Representations of Weyl groups.** In this subsection, we treat the case where $\lambda$ is regular. So we may assume $\lambda$ is dominant regular without loss of generality. Then it holds that

$$V(\chi_\lambda) = \mathfrak{A}'(\lambda) = \mathfrak{A}(\lambda) = \sum_{[H] \in \mathrm{Car}(G)}^{\oplus} \mathfrak{A}_H(\lambda).$$

We also write $\mathfrak{A}_H(\lambda) = \boldsymbol{T}(\mathfrak{B}(H; \lambda))$ as $V_H(\chi)$, where $\chi = \chi_\lambda$ (more generally, if $\lambda$ is not necessarily regular, we put $V_H(\chi) = \mathfrak{A}'_H(\lambda) = \boldsymbol{T}(\mathfrak{B}'(H; \lambda))$).

We define an action $\tau$ of $W_H(\lambda)$ on $V_H(\chi)$ by the formula:

$$\tau(w)\boldsymbol{T}\zeta(a_i, t\lambda; \cdot) = \boldsymbol{T}\zeta(a_i, tw^{-1}\lambda; \cdot) \qquad (0 \leqq i \leqq l,\ w \in W_H(\lambda),\ t \in W_{\tilde{H}}(\lambda)).$$

Then this autually gives a well-defined representation $(\tau, V_H(\chi))$ of $W_H(\lambda)$, since $\{\boldsymbol{T}\zeta(a_i, t\lambda; \cdot)\}$ generates $V_H(\chi)$ (see [34]). Since $W(\lambda) \subset W_H(\lambda)$ for any Cartan subgroup $H$ of $G$, we get a representation of $W(\lambda)$ on the space $V(\chi)$ by summing up $(\tau, V_H(\chi))$'s over $[H] \in \mathrm{Car}(G)$. We again denote this representation of $W(\lambda)$ by $(\tau, V(\chi))$.

Let $\Gamma_i \subset W_H^{\sim}(\lambda)$ be a complete system of representatives of a double coset space $W(G; H_i)\backslash W_H^{\sim}(\lambda)/W_H(\lambda)$. Remark that $W(G; H_i)$ acts naturally on $W_H^{\sim}(\lambda)$ by left multiplication. For $\gamma \in \Gamma_i$, put

$$W(i, \gamma) = W_H(\lambda) \cap \gamma^{-1} W(G; H_i)\gamma,$$
$$\varepsilon(i, \gamma; w) = \varepsilon(a_i; \gamma w \gamma^{-1}) \qquad (w \in W(i, \gamma)).$$

Then $\varepsilon(i, \gamma; \cdot)$ is a character of the group $W(i, \gamma)$.

**Theorem 2.4** ([34, Th. 5.1]). *The representation $(\tau, V_H(\chi))$ of $W_H(\lambda)$ is decomposed into a direct sum of induced representations*:

$$(\tau, V_H(\chi)) = \sum_{0 \leqq i \leqq l}^{\oplus} \sum_{\gamma \in \Gamma_i}^{\oplus} \mathrm{Ind}(\varepsilon(i, \gamma; \cdot); W(i, \gamma) \uparrow W_H(\lambda)),$$

*where* $\mathrm{Ind}(\varepsilon; A \uparrow B)$ *is a representation of $B$ induced from a representation $\varepsilon$ of $A$.*

Note that we can get easily the decomposition of the representation $(\tau, V(\chi))$ of $W(\lambda)$ from Theorem 2.4.

**Corollary 2.5.** *If $W_H(\lambda) = W$ for any $[H] \in \mathrm{Car}(G)$, we have a representation $(\tau, V(\chi))$ of $W$ by summing up $\{(\tau, V_H(\chi) \mid [H] \in \mathrm{Car}(G)\}$. This representation is decomposed as*

$$(\tau, V(\chi)) = \sum_{[T] \in \mathrm{Car}(G)}^{\oplus} \mathrm{Ind}(\varepsilon(a_T; \cdot); W(G; T) \uparrow W),$$

*where $T = H_i$ and $a_T = a_i$ for some $H$ and $H_i \subset H$.*

**2.2. Representations of shrinked Hecke algebras.** Let $\lambda \in \mathfrak{h}_C^*$ be singular dominant in this subsection, and put $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$. Then $W_\lambda$ is a subgroup of $W_H(\lambda)$ and we can define a Hecke algebra $\mathscr{H}_\lambda = \mathscr{H}(W_H(\lambda), W_\lambda)$ over the field of complex numbers $\boldsymbol{C}$[20, § 1]. We call this $\mathscr{H}_\lambda$ the (*shrinked*) *Hecke algebra with infinitesimal character* $\lambda$ in this paper. Let $\boldsymbol{C}[W]$ be a group ring of $W$ over $\boldsymbol{C}$ and we consider $\boldsymbol{C}[W_H(\lambda)]$ and $\boldsymbol{C}[W_\lambda]$ as subalgebras of $\boldsymbol{C}[W]$. Put

$$e_\lambda = (\sharp W_\lambda)^{-1} \sum_{w \in W_\lambda} w \in \boldsymbol{C}[W_\lambda].$$

Then by Proposition 1.1 of [20], we have an isomorphism $\mathscr{H}_\lambda \simeq e_\lambda \boldsymbol{C}[W_H(\lambda)] e_\lambda$. So, from now on, we consider $\mathscr{H}_\lambda$ as a subalgebra of $\boldsymbol{C}[W_H(\lambda)]$ (or $\boldsymbol{C}[W]$).

Take regular dominant $\lambda_0 \in \mathfrak{h}_C^*$ such that $\lambda_0 - \lambda$ is a sum of positive roots. Let $\Phi = \Phi_{\lambda_0}^{\lambda}$ and $\Psi = \Psi_{\lambda}^{\lambda_0}$ be Zuckerman's *translation functors* on the category of Harish-Chandra modules (see [44]), i.e.,

$$\Phi = \mathrm{Proj}(\lambda_0) \circ (F_\mu \otimes (\cdot)) \circ \mathrm{Proj}(\lambda),$$
$$\Psi = \mathrm{Proj}(\lambda) \circ (F_\mu^* \otimes (\cdot)) \circ \mathrm{Proj}(\lambda_0),$$

where $\mathrm{Proj}(\nu)$ is a projection to the component with infinitesimal character $\nu$, and $F_\mu$ is the irreducible finite dimensional representation of $G$ with highest weight $\mu$ and $F_\mu^*$ is its contragredient. Since $\Phi$ and $\Psi$ are exact functors, they naturally induce linear maps between the virtual character modules $V(\lambda)$ and $V(\lambda_0)$. We again denote these linear maps by $\Phi$ and $\Psi$.

We can take $\lambda_0$ so that $\xi_{t(\lambda_0-\lambda)}(a_i)=1$ for any $t \in W_{\tilde{H}}(\lambda)$ and $0 \leqq i \leqq l$ (see Lemma B.4 in [37]). Remark that $W_{\tilde{H}}(\lambda_0)=W_{\tilde{H}}(\lambda)$ and $W_H(\lambda_0)=W_H(\lambda)$. The following proposition holds.

**Proposition 2.6** (Zuckerman [28, Appendix]). (1) *$\mathrm{Ker}(\Psi\colon V(\chi_0)\to V(\chi))$ is equal to $\sum^{\oplus}_{[H]\in \mathrm{Car}(G)} \mathrm{Ker}(\tau(e_\lambda)\colon V_H(\chi_0)\to V_H(\chi_0))$, where $\chi_0=\chi_{\lambda_0}$ and the representation $\tau$ of $W_H(\lambda)$ is extended to a representation of the group ring $\boldsymbol{C}[W_H(\lambda)]$.*

(2) *For any $\Theta \in V_H(\chi_0)$, we have*

$$\Phi \circ \Psi(\Theta) = \sum_{s\in W_\lambda} \tau(s)\Theta = (\sharp W_\lambda)\tau(e_\lambda)\Theta.$$

For $e_\lambda w e_\lambda \in e_\lambda \boldsymbol{C}[W_H(\lambda)]e_\lambda = \mathscr{H}_\lambda$, we define an operator $\sigma(e_\lambda w e_\lambda)$ on $V_H(\chi)$ as

$$\sigma(e_\lambda w e_\lambda) = (\sharp W_\lambda)^{-1}(\Psi \circ \tau(e_\lambda w e_\lambda) \circ \Phi)(\Theta) \qquad (\Theta \in V_H(\chi)).$$

Then $\sigma$ turns out to be a representation of the Hecke algebra $\mathscr{H}_\lambda$ (see [37, Th. 5. 6]). Moreover we can define a representation $(\sigma, V(\chi))$ of $\mathscr{H}(W(\lambda), W_\lambda)$ similarly as in § 2.1.

Before describing the structure of the representation $(\sigma, V_H(\chi))$ of $\mathscr{H}_\lambda$, we define some notion about representations of Hecke algebras. Let $A$ be a finite group and $B \subset A$ a subgroup of $A$. Then we can form a Hecke algebra $\mathscr{H}(A, B)$ over $\boldsymbol{C}$ as in [20]. Let $(\tau, V)$ be a representation of $A$ on a finite dimensional vector space $V$. Put $e_B = (\sharp B)^{-1}\sum_{b\in B} b \in \boldsymbol{C}[A]$, then $\mathscr{H}(A, B) \simeq e_B \boldsymbol{C}[A] e_B$ as before. Therefore we identify $\mathscr{H}(A, B)$ with the subalgebra $e_B \boldsymbol{C}[A] e_B$ of $\boldsymbol{C}[A]$. We define a representation $(\sigma, V_1)$ of $\mathscr{H}(A, B)$ as follows. Put $V_1 = \mathrm{Ker}(\tau(e_B) - \mathrm{id}_V)$, where $\mathrm{id}_V$ is the identity operator on $V$. Then we define $\sigma(e_B a e_B)$ $(a \in A)$ by

$$\sigma(e_B a e_B)v = \tau(e_B a e_B)v \qquad \text{for } v \in V_1.$$

Then $(\sigma, V_1)$ becomes a well-defined representation of $\mathscr{H}(A, B)$. Moreover, if $(\tau, V)$ is irreducible then $(\sigma, V_1)$ is irreducible or zero. We denote this representation $(\sigma, V_1)$ by $\mathrm{Red}((\tau, V); A\downarrow B)$ or simply by $\mathrm{Red}(\tau; A\downarrow B)$.

If $\varepsilon$ is a representation of a subgroup $C \subset A$, we write

$$\mathrm{RI}(\varepsilon;\, C\uparrow A\downarrow B) = \mathrm{Red}\ (\mathrm{Ind}(\varepsilon;\, C\uparrow A);\, A\downarrow B).$$

Using Theorem 2.4 and Proposition 2.6, we have the following theorem.

**Theorem 2.7** ([37, Cor. 4.3]). *The representation $(\sigma, V_H(\chi))$ of $\mathcal{H}_\lambda$ is decomposed as follows*:

$$(\sigma, V_H(\chi)) = \sum_{0\leq i\leq l}^{\oplus} \sum_{\gamma\in\Gamma_i}^{\oplus} \mathrm{RI}(\varepsilon(i, \gamma;\, \cdot);\, W(i, \gamma)\uparrow W_H(\lambda)\downarrow W_\lambda),$$

*where $\Gamma_i$, $\varepsilon(i, \gamma;\, \cdot)$ and $W(i, \gamma)$ are the same as in Theorem* 2.4.

**Corollary 2.8.** *If $W_H(\lambda) = W$ for any $[H] \in \mathrm{Car}(G)$, we have a representation $(\sigma, V(\chi))$ of $\mathcal{H}_\lambda = \mathcal{H}(W, W_\lambda)$ by summing up $\{(\sigma, V_H(\chi)) \mid [H] \in \mathrm{Car}(G)\}$. This representation is decomposed as*

$$(\sigma, V(\chi)) = \sum_{[T]\in\mathrm{Car}^0(G)}^{\oplus} \mathrm{RI}(\varepsilon(a_T;\, \cdot);\, W(G;\, T)\uparrow W\downarrow W_\lambda).$$

**2.3. The case of complex groups.** Here we treat the case where $G$ is a complex semisimple Lie group. There are two good properties in dealing with complex Lie groups. (1) There is only one conjugacy class of Cartan subgroups and moreover all the Cartan subgroups are connected. (2) "Real" Weyl group is contained in "complex" Weyl group as the diagonal subgroup. We use these two properties to calculate out the representations $\tau$ and $\sigma$ explicitly. The case where $\lambda$ is integral is reported in [36].

Let $G$ be a connected and simply connected complex semisimple Lie group. Then $\mathfrak{g}$ has a root space decomposition with respect to a Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \boldsymbol{C} X_\alpha, \qquad \mathfrak{h} = \sum_{\alpha\in\Delta} \boldsymbol{C} H_\alpha,$$

where $\{H_\alpha, X_\alpha \mid \alpha \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$ over $\boldsymbol{Z}$([7, Th. 1]). We denote by $i$ an element in $\boldsymbol{C}$ such that $i^2 = -1$. Let $\mathfrak{h}_{\boldsymbol{R}}$ be a real subspace generated by $\{H_\alpha \mid \alpha \in \Delta\}$. We want to complexify $\mathfrak{g}$. We may call this the "second" complexification of $\mathfrak{g}$. We denote by $j$ a fixed square root in $\boldsymbol{C}$ of the second complexification. So, we have $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{g} + j\mathfrak{g}$. Then $\mathfrak{g}_{\boldsymbol{C}}$ is decomposed as $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $\mathfrak{g}_1 \simeq \mathfrak{g}_2 \simeq \mathfrak{g}$ as complex Lie algebras. Here $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are given explicitly as follows:

$$\mathfrak{g}_1 = (1+ij)\boldsymbol{C}\mathfrak{h}_{\boldsymbol{R}} + \sum_{\alpha\in\Delta} (1+ij)\boldsymbol{C} X_\alpha,$$

$$\mathfrak{g}_2=(1-ij)C\mathfrak{h}_R+\sum_{\alpha\in\Delta}(1-ij)CX_\alpha,$$

where $C=R+Ri$ (we usually denote this complex field by $C$ unless otherwise stated, while the complex field used in the second complexification is denoted by $C=(R+Rj)$). We put

$$X_\alpha^{(1)}=\tfrac{1}{2}(1+ij)X_\alpha, \qquad H_\alpha^{(1)}=\tfrac{1}{2}(1+ij)H_\alpha,$$
$$X_\alpha^{(2)}=\tfrac{1}{2}(1-ij)X_\alpha, \qquad H_\alpha^{(2)}=\tfrac{1}{2}(1-ij)H_\alpha.$$

Let $\mathfrak{g}_R^{(k)}=\langle H_\alpha^{(k)}, X_\alpha^{(k)} \mid \alpha\in\Delta\rangle/R$ be a real form of $\mathfrak{g}_k$ $(k=1, 2)$. Cartan subalgebras $\mathfrak{h}^{(1)}$, $\mathfrak{h}^{(2)}$ and their real forms $\mathfrak{h}_R^{(1)}$, $\mathfrak{h}_R^{(2)}$ are defined similarly. The explicit isomorphism between $\mathfrak{g}_C$ and $\mathfrak{g}_1\oplus\mathfrak{g}_2$ is given as:

$$(a+bj)X_\alpha\to(a-bi)X_\alpha^{(1)}+(a+bi)X_\alpha^{(2)},$$
$$(a+bj)H_\alpha\to(a-bi)H_\alpha^{(1)}+(a+bi)H_\alpha^{(2)},$$

where $a, b\in R$. For $x\in\mathfrak{g}_C$, we write $x=x^{(1)}+x^{(2)}\in\mathfrak{g}_1\oplus\mathfrak{g}_2$.

Let $W=W(\mathfrak{g}, \mathfrak{h})$ be a Weyl group. Then the complex Weyl group $W(\mathfrak{g}_C, \mathfrak{h})$ is a direct product of two copies of $W$: $W(\mathfrak{g}_C, \mathfrak{h}_C)\simeq W\times W$, and $W(\mathfrak{g}, \mathfrak{h})$ is imbedded into $W\times W$ as a diagonal subgroup $\Delta W$:

$$W(\mathfrak{g}, \mathfrak{h})\to\Delta W=\{(w, w)\in W\times W \mid w\in W\}\subset W\times W.$$

Take $\lambda\in\mathfrak{h}_C^*$ and put $\lambda|_{\mathfrak{h}_R}=\mu$ and $\lambda|_{i\mathfrak{h}_R}=\nu$ and extend them to $\mathfrak{h}^{(1)}\oplus\mathfrak{h}^{(2)}$ by $C=(R+Rj)$-linearity. Let us calculate $W_{\tilde{H}}(\lambda)$. We can assume $(e, e)\in W_{\tilde{H}}(\lambda)$ without loss of generality.

**Lemma 2.9.** *Suppose that* $(e, e)\in W_{\tilde{H}}(\lambda)$.

(1) *The number* $\sqrt{-1}\,\lambda(iH_\alpha)=\sqrt{-1}\,\nu(iH_\alpha)$ *is an integer for any* $\alpha\in\Delta$.

(2) *If* $\sqrt{-1}\,\lambda(iH_\alpha)$ *is an even integer for any* $\alpha\in\Delta$, *we have*

$$W_{\tilde{H}}(\lambda)=\{(s, t)\in W\times W \mid t^{-1}s\in W^0(\tfrac{1}{2}\mu)\},$$

*where* $W^0(\kappa)=\{w\in W \mid \langle w\kappa-\kappa, \alpha\rangle\in Z \text{ for } \forall\alpha\in\Delta(\mathfrak{g}, \mathfrak{h})\}$.

*Proof.* Since

$$\lambda(H_\alpha^{(1)})=\tfrac{1}{2}\{\lambda(H_\alpha)+\sqrt{-1}\,\lambda(iH_\alpha)\}=\tfrac{1}{2}\{\mu(H_\alpha)+\sqrt{-1}\,\nu(iH_\alpha)\},$$
$$\lambda(H_\alpha^{(2)})=\tfrac{1}{2}\{\lambda(H_\alpha)-\sqrt{-1}\,\lambda(iH_\alpha)\}=\tfrac{1}{2}\{\mu(H_\alpha)-\sqrt{-1}\,\nu(iH_\alpha)\},$$

we have for $(s, t)\in W\times W$,

$$(s, t)\lambda(H_\alpha^{(1)})=\tfrac{1}{2}\{s\mu(H_\alpha)+\sqrt{-1}\,s\nu(iH_\alpha)\},$$
$$(s, t)\lambda(H_\alpha^{(2)})=\tfrac{1}{2}\{t\mu(H_\alpha)-\sqrt{-1}\,t\nu(iH_\alpha)\}.$$

On the other hand, $\exp(s, t)\lambda$ is a well-defined character if and only if $\sqrt{-1}\,(s, t)\ \lambda(iH_\alpha) \in \boldsymbol{Z}$ for any $\alpha \in \Delta=\Delta(\mathfrak{g}, \mathfrak{h})$. Therefore the condition $(s, t) \in W_H^{\sim}(\lambda)$ is equivalent to

$$\begin{aligned}&\sqrt{-1}\,(s, t)\lambda(iH_\alpha) \in \boldsymbol{Z} \quad (\forall\alpha \in \Delta)\\ &\quad\Leftrightarrow \sqrt{-1}\,(s, t)\lambda(-jH_\alpha^{(1)}+jH_\alpha^{(2)}) \in \boldsymbol{Z} \quad (\forall\alpha \in \Delta)\\ &\quad\Leftrightarrow (s, t)\lambda(H_\alpha^{(1)})-(s, t)\lambda(H_\alpha^{(2)}) \in \boldsymbol{Z} \quad (\forall\alpha \in \Delta)\\ &\quad\Leftrightarrow \tfrac{1}{2}(s\mu-t\mu)(H_\alpha)+\tfrac{1}{2}\sqrt{-1}\,(s\nu+t\nu)(iH_\alpha) \in \boldsymbol{Z} \quad (\forall\alpha \in \Delta).\end{aligned}$$

From the assumption $(e, e) \in W_H^{\sim}(\lambda)$, we have $\sqrt{-1}\,\nu(iH_\alpha) \in \boldsymbol{Z}$ for any $\alpha \in \Delta$. This proves (1).

Let us prove (2). Since $\sqrt{-1}\,(s\nu+t\nu)(iH_\alpha) \in 2\boldsymbol{Z}$ by assumption, $\sqrt{-1}\,(s, t)\lambda(iH_\alpha) \in \boldsymbol{Z}$ is equivalent to

$$\tfrac{1}{2}(s\mu-t\mu)(H_\alpha)=t(t^{-1}s(\tfrac{1}{2}\mu)-\tfrac{1}{2}\mu)(H_\alpha)=(t^{-1}s(\tfrac{1}{2}\mu)-\tfrac{1}{2}\mu)(H_{t^{-1}\alpha}) \in \boldsymbol{Z}.$$

Since $\alpha$ is arbitrary, this means $t^{-1}s \in W^0(\tfrac{1}{2}\mu)$. Q.E.D.

From now on we assume that $\sqrt{-1}\,\nu(iH_\alpha)$ is an even integer for any $\alpha \in \Delta$. Put $I_\mu= W^0(\tfrac{1}{2}\mu)$.

**Lemma 2.10.** *Under the assumption above, we have*

$$W_H(\lambda)=\{(s, t) \in W\times W \mid t^{-1}s \in I_\mu,\ s, t \in N_W(I_\mu)\}\simeq N_W(I_\mu)\times I_\mu.$$

*Proof.* $W_H(\lambda)$ is the largest subgroup of $W\times W$ which leaves $W_H^{\sim}(\lambda)$ invariant under right multiplication. So we have

$$\begin{aligned}W_H(\lambda)&=\{(s, t) \in W\times W \mid W_H^{\sim}(\lambda)(s, t)=W_H^{\sim}(\lambda),\ W_H^{\sim}(\lambda)(s^{-1}, t^{-1})=W_H^{\sim}(\lambda)\}\\ &=\{(s, t) \mid t^{-1}I_\mu s=tI_\mu s^{-1}=I_\mu\}\\ &=\{(s, t) \mid t^{-1}s \in I_\mu,\ s, t \in N_W(I_\mu)\}.\end{aligned}$$

Q.E.D.

**Proposition 2.11.** *If $\lambda$ is regular, the representation $(\tau, V(\chi_\lambda))$ of $W_H(\lambda)$ is equivalent to the representation $(\tau', \boldsymbol{C}[I_\mu])$, where the action $\tau'$ is given by*

$$\tau'((s, t))v=svt^{-1} \qquad (v \in \boldsymbol{C}[I_\mu]).$$

*Proof.* We put $\chi=\chi_\lambda$. Remark that $V_H(\chi)=V(\chi)$ since $\mathrm{Car}(G)$ has the unique conjugacy class $[H]$. From Theorem 2.4, we have

$$(\tau, V(\chi)) \simeq \sum_{\gamma \in \Gamma}^{\oplus} \mathrm{Ind}(\varepsilon(\gamma; \cdot); W(0, \gamma) \uparrow W_H(\lambda)).$$

After easy calculations we know $\Gamma = \Delta W \backslash W_{\tilde{H}}(\lambda) = \{e\}$. Therefore we have

$$\begin{aligned}(\tau, V(\chi)) &\simeq \mathrm{Ind}(\varepsilon(e; \cdot); \Delta W \cap W_H(\lambda) \uparrow W_H(\lambda)) \\ &\simeq \mathrm{Ind}(\text{trivial}; N_W(I_\mu) \uparrow N_W(I_\mu) \times I_\mu) \simeq (\tau', C[I_\mu]),\end{aligned}$$

using $\varepsilon(e; s) = 1$ for any $s \in W$. Q.E.D.

Let $\lambda \in \mathfrak{h}_C^*$ be singular. Then the fixed subgroup breaks up into a direct product: $W(\mathfrak{g}_C, \mathfrak{h}_C)_\lambda = W_{\lambda_1} \times W_{\lambda_2}$, where $\lambda_k = \lambda|_{\mathfrak{h}^{(k)}}$ $(k=1, 2)$. Put

$$e_k = (\# W_{\lambda_k})^{-1} \sum_{s \in W_{\lambda_k}} s \qquad (k=1, 2),$$

then $e_\lambda = e_1 \cdot e_2 = e_2 \cdot e_1$ and $\mathscr{H}_\lambda = \{(e_1 s e_1, e_2 t e_2) \mid (s, t) \in W_H(\lambda)\}$.

**Corollary 2.12.** *The representation $(\sigma, V(\chi))$ of $\mathscr{H}_\lambda$ is equivalent to $(\sigma', e_1 C[I_\mu] e_2)$, where the action is given by*

$$\sigma'((e_1 s e_1, e_2 t e_2))v = e_1 s v t^{-1} e_2 \qquad (v \in e_1 C[I_\mu] e_2),$$

*for $(e_1 s e_1, e_2 t e_2) \in \mathscr{H}_\lambda$.*

*Proof.* This is a corollary of Theorem 2.7 and Proposition 2.11. Q.E.D.

**Corollary 2.13.** *Let $\lambda \in \mathfrak{h}_C^*$ and assume that $\sqrt{-1}\,\lambda(iH_\alpha) \in 2Z$ for any $\alpha \in \Delta$. Then the number of equivalence classes of irreducible admissible representations of $G$ with infinitesimal character $\chi_\lambda$ is equal to*

$$\dim e_1 C[I_\mu] e_2 = \dim \mathrm{Hom}_{I_\mu}(C[I_\mu] e_1, C[I_\mu] e_2).$$

*In particular, if $\lambda$ is regular, there are precisely $\#(I_\mu)$ irreducible admissible representations with infinitesimal character $\chi_\lambda$.*

*Proof.* Note that the number of irreducible representations with infinitesimal character $\chi_\lambda$ is equal to $\dim V(\chi_\lambda)$. Then Corollary 2.13 is an easy consequence of Proposition 2.11 and Corollary 2.12. Q.E.D.

## § 3. Character polynomials and Gelfand-Kirillov dimensions

**3.1. Character polynomials of a virtual character with regular infinitesimal character.** Take $[T] \in \mathrm{Car}^0(G)$ and assume that *$T$ is the neutral component of a Cartan subgroup $H$ of $G$.* Let $e \in T$ be the unit element. Moreover, in this and the next subsections §§ 3.1–3.2, we assume Assumption 2.3. We put

$$H'(\boldsymbol{R})=\{h \in H \mid \varepsilon_{\boldsymbol{R}}(h)\neq 0\}=\{h \in H \mid \xi_\alpha(h)\neq 1 \text{ for any } \alpha \in \Delta_{\boldsymbol{R}}\},$$

and $T'(\boldsymbol{R})=T\cap H'(\boldsymbol{R})$.

At first we assume $\lambda \in \mathfrak{h}_{\boldsymbol{C}}^*$ is regular. Put $\chi=\chi_\lambda$ as in § 2. Then $\Theta \in V(\chi)$ is expressed on $T'(\boldsymbol{R})$ as follows:

$$D\Theta(\exp x)=\sum_{w\in W} b_w(\exp x)\exp w\lambda(x) \qquad (x \in \mathfrak{h}),$$

where $b_w$ is a locally constant function on $T'(\boldsymbol{R})$ (see § 1.3). Let $F$ be a connected component of $T'(\boldsymbol{R})$ and $b_w$ a value of the function $b_w(\cdot)$ on $F$. For a non-negative integer $n$, we define a polynomial function on $\mathfrak{h}_{\boldsymbol{C}}^*\times\mathfrak{h}_{\boldsymbol{C}}$ as

$$c_n(\Theta, F; \mu, x)=(n!)^{-1}\sum_{w\in W} b_w(w\mu(x))^n \qquad (\mu \in \mathfrak{h}_{\boldsymbol{C}}^*,\ x \in \mathfrak{h}_{\boldsymbol{C}}).$$

Then $c_n$ is an intertwining map from the space of virtual characters to the space of polynomial functions on $\mathfrak{h}_{\boldsymbol{C}}^*\times\mathfrak{h}_{\boldsymbol{C}}$ on which $W$ acts by the natural action on the first variable. Namely, we have the following lemma.

**Lemma 3.1.** *Let $A$ be a Cartan subgroup of $G$. Then $c_n(\cdot, F; \mu, x)$ is a $W_A(\lambda)$-equivariant map of $V_A(\chi)$ to the space of homogeneous polynomials of degree $2n$ on $\mathfrak{h}_{\boldsymbol{C}}^*\times\mathfrak{h}_{\boldsymbol{C}}$.*

*Proof.* By the arguments in § 4 of [34], we have for $s \in W_A(\lambda)$,

$$\begin{aligned} D\cdot\tau(s)\Theta(\exp x) &= \sum_{w\in W} b_w(\exp x)\exp ws^{-1}\lambda(x) \\ &= \sum_{w\in W} b_{ws}(\exp x)\exp w\lambda(x) \qquad (x \in \mathfrak{h}) \end{aligned}$$

In [34, § 4], we only treat an integral $\lambda$ and $\xi_\lambda(h)$ was used instead of $\exp\lambda(x)$. However, since we only treat the neutral component here, it holds that $\xi_\lambda(\exp x)=\exp\lambda(x)$ and the arguments in [34, § 4] works well for non-integral $\lambda$.

By the above formula and the definition of a character polynomial, we have

$$\begin{aligned} c_n(s\Theta, F; \mu, x) &= (n!)^{-1}\sum_{w\in W} b_{ws}(w\mu(x))^n \\ &= (n!)^{-1}\sum_{w\in W} b_w(ws^{-1}\mu(x))^n = c_n(\Theta, F; s^{-1}\mu, x) \\ &\qquad\qquad (\mu \in \mathfrak{h}_{\boldsymbol{C}}^*,\ x \in \mathfrak{h}_{\boldsymbol{C}}), \end{aligned}$$

which means $c_n(\cdot, F; \mu, x)$ is $W_A(\lambda)$-equivariant. Q.E.D.

Take $s \in W(G; T)$. Since $s$ preserves the set of real roots $\Delta_{\boldsymbol{R}}$, $sF$ is also a connected component of $T'(\boldsymbol{R})$.

**Lemma 3.2.** *The spaces* $\{c_n(\Theta, F; \mu, x) \mid \Theta \in V_A(\chi)\}$ *and* $\{c_n(\Theta, sF; \mu, x) \mid \Theta \in V_A(\chi)\}$ *are isomorphic as* $W_A(\lambda)$*-modules.*

*Proof.* Since $\Theta \in V_A(\chi)$ is an IED, we have $\Theta(sh)=\Theta(h)$ for $h \in H'(\boldsymbol{R})$. Using this, we get $D\Theta(sh)=\operatorname{sgn}(s)D\Theta(h)$ for any $h \in F$. On the other hand, we have

$$\begin{aligned} D\Theta(s \exp x) &= D\Theta(\exp sx) \\ &= \sum_{w \in W} b_w(\exp sx) \exp w\lambda(sx) \\ &= \sum_{w \in W} b_w(sF) \exp s^{-1}w\lambda(x) \\ &= \sum_{w \in W} b_{sw}(sF) \exp w\lambda(x), \end{aligned}$$

for $\exp x \in F$. Therefore we get $b_{sw}(sF)=\operatorname{sgn}(s)b_w(F)$. Then a straightforward calculation leads us to

$$\begin{aligned} c_n(\Theta, sF; \mu, x) &= (n!)^{-1} \sum_{w \in W} b_w(sF)(w\mu(x))^n \\ &= (n!)^{-1} \sum_{w \in W} b_{sw}(sF)(sw\mu(x))^n \\ &= (n!)^{-1} \sum_{w \in W} \operatorname{sgn}(s) b_w(F)(w\mu(s^{-1}x))^n \\ &= \operatorname{sgn}(s) c_n(\Theta, F; \mu, s^{-1}x). \end{aligned}$$

Now it is clear that $c_n(\Theta, F; \mu, x) \rightarrow c_n(\Theta, sF; \mu, x)=\operatorname{sgn}(s)c_n(\Theta, F;, \mu, s^{-1}x)$ is an intertwining operator of $W_A(\lambda)$-modules. Q.E.D.

We remark that the components of $T'(\boldsymbol{R})$ are mutually conjugate under the action of $W(G; T)$, where $T$ is the neutral component of $H$. In fact, for any $\alpha \in \Delta_{\boldsymbol{R}}$ there is a reflection $s_\alpha \in W(G; T)$ corresponding to $\alpha$, and invariants of $s_\alpha$ is precisely the set $\{h \in T \mid \xi_\alpha(h)=1\}$.

From now on, we fix a component $F$ of $T'(\boldsymbol{R})$, so we may omit the notation $F$. Put $r=r(\Theta, F)=\min\{n \mid c_n(\Theta; \mu, x)$ is not identically zero$\}$. If $\Theta|_F$ is identically zero, then we put $r=\infty$ and $c_r(\Theta; \mu, x)=c_\infty(\Theta; \mu, x)=0$. We call $c_r(\Theta; \mu, x)$ a *character polynomial associated to* $\Theta$ *and* $F$.

Let $\theta$ be a Cartan involution which leaves $H$ invariant. Let $K$ be a maximal compact subgroup of $G$ corresponding to $\theta$. We call $\alpha \in \Delta_I$ a *compact* (hence imaginary) *root* if its root vector $X_\alpha$ lies in $\mathfrak{k}_C$. We denote by $\Delta_c$ the set of compact roots. Let $G=KP$ be a Cartan decomposition and put $H^+=H \cap K$ and $H^-=H \cap P$. Then $H=H^+ \cdot H^-$ (direct product) and $H^+$ is a Cartan subgroup of $M=Z_K(H^-)$. Remark that the Weyl group $W(\Delta_c)$ of compact roots is equal to $W(M_0; H_0^+)$, where $M_0$ and $H_0^+$ are neutral component of $M$ and $H^+$ respectively. We get the following proposition.

**Proposition 3.3.** (1) *A character polynomial $c_r(\Theta; \mu, x)$ is a $W\times W$-harmonic polynomial on $\mathfrak{h}_C^*\times\mathfrak{h}_C$.*

(2) *$r=r(\Theta, F)$ is contained in the set $\{i \mid \sharp\Delta_c^+\leqq i\leqq\sharp\Delta^+\}\cup\{\infty\}$.*

*Proof.* (1) is the consequence of Lemma 9.6 of [21] and Lemma 4.3 of [39] (see also 9A.2 of [21]).

Now we will prove (2). By Theorem 3.4 of [13], we know the degree of a $W$-harmonic polynomial on $\mathfrak{h}_C^*$ is less than or equal to the number of reflections in $W$. It is well-known that this number is equal to $\sharp\Delta^+$ (for example, see [3, Chap. 4, § 3]). Now we must prove $\sharp\Delta_c^+\leqq r$. This is rather complicated. Since the Weyl group $W(\Delta_c)$ fixes each real root, $W(\Delta_c)$ fixes each component $F$ of $H'(\boldsymbol{R})$. In fact for some partition of $\Delta_{\boldsymbol{R}}^+=\mathrm{Neg}\cup\mathrm{Pos}$, $F$ is expressed as

$$F=\{h\in T \mid \xi_\alpha(h)>1 \text{ for } \alpha\in \mathrm{Pos},\ \xi_\alpha(h)<1 \text{ for } \alpha\in\mathrm{Neg}\}.$$

Since $W(\Delta_c)=W(M_0; H_0^+)$ is contained in $W(G; H_0)$, we can argue as in the proof of Lemma 3.2, and get

$$\operatorname{sgn}(s)b_w(F)=b_{sw}(F) \qquad (s\in W(\Delta_c)).$$

Therefore $D\Theta$ is expressed as

$$\begin{aligned} D\Theta(\exp x)&=\sum_{w\in W} b_w(F)\exp w\lambda(sx)\\ &=(\sharp W(\Delta_c))^{-1}\sum_{w\in W}\sum_{s\in W(\Delta_c)} b_{sw}(F)\exp sw\lambda(x)\\ &=(\sharp W(\Delta_c))^{-1}\sum_{w\in W} b_w(F)\sum_{s\in W(\Delta_c)}\operatorname{sgn}(s)\exp sw\lambda(x). \end{aligned}$$

Since $\sum_{s\in W(\Delta_c)}\operatorname{sgn}(s)(sw\lambda(x))^n=0$ for $n<\sharp\Delta_c^+$, we have $\sharp\Delta_c^+\leqq r$. Q.E.D.

**3.2. Character polynomials of a virtual character with singular infinitesimal character.** Now we study the case where $\lambda$ is singular. Let $\lambda$ and $\lambda_0$ be the same as in § 2.2. We denote by $\Phi=\Phi_{\lambda_0}^{\lambda}$ and $\Psi=\Psi_{\lambda}^{\lambda_0}$ Zuckerman's translation functors. Put $\chi=\chi_\lambda$ and $\chi_0=\chi_{\lambda_0}$. Take $\Theta\in V(\chi)$, then there exists $\Theta_0\in V(\chi_0)$ such that $\Psi(\Theta_0)=\Theta$. Let $A$ be a Cartan subgroup of $G$. We remark that if $\Theta$ is contained in $V_A(\chi)$, then we can take $\Theta_0$ from $V_A(\chi_0)$. Put

$$e_\lambda=(\sharp W_\lambda)^{-1}\sum_{w\in W_\lambda} s\in \boldsymbol{C}[W_\lambda]$$

as in §2. Then $e_\lambda \in C[W_A(\lambda)]$ for any Cartan subgroup $A$ of $G$. Therefore we can consider the operator $\tau(e_\lambda)$ on the whole space $V(\chi_0)$. We define the polynomials $c_n(\Theta, F; \mu, x)$ by

$$c_n(\Theta, F; \mu, x) = c_n(\tau(e_\lambda)\Theta_0, F; \mu, x) \qquad (\mu \in \mathfrak{h}_C^*,\ x \in \mathfrak{h}_C),$$

where $F$ and $H$ are the same as in the former subsection §3.1. This definition of $c_n$ does not depend on the choice of $\Theta_0$. In fact, suppose that $\Psi(\Theta_1) = \Psi(\Theta_0) = \Theta$. Then we have $\Psi(\Theta_1 - \Theta_0) = 0$. By Proposition 2.6, this means $\tau(e_\lambda)(\Theta_1 - \Theta_0) = 0$, so we have $\tau(e_\lambda)\Theta_1 = \tau(e_\lambda)\Theta_0$. Similarly we put $r(\Theta, F) = r(\tau(e_\lambda)\Theta_0, F)$.

**Proposition 3.4.** (1) *Let $A$ be a Cartan subgroup of $G$. Then $c_n(\cdot, F; \mu, x)$ is an $\mathscr{H}_\lambda = \mathscr{H}(W_A(\lambda), W_\lambda)$-equivariant map of $V_A(\chi)$ to the space of homogeneous polynomials of degree $2n$ on $\mathfrak{h}_C^* \times \mathfrak{h}_C$.*

(2) *The representation of $\mathscr{H}_\lambda$ on the space $\{c_n(\Theta, F; \mu, x) \mid \Theta \in V_A(\chi)\}$ is equivalent to* $\mathrm{Red}(\{c_n(\Theta_0, F; \mu, x) \mid \Theta_0 \in V_A(\chi)\}; W_A(\lambda) \downarrow W_\lambda)$.

(3) *Put $r = r(\Theta, F)$. Then $c_r(\Theta, F; \mu, x)$ is a $W \times W$-harmonic polynomial on $\mathfrak{h}_C^* \times \mathfrak{h}_C$.*

*Proof.* By the definition of the representation $(\sigma, V_A(\chi))$ of $\mathscr{H}_\lambda$, we have

$$\sigma(e_\lambda w e_\lambda)(\Theta) = (\# W_\lambda)^{-1}(\Psi \circ \tau(e_\lambda w e_\lambda) \circ \Phi)(\Theta) = (\# W_\lambda)^{-1}(\Psi \circ \tau(e_\lambda w e_\lambda) \circ \Phi \circ \Psi)(\Theta_0).$$

By Proposition 2.6, we have $\Phi \circ \Psi(\Theta_0) = (\# W_\lambda)\tau(e_\lambda)\Theta_0$. Therefore the above formula becomes $\Psi \circ \tau(e_\lambda w e_\lambda)\Theta_0$. By the definition of $c_n$, we get

$$\begin{aligned} c_n(\sigma(e_\lambda w e_\lambda)\Theta, F; \mu, x) &= c_n(\tau(e_\lambda w e_\lambda)\Theta_0, F; \mu, x) \\ &= c_n(\Theta_0, F; (e_\lambda w^{-1} e_\lambda)\mu, x) = c_n(\Theta, F; (e_\lambda w^{-1} e_\lambda)\mu, x), \end{aligned}$$

which proves (1).

(2) is an easy consequence of (1) and the definition of Red. (3) is a corollary to Proposition 3.3. Q.E.D.

**3.3. Gelfand-Kirillov dimensions.** In this subsection, we only consider the connected component $F$ of $J_0'(\boldsymbol{R})$, where $J$ is a maximally split Cartan subgroup of $G$ and $J_0$ the neutral component. By the remark after Lemma 3.2, the choice of $F$ doesn't matter essentially.

Take $\lambda \in \mathfrak{j}_C^*$ which is not necessarily regular. For $\Theta \in V(\chi)$ $(\chi = \chi_\lambda)$, we define *Gelfand-Kirillov dimension* of $\Theta$ by

$$\mathrm{GKD}(\Theta) = \# \Delta^+ - r(\Theta, F).$$

Note that $\mathrm{GKD}(\Theta)$ does not depend on the choice of $F \subset J_0'(\boldsymbol{R})$ by the remark after Lemma 3.2.

**Lemma 3.5.** *For $\Theta \in V(\chi)$, we have either* $\#\Delta^+ - \#\Delta_c^+ \geqq \mathrm{GKD}(\Theta) \geqq 0$ *or* $\mathrm{GKD}(\Theta) = -\infty$.

*Proof.* This is a Corollary to Proposition 3.3. Q.E.D.

**Proposition 3.6.** *Let $\Theta_\pi$ be a character of an irreducible admissible representation $\pi$ of $G$. Then* $\mathrm{GKD}(\Theta_\pi)$ *is equal to the usual Gelfand-Kirillov dimension of $\pi$.*

For Gelfand-Kirillov dimensions of representations of $G$, we refer to [39] and [29]. Lemma 3.5, together with Proposition 3.6, contains another proof of Proposition 5.7 of [39]. Remark that we don't use subrepresentation theorem.

*Proof of Proposition* 3.6. If $\Theta_\pi$ has a regular infinitesimal character, then the result is obtained by D.R. King [24, Th. 1.1]. So we assume that $\Theta_\pi$ has a singular infinitesimal character $\lambda \in \mathfrak{j}_C^*$. Let $\lambda$ and $\lambda_0$ be as in § 2. Then there exists $\Theta_0 \in V(\chi_0)$ such that $\Psi(\Theta_0) = \Theta_\pi$. Moreover, we can take $\Theta_0$ to be an irreducible character (see [44, Th. 1.3]). Let us write $\Theta_0 = \Theta_\delta$, where $\delta$ is some irreducible admissible representation of $G$ with infinitesimal character $\lambda_0$. Since $\Theta_\pi = \Psi(\Theta_\delta)$ is obtained by tensor products with finite dimensional representations of $G$ and projections onto the components with the same infinitesimal character, it is easy to see GK-dim $\pi \leqq$ GK-dim $\delta$, where GK-dim denotes Gelfand-Kirillov dimension of a representation. On the other hand, we know $\Phi(\Theta_\pi) = \Phi\Psi(\Theta_\delta) = \tau(e_\lambda)\Theta_\delta$ is a genuine character and contains $\Theta_\delta$ from [44, Th. 1.3]. By the similar arguments as above, we conclude that GK-dim $\delta \leqq$ GK-dim $\pi$, and therefore GK-dim $\delta =$ GK-dim $\pi$. We get

$$\text{GK-dim}\, \pi \geqq \text{GK-dim}\,(\tau(e_\lambda)\Theta_\delta) = \mathrm{GKD}\,(\tau(e_\lambda)\Theta_\delta) \geqq \text{GK-dim}\, \delta = \text{GK-dim}\, \pi.$$

Now we know the above inequalities are equalities. By the definition of GKD, we have

$$\begin{aligned} \mathrm{GKD}\,(\Theta_\pi) &= \#\Delta^+ - r(\Theta_\pi, F) = \#\Delta^+ - r(\tau(e_\lambda)\Theta_\delta, F) \\ &= \mathrm{GKD}\,(\tau(e_\lambda)\Theta_\delta) = \text{GK-dim}\, \pi. \end{aligned}$$

Q.E.D.

Let $H$ be a Cartan subgroup of $G$. Put $V(\chi; d) = \{\Theta \in V(\chi) \mid \mathrm{GKD}(\Theta) \leqq d\}$ and $V_H(\chi; d) = V(\chi; d) \cap V_H(\chi)$.

**Proposition 3.7.** (1) *The subspace $V_H(\chi; d)$ is $\mathscr{H}(W_H(\lambda), W_\lambda)$-invariant.*

(2) *Similarly the subspace* $V(\chi; d)$ *is* $\mathscr{H}(W(\lambda), W_\lambda)$*-invariant, where* $W(\lambda)$ *is an integral Weyl group of* $\lambda$.

**Remark.** If $\lambda$ is regular, $\mathscr{H}(W_H(\lambda), W_\lambda)=C[W_H(\lambda)]$. So $V_H(\chi; d)$ is a $W_H(\lambda)$-invariant subspace. Similar remark is applicable to (2).

*Proof.* By Proposition 3.4, we know

$$c_n(\sigma(e_\lambda we_\lambda)\Theta, F; \mu, x)=c_n(\Theta, F; (e_\lambda w^{-1}e_\lambda)\mu, x) \qquad (e_\lambda we_\lambda \in \mathscr{H}(W_H(\lambda), W_\lambda)),$$

for $n\geqq 0$ and $\Theta \in V_H(\chi)$. If $\Theta \in V_H(\chi; d)$, then $c_n(\Theta, F; \mu, x)\equiv 0$ for $n\geqq r$, where $r=\sharp\Delta^+-d$. By the above equation we get $c_n(\sigma(e_\lambda we_\lambda)\Theta, F; \mu, x) \equiv 0$ for $n\geqq r$ and this means $\sigma(e_\lambda we_\lambda)\Theta \in V_H(\chi; d)$. This proves (1). The same arguments are available to the proof of (2), which we omit here.

Q.E.D.

**Corollary 3.8.** *Put* $r=\sharp\Delta^+-d$.

(1) $c_r(\cdot, F; \mu, x)$ *induces an* $\mathscr{H}(W_H(\lambda), W_\lambda)$*-isomorphism of* $V_H(\chi; d)/V_H(\chi; d-1)$ *into the space of* $W\times W$*-harmonic polynomials on* $\mathfrak{j}_C^*\times\mathfrak{j}_C$ *of homogeneous degree* $2r$.

(2) $c_r(\cdot, F; \mu, x)$ *induces an* $\mathscr{H}(W(\lambda), W_\lambda)$*-isomorphism of* $V(\chi; d)/V(\chi; d-1)$ *into the space of* $W\times W$*-harmonic polynomials on* $\mathfrak{j}_C^*\times\mathfrak{j}_C$ *of homogeneous degree* $2r$.

*Proof.* By the definition of $V(\chi; d)$, $\mathrm{Ker}\, c_r(\cdot, F; \mu, x)=V(\chi, d-1)$ (note that $V(\chi; -1)=V(\chi; -\infty)$). So this is an easy consequence of Proposition 3.7. Q.E.D.

**Corollary 3.9.** *Let* $\lambda$ *be regular. If* $V_H(\chi; -\infty)=(0)$, *then the representation* $(\tau, V_H(\chi))$ *of* $W_H(\lambda)$ *is a subrepresentation of the regular representation of* $W$ *restricted to* $W_H(\lambda)$. *Similarly, if* $V(\chi; -\infty)=(0)$, *then the representation* $(\tau, V(\chi))$ *of* $W(\lambda)$ *is a subrepresentation of the regular representation of* $W$ *restricted to* $W(\lambda)$.

*Proof.* $\{\mathrm{Im}\, c_r(\cdot, F; \mu, x)\,|\,0\leqq r\leqq\sharp\Delta^+\}$ is the set of linearly independent subspaces of $W\times W$-harmonic polynomials on $\mathfrak{j}_C^*\times\mathfrak{j}_C$, because they have different degrees. Now by the hypothesis, we have

$$\begin{aligned}\sum_{0\leqq r\leqq\sharp\Delta^+}^{\oplus} \mathrm{Im}\, c_r(\cdot, F; \mu, x) &\simeq \sum_{0\leqq d\leqq\sharp\Delta^+}^{\oplus} V_H(\chi; d)/V_H(\chi; d-1)\oplus V_H(\chi; -\infty)\\ &\simeq V_H(\chi).\end{aligned}$$

Since $W$-harmonic polynomials on $\mathfrak{j}_C^*$ carries the regular representation of $W$, we get the first statement. The second one is similarly obtained.

Q.E.D.

**Theorem 3.10.** *Let* $U \subset V_H(\chi))$ (*respectively* $U \subset V(\chi)$) *be an irreducible subspace of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ (*respectively* $\mathscr{H}(W(\lambda), W_\lambda)$). *For non-zero* $\Theta_1, \Theta_2 \in U$, *we have* $\mathrm{GKD}(\Theta_1) = \mathrm{GKD}(\Theta_2)$. *So we can define* $\mathrm{GKD}(U)$ *naturally.*

*Proof.* If $\mathrm{GKD}(\Theta_1) < \mathrm{GKD}(\Theta_2)$, then $U \cap V_H(\chi; \mathrm{GKD}(\Theta_1))$ is a non-trivial invariant subspace of $U$. Thus it contradicts to the irreducibility of $U$. Q.E.D.

This theorem will be a useful tool to determine the Gelfand-Kirillov dimensions of irreducible representations of $G$.

**Theorem 3.11.** *Let* $\lambda$ *be regular. If* $V(\chi; -\infty) \neq (0)$, *then there exists at least a pair of irreducible representations* $(\pi_1, \pi_2)$ *such that* $\pi_1$ *and* $\pi_2$ *have the same primitive ideal.*

*Proof.* From the hypothesis, there exists non-zero $\Theta \in V(\chi)$ such that $\Theta$ is identically zero on $J_0$. Write $\Theta = \Theta_1 - \Theta_2$, where $\Theta_1$ and $\Theta_2$ are positive linear sums of genuine characters. Clearly it holds that $\mathrm{GKD}(\Theta_1) = \mathrm{GKD}(\Theta_2)$. Put $d = \mathrm{GKD}(\Theta_1) = \mathrm{GKD}(\Theta_2)$ and let $\{\pi_i \mid 1 \leqq i \leqq l\}$ be irreducible components of $\Theta_1$ which has Gelfand-Kirillov dimension $d$ and $\{\delta_i \mid 1 \leqq i \leqq k\}$ be those of $\Theta_2$. If $\{\pi_i\} \cup \{\delta_i\}$ have different primitive ideals each other, then their Goldie rank polynomials are linearly independent (see [22, § 5.12] and [23, Th. 5.5]). By Theorem 6.1 of [24], linear independence of Goldie rank polynomials implies linear independence of character polynomials, which is a contradiction. Q.E.D.

**3.4. Goldie rank polynomials.** Let $J$ and $F$ be as in the former subsection § 3.3. Choose $x \in \mathfrak{j}_C$ such that $c_r(\Theta, F; \mu, x)$ is not identically zero as a polynomial on $\mathfrak{j}_C^*$ for any $\Theta \in V_H(\chi; d) \setminus V_H(\chi; d-1)$ $(d = \sharp\Delta^+ - r)$. Let $\mathscr{R}_x$ be a twisted projection map of the space of polynomials on $\mathfrak{j}_C^* \times \mathfrak{j}_C$ to the space of polynomials on $\mathfrak{j}_C^*$ defined by $(\mathscr{R}_x f)(\mu) = f(x, -\mu)$ $(\mu \in \mathfrak{j}_C^*, x \in \mathfrak{j}_C)$. By Corollary 3.8, it is clear that $\mathscr{R}_x \circ c_r(\cdot, F)$ is a $\mathscr{H}(W(\lambda), W_\lambda)$-isomorphism of $V(\chi; d)/V(\chi; d-1)$ into the space of $W$-harmonic polynomials on $\mathfrak{j}_C^*$ of homogeneous degree $r$.

**Lemma 3.12** (King [24, Theorem 6.1]). *If* $\Theta$ *is an irreducible character with regular infinitesimal character, then* $\mathscr{R}_x \circ c_r(\Theta, F)$ *is a Goldie rank polynomial up to a non-zero scalar multiple.*

**Proposition 3.13.** *Let* $r = \sharp\Delta_c^+$ *and put* $d = \sharp\Delta^+ - r$. *If* $\chi$ *is regular, then the space* $\mathscr{R}_x \circ c_r(\cdot, F)$ $(V(\chi))$ *has a basis consiting of Goldie rank polynomials.*

*Proof.* Since $V(\chi)$ has a basis consisting of irreducible characters, we can choose irreducible characters $\Theta_1, \cdots, \Theta_m$ such that $\{\Theta_i \mid 1 \leqq i \leqq m\}$ is a set of representatives of a basis for $V(\chi)/V(\chi; d-1)$. By the argument above, we know that $\{\mathscr{R}_x \circ c_r(\Theta_i, F) \mid 1 \leqq i \leqq m\}$ is a basis for the space $\operatorname{Im} \mathscr{R}_x \circ c_r(\cdot, F) = \mathscr{R}_x \circ c_r(\cdot, F)(V(\chi))$. Now Lemma 3.12 proves the proposition. Q.E.D.

**Lemma 3.14.** *The space $V_J(\chi)$ is generated by the characters of principal series representations induced from a minimal parabolic subgroup.*

*Proof.* This is an easy consequence of Lemma 5.4 and the proof of Theorem 2.8 in [35]. Q.E.D.

**Proposition 3.15.** *Let $r = \sharp \Delta_c^+$ and put $d = \sharp \Delta^+ - r$. Then the image of $V(\chi)$ and the image of $V_J(\chi)$ under the map $\mathscr{R}_x \circ c_r(\cdot, F)$ coincide with each other.*

*Proof.* At first, let $\chi = \chi_\lambda$ be a regular infinitesimal character. By subrepresentation theorem (see Theorem 3.2 in Part II), any irreducible admissible representation of $G$ appears as a subrepresentation of some principal series representation induced from a minimal parabolic subgroup. Let $\Theta$ be an irreducible character with Gelfand-Kirillov dimension $d$. We choose a principal series character $\Xi$ which contains $\Theta$ and write

$$\Xi = \Theta_0 + \Theta_1 + \cdots + \Theta_m,$$

where $\Theta_i$ is an irreducible character and $\Theta_0 = \Theta$. Let $\pi_i$ be an irreducible admissible representation whose character is $\Theta_i$. We can assume that $\pi_1, \cdots, \pi_k$ have the same primitive ideal as $\pi_0$ and $\pi_{k+1}, \cdots, \pi_m$ have different ones. Since $\Theta_0 + \Theta_1 + \cdots + \Theta_k$ is a genuine character and its Gelfand-Kirillov dimension is $d$, we conclude that

$$\begin{aligned} \mathscr{R}_x \circ c_r(\Xi, F) &= \mathscr{R}_x \circ c_r\Big(\sum_{i=0}^{k} \Theta_i, F\Big) + \sum_{i=k+1}^{m} \mathscr{R}_x \circ c_r(\Theta_i, F) \\ &= a\mathscr{R}_x \circ c_r(\Theta, F) + \sum_{i=k+1}^{m} \mathscr{R}_x \circ c_r(\Theta_i, F), \end{aligned}$$

where $a$ is a non-zero constant (remark that Lemma 3.12 remains valid for $\Theta$ a genuine character). By Theorem 5.5 in [23], the Goldie rank polynomials form a basis for some multiplicity-free representation of $W(\lambda)$ which are not equivalent each other and $\pi_0$ and $\pi_i$ have the same Goldie rank polynomial if and only if they have the same primitive ideal. Since $a\mathscr{R}_x \circ c_r(\Theta, F)$ is a Goldie rank polynomial multiplied by a non-zero constant, there is an element $s$ of $C[W(\lambda)]$ such that $s\mathscr{R}_x \circ c_r(\Xi, F) = \mathscr{R}_x \circ c_r(\tau(s)\Xi, F) = \mathscr{R}_x \circ c_r(\Theta, F)$. By Lemma 3.14, $\tau(s)\Xi$ is contained in $V_J(\chi)$ and we proved that $\mathscr{R}_x \circ c_r(\cdot, F)(V_J(\chi)) \supset \mathscr{R}_x \circ c_r(\cdot, F)(V(\chi))$. Inversed inclusion is trivial.

If $\chi$ is singular, proof of regular case and Proposition 3.4 (2) prove the result. Q.E.D.

## Part II

### § 1. Structure of the group $U(p, 1)$

In this Part II, we consider the group $G = U(p, 1)$, where

$$U(p, 1) = \{g \in GL(p+1, \boldsymbol{C}) \mid {}^t\bar{g} J_{p,1} g = J_{p,1}\},$$
$$J_{p,1} = \operatorname{diag}(1, \cdots, 1, -1).$$

We have an Iwasawa decomposition $G = K \exp \mathfrak{a} \exp \mathfrak{n}$, putting:

$$K = \{\operatorname{diag}(u, \lambda) \mid u \in U(p),\ \lambda \in U(1)\},$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0_{p-1} & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \middle|\ a \in \boldsymbol{R} \right\},$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0_{p-1} & c & -c \\ -{}^t\bar{c} & b & -b \\ -{}^t\bar{c} & b & -b \end{pmatrix} \middle|\ c \in \boldsymbol{C}^{p-1},\ b \in \sqrt{-1}\boldsymbol{R} \right\}.$$

Let $M = Z_K(A)$, where $A = \exp \mathfrak{a}$. Then we have

$$M = \{\operatorname{diag}(u, e^t, e^t) \mid u \in U(p-1),\ t \in \sqrt{-1}\,\boldsymbol{R}\}.$$

Put $\mathfrak{b}^+ = \{\operatorname{diag}(b_1, \cdots, b_{p-1}, b_p, b_p) \mid b_i \in \sqrt{-1}\,\boldsymbol{R}\}$. Then $\mathfrak{b}^+$ is a Cartan subalgebra of $\mathfrak{m}$ and $\mathfrak{j} = \mathfrak{b}^+ \oplus \mathfrak{a}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. Take

$$h = \begin{pmatrix} b_1 & & & & \\ & \ddots & & 0 & \\ & & b_{p-1} & & \\ & 0 & & b_p & a \\ & & & a & b_p \end{pmatrix} \in \mathfrak{j},$$

and put $\varepsilon_i(h) = b_i$ $(1 \leqq i \leqq p-1)$, $\varepsilon_p(h) = b_p + a$ and $\varepsilon_{p+1}(h) = b_p - a$. Then we have $\varepsilon_i \in \mathfrak{j}_C^*$ and $\Delta(\mathfrak{g}_C, \mathfrak{j}_C)=\{\varepsilon_i-\varepsilon_j \mid i \neq j\}$. We choose a positive system $\Delta^+=\{\varepsilon_i-\varepsilon_j \mid i>j\}$. Put

$$\operatorname{roth}'(a)=\begin{pmatrix}\operatorname{ch} a & \operatorname{sh} a\\ \operatorname{sh} a & \operatorname{ch} a\end{pmatrix}$$

and $\operatorname{roth}(a)=\operatorname{diag}(1_{p-1}, \operatorname{roth}'(a)) \in A$, where $1_p$ denotes the identity matrix of order $p$.

Let $J=B^+A$ be a maximally split Cartan subgroup and $B=\{\operatorname{diag}(b_1, \cdots, b_{p+1}) \mid b_i \in U(1)\}$ be a compact Cartan subgroup. We define a function $D_0$ on $J$ and $B$. Put

$$D_0(h)=\prod_{1 \leq i<j \leq p+1}(b_i-b_j)$$

for $h=\operatorname{diag}(b_1, \cdots, b_{p+1}) \in B$ and

$$D_0(h)=\prod_{1 \leq i<j \leq p+1}(b_i'-b_j')$$

for $h=\operatorname{diag}(b_1, \cdots, b_{p-1}, b_p, b_p)\cdot\operatorname{roth}(a) \in J$, where $b_i'=b_i$ $(1 \leqq i \leqq p-1)$, $b_p'=b_p\cdot\exp(-|a|)$ and $b_{p+1}'=b_p\cdot\exp|a|$.

## §2. Gelfand-Zetlin basis for $U(p, 1)$

In this section, we review the results of D.P. Zhelobenko [42, 43] for the group $U(p)$, and then construct admissible $(\mathfrak{g}_C, K)$-modules for $G=U(p, 1)$, using Gelfand-Zetlin basis.

**2.1. Gelfand-Zetlin basis for** $U(p)$. Let $V(\mu)$ be a finite dimensional representation of $U(p)$ with highest weight $\mu$. We put $\mathfrak{g}_p=\mathfrak{gl}(p, \boldsymbol{C})$ and $\mathfrak{h}_p=\{$diagonal matrices of $\mathfrak{g}_p\}$. Then we may consider as $\mu \in \mathfrak{h}_p^*$. Let us assume that $V(\mu)$ is unitary. Let $e_{i,j}$ be a matrix whose $(i, j)$-element is 1 and the other elements are 0. We consider

$$\mathfrak{g}_{p-1}=\sum_{1 \leqq i, j \leqq p-1}^{\oplus} \boldsymbol{C} e_{i,j} \subset \mathfrak{g}_p$$

as a Lie subalgebra of $\mathfrak{g}_p$. In the same way, we also consider $U(p-1)$ as a subgroup of $U(p)$. For $h=\operatorname{diag}(b_1, b_2, \cdots, b_p) \in \mathfrak{h}_p$, we put $\varepsilon_i(h)=b_i$ $(1 \leqq i \leqq p)$. Then $\varepsilon_i \in \mathfrak{h}_p^*$ and $\Delta(\mathfrak{g}_p, \mathfrak{h}_p)=\{\varepsilon_i-\varepsilon_i \mid 1 \leqq i \neq j \leqq p\}$ is a root system of $(\mathfrak{g}_p, \mathfrak{h}_p)$. We choose a positive system $\Delta^+=\{\varepsilon_i-\varepsilon_j \mid 1 \leqq i<j \leqq p\}$. If we write the highest weight $\mu$ as $\mu=\sum_{1 \leq i \leq p} \mu(i)\varepsilon_i$, then it holds that $\mu(1) \geqq \mu(2) \geqq \cdots \geqq \mu(p)$.

It is well-known how $V(\mu)$ decomposes into irreducible representations of $U(p-1)$, when $V(\mu)$ is restricted to a representation of $U(p-1)$.

An irreducible finite dimensional representation $V(\nu)$ of $U(p-1)$ with highest weight $\nu \in \mathfrak{h}_{p-1}^*$ appears in $V(\mu)$ if and only if $\mu(i) \geqq \nu(i) \geqq \mu(i+1)$ holds for $1 \leqq i \leqq p-1$. We express this situation as $\nu \subset \mu$. Since $V(\nu)$ has multiplicity one in $V(\mu)$, we have

$$V(\mu)=\sum_{\nu \subset \mu}^{\oplus} V(\nu).$$

Applying this step to the sequence $U(1) \subset U(2) \subset \cdots \subset U(p-1) \subset U(p)$ successively, we know that an orthonormal basis for $V(\mu)$ is obtained by the correction of $v(\beta)=v(\mu_p; \mu_{p-1}; \cdots; \mu_1)$, where $\mu_i$ is a highest weight for $\mathfrak{h}_i^*$ and satisfies that $\mu_i \supset \mu_{i-1}$ $(2 \leqq i \leqq p)$. The vector $v(\beta)$ is contained in $V(\mu_i)$, when $V(\mu=\mu_p)$ is considered as a representation of $U(i)$. Because, in each step, $V(\mu_i)$ decomposes into irreducible representations which have multiplicity one, $v(\beta)$ is completely determined by $\beta$ up to a scalar multiple. We assume that $\|v(\beta)\|=1$ without loss of generality. D.P. Zhelobenko wrote down the action of $\mathfrak{g}_p$ on $V(\mu)$ completely, using a basis $\{v(\beta)\}$ multiplied by suitable constants whose absolute values are 1. Originally $\{v(\beta)\}$ is considered by Gelfand-Zetlin for special linear groups and special orthogonal groups [9, 10], so we call $\{v(\beta)\}$ *Gelfand-Zetlin basis.*

Put $e^+=e_{p\ 1}^+=e_{p-1,p}$ and $e^-=e_{p-1}^-=e_{p,p-1}$.

**Proposition 2.1** (Zhelobenko [43, Th. 3.5]). *The actions of $e^+$ and $e^-$ on $\{v(\beta)\}$ are given as follows. For the formula of $e^+$, we have*

$$e^+v(\beta)=\sum_{1 \leqq j \leqq p-1} a_j^+(\beta) v(\beta+\varepsilon_j),$$

*where $v(\beta+\varepsilon_j)=v(\mu_p; \mu_{p-1}+\varepsilon_j; \cdots; \mu_1)$ and*

$$a_j^+(\beta)=\{\prod_{1 \leqq i \leqq p}(\mu_p(i)-\mu_{p-1}(j)+j-i) \prod_{1 \leqq i \leqq p-2}(\mu_{p-2}(i)-\mu_{p-1}(j)+j-i-1)\}^{1/2}$$
$$\times\{\prod_{\substack{1 \leqq i \leqq p-1 \\ i \neq j}}(\mu_{p-1}(i)-\mu_{p-1}(j)+j-i) \prod_{1 \leqq i \leqq p-1}(\mu_{p-1}(i)-\mu_{p-1}(j)+j-i-1)\}^{-1/2}.$$

*For e , we have*

$$e^-v(\beta)=\sum_{1 \leqq j \leqq p-1} a_j^-(\beta) v(\beta-\varepsilon_j),$$

*where $v(\beta-\varepsilon_j)=v(\mu_p; \mu_{p-1}-\varepsilon_j; \cdots; \mu_1)$ and $a_j^-(\beta)=a_j^+(\beta-\varepsilon_j)$.*

**2.2. Construction of representations for $U(p, 1)$.** Now we construct a series of representations of $U(p, 1)$, using the results described in the former subsection § 2.1. Let $m=(m_1, \cdots, m_{p-1})$ be a sequence of integers such that $m_1 \geqq m_2 \geqq \cdots \geqq m_{p-1}$ and $(c_1, c_2)$ a pair of complex numbers such that $c_1 + c_2$ is an integer. Let $\Xi = \Xi((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2)$ be a correction of $\mu_p = (\mu_p(1), \cdots, \cdots, \mu_p(p))$ such that $\mu_p \supset m$. Since $K = \operatorname{diag}(U(p), U(1)) \simeq U(p) \times U(1)$, an irreducible representation $\xi$ of $K$ is determined by the highest weight of $\xi|_{U(p)}$ and the representation $\xi|_{U(1)}$. Take $\mu_p \in \Xi$, and we put $\xi = V(\mu_p) \otimes \exp 2\pi i c(\mu_p)$, where $V(\mu_p)$ is a finite dimensional representation of $U(p)$ with highest weight $\mu_p$ and $c(\mu_p)$ is some integer determined by $\mu_p$ (see Corollary 2.4). By this correspondence, we identify the elements in $\Xi$ and the irreducible representations of $K$. Put

$$\mathscr{E} = \mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2) = \sum_{\mu_p \in \Xi}^{\oplus} V(\mu_p) \otimes \exp 2\pi i c(\mu_p),$$

So $\mathscr{E}$ is a $K$-module. Put $\mathfrak{g} = \mathfrak{u}(p, 1)$. We want to define an action of $\mathfrak{g}_C$ on this space $\mathscr{E}$ in order to get an admissible $(\mathfrak{g}_C, K)$-module. Put $e^+ = e_p^+ = e_{p,p+1}$ and $e^- = e_p^- = e_{p+1,p}$. Then $\mathfrak{g}_C$ is generated as a Lie algebra by $\mathfrak{u}(p)$ and $\{e^+, e^-\}$. Therefore we only need to give a formula for $e^{\pm}$. Put $l_i = m_i - i$ $(1 \leqq i \leqq p-1)$ and $\nu_j(i) = \mu_j(i) - i$ $(1 \leqq i \leqq j)$. For a vector $v(\beta) \in V_\xi$ in Gelfand-Zetlin basis for $V(\mu_p) = V_\xi$ (as a space), we define an action of $e^{\pm}$ by

$$e^{\pm} v(\beta) = \sum_{1 \leqq j \leqq p} b_j^{\pm}(\beta) v(\beta \pm \varepsilon_j),$$

where $v(\beta \pm \varepsilon_j) = v(\mu_p \pm \varepsilon_j; \mu_{p-1}; \cdots; \mu_1)$ and

$$\begin{aligned} b_j^+(\beta) = {} & (\nu_p(j) - (c_1 - 1)) \{ \prod_{1 \leqq i \leqq p-1} ((l_i - 1) - \nu_p(j))(\nu_{p-1}(i) - \nu_p(j) - 1) \}^{1/2} \\ & \times \{ \prod_{\substack{i \neq j \\ 1 \leqq i \leqq p}} (\nu_p(i) - \nu_p(j)) \prod_{1 \leqq i \leqq p} (\nu_p(i) - \nu_p(j) - 1) \}^{-1/2}, \end{aligned}$$

$$\begin{aligned} b_j^-(\beta) = {} & (\nu_p(j) - c_2) \{ \prod_{1 \leqq i \leqq p-1} (l_i - \nu_p(j))(\nu_{p-1}(i) - \nu_p(j)) \}^{1/2} \\ & \times \{ \prod_{1 \leqq j \leqq p} (\nu_p(i) - \nu_p(j) + 1) \prod_{\substack{1 \leqq i \leqq p \\ i \neq j}} (\nu_p(i) - \nu_p(j)) \}^{-1/2}. \end{aligned}$$

**Proposition 2.2.** *By actions of $e^{\pm}$ given above, $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1);$ $c_1, c_2)$ is an admissible $(\mathfrak{g}_C, K)$-module for $G = U(p, 1)$.*

*Proof.* We must check that $\mathscr{E}$ is a $\mathfrak{g}_C$-module. Then $K$-finiteness and compatibility of actions of $\mathfrak{g}_C$ and $K$ are automatically satisfied by the definition of $\mathscr{E}$. Let $h_i = e_{i,i}$ $(1 \leqq i \leqq p)$.

If we establish the following relations on $\mathscr{E}$, $\mathscr{E}$ is a $\mathfrak{g}_C$-module:

(1) $[[e_{i,p}, e^+], e^+] = 0 \qquad (1 \leqq i \leqq p-1),$

(2) $[[e_{p,i}, e^-], e^-]=0$ $(1\leqq i\leqq p-1)$,
(3) $[e^+, e_{i,j}]=0$ $(1\leqq i\leqq p,\ 1\leqq j\leqq p-1)$,
(4) $[e^-, e_{i,j}]=0$ $(1\leqq i\leqq p-1,\ 1\leqq j\leqq p)$,
(5) $[e_{i,j}, h_{p+1}]=0$ $(1\leqq i, j\leqq p)$, where $h_{p+1}=h_p-[e^+, e^-]$,
(6) $[h_p, e^\pm]=\pm e^\pm$,
(7) $[h_{p+1}, e^\pm]=\mp e^\pm$.

The formulae (3), (4) and (6) is a consequence of easy calculations. For the formulae (5) and (7), we use the following lemma.

**Lemma 2.3.** *Let $a=(a_i)$, $b=(b_i)$ and $c=(c_i)$ be vectors of dimension $p+1$, $p$ and $p-1$ respectively. Put*

$$d_j(a, b, c)=\{\prod_{1\leqq i\leqq p+1}(a_i-b_j)\prod_{1\leqq i\leqq p-1}(c_i-b_j-1)\}\times\{\prod_{\substack{1\leqq i\leqq p\\ i\neq j}}(b_i-b_j)\prod_{1\leqq i\leqq p}(b_i-b_j-1)\}^{-1}.$$

*Then we have*

$$\sum_{1\leqq j\leqq p}d_j(a, b, c)-\sum_{1\leqq j\leqq p}d_j(a, b-\varepsilon_j, c)=\sum_{1\leqq k\leqq p+1}a_k-2\sum_{1\leqq k\leqq p}b_k+\sum_{1\leqq k\leqq p-1}c_k+1.$$

*Proof.* If $a$, $b$ and $c$ are contained in the lattice of integers and satisfy

$$a_i>b_i>a_{i-1},\qquad b_i>c_i>b_{i-1},$$

then the formula is valid. In fact, put $a'=(a_i+i)$, $b'=(b_i+i)$ and $c'=(c_i+i)$. Consider the representation of $\mathfrak{u}(p+1)$ with highest weight $a'$. Now if $v=v(a'; b'; c'; \cdots)$ is a vector from Gelfand-Zetlin basis, we have

$$\sum_{1\leqq j\leqq p}d_j(a, b, c)=\|e_{p,p+1}v\|^2,\qquad \sum_{1\leqq j\leqq p}d_j(a, b-\varepsilon_j, c)=\|e_{p+1,p}v\|^2$$

and the right hand side of the formula is equal to $((h_{p+1}-h_p)v, v)$. Since $\|e_{p,p+1}v\|^2-\|e_{p+1,p}v\|^2=((h_{p+1}-h_p)v, v)$, we have the formula. Now the both hand sides of the formula are rational functions of $a$, $b$ and $c$, we conclude that it is valid for any $a$, $b$ and $c$. Q.E.D.

By this lemma, one can easily get the action of $h_{p+1}$:

$$h_{p+1}v(\beta)=(c_1+c_2+\sum_{1\leqq i\leqq p-1}l_i-\sum_{1\leqq j\leqq p}\nu_p(j))v(\beta).$$

Using this formula of $h_{p+1}$, we can check (5) and (7) soon. The formulae

(1) and (2) are rather complicated, but the straight forward calculations, using the results of [43], lead us to them. Q.E.D.

**Corollary 2.4.** *As a representation of $K$, $\mathscr{E}$ is decomposed as*

$$\mathscr{E}=\sum_{\mu_p\in\Xi}^{\oplus} V(\mu_p)\otimes \exp 2\pi i\, c(\mu_p),$$

*where* $c(\mu_p)=c_1+c_2+\sum_{1\leq i\leq p-1} l_i-\sum_{1\leq j\leq p}\nu_p(j)=c_1+c_2+\sum_{1\leq i\leq p-1} m_i-\sum_{1\leq j\leq p}\mu_p(j)+p$.

*Proof.* This is clear from the formula of $h_{p+1}$ in the above. Q.E.D.

**2.3. Decompositions of the representations for $U(p, 1)$.** We describe the structure of the representation $\mathscr{E}((m_i\,|\,1\leqq i\leqq p-1);\, c_1, c_2)$ in this subsection. At first, we assume that the elements of the set $\{l_i\,|\,1\leqq i\leqq p-1\}\cup\{c_1, c_2\}$ are all distinct. We call this situation "regular".

**Case 1.** If neither $c_1$ nor $c_2$ is an integer, $\mathscr{E}((m_i\,|\,1\leqq i\leqq p-1);\, c_1, c_2)$ is irreducible. In fact, any non-zero invariant subspace of $\mathscr{E}$ contains some $v(\beta)$. But $v(\beta)$ is clearly cyclic by the definition of the actions of $e^{\pm}$. Hence there is no nontrivial invariant subspace.

Next we assume both $c_1$ and $c_2$ are integers. Note that $c_1+c_2$ is an integer, hence $c_1$ is an integer if and only if so is $c_2$. Reorder $\{l_i\,|\,1\leqq i\leqq p-1\}$, $c_1$ and $c_2$, and put them $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_p)$, where $\alpha_0>\alpha_1>\cdots>\alpha_p$.

**Case 2.** If $\alpha_i=c_1$, $\alpha_{j-1}=c_2$ and $j\neq i, i+1, i+2$, then $\mathscr{E}((m_i\,|\,1\leqq i\leqq p-1);\, c_1, c_2)$ has four irreducible component. One can easily verify that $\mathscr{E}$ has the unique irreducible subrepresentation which we denote by $D_{i,j}(\alpha)$. The set of $K$-types of $D_{i,j}(\alpha)$ is given by

$$\Xi_{i,j}=\{\mu_p\,|\,\nu_p(k)\ (1\leqq k\leqq p) \text{ is not contained in the intervals } [\alpha_j, \alpha_{i-1}) \text{ nor } [\alpha_j, \alpha_{j-1})\},$$

where we put $\alpha_{-1}=-\infty$ and $\alpha_{p+1}=\infty$. Similarly, $\mathscr{E}$ has the unique irreducible quotient $D_{i+1,j-1}(\alpha)$ and two subquotient representations $D_{i+1,j}(\alpha)$ and $D_{i,j-1}(\alpha)$. Remark that $D_{i,j}(\alpha)$ and $D_{j,i}(\alpha)$ (i.e. the unique irreducible subrepresentation of $\mathscr{E}$ for $\alpha_j=c_1$ and $\alpha_{i-1}=c_2$) are isomorphic.

**Case 3.** If $\alpha_i=c_1$ and $\alpha_{i+1}=c_2$, then $\mathscr{E}$ has three irreducible component. In this case, $\mathscr{E}$ has the unique subrepresentation $D_{i,i+2}(\alpha)$ and two irreducible quotient representations $D_{i,i+1}(\alpha)$ and $D_{i+1,i+2}(\alpha)$. Their $K$-types are given by $\Xi_{i,i+2}$, $\Xi_{i,i+1}$ and $\Xi_{i+1,i+2}$ respectively.

**Case 4.** If $\alpha_i = c_2$ and $\alpha_{i+1} = c_1$, then $\mathscr{E}$ has three irreducible component as above. Contrary to Case 3, in this case $\mathscr{E}$ has two irreducible subrepresentations $D_{i,i+1}(\alpha)$ and $D_{i+1,i+2}(\alpha)$ and the unique irreducible quotient $D_{i,i+2}(\alpha)$.

Next we consider singular cases.

**Case S1.** The case where $c_1 \in \{l_i \mid 1 \leqq i \leqq p-1\}$ and $c_2 \notin \{l_i \mid 1 \leqq i \leqq p-1\}$. Let $\alpha$ be as above. Then $\alpha_{i-1} = \alpha_i = c_1$ and $\alpha_{j-1} = c_2$ for some $i$ and $j$. In this case $\mathscr{E}$ has two irreducible components $D_{i,j}(\alpha)$ and $D_{i,j-1}(\alpha)$. $D_{i,j}(\alpha)$ is the unique irreducible subrepresentation of $\mathscr{E}$, whose $K$-types are given by

$$\begin{aligned}\Xi_{i,j}(\alpha) &= \{\mu_p \in \Xi \mid \nu_p(k)\ (1 \leqq k \leqq p) \text{ is not contained in the intervals } [\alpha_i, \alpha_{i-1}) \text{ nor } [\alpha_j, \alpha_{j-1})\} \\ &= \{\mu_p \in \Xi \mid \nu_p(k)\ (1 \leqq k \leqq p) \text{ is not contained in the interval } [\alpha_j, \alpha_{j-1})\}.\end{aligned}$$

The representation $D_{i,j-1}(\alpha)$ is unique irreducible quotient of $\mathscr{E}$, whose $K$-types are given by $\Xi_{i,j-1}(\alpha)$.

**Case S2.** The case where $c_1 \notin \{l_i \mid 1 \leqq i \leqq p-1\}$ and $c_2 \in \{l_i \mid 1 \leqq i \leqq p-1\}$. Let $\alpha_i = c_1$ and $\alpha_{j-1} = \alpha_j = c_2$. Then $\mathscr{E}$ has two irreducible components $D_{i,j}(\alpha)$ and $D_{i+1,j}(\alpha)$. $D_{i,j}(\alpha)$ is the unique irreducible quotient with $K$-types $\Xi_{i,j}(\alpha)$ and $D_{i+1,j}(\alpha)$ is the unique irreducible subrepresentation with $K$-types $\Xi_{i+1,j}(\alpha)$.

**Case S3.** The case where $c_1 = c_2$ and $c_1 \notin \{l_i \mid 1 \leqq i \leqq p-1\}$. Let $\alpha_{i-1} = \alpha_i = c_1 = c_2$. Then $\mathscr{E}$ has two irreducible components $D_{i,i-1}(\alpha)$, and $D_{i,i+1}(\alpha)$, which are both subrepresentations of $\mathscr{E}$.

**Case S4.** The case where $c_1, c_2 \in \{l_i \mid 1 \leqq i \leqq p-1\}$. Let $\alpha_{i-1} = \alpha_i = c_1$ and $\alpha_{j-1} = \alpha_j = c_2$ $(i \neq j)$. Then $\mathscr{E}$ is irreducible and denoted by $D_{i,j}(\alpha)$.

Since the elements of $\{l_i \mid 1 \leqq i \leqq p-1\}$ are all distinct, the above cases S1–S4 exhaust all the singular cases.

**Lemma 2.5.** (1) *The infinitesimal character for the representation* $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1);\, c_1, c_2)$ *is* $(l_1, \cdots, l_{p-1}, c_1, c_2) + \frac{1}{2}(p, p, \cdots, p)$.

(2) *The infinitesimal character for the representations* $D_{i,j}(\alpha)$ $(0 \leqq i < j \leqq p+1)$ *is given by* $(\alpha_0, \alpha_1, \cdots, \alpha_p) + \frac{1}{2}(p, p, \cdots, p)$.

*Proof.* If both $c_1$ and $c_2$ are integers, then the infinitesimal character is calculated out, using the very same formula by Zhelobenko for $\mathfrak{g}_{p+1} =$

$\mathfrak{gl}(p+1, C)$. If neither $c_1$ nor $c_2$ is not an integer, we get the infinitesimal character by analytic continuation. Q.E.D.

## § 3. Classification of irreducible representations of $U(p, 1)$

**3.1. Principal series representations for** $U(p, 1)$. Let $P=MAN$ be a minimal parabolic subgroup of $G$. Take an irreducible representation $\sigma$ of $M$ and a (non-unitary) character $\exp \Upsilon$ of $A$, where $\Upsilon \in \mathfrak{a}_C^*$ and $\exp \Upsilon\,(\mathrm{roth}(a)) = \exp \Upsilon\,(a)$. Note that $\sigma$ may be taken to be a unitary finite dimensional representation since $M$ is compact. Let $T(\sigma, \Upsilon)$ be a $K$-finite induced representation $\mathrm{Ind}_K\,(\sigma \otimes \exp \Upsilon \otimes 1_N;\, P \uparrow G)$. We call the representation $T(\sigma, \Upsilon)$ a *(non-unitary) principal series representation induced from P*. We want to study the structure of $T(\sigma, \Upsilon)$.

Since $M$ is naturally isomorphic to $U(p-1) \times U(1)$, $\sigma$ is determined by a highest weight for $U(p-1)$ and a unitary character for $U(1)$. Let $m=(m_1, m_2, \cdots, m_{p-1})=(m_i \mid 1 \leqq i \leqq p-1)$ be the highest weight and $\exp 2\pi i(c_1+c_2+p)$ the unitary character for $\sigma$, where $\exp 2\pi ik$: $\mathrm{diag}(1, \cdots, 1, u, u) \to u^k$ $(u \in U(1))$. Put $\Upsilon = c_2 - c_1$.

**Lemma 3.1.** *When $T(\sigma, \Upsilon)$ is considered as a representation of K, $T(\sigma, \Upsilon)$ breaks up as follows:*

$$T(\sigma, \Upsilon)|_K = \sum_{\mu_p \supset m}^{\oplus} V(\mu_p) \otimes \exp 2\pi i\, c(\mu_p).$$

*where* $c(\mu_p) = c_1 + c_2 + \sum_{1 \leq i \leq p-1} m_i - \sum_{1 \leq i \leq p} \mu_p(j) + p$.

*Proof.* As a representation of $K$, $T(\sigma, \Upsilon)|_K$ is isomorphic to $\mathrm{Ind}\,(\sigma;\, M \uparrow K)$. By Frobenius' reciprocity theorem, $T(\sigma, \Upsilon)|_K$ is multiplcity free and $\xi \in K^\wedge$ appears in $T(\sigma, \Upsilon)|_K$ if and only if $\xi|_M$ contains $\sigma$. This is equivalent to the formula $\xi = V(\mu_p) \otimes \exp 2\pi i\, c(\mu_p)$, where $\mu_p$ satisfies $\mu_p(1) \geqq m_1 \geqq \mu_p(2) \geqq m_2 \geqq \cdots \geqq m_{p-1} \geqq \mu_p(p)$. Q.E.D.

Now we study the relation between $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1);\, c_1, c_2)$ and $T(\sigma, \Upsilon)$. Before this, we note the following, so-called *subrepresentation theorem.*

**Theorem 3.2** (Harish-Chandra, Casselman [5, Th. 8.21]). *Every irreducible admissible $(\mathfrak{g}_C, K)$-module appears as a subrepresentation of a certain principal series representation induced from a minimal parabolic subgroup.*

**Lemma 3.3.** *If neither $c_1$ nor $c_2$ is an integer, then $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1);\, c_1, c_2)$ is isomorphic to $T(\sigma, \Upsilon)$ as a $(\mathfrak{g}_C, K)$-module, where $\sigma \simeq V(m) \otimes \exp 2\pi i(c_1+c_2+p)$ and $\Upsilon = c_2 - c_1$.*

**Remark.** It is well-known that $T(\sigma, \gamma) \simeq T(\sigma, -\gamma)$ if $\gamma$ is not an integer, i.e., neither $c_1$ nor $c_2$ is an integer.

*Proof.* If neither $c_1$ nor $c_2$ is an integer, then $T(\sigma, \gamma)$ is irreducible with an infinitesimal character $(l_1, \cdots, l_{p-1}, c_1, c_2)+\frac{1}{2}(p, p, \cdots, p)$. By subrepresentation theorem there is no representation with an infinitesimal character $(l_1, \cdots, l_{p-1}, c_1, c_2)+\frac{1}{2}(p, p, \cdots, p)$ other than $T(\sigma, \gamma)$ and $T(\sigma, -\gamma)$. On the other hand, by Lemma 2.5, $\mathscr{E}$ has the same infinitesimal character $(l_1, \cdots, l_{p-1}, c_1, c_2)+\frac{1}{2}(p, p, \cdots, p)$. Hence $\mathscr{E}$ and $T(\sigma, \gamma)$ are isomorphic. Q.E.D.

Fix $m=(m_i \mid 1 \leqq i \leqq p-1)$ and $c_1+c_2$. Let $I(\gamma)$: $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2) \rightarrow T(\sigma, \gamma)$ be an intertwining operator between $(\mathfrak{g}_C, K)$-modules. We realize the representation space $\underline{T}(\sigma, \gamma)$ of $T(\sigma, \gamma)$ as follows. We define

$$T^{\sim}(\sigma, \gamma)=\{f\colon K \rightarrow V(\sigma) \mid f \in L^2(K; V(\sigma)) \text{ and } f(km)=\sigma(m^{-1})f(k) \text{ for } \forall m \in M\},$$

on which $g \in G$ acts as

$$(gf)(k)=\exp\{-(\gamma+p)a(g^{-1}k)\}f(\kappa(g^{-1}k)) \qquad (k \in K),$$

where $g^{-1}k=\kappa(g^{-1}k) \cdot \operatorname{roth}(a(g^{-1}k)) \cdot n$ is Iwasawa decomposition of $g^{-1}k$. Put $\underline{T}(\sigma, \gamma)=\{v \in T^{\sim}(\sigma, \gamma) \mid v \text{ is a } K\text{-finite vector}\}$. Fix a Gelfand-Zetlin basis $\{u(\beta)=u(\mu_p; \mu_{p-1}; \cdots; \mu_1)\}$ for $\underline{T}(\sigma, \gamma)$ such that

$$\int_K \|u(\beta)(k)\|^2_{V(\sigma)}\, dk=1.$$

Then clearly we get $I(\gamma)(v(\beta))=d(\beta, \gamma)u(\beta)$ for some constant $d(\beta, \gamma)$. Since representations of $K$ is written down actually in the same way both for the basis $\{v(\beta)\}$ and $\{u(\beta)\}$, the constant $d(\beta, \gamma)$ depends only on $\mu_p$ (the highest weight for a representation of $U(p)$) and $\gamma$.

**Proposition 3.4.** *Two $(\mathfrak{g}_C, K)$-modules $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1. c_2)$ and $T(\sigma, \gamma)$ have the same irreducible components, where $\gamma=c_2-c_1$.*

*Proof.* By Lemma 3.3, we can assume that both $c_1$ and $c_2$ are integers without loss of generality. Reorder $\{l_i \mid 1 \leqq i \leqq p-1\} \cup \{c_1, c_2\}$ and put them $\alpha=(\alpha_0, \cdots, \alpha_p)$, where $\alpha_0 \geqq \alpha_1 \geqq \cdots \geqq \alpha_p$. At first, we assume that $\alpha$ is regular. Put $e^+=e_{p,p+1}$ and $e^-=e_{p+1,p}$ as before. Then we can write

$$e^{\pm}u(\beta)=\sum_{1 \leqq j \leqq p} \Psi_j^{\pm}(\beta, \gamma)u(\beta \pm \varepsilon_j),$$

where $u(\beta\pm\varepsilon_j)=u(\mu_p\pm\varepsilon_j;\ \mu_{p-1};\ \cdots;\ \mu_1)$ and $\Psi_j^{\pm}(\beta, \gamma)$ are some constants. Moreover $\Psi_j^{\pm}(\beta, \gamma)$ are analytic functions of $\gamma$. In fact, put $E^+=(e^+ + e^-)/2$ and $E^-=i(e^+ - e^-)/2$. Then $E^{\pm}$ is contained in $\mathfrak{g}$. Now we have

$$\Psi_j^+(\beta, \gamma)=(e^+u(\beta),\ u(\beta+\varepsilon_j))=((E^+ - iE^-)u(\beta),\ u(\beta+\varepsilon_j)).$$

Put $\exp tE^{\pm}=g^{\pm}(t)$. For each $k\in K$, we decompose $g^{\pm}(-t)k$ along the Iwasawa decomposition: $kg^{\pm}(-t)=\kappa^{\pm}(k, t)\cdot \mathrm{roth}\,(a^{\pm}(k, t))\cdot n^{\pm}(k, t)\in KAN$. Then the right hand side of the above formula becomes

$$\begin{aligned}&\int_K \frac{d}{dt}\exp(-(\gamma+p)a^+(k, t))\cdot(u(\beta)(\kappa^+(k, t)),\ u(\beta+\varepsilon_j)(k))_{V(\sigma)}\Big|_{t=0}dk\\ &\quad -i\int_K \frac{d}{dt}\exp(-(\gamma+p)a^-(k, t))\cdot(u(\beta)(\kappa^-(k, t)),\ u(\beta+\varepsilon_j)(k))_{V(\sigma)}\Big|_{t=0}dk\\ &=-(\gamma+p)\int_K \{(a^+)'(k, 0)-i(a^-)'(k, 0)\}(u(\beta)(k),\ u(\beta+\varepsilon_j)(k))_{V(\sigma)}dk\\ &\quad +\int_K \{(u(\beta)((\kappa^+)'(k, 0)),\ u(\beta+\varepsilon_j)(k))_{V(\sigma)}\\ &\qquad\qquad -i(u(\beta)((\kappa^-)'(k, 0)),\ u(\beta+\varepsilon_j)(k))_{V(\sigma)}\}dk,\end{aligned}$$

where $(\ ,\ )_{V(\sigma)}$ is the inner product of $V(\sigma)$ and $'$ means differential with respect to $t$. Clearly $\Psi_j^{\pm}(\beta, \gamma)$ is a first order polynomial in $\gamma$. Since $I(\gamma)\colon \mathscr{E}((m_i\,|\,1\leqq i\leqq p-1);\ c_1, c_2)\to T(\sigma, \gamma)$ is an intertwining operator for $c_1, c_2\notin Z$, it holds that $I(\gamma)e^{\pm}v(\beta)=e^{\pm}I(\gamma)v(\beta)$. We have

$$\begin{aligned}I(\gamma)e^{\pm}v(\beta)&=I(\gamma)\sum_{1\leqq j\leqq p} b_j^{\pm}(\beta, \gamma)v(\beta\pm\delta_{jj})\\ &=\sum_{1\leqq j\leqq p} b_j^{\pm}(\beta, \gamma)I(\gamma)v(\beta\pm\varepsilon_j)\\ &=\sum_{1\leqq j\leqq p} b_j^{\pm}(\beta, \gamma)d(\beta\pm\varepsilon_j, \gamma)u(\beta\pm\varepsilon_j).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}e^{\pm}I(\gamma)v(\beta)&=e^{\pm}d(\beta, \gamma)u(\beta)\\ &=\sum_{1\leqq j\leqq p} d(\beta, \gamma)\Psi_j^{\pm}(\beta, \gamma)u(\beta\pm\varepsilon_j).\end{aligned}$$

Comparing the both formulae, we get

$$(*)\qquad b_j^{\pm}(\beta, \gamma)d(\beta\pm\varepsilon_j, \gamma)=d(\beta, \gamma)\Psi_j^{\pm}(\beta, \gamma).$$

Conversely, assume that an operator $I(\gamma)\colon \mathscr{E}=\mathscr{E}((m_i\,|\,1\leqq i\leqq p-1);\ c_1, c_2)\to T(\sigma, \gamma)$ is given by $I(\gamma)v(\beta)=d(\beta, \gamma)u(\beta)$, where $d(\beta, \gamma)$ depends only on $\mu_p$ and $\gamma$. Then $I(\gamma)$ is an intertwining operator if and only if $d(\beta, \gamma)$ satisfies (*) for any $\beta$. Now let both $c_1$ and $c_2$ be integers and put $\gamma_0=c_2-c_1$. We assume that $(\alpha_i, \alpha_{j-1})=(c_1, c_2)$ $(i<j)$. Take $\mu'_p \in \Xi_{i+1,j-1}$ such that $\nu'_p(i+1)=c_1$ and $\nu'_p(j-1)=c_2-1$ (recall that $\nu'_p(k)=\mu'_p(k)-k$). Starting from this $\mu'_p$, we define $d(\beta, \gamma)=d(\mu_p, \gamma)$ by the recurrence formula:

$$(**)\qquad \begin{aligned} &d(\mu'_p, \gamma)=1, \\ &d(\beta\pm\varepsilon_k, \gamma)=d(\beta, \gamma)\frac{\Psi_k^\pm(\beta, \gamma)}{b_k^\pm(\beta, \gamma)}. \end{aligned}$$

For $\gamma=\gamma_0$, we can avoid the zero of $b_k^\pm(\beta, \gamma_0)$ hence $d(\beta, \gamma_0)$ is well-defined for any $\beta \in \Xi((m_i \mid 1\leqq i\leqq p-1);\, c_1, c_2)$. Clearly $d(\beta, \gamma_0)$ satisfies (*) by analytic continuation. Therefore we have an intertwining operator $I(\gamma_0)\colon \mathscr{E}\to T(\sigma, \gamma_0)$. Since $d(\mu'_p, \gamma_0)\neq 0$, the image of $I(\gamma_0)$ must contain an irreducible component $D_{i+1,j-1}(\alpha)$. There exist the following four possibilities.

(a) The case where $\Psi_{i+1}^-(\mu'_p, \gamma_0)\neq 0$ and $\Psi_{j-1}^+(\mu'_p, \gamma_0)\neq 0$. In this case, we have

$$d(\mu'_p-\varepsilon_{i+1}, \gamma_0)\neq 0, \quad d(\mu'_p+\varepsilon_{j-1}, \gamma_0)\neq 0 \quad \text{and} \quad d(\mu'_p-\varepsilon_{i+1}+\varepsilon_{j-1}, \gamma_0)\neq 0,$$

using the formula (**). Since $\mu'_p-\varepsilon_{i+1} \in \Xi_{i,j-1}$, $\mu'_p+\varepsilon_{j-1} \in \Xi_{i+1,j}$ and $\mu'_p-\varepsilon_{i+1}+\varepsilon_{j-1} \in \Xi_{i,j}$, we conclude that the image of $I(\gamma_0)$ contains $D_{i,j-1}(\alpha)$, $D_{i+1,j}(\alpha)$ and $D_{i,j}(\alpha)$. Therefore the image of $I(\gamma_0)$ is isomorphic to $\mathscr{E}$. Comparing $K$-types, we get $\mathscr{E}\simeq T(\sigma, \gamma_0)$. Thus we proved the assertion.

(b) The case where $\Psi_{i+1}^-(\mu'_p, \gamma_0)=\Psi_{j-1}^+(\mu'_p, \gamma_0)=0$. We show the following lemma.

**Lemma 3.5.** *Let $\gamma_0\neq 0$ be a real number. If $\Psi_k^+(\beta, \gamma_0)=0$, then we have $\Psi_k^-(\beta+\varepsilon_k, -\gamma_0)=0$. If $\Psi_k^-(\beta, \gamma_0)=0$, then we have $\Psi_k^+(\beta-\varepsilon_k, -\gamma_0)=0$.*

*Proof.* We prove only the first assertion because the second one can be treated in the similar way. Let $\Psi_k^+(\beta, \gamma)=a\gamma+b$ be a first order polynomial in $\gamma$. Then we get

$$\Psi_k^-(\beta+\varepsilon_k, \gamma)=-\bar{a}\gamma+\bar{b}.$$

Since $\gamma_0=-(b/a)$ is real by the assumption, we have

$$\Psi_k^-(\beta+\varepsilon_k, -\gamma_0)=-\bar{a}\frac{b}{a}+\bar{b}=\bar{a}\left(-\frac{b}{a}+\frac{\bar{b}}{\bar{a}}\right)=0. \qquad \text{Q.E.D.}$$

Using an intertwining operator $J(\gamma)$: $\mathscr{E} \rightarrow T(\sigma, -\gamma)$ instead of $I(\gamma)$, we have a relation

$$b_k^{\pm}(\beta, \gamma) d_J(\beta \pm \varepsilon_k, \gamma) = d_J(\beta, \gamma) \Psi_k^{\pm}(\beta, -\gamma),$$

where $d_J(\beta, \gamma)$ is given by $J(\gamma)v(\beta) = d_J(\beta, \gamma)u(\beta)$. Since $b_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, \gamma_0) = \Psi_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, -\gamma_0) = 0$ and $b_{j-1}^{-}(\mu_p' + \varepsilon_{j-1}, \gamma_0) = \Psi_{j-1}^{-}(\mu_p' + \varepsilon_{j-1}, -\gamma_0) = 0$ by Lemma 3.5,

$$\frac{\Psi_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, -\gamma)}{b_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, \gamma)} \quad \text{and} \quad \frac{\Psi_{j-1}^{-}(\mu_p' + \varepsilon_{j-1}, -\gamma)}{b_{j-1}^{-}(\mu_p' + \varepsilon_{j-1}, \gamma)}$$

are non-zero constants. Therefore, if we normalize $d_J(\mu_p', \gamma) = 1$, we see that

$$d_J(\mu_p' - \varepsilon_{i+1}, \gamma) = d_J(\mu_p', \gamma) \frac{b_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, \gamma)}{\Psi_{i+1}^{+}(\mu_p' - \varepsilon_{i+1}, -\gamma)}$$

is non-zero at $\gamma_0$. Similarly we can get $d_J(\mu_p' + \varepsilon_{j-1}, \gamma_0) \neq 0$ and $d_J(\mu_p' - \varepsilon_{i+1} + \varepsilon_{j-1}, \gamma_0) \neq 0$. Therefore $J(\gamma_0)$ is well-defined and is an isomorphism between $\mathscr{E}$ and $T(\sigma, -\gamma_0)$. Since $T(\sigma, \gamma_0)$ and $T(\sigma, -\gamma_0)$ have the same irreducible components, we proved the assertion.

(c) The case where $\Psi_{i+1}^{-}(\mu_p', \gamma_0) = 0$ and $\Psi_{j-1}^{+}(\mu_p', \gamma_0) \neq 0$. In this case, we see that the image of $I(\gamma_0)$ (hence $T(\sigma, \gamma_0)$) contains $D_{i+1, j-1}(\alpha)$ and $D_{i+1, j}(\alpha)$ as irreducible components. Since Image $I(\gamma_0)$ does not contain the $K$-type $\mu_p' - \varepsilon_{i+1} + \varepsilon_{j-1}$ but contains the $K$-type $\mu_p' + \varepsilon_{j-1}$, we get $\Psi_{i+1}^{-}(\mu_p' + \varepsilon_{j-1}, \gamma_0) = 0$. By Lemma 3.5, we get $\Psi_{i+1}^{+}(\mu_p' - \varepsilon_{i+1} + \varepsilon_{j-1}, -\gamma_0) = 0$. If $\Psi_{j-1}^{-}(\mu_p' - \varepsilon_{i+1} + \varepsilon_{j-1}, -\gamma_0) = 0$, then we can prove $\mathscr{E} \simeq T(\sigma, -\gamma_0)$ as in the case (b) and $\mathscr{E}$ has the same irreducible components as $T(\sigma, \gamma_0)$. If $\Psi_{j-1}^{-}(\mu_p' - \varepsilon_{i+1} + \varepsilon_{j-1}, -\gamma_0) \neq 0$, then we can prove that $T(\sigma, -\gamma_0)$ has irreducible components $D_{i, j-1}(\alpha)$ and $D_{i,j}(\alpha)$, using the similar arguments as in the former part of (c). Because $T(\sigma, \gamma_0)$ and $T(\sigma, -\gamma_0)$ have the same irreducible components, $T(\sigma, \gamma_0)$ has irreducible components $D_{i+1, j-1}(\alpha)$, $D_{i+1, j}(\alpha)$, $D_{i, j-1}(\alpha)$ and $D_{i,j}(\alpha)$, which are precisely all the irreducible components of $\mathscr{E}$.

(d) The case where $\Psi_{i+1}^{-}(\mu_p', \gamma_0) \neq 0$ and $\Psi_{j-1}^{+}(\mu_p', \gamma_0) = 0$. This case is similar to (c), so we omit the proof for this case.

Singular cases except the case where $\gamma_0 = 0$ can be treated similarly. So we only prove the case where $\gamma_0 = 0$, i.e., $c_1 = c_2$. Assume that $\alpha_i = \alpha_{i+1} = c_1 = c_2$ and take $\mu_p' \in \Xi((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2)$ such that $\nu_p'(i+1) = c_1 - 1$. If $\Psi_{i+1}^{+}(\mu_p', 0) = 0$, then we define $d(\beta, \gamma)$ by the recurrence formula (**). We can avoid the zero of $b_k^{\pm}(\beta, \gamma)$ except $b_{i+1}^{+}(\mu_p', 0) = 0$. However, since $\Psi_{i+1}^{+}(\mu_p', 0) = 0$, we have

$$\frac{\Psi^+_{i+1}(\mu'_p, \gamma)}{b^+_{i+1}(\mu'_p, \gamma)} \neq 0$$

is a non-zero constant and hence

$$d(\mu'_p+\varepsilon_{i+1}, 0)=d(\mu'_p, 0)\frac{\Psi^+_{i+1}(\mu'_p, \gamma)}{b^+_{i+1}(\mu'_p, \gamma)} \neq 0.$$

Remark that $\mu'_p \in \Xi_{i,i+1}$ and $\mu'_p+\varepsilon_{i+1} \in \Xi_{i+1,i+2}$, and that $\mathscr{E}$ is a direct sum of representations $D_{i,i+1}(\alpha)$ and $D_{i+1,i+2}(\alpha)$. Now we can see that the image of $I(0)$ is isomorphic to $\mathscr{E}$ and $T(\sigma, 0) \simeq \mathscr{E}$.

Next we assume that $\Psi^+_{i+1}(\mu'_p, 0) \neq 0$. Then similar arguments as in the proof of Lemma 3.5 tell us that $\Psi^-_{i+1}(\mu'_p+\varepsilon_{i+1}, 0) \neq 0$. We can define

$$d(\mu'_p+\varepsilon_{i+1}, \gamma)=d(\mu'_p+\gamma)\frac{b^-_{i+1}(\mu'_p+\varepsilon_{i+1}, \gamma)}{\Psi^-_{i+1}(\mu'_p+\varepsilon_{i+1}, \gamma)}.$$

Since $b^-_{i+1}(\mu'_p+\varepsilon_{i+1}, 0)=0$, we have $d(\mu'_p+\varepsilon_{i+1}, 0)=0$. We conclude that the image of $I(0)$ is isomorphic to $D_{i,i+1}(\alpha)$ which is a subrepresentation of $T(\sigma, 0)$. But this means $\Psi^+_{i+1}(\mu'_p, 0)=0$ and we have a contradiction.

Q.E.D.

For equivalence between $\mathscr{E}$ and $T(\sigma, \gamma)$, we have the following.

**Proposition 3.6.** *Assume that* $\gamma_0=c_2-c_1 \neq 0$ *satisfies one of the following conditions* (a)–(d):

(a) $c_1 \geqq m_1$, $m_{p-1} \geqq c_2+p$, (b) $c_2 \geqq m_1$, $m_{p-1} \geqq c_1+p$,

(c) $c_1 \geqq m_1$, $c_2 \geqq m_1$, (d) $m_{p-1} \geqq c_1+p$, $m_{p-1} \geqq c_2+p$.

*Then* $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2)$ *is isomorphic to* $T(\sigma, \gamma_0)$ *as a* $(\mathfrak{g}_C, K)$-*module,*

*Proof.* At first we prove the following lemmas.

**Lemma 3.7.** *Take integers* $c_1$ *and* $c_2$ *such that* $c_1 \geqq m_1 \geqq m_2 \geqq \cdots \geqq m_{p-1} \geqq c_2+p$. *Then the representations* $\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2)$ *and* $T(\sigma, c_2-c_1)$ *contains the same irreducible finite dimensional representation of* $U(p, 1)$ *with highest weight* $(c_1, m_1, m_2, \cdots, m_{p-1}, c_2+p)$ *as a subrepresentation.*

*Proof.* For the representation $\mathscr{E}=\mathscr{E}((m_i \mid 1 \leqq i \leqq p-1); c_1, c_2)$, we know from § 2.3 that $\mathscr{E}$ contains an irreducible finite dimensional representation denoted by $D_{0,p+1}(\alpha)$. The highest weight of $D_{0,p+1}(\alpha)$ is precisely $(c_1, m_1, m_2, \cdots, m_{p-1}, c_2+p)$.

Let us consider the representation $T(\sigma, c_2-c_1)$. Denote $F$ the irreducible finite dimensional representation with highest weight $(c_1, m_1, m_2, \cdots, m_{p-1}, c_2+p)$. By Frobenius' reciprocity theorem in [6], We can tell which principal series representation $F$ is embedded into, if we know the $\mathfrak{n}$-cohomology of $F$. Put

$$\mathfrak{u}_C = \sum_{1\leq i<j\leq p}^{\oplus} Ce_{i,j} + \sum_{1\leq j\leq p}^{\oplus} Ce_{p+1,j}.$$

$\mathfrak{u}_C$ is a nilpotent subalgebra which determines a positive system $\Delta_{\mathfrak{u}}^+ = \{\varepsilon_i-\varepsilon_j \mid 1\leqq i\leqq j\leqq p\} \cup \{\varepsilon_{p+1}-\varepsilon_j \mid 1\leqq j\leqq p\}$ of roots. Let $v\in F$ be a "highest weight" vector for $\Delta_{\mathfrak{u}}^+$, i.e., $xv=0$ for any $x\in\mathfrak{u}_C$. Remark that the weight of $v$ is given by $(m_1, m_2, \cdots, m_{p-1}, c_2+p, c_1)$. Since $\mathfrak{g}_C = \mathfrak{m}_C + \mathfrak{a}_C + \mathfrak{n}_C + \mathfrak{u}_C$, we can easily check that $U(\mathfrak{g}_C)v = U(\mathfrak{m}_C)v + \mathfrak{n}_C F$. This means $F/\mathfrak{n}_C F \simeq U(\mathfrak{m}_C)v$ and we know that the $\mathfrak{n}$-cohomology $F/\mathfrak{n}_C F$ is isomorphic to an irreducible finite dimensional representation $V(m_1, m_2, \cdots, m_{p-1}) \otimes \exp 2\pi i\,(c_1+c_2+p)$ as an $\mathfrak{m}_C$-module. As an $\mathfrak{a}_C$-module, $F/\mathfrak{n}_C F$ is isomorphic to $\exp(c_2+p-c_1)$. In fact, put

$$A = \begin{pmatrix} 0_{p-1} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \in \mathfrak{a}.$$

We decompose $A$ as

$$A = \begin{pmatrix} 0_{p-1} & & \\ & 0 & 0 \\ & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0_{p-1} & & \\ & -1 & 1 \\ & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0_{p-1} & & \\ & 1 & 0 \\ & 0 & -1 \end{pmatrix}$$

and then operate $A$ on $v$. The first matrix is contained in $\mathfrak{u}_C$ and the second is contained in $\mathfrak{n}_C$. Therefore we have $Av = (c_2+p-c_1)v$ modulo $\mathfrak{n}_C F$.

Now Frobenius' reciprocity theorem due to Casselman and Osborne tells us that $F$ is imbedded into the principal series representation $\mathrm{Ind}_K(V(m_1, m_2, \cdots, m_{p-1}) \otimes \exp 2\pi i(c_1+c_2+p) \otimes \exp(c_2-c_1) \otimes 1_N; P\uparrow G) = T(\sigma, c_2-c_1)$ (note that trace $\mathrm{ad}(x)|_{\mathfrak{n}_C}=0$ for any $x\in\mathfrak{m}_C$ and trace $\mathrm{ad}(A)|_{\mathfrak{n}_C}=2p$). Q.E.D.

**Lemma 3.8.** *Take integers $c_1$ and $c_2$ such that $c_2 > c_1 \geqq m_1$. Then the representations $\mathscr{E}((m_i \mid 1\leqq i\leqq p-1); c_1, c_2)$ and $T(\sigma, c_2-c_1)$ contains the same irreducible representation $D_{0,1}(\alpha)$ as a subrepresentation.*

*Proof.* We use the notations in Lemma 3.7. Since $D_{0,1}(\alpha)$ is a highest weight module with respect to $\Delta_{\mathfrak{u}}^+$ with highest weight $(c_1, m_1, m_2, \cdots, m_{p-1}, c_2+p)$, the proof of Lemma 3.7 can be applied to this case exactly the same way. Q.E.D.

We return to the proof of Proposition 3.6. Let us consider the case (a). By Lemma 3.7, we can see $\Psi_1^+(\mu_p'', \gamma_0)=\Psi_p^-(\mu_p'', \gamma_0)=0$, where $\mu_p''=(c_1, m_1, m_2, \cdots, m_{p-2}, c_2+p)$. By the arguments in the proof of Proposition 3.4, we have $\mathscr{E}\simeq T(\sigma, \gamma_0)$.

Using Lemma 3.8 instead of Lemma 3.7, we can prove the case $c_2>c_1\geqq m_1$. The cases (b) and $c_1>c_2\geqq m_1$ are contragredient to the cases (a) and $c_2>c_1\geqq m_1$ respectively. The case (d) is proved by the similar lemma as Lemma 3.8 dealing with anti-holomorphic discrete series $D_{p,p+1}(\alpha)$. Q.E.D.

**Theorem 3.9.** *Let $\lambda$ be a dominant infinitesimal character for $U(p, 1)$ and put $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_p)=\lambda-\frac{1}{2}(p, p, \cdots, p)$.*

(1) *If $\{\alpha_i \mid 0\leqq i\leqq p\}$ contains exactly $p-1$ distinct integers and the sum of the rest two is an integer, then there is only one irreducible representation with infinitesimal character $\lambda$, which is an irreducible principal series representation.*

(2) *If $\alpha_i$'s are all distinct integers, then there are $(p+1)(p+2)/2$ irreducible representations with infinitesimal character $\lambda$. They are denoted by $\{D_{i,j}(\alpha) \mid 0\leqq i<j\leqq p+1\}$ as above. $D_{0,p+1}(\alpha)$ is a finite dimensional representation and $D_{i,i+1}(\alpha)$ $(0\leqq i\leqq p)$ is a discrete series representation of $U(p, 1)$. Moreover $D_{0,1}(\alpha)$ and $D_{p,p+1}(\alpha)$ are holomorphic or anti-holomorphic discrete series representations.*

(3) *If $\{\alpha_i \mid 0\leqq i\leqq p\}$ is the set of distinct integers except $\alpha_{i-1}=\alpha_i$, then there are $p+1$ irreducible representations $\{D_{i,j}(\alpha) \mid j\neq i, 0\leqq j\leqq p+1\}$ with infinitesimal character $\lambda$. $D_{i,i-1}(\alpha)$ and $D_{i,i+1}(\alpha)$ are limits of discrete series representations.*

(4) *If $\{\alpha_i \mid 0\leqq i\leqq p\}$ is the set of distinct integers except $\alpha_{i-1}=\alpha_i$ and $\alpha_{j-1}=\alpha_j$ $(i\neq j)$, then there is only one irreducible representation with infinitesimal character $\lambda$, which is an irreducible principal series representation.*

(5) *If $\{\alpha_i \mid 0\leqq i\leqq p\}$ does not satisfies the conditions* (1)–(4), *then there is no representation with infinitesimal character $\lambda$.*

*Proof.* By the subrepresentation theorem Theorem 3.2, irreducible $T(\sigma, \gamma)$'s and $D_{i,j}(\alpha)$'s exhaust all the irreducible admissible representations of $U(p, 1)$. Therefore it is sufficient to prove the statements for finite dimensional rerpesentations and (limits of) discrete series representations.

Since the set of $K$-types of $D_{0,p+1}(\alpha)$ is finite, $D_{0,p+1}(\alpha)$ is a finite dimensional representation.

Let us consider the representation $D_{0,1}(\alpha)$. It contains a highest weight vector $v(\beta)$ for the $K$-type $V(\mu_p)\otimes\exp 2\pi i\, c(\mu_p)$, where $\mu_p=(\alpha_1, \alpha_2+1, \cdots, \alpha_p+p-1)$. The weight of $v(\beta)$ is given by $(\alpha_1, \alpha_2+1, \cdots, \alpha_p+p-1, \alpha_0+p)$. Moreover we can see directly that it is the highest weight for $D_{0,1}(\alpha)$. We have

$$\begin{aligned}(\alpha_1, \alpha_2+1, \cdots, \alpha_p+p-1, \alpha_0+p)+\rho \\ =(\alpha_1, \alpha_2, \cdots, \alpha_p, \alpha_0)+\tfrac{1}{2}(p, p, \cdots, p)\end{aligned}$$

and this clearly satisfies the condition for the highest weight of a holomorphic discrete series representation (see, for example, [38, Prop. 2.3.5]). Similar arguments lead us to that $D_{p,p+1}(\alpha)$ is an (anti-)holomorphic discrete series representation.

The representation $D_{i,i+1}(\alpha)$ $(0\leqq i\leqq p)$ has the minimal $K$-type $V(\mu_p)\otimes\exp 2\pi i\, c(\mu_p)$ where $\mu_p=(\alpha_0, \alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}-1, \alpha_{i+2}-1, \cdots, \alpha_p-1)+(1, 2, \cdots, p)$ and $c(\mu_p)=\alpha_i+p-i$. By Theorem 8.5 of [1], the discrete series representation with Harish-Chandra parameter $(\alpha_0, \alpha_1, \cdots, \hat{\alpha}_i, \cdots, \alpha_p, \alpha_i)+\frac{1}{2}(p, p, \cdots, p)$ contains the same $K$-type, where $\wedge$ means elimination. Since we can easily check that the other representations $D_{i,j}(\alpha)$ does not contain the $K$-type above, $D_{i,i+1}(\alpha)$ is a discrete series representation.

The case of limits of discrete series representation is similar. Q.E.D.

**Remark.** We know the $K$-types of the representation $D_{i,j}(\alpha)$ explicitly. If we sum up the characters for $K$, using Weyl's character formula, then we have the value of the irreducible character of $D_{i,j}(\alpha)$ on the compact Cartan subgroup $B$, at least formally. By Harish-Chandra's famous theorem [12], we know the explicit form of discrete series characters on a compact Cartan subgroup. In this way, we can also see that $D_{i,i+1}(\alpha)$ is a discrete series representation.

For the case (2) of Theorem 3.9, we often use the following convenient diagram.

| | | | | | |
|---|---|---|---|---|---|
| $D_{0,1}(\alpha)$ | $D_{0,2}(\alpha)$ | $D_{0,3}(\alpha)$ | $\cdots$ | $D_{0,p}(\alpha)$ | $D_{0,p+1}(\alpha)$ |
| | $D_{1,2}(\alpha)$ | $D_{1,3}(\alpha)$ | $\cdots$ | $D_{1,p}(\alpha)$ | $D_{1,p+1}(\alpha)$ |
| | | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| | | | $\cdots$ | $\cdots$ | $\cdots$ |
| | | | | $D_{p-1,p}(\alpha)$ | $D_{p-1,p+1}(\alpha)$ |
| | | | | | $D_{p,p+1}(\alpha)$ |

**Figure 3.10.**

In this diagram, four adjacent squares $D_{i,j}(\alpha)$, $D_{i-1,j}(\alpha)$, $D_{i,j+1}(\alpha)$ and $D_{i-1,j+1}(\alpha)$ are the irreducible components of a principal series representation $T(\sigma, \gamma)$, where $\sigma$ and $\gamma$ is given as follows. The representation $\sigma = V(m)\otimes \exp 2\pi i(\alpha_{i-1}+\alpha_j+p)$ is given by $m=(\alpha_0, \cdots, \hat{\alpha}_{i-1}, \cdots, \hat{\alpha}_j, \cdots, \alpha_p)+(1, 2, \cdots, p-1)$ and $\gamma=\alpha_{i-1}-\alpha_j$ $(i<j)$, where $\wedge$ denotes elimination. We also denote this principal series representation $T(\sigma, \gamma)$ as $T_{i-1,j}(\alpha)$. Then we may write as

$$T_{i-1,j}(\alpha)=\begin{pmatrix} D_{i-1,j}(\alpha) & D_{i-1,j+1}(\alpha) \\ D_{i,j}(\alpha) & D_{i,j+1}(\alpha) \end{pmatrix}.$$

In the diagonal of the diagram, putting $D_{i,i}(\alpha)=0$, the above arguments go well, too, i.e.,

$$T_{i-1,i}(\alpha)=\begin{pmatrix} D_{i-1,1}(\alpha) & D_{i-1,i+1}(\alpha) \\ 0 & D_{i,i+1}(\alpha) \end{pmatrix}.$$

## § 4. Irreducible characters for $U(p, 1)$

**4.1. Description of irreducible characters.** In this section, we give irreducible characters for the representations listed in Theorem 3.9. For $(a_k \mid 1\leqq k\leqq m)$ and $(b_k \mid 1\leqq k\leqq n)$, we denote by $((b_k \mid 1\leqq k\leqq n); (a_k \mid 1\leqq k\leqq m))$ an $n\times m$ matrix whose $(k, l)$-element is $b_k^{a_l}$.

**Lemma 4.1.** *Let $\sigma=V(m)\otimes \exp 2\pi i(c_1+c_2+p)$ and $\gamma=c_2-c_1$ as before, where $m=(m_k \mid 1\leqq k\leqq p-1)$ is a highest weight for $U(p-1)$ and $c_1+c_2$ is an integer. Then the character of $T(\sigma, \gamma)$ is given as follows. On the compact Cartan subgroup $B$, it is identically zero. On the maximally split Cartan subgroup $J=B^+A$, it is given by*

$$\begin{aligned}\Theta(\sigma, \gamma)(h)=&(-1)^p D_0(h)^{-1}\\ &\times \det((b_k \mid 1\leqq k\leqq p-1); (l_k \mid 1\leqq k\leqq p-1))b_p^{c_1+c_2}(\exp \gamma a+\exp(-\gamma a)),\end{aligned}$$

*where $h=\mathrm{diag}(b_1, \cdots, b_{p-1}, b_p, b_p)\cdot \mathrm{roth}(a) \in J$ and $l_k=m_k-k$.*

*Proof.* In general, the character of a principal series representation is given on $J$ as

$$\begin{aligned}\Theta(\sigma, \gamma)(h)=&(\sharp W(G; B^+))^{-1}\sum_{w\in W(G;J)}\Theta(\sigma\otimes\exp\gamma)(wh)\\ &\times\prod_{\alpha\in R}|\xi_{\frac{1}{2}\alpha}(wh)-\xi_{-\frac{1}{2}\alpha}(wh)|^{-1},\end{aligned}$$

where $R=\{\alpha \in \Delta^+ \mid \alpha|_{\mathfrak{a}}\neq 0\}$ and $\xi_{\frac{1}{2}\alpha}=\exp(\frac{1}{2}\alpha)$ (see Theorem 2 of [15]). Now straightforward calculations lead us to the result. Q.E.D.

**Remark.** In the formula for a principal series character given in [14, p. 910], the sign is wrong. The sign $(-1)^{p+1}$ there should be replaced by $(-1)^p$.

**Theorem 4.2** (Hirai [14]). *Let $\lambda$ be an infinitesimal character and put $\alpha=\lambda-\frac{1}{2}(p, p, \cdots, p)$. We assume that all the $\alpha_i$'s are integers and $\alpha$ is regular. Then the character of $D_{i,j}(\alpha)$ is given as follows. On the compact Cartan subgroup B, it is given by*

$$D_0\cdot\Theta(D_{i,j}(\alpha))(h)=(-1)^{p+(j-i-1)}\det\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ \underbrace{0,\cdots,0}_{i}\ \underbrace{b_{p+1}^{\alpha_i},\cdots,b_{p+1}^{\alpha_{j-1}}}_{j-i}\ \underbrace{0,\cdots,0}_{p+1-j}\end{pmatrix}$$

*where $h=\mathrm{diag}(b_1, b_2, \cdots, b_{p+1})\in B$. Here we denote the matrix whose first $p$ rows are $((b_k\,|\,1\leqq k\leqq p-1);\ (\alpha_k\,|\,0\leqq k\leqq p))$ and whose $(p+1)$-th row is $(0, \cdots, 0\ b_{p+1}^{\alpha_i}, \cdots, b_{p+1}^{\alpha_{j-1}}\ 0, \cdots, 0)$ by*

$$\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ 0,\cdots,0\ b_{p+1}^{\alpha_i},\cdots,b_{p+1}^{\alpha_{j-1}}\ 0,\cdots,0\end{pmatrix}.$$

*Similarly we denote the matrix whose first $p-1$ rows are $((b_k\,|\,1\leqq k\leqq p-1);$ $(\alpha_k\,|\,0\leqq k\leqq p))$ and whose p-th row (respectively $(p+1)$-th row) is $(c_1, c_2, \cdots, c_{p+1})$ (respectively $(d_1, d_2, \cdots, d_{p+1})$) by*

$$\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p-1);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ c_1, c_2, \cdots, c_{p+1} \\ d_1, d_2, \cdots, d_{p+1}\end{pmatrix}.$$

*On the maximally split Cartan subgroup J, $\Theta(D_{i,j}(\alpha))$ is given by*

$$\begin{aligned} &D_0\cdot\Theta(D_{i,j}(\alpha))(h) \\ &=(-1)^{p+(j-i-1)}\left\{\det\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p-1);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ 0,\cdots,0\ \ h_p^{\alpha_i},\cdots,h_p^{\alpha_{j-1}}\ \ 0,\cdots,0 \\ 0,\cdots,0\ \ h_{p+1}^{\alpha_i},\cdots,h_p^{\alpha_{j-1}}\ \ 0,\cdots,0\end{pmatrix}\right. \\ &\qquad+\det\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p-1);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ h_p^{\alpha_0},\cdots,h_p^{\alpha_{i-1}}\ \ 0,\cdots,0\ \ 0,\cdots,0 \\ 0,\cdots,0\ \ h_{p+1}^{\alpha_i},\cdots,h_{p+1}^{\alpha_{j-1}}\ \ 0,\cdots,0\end{pmatrix} \\ &\qquad\left.+\det\begin{pmatrix} ((b_k\,|\,1\leqq k\leqq p-1);\ (\alpha_k\,|\,0\leqq k\leqq p)) \\ 0,\cdots,0\ \ h_p^{\alpha_i},\cdots,h_p^{\alpha_{j-1}}\ \ 0,\cdots,0 \\ 0,\cdots,0\ \ 0,\cdots,0\ \ h_{p+1}^{\alpha_j},\cdots,h_{p+1}^{\alpha_p}\end{pmatrix}\right\}\end{aligned}$$

*where* $h=\mathrm{diag}(b_1, \cdots, b_{p-1}, b_p, b_p)\cdot\mathrm{roth}(a)\in J$, $h_p=b_p\cdot\exp(-|a|)$ *and* $h_{p+1}=b_p\cdot\exp|a|$.

*Moreover, even if the parameter* $\alpha$ *is singular, the character of* $D_{i,j}(\alpha)$ *is given by the same formula.*

*Proof.* By Lemma 4.1, we know the principal series characters. Therefore, using Figure 3.10 and the comments below it, we know that the character $\Theta(D_{i,j}(\alpha))$ can be calculated from the discrete series characters $\Theta(D_{i,i+1}(\alpha))$ $(0\leqq i\leqq p)$ and principal series characters by additions and subtructions. On the compact Cartan subgroup $B$, a formula for the discrete series characters is given by Harish-Chandra. On a maximally split Cartan subgroup $J$, it is given by T. Hirai [16, 19] (note that their results are applicable to general semisimple Lie groups). Q.E.D.

**4.2. Linear independence of irreducible characters on a maximally split Cartan subgroup.**

**Theorem 4.3.** *Let* $\Theta$ *be a virtual character of* $U(p, 1)$ $(p\geqq 2)$. *If the restriction of* $\Theta$ *to* $J$ *is zero then* $\Theta$ *is identically zero on the whole group* $U(p, 1)$.

*Proof.* Let $\Theta$ be a virtual character with infinitesimal character $\lambda$. If $\lambda$ satisfies the condition of (1) in Theorem 3.9, then the theorem clearly holds. Next we assume that $\lambda$ satisfies the condition of (2) in Theorem 3.9, i.e., $\lambda$ is regular integral. Put $\alpha=\lambda-\frac{1}{2}(p, p, \cdots, p)$ and $\Theta_{ij}=\Theta(D_{i,j}(\alpha))$. Then $\Theta$ is expanded by $\Theta_{i,j}$'s:

$$\Theta=\sum_{0\leq i<j\leq p+1} x_{i,j}\Theta_{i,j} \qquad (x_{i,j}\in \boldsymbol{C}). \tag{4.1}$$

Accodrding to Theorem 4.2, $\Theta|_J$ is written down as

$$D_0\cdot\Theta(h)=\sum_{s\in\mathfrak{S}_{p+1}} c(s)h_{s(1)}^{\alpha_0}h_{s(2)}^{\alpha_1}\cdots h_{s(p+1)}^{\alpha_p},$$

where $h=\mathrm{diag}(b_1, \cdots, b_{p-1}, b_p, b_p)\cdot\mathrm{roth}(a)\in J$, $h_i=b_i$ $(1\leqq i\leqq p-1)$, $h_p=b_p\cdot\exp(-|a|)$ and $h_{p+1}=b_p\cdot\exp|a|$. Note that the Weyl group $W(\mathfrak{g}_C, \mathfrak{j}_C)$ is isomorphic to $\mathfrak{S}_{p+1}$, the permutation group of $\{1, 2, \cdots, p+1\}$. Choose elements $s(i,j)$ and $t(i,j)$ $(0\leqq i<j\leqq p)$ in $\mathfrak{S}_{p+1}$ as

$$s(i,j)=\begin{pmatrix} 1 & 2\cdots i & i+1 & i+2\cdots & j & j+1 & j+2\cdots & p & p+1 \\ 1 & 2\cdots i & p & i+1 & \cdots j-1 & p+1 & j & \cdots p-2 & p-1 \end{pmatrix},$$

$$t(i,j)=(p, p+1)s(i,j).$$

**Lemma 4.4.** *Coefficient* $c(s)$ *is expressed for* $s=s(i,j)$ *or* $t(i,j)$ *as* *follows*:

$$c(t(i,j))=(-1)^{p-1+i+j}\sum_{0\leq l\leq i}\sum_{j+1\leq k\leq p+1}(-1)^{k+l}x_{l,k} \qquad (0\leqq i<j\leqq p),$$

$$c(s(i,i+1))=(-1)^{p-1}\sum_{0\leq l\leq i}\sum_{i+1\leq k\leq p+1}(-1)^{k+l}x_{l,k}$$
$$+(-1)^{p}\sum_{i+2\leq k\leq p+1}(-1)^{k+i}x_{i+1,k} \qquad (0\leqq i\leqq p-1),$$

$$c(s(i,i+2))=(-1)^{p-1}\sum_{0\leq l\leq i}\sum_{i+1\leq k\leq p+1}(-1)^{k+l}x_{l,k}$$
$$+(-1)^{p}\sum_{i+1\leq l\leq i+2}\sum_{i+3\leq k\leq p+1}(-1)^{k+l}x_{l,k} \qquad (0\leqq i\leqq p-2).$$

*Proof.* Direct calculations using Theorem 4.2 lead us to the results. Q.E.D.

The sysetm of equations listed in Lemma 4.4 is invertible. In fact, we have the following.

**Lemma 4.5.** *Put* $c(t(i,j))=0$ *if* $i\leqq -1$ *or* $p+1\leqq j$. *Then we have*

$$x_{0,j}=(-1)^{p}\{c(t(0,j))+c(t(0,j-1))\} \qquad (2\leqq j\leqq p+1),$$

$$x_{i,j}=(-1)^{p}\{c(t(i,j-1))+c(t(i-1,j-1))+c(t(i,j))+c(t(i-1,j))\}$$
$$(i+2\leqq j\leqq p+1),$$

$$x_{0,1}=\tfrac{1}{2}(-1)^{p}\{c(s(0,1))-c(s(0,2))-c(s(1,2))$$
$$+c(t(0,1))+c(t(0,2))+c(t(1,2))\},$$

$$x_{i,i+1}=\tfrac{1}{2}(-1)^{p}\{c(s(0,1))+c(s(0,2))+c(s(1,2))$$
$$-c(t(0,1))-c(t(0,2))-c(t(1,2))\}$$
$$+(-1)^{p-1}\{\sum_{0\leq k\leq i-2}\{c(s(k,k+1))+c(s(k,k+2))\}$$
$$-\sum_{0\leq k\leq i}c(t(k,k+1))-\sum_{0\leq k\leq i-1}c(t(k,k+2))\} \qquad (i\geqq 1).$$

Now we return to the proof of Theorem 4.3. If $\Theta=0$ on $J$, then clearly $c(s)=0$ for any $s\in\mathfrak{S}_{p+1}$. Using Lemma 4.5, we have $x_{i,j}=0$ for any $(i,j)$ and this means $\Theta=0$ on $G=U(p,1)$.

Singular cases can be treated in the similar way, so we omit them. Q.E.D.

## § 5. Weyl group representations on virtual character modules

We want to express explicitly the representations $(\tau, V_H(\chi_\lambda))$ of $W_H(\lambda)$ and $(\sigma, V_H(\chi_\lambda))$ of $\mathscr{H}_\lambda$ in Part I in this section. For this, the regular integral cases are essential. However, we start from non-integral cases.

**5.1. Regular non-integral cases.** Let $\lambda$ be a regular infinitesimal character and put $\alpha=\lambda-\frac{1}{2}(p, p, \cdots, p)$. We assume that there are exactly $p-1$ integers among $\alpha_k$'s and the sum of the rest two is an integer. Then after the definition in § 2 of Part I, we get that $W_B(\lambda)=\{e\}$ (a trivial group), and $W_J(\lambda)=\mathfrak{S}_{p-1}\times\mathfrak{S}_2$, where $\mathfrak{S}_{p-1}$ acts on $p-1$ integers and $\mathfrak{S}_2$ acts on the rest two $\alpha_k$'s. So it is sufficient to treat only $W_J(\lambda)$.

**Proposition 5.1.** *Let $\lambda$ be as above. Then the virtual character module $V(\chi)$ $(\chi=\chi_\lambda)$ with infinitesimal character $\lambda$ is one dimensional and is equal to $V_J(\chi)$. The representation of the integral Weyl group $W_J(\lambda)$ on $V_J(\chi)$ is given by*

$$\tau((s, t))\Theta=\operatorname{sgn}(s)\Theta \qquad ((s, t)\in\mathfrak{S}_{p-1}\times\mathfrak{S}_2,\ \Theta\in V_J(\chi)=V(\chi)).$$

*Proof.* By Theorem 3.9 (1), there is only one irreducible representation with infinitesimal character $\lambda$ and it is a principal series representation. So we get $V(\chi)=V_J(\chi)$ is one dimensional. Now, using Lemma 4.1, we can easily calculate out the representation $\tau$. Q.E.D.

**5.2. Regular integral cases.** Let $\lambda$ and $\alpha$ be as in § 5.1. We assume that all the $\alpha_k$'s are integers which are mutually different. By Theorem 3.9 (2), there are $(p+1)(p+2)/2$ irreducible representations $\{D_{i,j}(\alpha)\,|\,0\leqq i\leqq j\leqq p+1\}$. Let $\Theta_{i,j}$ be the irreducible character of $D_{i,j}(\alpha)$. Then it holds that

$$V(\lambda)=\sum_{0\leq i<j\leq p+1} C\Theta_{i,j}.$$

Let $\Phi_{i,j}$ be the character of a principal series representation $T_{i,j}(\alpha)$, then

$$\begin{aligned}&\Phi_{i,j}=\Theta_{i,j}+\Theta_{i,j+1}+\Theta_{i+1,j}+\Theta_{i+1,j+1} && \text{if } i+2\leqq j,\\ &\Phi_{i,i+1}=\Theta_{i,i+1}+\Theta_{i+1,i+2}+\Theta_{i,i+2} && \text{if } 0\leqq i\leqq p,\end{aligned}$$

and we have

$$V_J(\lambda)=\sum_{0\leq i<j\leq p} C\Phi_{i,j}.$$

Since $\lambda$ is integral, $W_B(\lambda)=W_J(\lambda)$ is equal to $W=W(\mathfrak{g}_C, \mathfrak{j}_C)\simeq\mathfrak{S}_{p+1}$.

**Theorem 5.2.** *We identify $W$ with the group of permutations of $\{0, 1, 2, \cdots, p\}$. Let $s_i=(i-1, i)$ be a transposition of figures $i-1$ and $i$. Then the representation $(\tau, V(\chi))$ of $W$ is given by*

$$\tau(s_k)\Theta_{ij}=\begin{cases} -\Theta_{i,j} & \textit{if } k\neq i,j, \\ \Theta_{i-1,j}+\Theta_{i,j}+\Theta_{i+1,j} & \textit{if } k=i, \\ \Theta_{i,j-1}+\Theta_{i,j}+\Theta_{i,j+1} & \textit{if } k=j, \end{cases}$$

*where $\Theta_{i,i}$ is considered to be* 0. *Moreover, we have an irreducible decomposition of* $(\tau, V(\chi))$:

$$(\tau, V(\chi))\simeq[1^{p+1}]\oplus 2[2\cdot 1^{p-1}]\oplus[3\cdot 1^{p-2}].$$

**Remark.** $[1^{p+1}]$ is the sign representation of $\mathfrak{S}_{p+1}$ and $[2\cdot 1^{p-1}]$ (respectively $[3\cdot 1^{p-2}]$) is an irreducible representation of $\mathfrak{S}_{p+1}$ whose dimension is $p$ (respectively $p(p-1)/2$). See [31] for notations $[k^{n_k}\cdot(k-1)^{n_{k-1}}\cdot 2^{n_2}\cdot 1^{n_1}]$ for irreducible representations of a symmetric group.

*Proof.* Since it is easy to see that

$$\tau(s_k)\Phi_{ij}=\begin{cases} -\Phi_{i,j} & \text{if } k\neq i,\ i+1, j \text{ and } j+1, \\ \Phi_{i-1,j} & \text{if } k=i, \\ \Phi_{i+1,j} & \text{if } k=i+1, \\ \Phi_{i,j-1} & \text{if } k=j, \\ \Phi_{i,j+1} & \text{if } k=j+1, \end{cases}$$

by direct calculations, it is enough to get the formula for discrete series representations (see Figure 3.10 and comments below it). By Theorem 4.3, we only need to consider $\Theta_{i,i+1}$ on $J$. It is clear from the formula for $\Theta_{i,i+1}|_J$ listed in Theorem 4.2 that

$$\tau(s_k)\Theta_{i,i+1}=-\Theta_{i,i+1} \qquad \text{if } k\neq i,\ i+1.$$

We use the following simplified notation:

$$\det\begin{pmatrix} & ((b_i\,|\,1\leqq i\leqq p-1);\ (\alpha_i\,|\,0\leqq i\leqq p)) & \\ h_p^{\alpha_0}, \cdots, h_p^{\alpha_{i-1}} & 0, \cdots, 0 & 0, \cdots, 0 \\ 0, \cdots, 0 & h_{p+1}^{\alpha_i}, \cdots, h_{p+1}^{\alpha_{j-1}} & 0, \cdots, 0 \end{pmatrix}$$
$$=\begin{pmatrix} \bar{0} & \bar{1}\cdots\overline{i-1} & 0 & 0 & \cdots & 0 & 0\cdots 0 \\ 0 & 0\cdots & 0 & \bar{i} & \overline{i+1} & \cdots\overline{j-1} & 0\cdots 0 \end{pmatrix}, \quad \text{etc.}$$

Then we have

$$(-1)^{p-1}D_0\cdot\tau(s_i)\Theta_{i,i+1}|_J$$
$$=\begin{pmatrix} \bar{0} & \bar{1}\cdots\overline{i-2} & 0 & \bar{i} & 0\cdots 0 \\ 0 & 0\cdots & 0 & \overline{i-1} & 0 & 0\cdots 0 \end{pmatrix}$$

$$
+\begin{pmatrix} 0\cdots 0 & \overline{i-1} & 0 & 0 & 0 & \cdots 0 \\ 0\cdots 0 & 0 & 0 & \overline{i+1} & \overline{i+2} & \cdots \bar{p} \end{pmatrix}
$$

$$
=\begin{pmatrix} \bar{0} & \bar{1}\cdots\overline{i-2} & 0 & 0 & 0\cdots 0 \\ 1 & 0\cdots\ 0 & \overline{i-1} & 0 & 0\cdots 0 \end{pmatrix}+\begin{pmatrix} 0\cdots 0 & 0 & \bar{i} & 0 & 0\cdots 0 \\ 0\cdots 0 & \overline{i-1} & 0 & 0 & 0\cdots 0 \end{pmatrix}
$$

$$
+\begin{pmatrix} 0\cdots 0 & \overline{i-1} & \bar{i} & 0 & 0 & \cdots 0 \\ 0\cdots 0 & 0 & 0 & \overline{i+1} & \overline{i+2} & \cdots \bar{p} \end{pmatrix}-\begin{pmatrix} 0\cdots 0 & 0 & \bar{i} & 0 & 0 & \cdots 0 \\ 0\cdots 0 & 0 & 0 & \overline{i+1} & \overline{i+2} & \cdots \bar{p} \end{pmatrix}
$$

$$
=\begin{pmatrix} \bar{0} & \bar{1}\cdots\overline{i-2} & 0 & 0 & 0\cdots 0 \\ 0 & 0\cdots\ 0 & \overline{i-1} & \bar{i} & 0\cdots 0 \end{pmatrix}+\begin{pmatrix} 0\cdots 0 & \overline{i-1} & 0 & 0 & 0\cdots 0 \\ 0\cdots 0 & 0 & \bar{i} & 0 & 0\cdots 0 \end{pmatrix}
$$

$$
-\begin{pmatrix} \bar{0} & \bar{1}\cdots\overline{i-2} & \overline{i-1} & 0 & 0\cdots 0 \\ 0 & 0\cdots\ 0 & 0 & \bar{i} & 0\cdots 0 \end{pmatrix}+\begin{pmatrix} 0\cdots 0 & 0 & \bar{i} & 0 & 0\cdots 0 \\ 0\cdots 0 & \overline{i-1} & 0 & 0 & 0\cdots 0 \end{pmatrix}
$$

$$
+\begin{pmatrix} 0\cdots 0 & \overline{i-1} & \bar{i} & 0 & 0 & \cdots 0 \\ 0\cdots 0 & 0 & 0 & \overline{i+1} & \overline{i+2} & \cdots \bar{p} \end{pmatrix}-\begin{pmatrix} 0\cdots 0 & 0 & \bar{i} & 0 & 0 & \cdots 0 \\ 0\cdots 0 & 0 & 0 & \overline{i+1} & \overline{i+2} & \cdots \bar{p} \end{pmatrix}
$$

$$
=(-1)^{p-1}D_0\cdot\Theta_{i-1,i+1}|_J+(-1)^{p-1}D_0\cdot\Theta_{i,i+1}|_J.
$$

Hence $\tau(s_i)\Theta_{i,i+1}=\Theta_{i-1,i+1}+\Theta_{i,i+1}$. We get $\tau(s_i)\Theta_{i,i+1}+\Theta_{i,i+2}$ similarly.

The irreducible decomposition of $(\tau, V(\chi))$ is stated in [34, Lemma 6.2]. Q.E.D.

From [34, Lemma 6.2], we know

$$
V_J(\lambda)=\sum_{0\leq i<j\leq p} C\Phi_{i,j}\simeq[2\cdot 1^{p-1}]\oplus[3\cdot 1^{p-2}]
$$

and

$$
V_B(\lambda)=\sum_{0\leq i\leq p} C\overline{\Theta_{i,i+1}}\simeq[1^{p+1}]\oplus[2\cdot 1^{p-1}],
$$

where $\overline{\Theta_{i,i+1}}=\boldsymbol{T}(\varepsilon_{\boldsymbol{R}}\cdot D\Theta_{i,i+1}|_B)$ (see § 1.3 of Part I).

**5.3. Singular cases.** All the singular cases can be deduced from regular cases (see [37, Cor. 4.3]). So we only treat here the case (3) of Theorem 3.9. Let $\lambda$ and $\alpha$ be as in Theoerm 3.9 (3). Then the fixed subgroup of $\lambda$ is $W_\lambda=\{1, s_i\}$. Put $e_\lambda=(1+s_i)/2$ and we have $\mathscr{H}_\lambda=\mathscr{H}(W, W_\lambda)\simeq e_\lambda\boldsymbol{C}[W]e_\lambda$. For generators of a Hecke algebra $\mathscr{H}_\lambda$, we can take $t_k=e_\lambda s_k e_\lambda$ $(1\leqq k\leqq p)$ (note that $t_i=e_\lambda$ is the identity for $\mathscr{H}_\lambda$). Now we describe the actions of $t_k$ on $V(\chi_\lambda)$.

**Lemma 5.3.** *Let* $V(\chi_\lambda)=\sum_{0\leq j<i}\boldsymbol{C}\theta_{j,i}+\sum_{i<j\leq p}\boldsymbol{C}\theta_{i,j}$ *be a virtual character module with infinitesimal character* $\lambda$, *where* $\theta_{i,j}$ *is an irreducible character for degenerate representation* $D_{i,j}(\alpha)$. *Then* $t_k$ $(1\leqq k\leqq p)$ *acts*

*on* $V(\chi)$ *as follows.*

(1) *Case* $\alpha_0=\alpha_1$. *In this case* $i=1$ *and we have*

$$\sigma(t_k)\theta_{1,j}=\begin{cases} -\theta_{1,j} & \text{if } k\neq 1, 2 \text{ and } j, \\ \theta_{1,j-1}+\theta_{1,j}+\theta_{1,j+1} & \text{if } k=j\neq 1, 2, \\ \theta_{1,2}+\theta_{1,3}+\frac{1}{2}\theta_{0,1} & \text{if } k=j=2, \\ -\frac{1}{2}\theta_{1,j} & \text{if } k=2\neq j, \end{cases}$$

$$\sigma(t_k)\theta_{0,1}=\begin{cases} -\theta_{0,1} & \text{if } k\neq 1, 2, \\ -\frac{1}{2}\theta_{0,1} & \text{if } k=2. \end{cases}$$

(2) *Case* $\alpha_{i-1}=\alpha_i$ $(2\leqq i\leqq p-1)$. *We have*

$$\sigma(t_k)\theta_{i,j}=\begin{cases} -\theta_{i,j} & \text{if } k\neq i-1, i, i+1 \text{ and } j, \\ \theta_{i,j-1}+\theta_{i,j}+\theta_{i,j+1} & \text{if } k=j\neq i-1, i, i+1, \\ \frac{1}{2}\theta_{i-1,i}+\theta_{i,i+1}+\theta_{i,i+2} & \text{if } k=j=i+1, \\ -\frac{1}{2}\theta_{i,j} & \text{if } k=i-1 \text{ or } k=i+1\neq j, \end{cases}$$

$$\sigma(t_k)\theta_{j,i}=\begin{cases} -\theta_{j,i} & \text{if } k\neq i-1, i, i+1 \text{ and } j, \\ \theta_{j-1,i}+\theta_{j,i}+\theta_{j+1,i} & \text{if } k=j\neq i-1, i, i+1, \\ \frac{1}{2}\theta_{i,i+1}+\theta_{i-2,i}+\theta_{i-1,i} & \text{if } k=j=i-1, \\ -\frac{1}{2}\theta_{j,i} & \text{if } k=i+1 \text{ or } k=i-1\neq j. \end{cases}$$

(3) *Case* $\alpha_{p-1}=\alpha_p$. *In this case* $i=p$ *and we have*

$$\sigma(t_k)\theta_{j,p}=\begin{cases} -\theta_{j,p} & \text{if } k\neq p-1, p \text{ and } j, \\ \theta_{j-1,p}+\theta_{j,p}+\theta_{j+1,p} & \text{if } k=j\neq p-1, p, \\ \frac{1}{2}\theta_{p,p+1}+\theta_{p-2,p}+\theta_{p-1,p} & \text{if } k=j=p-1, \\ -\frac{1}{2}\theta_{j,p} & \text{if } k=p-1\neq j. \end{cases}$$

$$\sigma(t_k)\theta_{p,p+1}\begin{cases} -\theta_{p,p+1} & \text{if } k\neq p-1, p, \\ -\frac{1}{2}\theta_{p,p+1} & \text{if } k=p-1. \end{cases}$$

*Proof.* Here we show the formula for $\sigma(t_{i+1})\theta_{i,i+1}$ in (2). The rest of the formulae can be obtained in the similar way. Let $\lambda'$ be a regular integral infinitesimal character and put $\alpha'=\lambda'-\frac{1}{2}(p, p, \cdots, p)$. Let $\Psi$: $V(\chi')\rightarrow V(\chi)$ be a Zuckerman's translation functor (see § 2 of Part I). Then we have $\Psi(\Theta_{i,j})=\theta_{i,j}$. By the definition of the representation $(\sigma, V(\chi))$,

$$\sigma(t_{i+1})\theta_{i,i+1}=\Psi(\tau(e_\lambda s_{i+1}e_\lambda)\Theta_{i,i+1}).$$

By Theorem 5.2, we get $\tau(e_\lambda s_{i+1}e_\lambda)\Theta_{i,i+1}$ as follows:

$$\begin{aligned}\tau(e_\lambda s_{i+1}e_\lambda)\Theta_{i,i+1}&=\tfrac{1}{2}\tau(e_\lambda s_{i+1})(2\Theta_{i,i+1}+\Theta_{i-1,i+1})\\&=\tfrac{1}{2}\tau(e_\lambda)(2\Theta_{i,i+1}+2\Theta_{i,i+2}+\Theta_{i-1,i}+\Theta_{i-1,i+1}+\Theta_{i-1,i+2})\\&=\tau(e_\lambda)\Theta_{i,i+1}+\tau(e_\lambda)\Theta_{i,i+2}+\tfrac{1}{2}\tau(e_\lambda)\Theta_{i-1,i}.\end{aligned}$$

Now, since $\Psi(\tau(e_\lambda)\Theta_{i,i+1})=\theta_{i,i+1}$ etc., we have

$$\sigma(t_{i+1})\theta_{i,i+1}=\theta_{i,i+1}+\theta_{i,i+2}+\tfrac{1}{2}\theta_{i-1,i}. \qquad \text{Q.E.D.}$$

## § 6. Gelfand-Kirillov dimensions and $\tau$-invariants

In this section, we calculate out the Gelfand-Kirillov dimensions and $\tau$-invariants for the representations $D_{i,j}(\alpha)$ $(0\leqq i<j\leqq p+1)$.

**6.1. Gelfand-Kirillov dimensions.** At first, we assume that $\lambda$ is regular and put $\alpha=\lambda-\frac{1}{2}(p,p,\cdots,p)$.

**Theorem 6.1.** *If $\lambda$ is regular integral, then we have*

$$\text{GK-dim}\,(D_{i,j}(\alpha))=\begin{cases}2p-1 & \text{if } 0<i<j<p+1,\\ p & \text{if } (i,j)\neq(0,p+1) \text{ but } i=0 \text{ or } j=p+1,\\ 0 & \text{if } (i,j)=(0,p+1),\end{cases}$$

*where* GK-dim $(\pi)$ *is Gelfand-Kirillov dimension of* $\pi$.

*Proof.* By Proposition 6.1 in [25], we know GK-dim $(D_{i,i+1}(\alpha))=2p-1$ for $0<i<p$ and GK-dim $(D_{0,1}(\alpha))=$ GK-dim $(D_{p,p+1}(\alpha))=p$. Since Gelfand-Kirillov dimension of a finite dimensional representation is zero by definition, we have GK-dim $(D_{0,p+1}(\alpha))=0$.

**Lemma 6.2.** *Let $\Theta_{i,j}$ be the irreducible character of $D_{i,j}(\alpha)$.*

(1) *Put $V_0=C\Theta_{0,p+1}$. Then $V_0$ is $W$-invariant and $V_0\simeq[1^{p+1}]$.*

(2) *Put $B_i=\Theta_{0,i}+\Theta_{0,i+1}-(-1)^{p+i}(p+1)^{-1}\Theta_{0,p+1}$. Then $V_1=\sum_{1\leq i\leq p}CB_i$ is $W$-invariant subspace which is isomorphic to $[2\cdot 1^{p-1}]$.*

(3) *Put $C_i=\Theta_{i,p+1}+\Theta_{i+1,p+1}-(-1)^i(p+1)^{-1}\Theta_{0,p+1}$. Then $V_2=\sum_{1\leq i\leq p}CC_i$ is $W$-invariant subspace which is isomorphic to $[2\cdot 1^{p-1}]$.*

*Proof.* We can check easily that $V_i$ $(i=0, 1, 2)$ is $W$-invariant, using Theorem 5.2. Now, comparing dimensions of invariant subspaces $[1^{p+1}]$, $[2\cdot 1^{p-1}]$ and $[3\cdot 1^{p-2}]$, we conclude that $V_0\simeq[1^{p+1}]$ and $V_1\simeq V_2\simeq[2\cdot 1^{p-1}]$. Q.E.D.

Remark that $D_{0,1}(\alpha)$ is cyclic for $V_0+V_1$. Then Theorem 3.10 in Part I implies that $\mathrm{GKD}(V_0)$ and $\mathrm{GKD}(V_1)$ are less than or equal to $\mathrm{GKD}(\Theta_{0,1})=p$. Since $\mathrm{GKD}(V_0)=0$, $\mathrm{GKD}(V_1)$ must be $p$. Hence we have $\mathrm{GKD}(\Theta_{0,j})=p$ $(1\leqq j\leqq p)$. Similar reasoning proves that $\mathrm{GKD}(\Theta_{i,p+1})=p$ $(1\leqq i\leqq p)$. Since there is an irreducible representation which has Gelfand-Kirillov dimension $2p-1$, we know that $\mathrm{GKD}(V_3)=2p-1$, where $V_3$ is the $W$-invariant subspace which is isomorphic to $[3\cdot 1^{p-2}]$. Therefore irreducible characters which are not contained in the space $V_0+V_1+V_2$ have Gelfand-Kirillov dimension $2p-1$. Q.E.D.

By Proposition 3.6 in Part I, we get

**Corollary 6.3.** *If $\lambda$ is singular integral, then we have*

$$\text{GK-dim}\,(D_{i,j}(\alpha))=\text{GK-dim}\,(D_{i,j}(\alpha')),$$

*where $\alpha'$ is regular integral.*

Since Gelfand-Kirillov dimension of a principal series is equal to $2p-1$ ([39, Cor. 5.3]), we get all the Gelfand-Kirillov dimensions for irreducible representations of $U(p, 1)$.

**6.2. $\tau$-invariants.** At first we assume $\lambda$ is regular integral. Let $R=\{s_i=(i-1, i)\,|\,1\leqq i\leqq p\}$ be a set of simple reflections of $W=\mathfrak{S}_{p+1}$.

**Definition 6.4** ([40, Def. 7.3.8]). For $\Theta\in V(\chi)$, put

$$\tau\text{-inv}\,(\Theta)=\{s\in R\,|\,\tau(s)\Theta=-\Theta\}$$

and call a reflection in $\tau$-inv $(\Theta)$ a $\tau$-invariant of $\Theta$.

**Theorem 6.5.** *We get* $\tau$-inv $(\Theta_{ij})=\{s_k\,|\,k\neq i, j\}$.

*Proof.* This is a Corollary to Theorem 5.2. Q.E.D.

Next assume that $\lambda$ is not integral but regular. Then we can assume that $W(\lambda)=\langle s_i\,|\,1\leqq i\leqq p-2\rangle\simeq\mathfrak{S}_{p-1}$ without loss of generality. We define $\tau$-invariants using $R(\lambda)=\{s_i\,|\,1\leqq i\leqq p-2\}$ instead of $R$. We have

**Theorem 6.5′.** *If $\lambda$ is not integral but regular, then we get $\tau$-inv $(\Theta)$ $=R(\lambda)$ for any $\Theta\in V(\chi)$.*

*Proof.* Since $V(\chi)$ is of dimension one and generated by a principal series character, Proposition 5.1 proves the theorem. Q.E.D.

**6.3. Character polynomials.** Let $\lambda$ be regular integral. Remark that once we get character polynomials for regular $\lambda$, Proposition 3.4 in Part I gives character polynomials for singular infinitesimal character. Let $\alpha$ be as above.

At first, we treat the compact Cartan subgroup $B$. By Theorem 4.2, it is clear that the discrete series characters restricted to $B$ are linearly independent on $B$ and, moreover, they form actually a base for virtual character module restricted to $B$. Since $r(\Theta_{i,i+1}; F)=\sharp\Delta_c^+=p(p-1)/2$ for $0\leqq i\leqq p$ and $F\subset B'(\boldsymbol{R})$ a connected component of $B'(\boldsymbol{R})$, $r(\Theta; F)$ is less than or equal to $p(p-1)/2$ for any $\Theta\in V(\chi)$.

Let $F$ be a component of $B'(\boldsymbol{R})$ which contains the positive Weyl chamber. Then, after easy calculations, we get

$$c_r(\Theta_{p,p+1}, F; \mu, x)=(-1)^p(r!)^{-1}\prod_{1\leq i<j\leq p}(\varepsilon_i-\varepsilon_j, \mu)\prod_{1\leq i<j\leq p}(h_i-h_j, x),$$

where $r=p(p-1)/2$ and $h_i$ is a diagonal matrix whose $(i, i)$-element is 1 and the other elements are zero. The representation of $W=\mathfrak{S}_{p+1}$ generated by this polynomial is irreducible (see [32]) and must be isomorphic to $[1^{p+1}]$, $[2\cdot 1^{p-1}]$ or $[3\cdot 1^{p-2}]$. Comparing dimensions, we know the representation is of dimension $p$ and isomorphic to $[2\cdot 1^{p-1}]$. Since $r(\Theta_{0,p+1}; F)=\sharp\Delta^+=p(p+1)/2$, we get $V_B(\chi; p)=V_B(\chi)$ and $V_B(\chi; 0)=\boldsymbol{C}\Theta_{0,p+1}$. The character polynomial of $\Theta_{0,p+1}$ is given by

$$c_r(\Theta_{0,p+1}, F; \mu, x)=(r!)^{-1}\prod_{1\leq i<j\leq p+1}(\varepsilon_i-\varepsilon_j, \mu)\prod_{1\leq i<j\leq p+1}(h_i-h_j, x),$$

where $r=p(p-1)/2$. Moreover we get, for $F\subset B'(\boldsymbol{R})$, $V(\chi; p)=V(\chi)$, $V(\chi; 0)=\boldsymbol{C}\Theta_{0,p+1}+V_J(\chi)$ and $V(\chi; -\infty)=V_J(\chi)$.

Secondly, we treat the maximally split Cartan subgroup $J=B^+A$. However, this case is essentially due to D.R. King [25, Pro. 6.1]). Character polynomials of discrete series characters are given there, so we omit them. Filtration of the space $V(\chi)$ (or $V_J(\chi)$) by Gelfand-Kirillov dimensions is given in § 6.1.

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*Added in proof.* A. Prof. D. H. Collingwood kindly suggested the author that some theorems in his book:

[45] D. H. Collingwood, Representations of rank one Lie groups, Pitman Publ., 1986.

are very useful to Part II of this article. Indeed there are complete determinations of subquotient structures of principal series representations of rank one Lie groups (Chap. 5.3). Using his results, one can prove that $\mathscr{E}((m_i|1\leqq i\leqq p-1); c_1, c_2)$ in § II-3 is infinitesimally equivalent to a principal series representation.

The author thanks his suggestion very much.

B. Hecke algebra structure using in § II-5.3 is more presicely studied in

[46] K. Nishiyama, Generators and relations of a certain Hecke algebra, Research Activities of Fac. of Sci. and Engineering, Tokyo Denki Univ., **8&9** (1987), 9–14.

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