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Schur Orthogonality Relations for Non Square Integrable Representations of Real Semisimple Linear Group and Its Application

Hisaichi Midorikawa

Introduction

In the previous paper [20], we discuss the Schur orthogonality relations for certain non square integrable representations of a given connected real semisimple linear group G. Those representations are the subrepresentations of unitary principal series of G induced from a maximal cuspidal parabolic subgroup, although I did not state explicitly this fact in [20]. We formulate our results as follows.

Let $C^\infty(G)$ be the set of all complex valued C^∞-functions on G and $\mathfrak{g}_C$ the complexification of the Lie algebra $\mathfrak{g}$ of G. The universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}_C$ acts on $C^\infty(G)$. The left (resp. right) action of b in $\mathfrak{u}(\mathfrak{g})$ will be denoted by bf (resp. fb) for f in $C^\infty(G)$. Let $\mathfrak{z}$ be the center of $\mathfrak{u}(\mathfrak{g})$ and $d(p, q)$ the Riemannian distance on the symmetric space G/K where K is a maximal compact subgroup of G. Define a function d on G and a seminorm $\| \ \|_p$ on $C^\infty(G)$ by

$$d(x)=d(xo, o), \ o \text{ is the origin in } G/K$$

and

$$\|f\|_p^2=\lim_{\varepsilon\to+0} \varepsilon^p \int_G |f(x)|^2 e^{-\varepsilon d(x)} dx \quad \text{for } f \text{ in } C^\infty(G)$$

where p is a nonnegative real number and dx is the Haar measure on G.

Definition I. Let χ be a character of $\mathfrak{z}$. The space $H_p(G, \chi)$ is defined as the set of all C^∞-functions f satisfying $\|b_1 f b_2\|_p<\infty$ and $(z-\chi(z))f=0$ for all b_i in $\mathfrak{u}(\mathfrak{g})$ and z in $\mathfrak{z}$. $H_p(G, \chi)$ is a topological G-module with the canonical actions. Furthermore $\|R_x f\|_p=\|L_x f\|_p=\|f\|_p$ for x in G and f in $H_p(G, \chi)$ where R and L are respectively the right and left actions of G on $H_p(G, \chi)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote the root space decomposition of $\mathfrak{g}_C$ by $\mathfrak{g}_C=\mathfrak{h}_C\oplus\sum_{\alpha\in\Phi}\mathfrak{g}_\alpha$ where Φ is the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Select, for each α in Φ, X_α in $\mathfrak{g}_\alpha$ satisfying $B(X_\alpha, X_{-\alpha})=1$

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(B is the Killing form on $\mathfrak{g}_C$). Each element $H_\alpha = ad(X_\alpha)X_{-\alpha}$ belongs to $\mathfrak{h}_C$. Using the canonical isomorphism of $\mathfrak{z}$ into the ring of polynomial functions on the dual space of $\mathfrak{h}_C$, we can parametrize all characters χ of $\mathfrak{z}$ by the linear forms on $\mathfrak{h}_C$. We shall denote this parametrization by $\chi = \chi_\lambda$, λ is a linear form on $\mathfrak{h}_C$.

Definition II. The number $i(\chi)$ is defined by $i(\chi) = \sharp\{\alpha \in \Psi;\ \lambda(H_\alpha) \in \boldsymbol{R} - \{0\}\}$ where Ψ is a fundamental root system of Φ, $\sharp S$ is the cardinality of a given set S. The number $i(\chi)$ is called the index of χ.

Theorem I. *Let χ be a character of $\mathfrak{z}$. Assume that $H_{i(\chi)}(G, \chi)$ is nontrivial. Then $H_{i(\chi)}(G, \chi)$ is a pre-Hilbert space with the norm $\|\ \|_{i(\chi)}$.*

The theorem will be proved by using Harish-Charandra's classification theorem for discrete series representations and the asymptotic expansion theorems (for the K-finite eigenfunctions on G) obtained by Harish-Chandra [8], W. Casselman and Miličić [4], [5], [21] (see also M. Kashiwara et al. [17], N.R. Wallach [25]).

We shall denote the completion of $H_{i(\chi)}(G, \chi)$ and its norm by $H(G, \chi)$ and $\|\ \|$ respectively. The regular representations R and L on $H(G, \chi)$ are unitary, and all K-finite functions in $H(G, \chi)$ are real analytic.

Definition III. An irreducible unitary representation (π, H) of G is realized on $H(G, \chi)$ if there exists an isometric linear operator η of H into $H(G, \chi)$ such that $R_x \circ \eta = \eta \circ \pi(x)$ for all x in G.

Theorem II. *An irreducible unitary representation (π, H) of G is realized on $H(G, \chi)$ if and only if there exists a K-finite vector ϕ in H such that $(\pi(x)\phi, \phi)$ belongs to $H(G, \chi)$.*

We remark that if $i(\chi) = 0$, then $H(G, \chi) \subset L^2(G)$ where $L^2(G)$ is the space consisting of all square integrable functions on G. Therefore $H(G, \chi)$ is a closed invariant subspace of $L^2(G)$, and the representation π realized on $H(G, \chi)$ belongs to the discrete series in this case.

By using Theorem I and Theorem II, the standard arguments for the proof of Schur orthogonality relations of square integrable representations of G imply the following theorem.

Theorem III. *Let (π, H) and (π', H') be two irreducible unitary representations of G realized on $H(G, \chi)$. Then there exists a positive constant d_π such that*

$$\lim_{\varepsilon \to +0} \varepsilon^{i(\chi)} \int_G (\pi(x)\phi, \psi)\overline{(\pi'(x)\phi', \psi')} e^{-\varepsilon d(x)}\, dx = \begin{cases} d_\pi^{-1}(\phi, \phi')\overline{(\psi, \psi')} & \text{if } \pi \cong \pi' \\ 0 & \text{otherwise} \end{cases}$$

for all K-finite vectors $\phi, \psi \in H$ *and* $\phi', \psi' \in H'$.

The constant d_π is called the formal degree of π. In case of $i(\chi)=0$, the relations in the above theorem are well known as a result of R. Godement [6] (see Theorem 4.5.9.3, [26]). For the case $i(\chi)=1$, we proved the similar theorem in [20].

In the following we shall assume that $i(\chi)>0$. Let $P=MAN$ be a proper cuspidal parabolic subgroup of G. Consider a discrete series representation σ of M and a unitary character $a \to e^{\nu(\log a)}$ of A where ν is a purely imaginary valued linear form on the Lie algebra $\mathfrak{a}$ of A. The representation $\sigma \otimes e^\nu$ of MA is extended to P by $(\sigma \otimes e^\nu \otimes 1)(\mathrm{man}) = e^{\nu(\log a)}\sigma(m)$ for $a \in A$, $m \in M$ and $n \in N$. Let $\pi(\sigma, \nu) = \mathrm{ind}_P^G(\sigma \otimes e^\nu \otimes 1)$ be the induced representation of G from P constructed by canonical procedure. $\pi(\sigma, \nu)$ is called a principal series unitary representation of G induced from P. The following theorem is proved by Schur orthogonality relations in Theorem III.

Theorem IV. *Assume that* $i(\chi)>0$. *Then each irreducible unitary representation of G realized on* $H(G, \chi)$ *is equivalent to a subrepresentation of a principal series of G induced from a certain cuspidal parabolic subgroup* $P=MAN$ *with* $i(\chi)=\dim A$.

Definition IV. Let notations be as above. A principal series representation $\pi(\sigma, \nu)$ of G induced from $P=MAN$ is regular if the linear form ν on $\mathfrak{a}$ is regular.

Theorem V. *Each regular principal series unitary representation* $\pi(\sigma, \nu)$ *of G with infinitesimal character* χ *is realized on* $H(G, \chi)$.

As an application of Schur orthogonality relations for non square integrable representation of G, we give a proof of irreducibility of the regular principal series in the following.

Theorem VI (Bruhat and Harish-Chandra). *All regular principal series unitary representations of G are irreducible.*

Our proof of this theorem is based on the character theory due to T. Hirai [13], the lowest (minimal) K-type theorem for principal series representation of G obtained by D. Vogan [24] (see also A.W. Knapp [15], J. Carmona [3]) and Schur orthogonality relations. By [13], we see that all tempered invariant eigendistributions on G with the same regular infinitesimal character are uniquely determined up to constant. To apply Hirai's theorem we use the following theorem.

Theorem VII (Knapp and Zuckerman). *Let $\pi(\sigma, \nu)$ be a principal series representation of G. Then the character of each subrepresentation of $\pi(\sigma, \nu)$ is tempered.*

In [14], there is a character table of all irreducible components of principal series representations of G. Since their characters are determined explicitly, we can observe that the character of each irreducible component of $\pi(\sigma, \nu)$ is tempered. However, in this paper, we shall prove directly the temperedness as in the above theorem by using uniform estimation, which is a result of P.C. Trombi and V.S. Varadarajan [22], for the matrix coefficients of discrete series representation σ of M.

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Contents

§ 1. Preliminaries and notational definitions

We first state, in this section, two lemmas for elementary spherical function Ξ on a connected real semisimple linear group G. Let K be a fixed maximal compact subgroup of G and $P_0=M_0A_0N_0$ a minimal parabolic subgroup of G with θ-stable split component A_0 where θ is the Cartan involution of (G, K). Therefore $G=KA_0N_0$ is the Iwasawa decomposition. Each element x in G is uniquely written by $x=k(x)\exp H(x)n(x)$, $k(x) \in K$, $H(x) \in \mathfrak{a}_0$ and $n(x) \in N_0$ where $\mathfrak{a}_0$ is the Lie algebra of A_0. Let $\mathfrak{g}$ and $\mathfrak{n}_0$ be the Lie algebras of G and N_0 respectively. The action $\mathrm{Ad}(p)$ $(p \in P_0)$ on $\mathfrak{n}_0$ will be denoted by $\mathrm{Ad}(p)|_{\mathfrak{n}_0}$. Then there exists a linear form ρ on $\mathfrak{a}_0$ such that $e^{\rho(\log a)}=\sqrt{|\det \mathrm{Ad}(a)|_{\mathfrak{n}_0}|}$ for all a in A_0. We define a function on G by $\Xi(x)=\int_K e^{-\rho(H(x^{-1}k))}dk$, $x \in G$ where dk is the Haar measure on K normalized as $\int_K dk=1$. Let $d(p, q)$ $(p, q \in G/K)$ be the Riemannian distance on the symmetric space G/K and o the origin in the space. Then we have the following (see, for the proofs, Lemma 8.5.2.6 and Lemma in p. 239 [26]).

Lemma 1. *The function satisfies the properties below;*
(1) $\Xi(kxk')=\Xi(x)$ *for all* $x \in G$, $k, k' \in K$,
(2) $\Xi(x^{-1})=\Xi(x)$,
(3) *there exists a nonnegative integer* p *such that*

$$e^{-\rho(\log a)} \leqq \Xi(a) \leqq a \text{ const. } e^{-\rho(\log a)}(1+d(xo, o))^p$$

for all a *in the positive Weyl chamber* A_0^+ *of* A_0 *and*
(4) *choosing a positive number* p' *suitably*

$$\Xi(an)\,(1+d(ano, o))^{-p'} \leqq a \text{ const. } e^{-(\rho(\log a)+\rho(H(\theta(n^{-1}))))}$$

for all a *in* A_0 *and* n *in* N_0.

Remark 1. The function $\rho(H(\theta(n^{-1})))$ on N_0 is nonnegative.

Secondly we define the Schwarz space on G following Harish-Chandra. Let $\mathfrak{u}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_C$. The actions on the ring of all C^∞-functions $C^\infty(G)$ on G are defined by

$$(Xf)(x)=\frac{d}{dt}f(\exp -tXx)|_{t=0} \quad \text{and} \quad (fX)(x)=\frac{d}{dt}f(x\exp tX)|_{t=0}$$

for x in G, f in $C^\infty(G)$ and X in $\mathfrak{g}$. We shall denote the actions to the left and right by bf and fb respectively for all b in $\mathfrak{u}(\mathfrak{g})$ and f in $C^\infty(G)$. Let b_1, b_2 be two elements in $\mathfrak{u}(\mathfrak{g})$ and r a real number. We put a seminorm $\nu_{b_1,b_2,r}$ on $C^\infty(G)$ by

$$\nu_{b_1,b_2,r}(f)=\sup_{x\in G}|(b_1fb_2)(x)|\,\Xi(x)^{-1}(1+d(x))^{-r}$$

where $d(x)=d(xo, o)$.

Definition 1. The Schwarz space $\mathscr{C}(G)$ on G is consists of all C^∞-functions f on G with the following properties; $\nu_{b_1,b_2,r}(f)<\infty$ for all b_1, b_2 in $\mathfrak{u}(\mathfrak{g})$ and positive real numbers r.

Definition 2. A distribution T on G is called tempered if T is extended to a continuous linear form on $\mathscr{C}(G)$. To study the tempered distributions on G the following integral formula on G is crucial. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$ and Φ the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. For the root system $\Phi(A_0)$ of (P_0, A_0), we induce a linear order of Φ as the following; if α is positive on $\mathfrak{a}_0$, then α is also positive on $\mathfrak{h}$. Let Φ_+ be the set of all positive roots in Φ which does not vanish on $\mathfrak{a}_0$. We define a function D on A_0 by

$$D(a)=\prod_{\alpha\in\Phi_+}|\exp\alpha(\log a)-\exp(-\alpha(\log a))|, \qquad a\in A_0.$$

Lemma 2. *There exists a positive constant $C=C_G$ such that*

$$\int_G f(x)dx=C\int_{A_0^+}da\iint_{K\times K}f(kak')D(a)dkdk'$$

for all f in $C_c^\infty(G)$ where A_0^+ is the positive Weyl chamber of A_0 and $C_c^\infty(G)$ is the set of all C^∞-functions on G with compact support.

(See Proposition 10.17, [11]).

Let $\mathfrak{z}$ be the center of $\mathfrak{u}(\mathfrak{g})$. A function f in $C^\infty(G)$ is $\mathfrak{z}$- (resp. K-) finite if $\dim \mathfrak{z}f$ (resp. the dimension of linear span $\{L_k\circ R_{k'}f;\ k, k'\in K\}$) is finite, where L and R are respectively the canonical actions on $C^\infty(G)$ to the left and right respectively.

Finally, we shall state for the character of a given admissible unitary representation of G after the following preparations. Let $\mathscr{E}(K)$ be the set of all equivalence classes of irreducible unitary representations of K. We put, for each $[\tau]$ in $\mathscr{E}(K)$, $\chi_\tau(k)=d_\tau\,\mathrm{Trace}\,\tau(k)$, $k\in K$, $d_\tau=$the dimension (degree) of τ. Let (π, H) be a unitary representation of G. We define a projection operator $E(\tau)$ on H as follows; $E(\tau)v=\int_K\overline{\chi_\tau(k)}\pi(k)dk$, $v\in H$.

Definition 3. A unitary representation (π, H) of G is admissible if there exist two positive numbers N and m such that $\dim E(\tau)H\leqq N(d_\tau)^m$ for all $[\tau]$ in $\mathscr{E}(K)$.

For an admissible unitary representation π of G the operator $\pi(f)=\int_G f(x)\pi(x)dx$ is of trace class, $\Theta_\pi(f)=\mathrm{Trace}\ \pi(f)$ is a distribution on G where f is a function in $C_c^\infty(G)$. Furthermore if π is irreducible, then there exists a character χ of $\mathfrak{z}$ such that $(z-\chi(z))\Theta_\pi=0$ for all z in $\mathfrak{z}$. Θ_π (resp. χ) is the character (resp. the infinitesimal character) of π.

§ 2. Principal P-series representation

In this section, we shall define a principal series representation of G induced from a given cuspidal parabolic subgroup, and state for the admissibility of the representation.

Let $P=MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup P of G. Throughout of this paper, we always assume that the split component A of P is θ-stable. The Lie algebras of M, A and N respectively are denoted by $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$. We define a function d_P on P and linear form ρ_P on $\mathfrak{a}$ as follows; $d_P(p)=\sqrt{|\det \mathrm{Ad}\,(p)|_{\mathfrak{n}}|}$ and $\exp \rho_P\,(\log a)=d_P(a)$ for p in P and a in A.

Let (σ, H_σ) be a square integrable (discrete series) representation of M and $a\mapsto e^{\nu\,(\log a)}$ a unitary character of A where ν is a purely imaginary valued linear form on $\mathfrak{a}$. We extend the representation $\sigma\otimes e^\nu$ of MA to P by $(\sigma\otimes e^\nu\otimes 1)\,(man)=(\sigma\otimes e^\nu)\,(ma)$, $m\in M$, $a\in A$, $n\in N$. A H_σ-valued C^∞-function f on G belongs to $C^\infty(G, H_\sigma)$ if f satisfies that

(2.1) $f(xp)=d_P(p)^{-1}(\sigma\otimes e^\nu\otimes 1)(p)^{-1}f(x)$ for all x in G and p in P. The space $C^\infty(G, H_\sigma)$ is a pre-Hilbert space with the following positive definite Hermitian structure (,);

$$(2.2)\qquad (\phi, \psi)=\int_K(\phi(k), \psi(k))dk \quad \text{for, } \phi, \psi \quad \text{in } C^\infty(G, H_\sigma).$$

The completion of $C^\infty(G, H_\sigma)$ will be denoted by $H(\sigma, \nu)$. We see that the left regular representation $\pi(\sigma, \nu)\equiv\mathrm{ind}_P^G\,(\sigma\otimes e^\nu\otimes 1)$ of G on the space $H(\sigma, \nu)$ is unitary. $\pi(\sigma, \nu)$ is called a principal series representation of G induced from the cuspidal parabolic subgroup P (or simply principal P-series representation of G). Let $H(\sigma)$ be the set of all restriction of functions in $H(\sigma, \nu)$ to K. $H(\sigma)$ can be identified to the subspace $(L^2(K)\otimes H_\sigma)_\sigma$ of $L^2(K)\otimes H_\sigma$, $L^2(K)$ is the space of all square integrable functions on K and

(2.3) $(L^2(K)\otimes H_\sigma)_\sigma=$the set of all $\sum_i f_i\otimes v_i$ in $L^2(K)\otimes H_\sigma$ satisfying $\sum_i f_i\otimes v_i(km)=\sum_i f_i(k)\otimes\sigma(m)^{-1}v_i$ for all k in K and m in $K_M\equiv K\cap M$ where the summation runs over a finite members of i.

Let us give an another realization of $\pi(\sigma, \nu)$ as following.

Define a representation $\pi'(\sigma, \nu)$ of G on $H(\sigma)$ by

$$(2.4)\qquad (\pi'(\sigma, \nu)(x)\phi)(k)=e^{-(\nu+\rho_P)(H(x-1k))}\sigma(m(x^{-1}k))^{-1}\phi(k(x^{-1}k))$$
$$\text{where } k(x^{-1}k)\in K, \quad m(x^{-1}k)\in M, \quad H_P(x^{-1}k)\in\mathfrak{a}$$

determined by $x^{-1}k\in k(x^{-1}k)m(x^{-1}k)\exp H_P(x^{-1}k)N$ for k in K, x in G.

Let η be a linear mapping of $H(\sigma, \nu)$ onto $H(\sigma)$ defined by $\eta(\phi)(k)=\phi(k)$, $\phi \in H(\sigma, \nu)$. Then $\pi'(\sigma, \nu)(x)\circ\eta=\eta\circ\pi(\sigma, \nu)(x)$ for all x in G. We shall denote $\pi'(\sigma, \nu)=\pi(\sigma, \nu)$ under this identification. Let us state the admissibility for $\pi(\sigma, \nu)$. Let $\mathscr{E}(K)$ be the set of all equivalence classes of irreducible unitary representations of K and $\pi(\sigma, \nu)|_K$ the restriction of $\pi(\sigma, \nu)$ to K. For each class $[\tau]$ in $\mathscr{E}(K)$, we denote the multiplicity of τ appearing in $\pi(\sigma, \nu)|_K$ by $[\pi(\sigma, \nu)|_K: \tau]$. Similarly we also denote by $[\sigma|_{K_M}: \xi]$ for $[\xi]$ in $\mathscr{E}(K_M)$ the same as K. Since $\pi(\sigma, \nu)|_K$ is the left regular representation of compact group K, the Frobenius reciprocity theorem implies that

$$(2.5)\qquad [\pi(\sigma, \nu)|_K: \tau]=\sum_{[\xi]\in\mathscr{E}(K_M)}[\sigma|_{K_M}: \xi][\tau|_{K_M}: \xi]$$
$$\text{for all } [\tau] \text{ in } \mathscr{E}(K).$$

By our assumption for σ, σ is realized on a closed invariant subspace of $L^2(M)$. Consequently, by using Peter-Weyl theorem, we have

$$(2.6)\qquad [\sigma|_{K_M}: \xi]\leqq(d_\xi)^2 \qquad \text{for all } [\xi] \text{ in } \mathscr{E}(K_M).$$

Combining (2.5) with (2.6), we have the following.

Lemma 1. *Let notations and assumptions being as above. Then* $[\pi(\sigma, \nu)|_K: \tau]\leqq(d_\tau)^4$ *for all* $[\tau]$ *in* $\mathscr{E}(K)$.

Thus by the above lemma, each subrepresentation π of $\pi(\sigma, \nu)$ is admissible.

Lemma 2. *There exists a character* χ *of* $\mathfrak{z}$ *such that* $(z-\chi(z))\Theta_\pi=0$ *for all subrepresentations* π *of* $\pi(\sigma, \nu)$ *and* z *in* $\mathfrak{z}$.

Proof. In view of the explicit formura of the character $\Theta_{\pi(\sigma,\nu)}$ (see [12]), there exists a character χ of $\mathfrak{z}$ such that $(z-\chi(z))\Theta_{\pi(\sigma,\nu)}=0$ for all z in $\mathfrak{z}$. We define for $[\tau]$ in $\mathscr{E}(K)$ and f in $C^\infty(G)$, $\chi_\tau*f$ and $f*\chi_\tau$ by

$$(2.7)\qquad (\chi_\tau*f)(x)=\int_K\overline{\chi_\tau(k)}f(k^{-1}x)dk,$$
$$(f*\chi_\tau)(x)=\int_K\overline{\chi_\tau(k)}f(xk)dk, \qquad x\in G.$$

Let $E(\tau)$ be the projection operator as in Section 1. By the definition $\Theta_{\pi(\sigma,\nu)}(f)=\sum_{[\tau]\in\mathscr{E}(K(}\int_G f(x)\phi_\tau(x)dx$ for all f in $C_c^\infty(G)$ where $\phi_\tau(x)=\mathrm{Trace}(E(\tau)\pi(\sigma, \nu)(x)E(\tau))$. Therefore $\Theta_{\pi(\sigma,\nu)}(\chi_\tau*f*\chi_\tau)=\int_G f(x)\phi_\tau(x)dx$. Since $\Theta_{\pi(\sigma,\nu)}$ is contained the kernel of $z-\chi(z)$ we have

$$(2.8) \qquad (z-\chi(z))\phi_\tau=0 \qquad \text{for all } z \text{ in } \mathfrak{z}.$$

We choose $[\tau]$ in $\mathscr{E}(K)$ satisfying $[\pi|_K\colon \tau]>0$ for a given irreducible subrepresentation π of $\pi(\sigma, \nu)$. Then there exist a finite number of irreducible subrepresentations $\pi=\pi_1, \pi_2, \cdots, \pi_n$ of $\pi(\sigma, \nu)$ such that

$$(2.9) \qquad \phi_\tau=\sum_{i=1}^{n}[\pi(\sigma, \nu)|_K\colon \pi_i]\phi_i, \ \phi_i(x)=\mathrm{Trace}\,(E(\tau)\pi_i(x)E(\tau)).$$

Let χ_i be the infinitesimal character of π_i. Then by (2.8) and (2.9), we have $\sum_{i=1}^{n}[\pi(\sigma, \nu)|_K\colon \pi_i](\chi(z)-\chi_i(z))\phi_i=0$. Since all π_i's are inequivalent to each other, $\{\phi_i\}$ is linearly independent. Thus $(z-\chi(z))\Theta_\pi=0$ for all z in $\mathfrak{z}$ and subrepresentations π of $\pi(\sigma, \nu)$ as claimed.

§ 3. Temperedness for the character of subrepresentation of principal *P*-series

We keep the same notations as in previous section. Choose an orthonormal basis $\phi_1, \phi_2, \cdots$ of $H(\sigma)\cong(L^2(K)\otimes H_\sigma)_\sigma$ satisfying $E(\tau_i)\phi_i=\phi_i$ for some $[\tau_i]$ in $\mathscr{E}(K)$ and $v_1, v_2, \cdots$ of H_σ with properties $E(\xi_i)v_i=v_i$ for $[\xi_i]$ in $\mathscr{E}(K_M)$. We now fix $\phi=\phi_p$ and $\tau=\tau_p$. Then ϕ is of the form

$$(3.1) \qquad \phi(k)=\sum_{j,l,m}c_{j,l,m}(\tau(k)\psi_l, \psi_m)\otimes v_j$$

where the summation runs over the set

$$W_{\tau,\sigma}=\{(j, l, m);\ [\tau|_{K_M}\colon \xi_j]>0,\ j\in N \text{ and } 1\leqq i,\ m\leqq d_\tau\},$$

$N=$the set of all natural numbers and $\psi_1, \psi_2, \cdots, \psi_{d_\tau}$ is an orthonormal basis of the space on which τ acts.

Lemma 1. *Let $c_{j,l,m}$ be the constant as in* (3.1). *Then we have $|c_{j,l,m}|^2\leqq d_\tau$ for (j, l, m) in $W_{\tau,\sigma}$.*

Proof. Since $|\phi|=1$, we have

$$1=\sum_j\int_K\sum_{(j,l,m)\in W_{\tau,\sigma}}|(\tau(k)\psi_l, \psi_m)c_{j,l,m}|^2dk$$
$$=(d_\tau)^{-1}\sum_{(j,l,m)\in W_{\tau,\sigma}}|c_{j,l,m}|^2.$$

Hence the lemma follows.

We put $f(x)=(\pi(\sigma, \nu)(x)\phi, \phi)$ for x in G. In view of the formula in (2.4), we have

$$|f(x)|\leqq\int_K e^{-\rho_P(H(x^{-1}k))}|(\sigma(m(x^{-1}k))^{-1}\phi(k(x^{-1}k)),\ \phi(k))|\,dk$$

$$\leqq\sum_{j,l,m}\sum_{i,s,t}\int_K e^{-\rho_P(H(x^{-1}k))}|(\sigma(m(x^{-1}k)v_i,\ v_j)|\,dk\,d_\tau|c_{j,l,m}c_{i,s,t}|,$$

hence by the above lemma we get the following.

(3.2) $$|f(x)|\leqq\sum_{j,l,m}\sum_{i,s,t}(d_\tau)^2\int_K e^{-\rho_P(H(x^{-1}k))}|(\sigma(m(x^{-1}k))^{-1}v_j,\ v_i)|\,dk$$

where (j, l, m) and (i, s, t) run over the set $W_{\tau,\tau}$.

Let m be an element in M. We put $g_{i,j}(m)=(\sigma(m)^{-1}v_i, v_j)$ for all $i, j=1, 2, \cdots$, (v_i's are the orthonormal basis of H_σ). For a fixed (i, j) we put $V_{i,j}=$the linear span of the set $\{L_k\circ R_{k'}g_{i,j};\ k, k'\in K_M\}$. Since v_i and v_j are K_M-finite $V_{i,j}$ is finite dimensional. Let Ω_{K_M} be the Casimir operator on K_M. Then there exists a constant $\chi_{\xi_i}(\Omega_{K_M})$ such that $\Omega_{K_M}\xi_i=\chi_{\xi_i}(\Omega_{K_M})\xi_i$. Therefore Ω_{K_M} acts on $V_{i,j}$ to the left (resp. right) as a scalar operator $\xi_i(\Omega_{K_M})$ (resp. $\xi_j(\Omega_{K_M})$). Consequently by the uniform estimation, which is due to P.C. Trombi and V.S. Varadrarajan (see for instance, Theorem 16.1.9, II, [22]),

(3.3) there exist two positive constants C, κ and a positive number q such that

$$|g_{i,j}(m)|\leqq C((1+|\xi_i(\Omega_{K_M})|)(1+|\xi_j(\Omega_{K_M})|))^q\,\|g_{i,j}\|\,\Xi_M(m)^{1+\kappa}$$

for all m and $i, j=1, 2, \cdots$, where C, κ, q are independent on i, j and m in M, $\|g_{i,j}\|$ is the L^2-norm on M, $|\xi_i(\Omega)_{K_M})|$ is the operator norm of $\xi_i(\Omega_{K_M})$. Using the Schur orthogonality relations for square integrable representation σ, there exists a positive constant d_σ (which is called the formal degree of σ) such that $\|g_{i,j}\|^2=d_\sigma^{-1}|v_i||v_j|$ for all $i, j=1, 2, \cdots$. Therefore (3.3) is rewritten as follows;

(3.4) $|g_{i,j}(m)|\leqq C(1+d_\tau|\chi_\tau(\Omega_K)|)^{2q}\Xi_M(m)^{1+\kappa}$ for all (i, j) satisfying $[\tau:\ \xi_j]>0$ and $[\tau:\ \xi_i]>0$ where $\tau=\tau_p$ is the fixed representation of K as in (3.2), $\chi_\tau(\Omega_K)1=\tau(\Omega_K)$ and C, κ, q are constant (positive) independent on m in M and (i, j).

Combining (3.4) with (3.2) we have

(3.5) $$|f(x)|\leqq C'(1+d_\tau|\chi_\tau(\Omega_K)|)^{2q}(\sharp W_{\tau,\sigma})^2\int_K e^{-\rho_P(H(x^{-1}k))}\Xi_M(m(x^{-1}k))^{1+\kappa}dk$$

where C' does not depend on m in M, $\tau=\tau_p$, $\phi=\phi_p$, and f is the function defined by $f(x)=(\pi(\sigma,\nu)(x)\phi,\ \phi)$.

By the definition of $W_{\tau,\sigma}$ as in (3.1), $\sharp W_{\tau,\sigma}$ is estimated by

$$\sharp W_{\tau\,\sigma} \leqq (d_\tau)^2 \sum_j [\tau|_{K_M} : \xi_j][\sigma|_{K_M} : \xi_j] \leqq (d_\tau)^5. \tag{3.6}$$

Let us estimate $\Xi_M(m(x^{-1}k))^{1+\kappa}$. Let $P_0^* = M_0^* A_0^* N_0^*$ be a minimal parabolic subgroup of M. Choosing P_0^* suitably, we can assume $A_0 = AA_0^*$. Define ρ^*, $k^*(m)$, $H^*(m)$ and $n^*(m)$ for M by the same as in Section 1. Then we have (see Lemma 1.1) $\Xi_M(m) = \Xi_M(\exp H^*(m) n^*(m))$, $m \in M$. Furthermore by (4) in Lemma 1.1, we have $\Xi_M(m(x^{-1}k))^{1+\kappa} \leqq$ a const. $e^{-\rho^*(H^*(m(x^{-1}k)))}$ for all $x \in G$ and $k \in K$. Hence by (3.6) and (3.5), we have the following lemma.

Lemma 2. *There exist two positive numbers p, q and a positive constant C such that $|(\pi(\sigma, \nu)(x)\phi_i, \phi_i)| \leqq C(1+d_{\tau_i})^p |\chi_{\tau_i}(\Omega_K)|^q \Xi_M(x)$ for all $i = 1, 2, \cdots$, and x in G where Ω_K is the Casimir operator on K, $\chi_{\tau_i}(\Omega_K)$ is the constant determined by $\tau_i(\Omega_K) = \chi_{\tau_i}(\Omega_K)1$, $\phi_1, \phi_2, \cdots$ is an orthonormal basis of $H(\sigma, \nu)$ satisfying $E(\tau_i)\phi_i = \phi_i$ for some τ_i in $\mathscr{E}(K)$.*

Theorem 1. *Let (π, H) be an irreducible component of principal P-series representation $\pi(\sigma, \nu)$ of G where $P = MAN$ is a parabolic subgroup which is cuspidal, σ is a discrete series representation of M and e^ν is a unitary character of A. Then the character Θ_π of π is tempered.*

Remark. There is a table of characters of all irreducible components of principal P-series representations of G which is obtained by A.W. Knapp and G. Zuckerman ([14]). In view of the table, we see that all character of subrepresentations are tempered. In this paper we give a proof which is different from [14].

Proof of Theorem 1. Let $\phi_1, \phi_2, \cdots$ be an orthonormal basis of H. We choose ϕ_i which has the same property as in Lemma 2. Let p and q be the same as in Lemma 2. Then there exists a positive number m such that the series $c_m = \sum_{[\tau] \in \mathscr{E}(K)} (1+d_\tau)^p d\tau^4 (\chi_\tau(\Omega_K))^{2(q-m)}$ is convergent. We fix such a number m. By definition

$$\begin{aligned}
|\Theta_\pi(f)| &\leqq \sum_{i=1}^{\infty} \left| \int_G f(x)(\pi(x)\phi_i, \phi_i)dx \right| \\
&\leqq \sum_{i=1}^{\infty} (\chi_{\tau_i}(\Omega_K))^{-2m} \int_G (f\Omega_K^{2m})(x)| |(\pi(x)\phi_i, \phi_i)| dx \\
&= \sum_{[\tau] \in \mathscr{E}(K)} \sum_{E(\tau)\phi_i = \phi_i} (\chi_\tau(\Omega_K))^{-2m} \int_G |(f\Omega_K^{2m})(x)| |(\pi(x)\phi_i, \phi_i)| dx.
\end{aligned}$$

Hence by Lemma 2.1 and Lemma 3, we have

$$|\Theta_\pi(f)| \leqq c_m \int_G |(f\Omega_K^{2m})(x)| \Xi(x) dx.$$

Let r be a positive number satisfying $c=\int_G \Xi(x)^2(1+d(x))^{-r}dx<\infty$. Then we have $|\Theta_\pi(f)| \leqq cc_m\nu_{1,\Omega_K^{2m},r}(f)$ for all f in $C_c^\infty(G)$. Thus the character Θ_π is tempered. This completes our proof.

§ 4. Pre-Hilbert structure on $H_{i(\chi)}(G, \chi)$

First of all, in this section, we define a topological G-module $H_{i(\chi)}(G, \chi)$ for an infinitesimal character χ of $\mathfrak{z}$. Let $P_0=M_0A_0N_0$ be the minimal parabolic subgroup of G and $\mathfrak{h}$ a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$. The root system and Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$ will be denoted by Φ and W respectively. Canonically W acts on the universal enveloping algebra $\mathfrak{u}(\mathfrak{h})$ of $\mathfrak{h}_C$. We regard $\mathfrak{u}(\mathfrak{h})$ as an algebra of polynomial functions on the dual space of $\mathfrak{h}_C$, and denote $I(\mathfrak{h})$ the stabilizer of W in $\mathfrak{u}(\mathfrak{h})$. Let Φ^+ be a positive root system of Φ. Therefore $\mathfrak{g}_C=\mathfrak{h}_C\oplus\sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}_C; ad(H)X=\alpha(H)X$ for all H in $\mathfrak{h}\}$. We put $\mathfrak{n}^+=\sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$. Then there exists a unique isomorphism γ of $\mathfrak{z}$ into $\mathfrak{u}(\mathfrak{h})$ such that $z-\gamma(z)\in\mathfrak{u}(\mathfrak{g})\mathfrak{n}^+$ for z in $\mathfrak{z}$. Let ρ be one half the sum of all positive roots in Φ, and define μ of $\mathfrak{z}$ into $\mathfrak{u}(\mathfrak{h})$ by $\mu(z)(\lambda)=\gamma(z)(\lambda-\rho)$ for z in $\mathfrak{z}$ and linear form λ on $\mathfrak{h}_C$. μ is an isomorphism of $\mathfrak{z}$ onto $I(\mathfrak{h})$. Therefore each character χ of $\mathfrak{z}$ is parametrized by $\chi=\chi_\lambda$ where $\chi(z)=\mu(z)(\lambda)$ for some linear form λ on $\mathfrak{h}_C$. By the definition, $\chi_{s\lambda}=\chi_\lambda$ for all s in W. Let X_α and $X_{-\alpha}$ be the basis of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ respectively satisfying $B(X_\alpha, X_{-\alpha})=1$, and put $H_\alpha=[X_\alpha, X_{-\alpha}]$ where B is the Killing form on $\mathfrak{g}_C\cdot A$ character χ_λ of $\mathfrak{z}$ is regular if $\lambda(H_\alpha)\neq 0$ for all α in Φ and real if $\lambda(H_\alpha)\in\boldsymbol{R}$ for all α in Φ.

Definition 1. Let Ψ be a fundamental root system of Φ. We put $\Psi(\chi)=\{\alpha\in\Psi; \lambda(H_\alpha)\in\boldsymbol{R}-\{o\}\}$. The number $i(\chi)=\sharp\Psi-\sharp\Psi(\chi)$ is called the index of $\chi=\chi_\lambda$.

Definition 2. Let χ be a character of $\mathfrak{z}$. A function f in $C^\infty(G)$ belongs to $H_{i(\chi)}(G, \chi)$ if f satisfies $(z-\chi(z))f=0$ and $\|b_1fb_2\|<\infty$ for all z in $\mathfrak{z}$ and b_i in $\mathfrak{u}(\mathfrak{g})$, where $\|\ \|$ is the seminorm on $C^\infty(G)$ defined by

$$(4.1)\qquad \|f\|^2=\lim_{\varepsilon\to+0}\varepsilon^{i(\chi)}\int_G |f(x)|^2e^{-\varepsilon d(x)}dx, \qquad d(x)=d(xo, o).$$

We restate the properties for $H_{i(\chi)}(G, \chi)$ in the following two lemmas (see Lemma 2.1 and Lemma 2.2 in [20]).

Lemma 1. *$H_{i(\chi)}(G, \chi)$ is a topological G-module with seminorm* $\|\ \|$

under the canonical left (resp. right) action L and R. Furthermore for each f in $H_{i(\chi)}(G, \chi)$ and x in G, $\|L_x f\| = \|R_x f\| = \|f\|$.

Let τ be an irreducible unitary representation of K. We define two actions $\chi_\tau *$ and $*\chi_\tau$ on $C^\infty(G)$ as in (2.7). Then by Peter-Weyl theorem on the compact group K, we have

$$f(x) = \sum_{\tau, \tau' \in \mathcal{E}(K)} d_\tau d_{\tau'} (\chi_\tau * f * \chi_{\tau'})(x) \qquad \text{for } f \text{ in } C^\infty(G). \tag{4.2}$$

Lemma 2. *Let f be an element in $H_{i(\chi)}(G, \chi)$. Then we have*

(1) $\|\chi_\tau * f * \chi_{\tau'}\| \leqq (d_\tau d_{\tau'})^{1/2} \|f\|$ *for all τ and τ' in $\mathcal{E}(K)$,*

(2) *the expansion of f in* (4.2) *converges to f in the topology $H_{i(\chi)}(G, \chi)$.*

Remark 1. Let $H_{i(\chi), K}$ be the set of all K-finite (left and right) functions in $H_{i(\chi)}(G, \chi)$. $H_{i(\chi), K}$ is an algebraic $\mathfrak{u}(\mathfrak{g})$-module (see for a proof, Lemma 3.5 in [19]).

The purpose of this section is to prove $H_{i(\chi)}(G, \chi)$ is a pre-Hilbert space with norm $\| \ \|$. This will be proved by using two asymptotic expansion theorems for τ-spherical eigenfunctions on G.

Definition. A unitary representation (τ, U) of $K \times K$ is a double representation of K if there exist two unitary representations τ_1 and τ_2 of K such that $\tau(k_1, k_2)\phi = \tau_1(k_1)\phi\tau_2(k_2)$ for all k_i in K and ϕ in U.

For the double unitary representation of K, we shall denote $\tau = (\tau_1, \tau_2)$.

Let f be a C^∞-function on G. We define for each x in G,

$$F(x)(k_1, k_2) = f(k_1 x k_2). \tag{4.3}$$

We see that $F(x)$ belongs to $L^2(K \times K)$ for a fixed x in G.

Lemma 3. *Let f be a K-finite C^∞-function on G and $F = F_f$ the same as in* (4.3). *Then there exists a finite dimensional double unitary representation (τ, U) of K such that $F(x) \in U$ and $F(kxk') = \tau_1(k)F(x)\tau_2(k')$ for all x in G, k, k′ in K.*

Proof. We define two unitary representations of K on $L^2(K \times K)$ by $(\zeta_1(k)\phi)(k_1, k_2) = \phi(k_1 k, k_2)$, $(\zeta_2(k)\phi)(k_1, k_2) = \phi(k_1, kk_2)$ for k in K and ϕ in $L^2(K \times K)$. Then $\zeta = (\zeta_1, \zeta_2)$ is a double unitary representation of K. Furthermore we have $F(kxk') = \zeta_1(k)F(x)\zeta_2(k')$ for all x in G and k, k' in K. Let U be the subspace of $L^2(K \times K)$ generated by the set $\{\zeta_1(k)F(x)\zeta_2(k'); k, k' \in K \text{ and } x \in G\}$. Since f is K-finite, the dimension of U is finite, Let $\tau = (\tau_1, \tau_2)$ be the restriction of ζ to U. Then F and τ have the property as claimed.

Remark 2. *By definition of $F=F_f$, we see that there exists ϕ in U such that $f(x)=(F(x), \phi)$ for all x in G where f is a K-finite C^∞-function on G.*

Definition 4. Let $\tau=(\tau_1, \tau_2)$ be a finite dimensional double unitary representation of K realized on U. A U-valued C^∞- (resp. L^2-) function F on G is τ-spherical if F satisfies $F(kxk')=\tau_1(k)F(x)\tau_2(k')$ for all k, k' in K and x in G.

Let f be a K-finite function in $H_{i(\chi)}(G, \chi)$. We define $F=F_f$ as in (4.3). By Lemma 3, F is (τ, U)-spherical on G. Furthermore by using the integral formula of Lemma 1.2, we have

$$\|f\|^2=C_G \lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_{A_0^+} |F(a)|^2 D(a) e^{-\varepsilon d(a)} da. \tag{4.4}$$

Since $(z-\chi(z))f=0$, we have also

$$(z-\chi(z))F=0 \qquad \text{for all } z \text{ in } \mathfrak{z}. \tag{4.5}$$

Thus the function $F=F_f$ is a τ-spherical eigenfunction of $\mathfrak{z}$. Concerning with the integral of (4.4), we give the following estimations for d;

(4.6) there exist two positive constants c_1 and c_2 such that $c_1 e^{\rho(\log a)} \leqq d(a) \leqq c_2 e^{\rho(\log a)}$ for all a in A_0^+ (we remark that $d(a)^2=B(\log a, \log a)$ for all a in A_0).

Let $\Psi(A_0)$ be the simple root system of (P_0, A_0). We choose the dual basis $\omega_1, \omega_2, \cdots, \omega_l$ of $\mathfrak{a}_0$ with respect to $\Psi(A_0)=\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ satisfying $\alpha_i(\omega_j)=\delta_{i,j}$. We put $A_0^+(R)=\{a \in A_0^+ ; \alpha(\log a)>R \text{ for } \alpha \in \Psi(A_0)\}$ for a given positive real number R. $\boldsymbol{Z}^+=$the set of all nonnegative integers. We state the first expansion theorem for a τ-spherical eigenfunction on G.

Lemma 4. *Let F be a τ-spherical $\mathfrak{z}$-finite function on G where τ is realized on a finite dimensional vector space U. Then F has the following expansion on $A_0^+(R)$; there exist a finite number of linear forms $\nu_1, \nu_2, \cdots, \nu_p$ and polynomials $p_1, p_2, \cdots, p_q$ on $\mathfrak{a}_0$ and $F_{i,j}$ $(1\leqq i\leqq q,\ 1\leqq j\leqq p)$ such that*

$$(d_{P_0}F)(a)=\sum_{i=1}^{q}\sum_{j=1}^{p} p_i(\log a) e^{\nu_j(\log a)} F_{i,j}(a),$$

$$F_{i,j}(a)=\sum_{m=(m_1,m_2,\cdots,m_l)\in(\boldsymbol{Z}^+)^l} c_{i,j,m} e^{-(m_1\alpha_1+m_2\alpha_2+\cdots+m_l\alpha_l)(\log a)}$$

where $c_{i,j,m} \in U$,

Furthermore the series $F_{i\,j}$ *is uniform and absolute convergence on* $A_0^+(R)$.

(For the proof of this lemma, see Theorem 8.32, [16]).

We now parametrize A_0 by $A_0=\{a_t;\ t=(t_1, t_2, \cdots, t_l)\in \boldsymbol{R}^l\}$ where $a_t=\exp(\sum_{i=1}^{l} t_i\omega_i)$. Therefore $A_0^+(R)\in a_t$ if and only if $t_i>R$ for all $i=1, 2, \cdots, l$.

Lemma 5. *Each K-finite function in* $H_{i(\chi)}(G, \chi)$ *is tempered.*

Proof. Let f be a K-finite function in $H_{i(\chi)}(G, \chi)$. We define the τ-spherical eigenfunction $F=F_f$ of $\mathfrak{z}$ as in Lemma 3. In view of (4.4) and (4.6), we have

$$\lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_{A_0^+(R)} |(d_{P_0}F)(a)|^2 e^{-\varepsilon\rho(\log a)}da$$

$$\leqq a \text{ const.} \lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_{A_0^+} |F(a)|^2 D(a) e^{-\varepsilon d(a)}da<\infty.$$

Using the expansion for F as in Lemma 4, an elementary calculation verifies that $(\operatorname{Re}\nu_j)(\omega_k)\leqq 0$ for all (j, k) where ν_j's are the same as in the expansion of F.

Consequently, it follows from a result of Casselman and Miličić (see Theorem 8.47, [16]) that $|(d_{P_0}F)(a)|\leqq a$ const. $(1+d(a))^n$ for all a in A_0 for a suitable nonnegative number n. Therefore F is tempered, and hence f is also tempered (see Remark 2).

We shall state the second theorem for the asymptotic expansion (which is due to Harish-Chandra) for a τ-spherical eigenfunction of $\mathfrak{z}$ on G. Let $P=MAN$ be a fixed parabolic subgroup of G. We denote the Lie algebras of A and M respectively by $\mathfrak{a}$ and $\mathfrak{m}$. $\mathfrak{m}_1=\mathfrak{m}\oplus\mathfrak{a}$ is the Lie algebra of reductive group $M_1=MA$.

Notations: $\mathfrak{u}(\mathfrak{m}_1)=$the universal enveloping algebra of $(\mathfrak{m}_1)_C$,
$\mathfrak{z}_{M_1}=$the center of $\mathfrak{u}(\mathfrak{m}_1)$,
$W_1=$the Weyl group of $((\mathfrak{m}_1)_C, \mathfrak{h}_C)$,
$I_1(\mathfrak{h})=$the ring of all W_1-invariants of $u(\mathfrak{h})$,
$\mu_1=$the canonical isomorphism of $\mathfrak{z}_{M_1}$ to $I_1(\mathfrak{h})$ and
$\mu=$the canonical isomorphism of $\mathfrak{z}$ to $I(\mathfrak{h})$.

We see that there exists a unique isomorphism μ_P of $\mathfrak{z}$ into $\mathfrak{z}_{M_1}$ such that $\mu=\mu_1\circ\mu_P$.

Let A^+ and $\Psi(A)$ be the positive Weyl chamber and the simple root system of (P, A) respectively. We define a function β on $\operatorname{cl}(A^+)$ by

$\beta(a)=\min_{\alpha\in\Psi(A)}\alpha(\log a)$, $a\in\mathrm{cl}(A^+)$ where $\mathrm{cl}(A^+)$ is the closure of A^+. We denote $(\tau_{K_M}, U)=$ the restriction of τ to K_M.

Lemma 6. *Let F be a tempered $\mathfrak{z}$-finite (τ, U)-spherical function on G. Then there exists a $\mathfrak{z}_{M_1}$-finite tempered τ_{K_M}-spherical function F_P on M_1 and a nonnegative real number r such that*

(1) $|(d_PF)(am)-F_P(am)|\leqq a$ const. $\Xi_M(m)e^{-\beta(a)}(1+d(ma))^r$ *for all a in* $\mathrm{cl}(A^+)$ *and m in Ω where Ω is a compact subset in M_1,*

(2) $\mu_P(z)F_P=(zF)_P$ *for all z in $\mathfrak{z}$.*

(For a proof this lemma, see Chapter 14, II, [23]).

The function F_P is called the constant term of F along P. By (1) in the above lemma, we see that F_P is uniquely determined by F. Furthermore since $\mathfrak{z}_{M_1}$ is a free $\mu_P(\mathfrak{z})$-module with finite rank (see Corollary 4.2.10, I, [23]), it follows from (2) in the lemma that F_P is of the form

(4.7) $F_P(am)=\sum_{i=1}^q p_i(\log a)e^{\lambda_i(\log a)}F_i(m)$ where p_i is a polynomial and λ_i is a purely imaginary valued linear form on $\mathfrak{a}$, F_i is a tempered τ_{K_M}-spherical eigenfunction of $\mathfrak{z}_M$ on M for some character χ_i^* of $\mathfrak{z}_M$.

Let Θ be a fixed subset of $\Psi(A_0)$. We put

(4.8) $$A_\Theta=\{a\in A_0;\ \beta(\log a)=0\}\quad\text{for all }\beta\text{ in }\Psi(A_0)-\{\alpha\}\}.$$

Then there exists a parabolic subgroup P_Θ of G such that $P_\Theta=M_\Theta A_\Theta N_\Theta$ (see for precise descriptions [1] or [26]). Let α be an element in Θ, and put $\Theta_\alpha=\Theta-\{\alpha\}$. Then the parabolic subgroup $P_{\Theta_\alpha}=M_{\Theta_\alpha}A_{\Theta_\alpha}N_{\Theta_\alpha}$ satisfies $M_{\Theta_\alpha}\subset M_\Theta$, $A_{\Theta_\alpha}\supset A_\Theta$ and $N_{\Theta_\alpha}\supset N_\Theta$. We put $P_{\Theta,\alpha}=M_\Theta\cap P_{\Theta_\alpha}$. Let $(P_\Theta^*)_0=(M_\Theta^*)_0(A_\Theta^*)_0(N_\Theta^*)_0$ be the minimal parabolic subgroup of M_Θ. Then we have $A_0=A_\Theta(A_\Theta^*)_0$.

Let r be a positive real number and $\mathrm{cl}(A_0^+)$ the closure of A_0^+ in A_0. We put for each α in $\Psi(A_0)$.

(4.9) $$A(\alpha, r)=\{a\in\mathrm{cl}(A_0^+);\ \alpha(\log a)\geqq r\rho(\log a)\}.$$

Lemma 7. *For a sufficiently small real positive number r, we have that* $\mathrm{cl}(A_0^+)\subset\bigcup_{\alpha\in\Psi(A_0)}A(\alpha, r)$.

Proof. We put $S^+=\{a\in\mathrm{cl}(A_0^+);\ d(a)=1\}$, and define two functions f, g by $f(a)=\max_{\beta\in\Psi(A_0)}\beta(\log a)$, $g(a)=\rho(\log a)$. S^+ is compact and f (resp. g) is continuous on $\mathrm{cl}(A_0^+)$. Therefore g (resp. f) has the maximal (resp. minimal) value r_2 (resp. r_1) on S^+. Since f and g are positive on

S^+, r_1 and r_2 are positive. Let r be a real number satisfying $0<r<(r_1/r_2)$. We claim that $\mathrm{cl}(A_0^+)\subseteqq \bigcup_{\alpha\in\Psi(A_0)} A(\alpha, r)$. Let a be an element in A_0^+. We put $H'=d(a)^{-1}\log a$. Then $a'=\exp H'$ belongs to S^+. Consequently $r_1\leqq f(a')$ and $g(a')\leqq r_2$. Choose an element α in $\Psi(A_0)$ satisfying $\alpha(\log a')=f(a')$. Then we have $\alpha(\log a')>r\rho(\log a')$. Hence the lemma follows.

Lemma 8. *Let F be a tempered $\mathfrak{z}$-finite (τ, U)-spherical function on G. Assume that $F_P=0$ for all maximal proper parabolic subgroup P of G. Then F is square integrable on G.*

Proof. Let $A(\alpha, r)$ be the same as in Lemma 7. Then we have

$$\int_G |F(x)|^2 dx \leqq C_G \sum_{\alpha\in\Psi(A_0)} \int_{A(\alpha,r)} |(d_{P_0}F)(a)|^2 da. \tag{4.10}$$

We now fix an element α in $\Psi(A_0)$, and consider the maximal parabolic subgroup $P=MAN$ corresponding to the set $\Theta=\{\alpha\}$. For the minimal parabolic subgroup $P_0^*=M_0^*A_0^*N_0^*$ of M, we define ρ^* by the same as in Section 1. By (1) in Lemma 6 and our assumption $F_P=0$, the function d_PF is estimated by

$$|(d_PF)(a)| \leqq a \text{ const. } \Xi_M(a)(1+d(a))^p e^{-r\rho(\log a)} \tag{4.11}$$

for all a in $A(\alpha, r)$ where p is a nonnegative integer. Hence by Lemma 1.1, we get $|d_PF(a)|\leqq c'e^{-c\rho(\log a)}e^{-\rho^*(\log a)}$ for all a in $A(\alpha, r)$ where c and c' are positive constants. Combining (4.10) with this inequality, we have our conclusion.

Remark 3. Let F be a square integrable τ-spherical function on G and χ a character of $\mathfrak{z}$. If F satisfies the differential equation $(z-\chi(z))F=0$ and F is nontrivial. Then χ is real regular. For this proof, see Harish-Chandra's classification for discrete series representations of G ([8] or Theorem 14.4.9 and Theorem 16.3.19, II, [23]).

We now prove our main purpose of this section.

Theorem 1. *Let $H_{i(\chi)}(G, \chi)$ be the topological vector space as in Definition 2. Assume that $H_{i(\chi)}(G, \chi)\neq\{0\}$. Then the space has a pre-Hilbert structure with norm $\|\ \|$.*

Proof. Let f be a nontrivial element in $H_{i(\chi)}(G, \chi)$. It is enough to show that if $\|f\|=0$, then there is a contradiction. By Lemma 2, the series $\sum_{\tau,\tau'\in\mathscr{E}(K)}(\chi_\tau * f * \chi_\tau)$ converges to f in the topology of $H_{i(\chi)}(G, \chi)$. Consequently we have $\|f\|^2=\sum_{\tau,\tau'\in\mathscr{E}(K)}\|\chi_\tau * f * \chi_{\tau'}\|^2$. Therefore we can assume that f is K-finite and nontrivial. Define a τ-spherical eigenfunction $F=F_f$ as in (4.3). Let p_i, ν_j and $F_{i,j}$ be the same appearing in the expansion of F on $A^+(R)$ as in Lemma 4. By our assumption $F\neq 0$, we can assume that $p_iF_{i,j}\neq 0$ for all (i,j). Furthermore since

$$\|f\|^2=\lim_{\varepsilon\to+0} C_G\varepsilon^{i(\chi)}\int_{A_0^+}|F(a)|^2D(a)e^{-\varepsilon d(a)}da=0,$$

we get for each $\nu=\nu_i$, (1) $(\mathrm{Re}\,\nu)(\omega_k)\leqq 0$ for all $k=1,2,\cdots,l$, (2) $\#\Theta_i<i(\chi)$ for all i where $\Theta_i=\{\alpha_k\in\Psi(A_0);\ \mathrm{Re}\,\nu_i(\omega_k)=0\}$. We choose Θ_{i_0} satisfying $\#\Theta_i\leqq\#\Theta_{i_0}$ for all $i=1,2,\cdots,p$. Put $\Theta=\Theta_{i_0}$. Then we have

$$i(\chi)>\#\Theta. \tag{4.12}$$

Let $P_\Theta=M_\Theta A_\Theta N_\Theta$ be the parabolic subgroup of G corresponding to Θ. and F_{P_Θ} the constant term of F along P_Θ. Combining (1) in Lemma 6 with the expansion of F in Lemma 4, the choice of Θ implies that $F_{P_\Theta}\neq 0$. Let $F_{P_\Theta}=\sum_k p_ke^{\lambda_k}F_k$ and χ_k^* be the same as in (4.7).

We put for each α in Θ, $\Theta_\alpha=\Theta-\{\alpha\}$, and consider the parabolic subgroup $P^*_{\Theta,\alpha}=M_\Theta\cap P_{\Theta_\alpha}=M^*_{\Theta,\alpha}A^*_{\Theta,\alpha}N^*_{\Theta,\alpha}$. Then we have $A_{\Theta_\alpha}=A^*_{\Theta,\alpha}A_\Theta$. Define a function $F_{P_\Theta,a}$ on M_Θ for a fixed a in A_Θ by $(F_{P_\Theta,a})(m)=F_{P_\Theta}(am)$. Then we have $(F_{P_\Theta,a})_{P^*_{\Theta,\alpha}}(a^*m^*)=F_{P_{\Theta_\alpha}}(aa^*m^*)$ for all a in A_Θ, a^* in $A^*_{\Theta,\alpha}$ and m^* in $M^*_{\Theta,\alpha}$. Therefore

$$F_{P_{\Theta_\alpha}}(aa^*m^*)=\sum_k e^{\lambda_k}\Big(\sum_j p^*_{j,k}e^{\lambda_{j,k}}F_{j,k}\Big) \tag{4.13}$$

where λ_k and $\lambda_{j,k}$ are purely imaginaly valued linear forms on $\mathfrak{a}_\Theta$ and $\mathfrak{a}^*_{\Theta,\alpha}$ respectively. In the expression of $d_{P_0}F=\sum_{i=1}^q\sum_{j=1}^p p_ie^{\nu_j}F_{i,j}$ on $A_0^+(R)$ as in Lemma 4, we have $\#\Theta_j\leqq\#\Theta=\dim A_\Theta$. However by the estimation for $(d_{P_\Theta}F-F_{P_\Theta})$ as in Lemma 6 and the fact $\dim A_{\Theta_\alpha}=\dim A_\Theta+1$, it follows from the uniqueness for expansion of F on $A_0^+(\boldsymbol{R})$ that $F_{P_{\Theta_\alpha}}=0$ for all α in Θ. Hence by Lemma 8 and Remark 3, χ_i^* is real regular. Consequently we have a contradiction;

$$i(\chi)=\#\Psi-\#\Psi(\chi)\leqq\#\Psi-\mathrm{rank}\,(M_\Theta)=\dim A_\Theta<i(\chi).$$

This completes our proof.

Lemma 9. *Let notations and assumptions being as in above theorem. In the term of expansion of* $F=F_f=\sum_i\sum_j p_ie^{\nu_j}F_{i,j}$ *on* $A_0^+(R)$, *we have* $i(\chi)=\#\Theta_j$ *and* $p_i=a$ *constant where* f *is a nontrivial function* (K-*finite*) *in* $H_{i(\chi)}(G,\chi)$ *and* $\Theta_j=\{\alpha_k\in\Psi(A_0);\ \mathrm{Re}\,\nu_j(\omega_k)=0\}$.

Proof. In view of the proof for Theorem 1, we see that $i(\chi)\leqq\#\Theta_j$.

On the other hand since

(4.14) $\|f\|^2 \geqq a$ positive const. $\lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_{A_0^+} |(d_P F)(a)|^2 e^{-\varepsilon\rho(\log a)} da$,

we have $i(\chi) \geqq \sharp\Theta_j$. Consequently $i(\chi) = \sharp\Theta_j$ for all $j = 1, 2, \cdots, p$. Again by (4.14), we have also $p_i = a$ const. for all $i = 1, 2, \cdots, q$.

Lemma 10. *Let f be a nontrivial K-finite function in $H_{i(\chi)}(G, \chi)$. Then there exists a cuspidal parabolic subgroup P of G such that $F_P \neq 0$ where $F = F_f$ and F_P is the constant term of F along P.*

Proof. Let Θ_j be the same as in the above lemma, and put $\Theta = \Theta_j$. We denote $P_\Theta = M_\Theta A_\Theta N_\Theta$, $F_{P_\Theta} = \sum_k p_k e^{\lambda_k} F_k$. By the choice of Θ, we have $F_{P_\Theta} \neq 0$ and $(F_k)_{P^*_{\Theta,\alpha}} = 0$ for all α in Θ, where $\Theta_\alpha = \Theta - \{\alpha\}$, $P^*_{\Theta,\alpha} = M_\Theta \cap P_{\Theta_\alpha}$. Hence F_k is square integrable on M. Since F_k is nontrivial, it follows from a result of Harish-Chadra that rank $M =$ rank $M \cap K$. Thus P_Θ is a parabolic cuspidal subgroup of G.

Lemma 11. *Let f be a nontrivial K-finite function in $H_{i(\chi)}(G, \chi)$ and $F_{P_\Theta} = \sum_k p_k e^{\lambda_k} F_k$ the constant term of $F = F_f$ along P_Θ where Θ is a given subset of $\Psi(A_0)$ and F_k is a tempered τ_{K_M}-spherical function on M_Θ satisfying $(z - \chi_k(z))F_k = 0$ ($z \in \mathfrak{z}_{M_\Theta}$) for some character χ_k of $\mathfrak{z}_M$, p_k is a polynomial function on $\mathfrak{a}_\Theta$ and λ_k is a purely imaginary valued linear form on $\mathfrak{a}_\Theta$. Assume that $F_{P_\Theta} \neq 0$. Then we have $i(\chi) = \dim A_\Theta + i(\chi_k)$ for all k.*

Proof. Let $P_0^* = M_0^* A_0^* N_0^*$ be the minimal parabolic subgroup of M_Θ and $d_{P_0^*} F_k = \sum_{i=1}^p \sum_{j=1}^q p_{k,i} e^{\nu_{k,j}} F_{i,j}^k$ be the expansion of F_k on $(A_0^*)^+(R)$ as in Lemma 4. We put $\Theta_{k,j} = \{\alpha_u \in \Theta = \Psi((A_0^*)^+); \operatorname{Re} \nu_{k,j}(\omega_u) = 0\}$. Let $P^*_{\Theta_{k,j}}$ be the parabolic subgroup of M_Θ corresponding to the set $\Theta_{k,j}$. Then we have $(F_k)_{P^*_{\Theta_{k,j}}} \neq 0$. We now fix a number k and denote $(F_u^*)_{P^*_{\Theta_{k,j}}} = \sum_t p_{u,j,t} e^{\lambda_{u,j,t}} F^*_{u,j,t}$ where $F^*_{u,j,t}$ is a solution of the differential equations $zF^*_{u,j,t} = \chi^*_{u,j,t}(z) F^*_{u,j,t}$ ($z \in \mathfrak{z}_{M^*_{\Theta_{j,k}}}$) for some character $\chi^*_{u,j,t}$ of $\mathfrak{z}_{M^*_{\Theta_{j,k}}}$. By the choice of $\Theta_{k,j}$, $\chi^*_{k,j,t}$ is real regular. Therefore $i(\chi_k) = \sharp\Theta_{k,j}$.

Since $(F_{P_{\Theta,a}})_{P^*_{\Theta_{k,j}}}(a^*m) = F_{P_{\Theta_{k,j}}}(aa^*m)$ for all a in A_Θ, a^* in $A^*_{\Theta_{k,j}}$ and m in $M^*_{\Theta_{k,j}}$,

(4.15) $$F_{P_{\Theta_{k,j}}}(aa^*m) = \sum_i p_i e^{\lambda_i} \left(\sum_{u,t} p_{u,j,t} e^{\lambda_{u,j,t}} F^*_{u,j,t}\right)$$

where λ_k and $\lambda_{u,j,t}$ are purely imaginary valued linear forms. Since $\chi^*_{k,j,t}$ is real, it follows from the expressions for $F_{P_{\Theta_{k,j}}}$ and the expansion of F in Lemma 4, that $i(\chi) = \sharp\Psi -$ rank $M_{\Theta_{k,j}} = \dim A_{\Theta_{k,j}} = \dim A_\Theta + \dim A^*_{\Theta_{k,j}} = \dim A_\Theta + i(\chi_k)$ and $p_k p_{u,j,t} = a$ const. (see Lemma 9). Thus the lemma follows.

§ 5. Schur orthogonality relations

Let χ be a character of $\mathfrak{z}$ and $H_{i(\chi)}(G, \chi)$ the same as in (4.1). We define a Hermitian form (,) on $H_{i(\chi)}(G, \chi)$ by

$$(5.1) \qquad (f, g)=\lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_G f(x)\overline{g(x)}e^{-\varepsilon d(x)}dx$$

where $d(x)=d(xo, o)$, $d(\ ,\ )$ is the Riemannian distance on the symmetric space G/K and o is the origin.

By Theorem 4.1, the form (,) is a positive definite Hermitian form on $H_{i(\chi)}(G, \chi)$.

Definition 1. $H(G, \chi)$: the completion of $H_{i(\chi)}(G, \chi)$,
$H_K(G, \chi)$: the set of all K-finite elements in $H(G, \chi)$,
$\mathfrak{z}^\vee$: the set of all characters of $\mathfrak{z}$ satisfying $H(G, \chi)\neq \{0\}$.

Remark 1. Let $H_{i(\chi),K}$ be the set of all K-finite functions in $H_{i(\chi)}(G, \chi)$. Since $H_{i(\chi),K}$ is dense in $H_{i(\chi)}(G, \chi)$, we have $H_K(G, \chi)=H_{i(\chi),K}$. Therefore all functions in $H_K(G, \chi)$ are real analytic and tempered (see Lemma 4.5).

Let R be the right regular representation of G on $H(G, \chi)$. We see that the representation $(R, H(G, \chi))$ is unitary.

Definition 2. An irreducible unitary representation (π, H) of G is realized on $H(G, \chi)$ if (π, H) is unitary equivalent to a subrepresentation of $(R, H(G, \chi))$.

Let (π, H) be an irreducible unitary representation of G and $C_c^\infty(G)$ the set of all C^∞-functions on G with compact support. For a fixed K-finite vector ϕ in H, we put

$$(5.2) \qquad H(\phi)=\{\pi(f); f\in C_c^\infty(G)\} \quad \text{where } \pi(f)=\int_G f(x)\pi(x)dx.$$

Then all vectors in $H(\phi)$ are differentiable. Furthermore since π is irreducible the space $H(\phi)$ is a G-invariant dense subspace of H. Let ϕ_0, ψ_0 be two fixed K-finite vectors in H. We define a linear operator S_{ψ_0} of $H(\phi_0)$ to $C^\infty(G)$ by

(5.3) $S_{\psi_0}(\pi(f)\phi_0)(y)=(\pi(y)\pi(f)\phi_0, \psi_0)$ for y in G.

Immediately we have

(5.4) S_{ψ_0} is injective, $R_x\circ S_{\psi_0}=S_{\psi_0}\circ\pi(x)$ for all x in G.

Lemma 1, *Let (π, H) be an irreducible unitary representation of G. Suppose that there exist two K-finite vectors ψ_0 and ϕ_0 such that $S_{\psi_0}(\phi_0) \in H(G, \chi)$ for some χ in $\mathfrak{z}^\vee$. Then we have $S_{\psi_0}(\pi(f)\phi_0) \in H(G, \chi)$ for all f in $C_c^\infty(G)$.*

Proof. Let χ_π be the infinitesimal character of π. Then we have $\chi=\chi_\pi$ and $zS_{\psi_0}(\pi(f)\phi_0)=\chi(z)S_{\psi_0}(\pi(f)\phi_0)$ for all f in $C_c^\infty(G)$. It remains to prove $\|S_{\psi_0}(\pi(f)\phi_0)\|<\infty$. Let W be the support of f. We put

$$c_f=\int_G |f(x)|^2 dx=\int_W |f(x)|^2 dx.$$

By using Schwarz inequality, we have

$$\begin{aligned}\|S_{\psi_0}(\pi(f)\phi_0)\|^2 &\leqq c_f \lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_G\int_W |(\pi(yx)\phi_0, \psi_0)|^2 e^{-\varepsilon d(x)}dx \\ &\leqq c_f \lim \varepsilon^{i(\chi)}\int_W\int_G |(\pi(yx)\phi_0, \psi_0)|^2 e^{-\varepsilon d(x)}dx \\ &\leqq c_f \operatorname{vol}(W)\|S_{\psi_0}(\phi_0)\|^2\end{aligned}$$

where $\operatorname{vol}(W)$ is the volume of W. Hence the lemma follows.

Lemma 2. *Let notations and assumptions being as above lemma. Then the representation (π, H) is realized on $H(G, \chi)$.*

Proof. Let H' be the minimal closed invariant subspace of $(R, H(G,\chi))$ containing $S_{\psi_0}(\phi_0)$. We put $\pi'=$the restriction of R to H'. By Lemma 1, we have $S_{\psi_0}(H(\phi_0))\subset H'$. We shall prove that (π', H') is irreducible. Choosing ϕ_0 suitably we can assume that $E(\tau)\phi_0=\phi_0$ for an element $[\tau]$ in $\mathscr{E}(K)$. We put $H(\tau)=E(\tau)H$ and

$$R(\tau)=\left\{f \in C_c^\infty(G);\ \chi_\tau * f=f,\ \int_K f(kxk^{-1})dx=f(x) \text{ for all } x \text{ in } G\right\}.$$

$R(\tau)$ is an algebra with convolution product. Furthermore the representation of algebra $R(\tau)$ on $H(\tau)$ is irreducible (see [7], Theorem 6). Consequently since $\dim H(\tau)=\dim S_{\psi_0}(H(\tau))$ is finite, the algebra representation of $R(\tau)$ on $S_{\psi_0}(H(\tau))$ is irreducible. Let W be a nontrivial closed invariant subspace of H' and $W^\perp$ the orthogonal complement of W. Then we have $S_{\psi_0}(H(\tau))\subset E(\tau)W+E(\tau)W^\perp$. Consequently the irreducibility of the representation of $R(\tau)$ on $S_{\psi_0}(H(\tau))$ implies $S_{\psi_0}(H(\tau))\subset E(\tau)W$ or $S_{\psi_0}(H(\tau))\subset E(\tau)W$. Since $S_{\psi_0}(H(\tau))$ contains $S_{\psi_0}(\phi_0)$, it follows from this fact that $S_{\psi_0}(\phi_0)$ belongs to W or $W^\perp$. However H' is the minimal invariant subspace of $H(G, \chi)$. Hence $W=H'$ and $W^\perp=\{0\}$. Thus π' is irreducible as claimed. Therefore π and π' are irreducible and infinitesimal equivalent to each other. We now apply Corollary 4.5.5.3 in [26] to those of representations. Then π and π' are unitary equivalent.

The following theorem will be proved in Section 6.

Theorem 1. *An irreducible unitary representation* (π, H) *of* G *is realized on* $H(G, \chi)$ *if and only if there exists a K-finite vector* ϕ *in* H *such that* $S_\phi(\phi) \in H(G, \chi)$.

We now establish the Schur orthogonality relations of a representation of G realized on $H(G, \chi)$.

Theorem 2. *Let* χ *be an element in* $\mathfrak{z}^\vee$. *Then for each two irreducible unitary representations* (π, H) *and* (π', H') *of* G *realized on* $H(G, \chi)$, *we have the following.*

There exists a positive constant d_π *such that*

$$\lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\int_G (\pi(x)\phi, \psi)\overline{(\pi'(x)\phi', \psi')}e^{-\varepsilon d(x)}\,dx = \begin{cases} d_\pi^{-1}(\phi, \phi')\overline{(\psi, \psi')} & \text{if } \pi\cong\pi' \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let ϕ and ψ be K-finite vectors in H. By Lemma 1 and Lemma 2, we have $S_\psi(\phi) \in H(G, \chi)$. Let H^* be the closure of $S_\psi(H(\phi))$ in $H(G, \chi)$ and π^* the restriction of R to H^*. Then by the proof of Lemma 2, (π, H) and (π^*, H^*) are unitary equivalent. Applying the same arguments as in the proof of Theorem 4.5.9.1 and Theorem 4.5.9.3, [26] to those representations, the conclusion in this theorem follows.

Remark 2. When the case $i(\chi)=0$, π and π' are square integrable. Therefore the relations in Theorem 1 is well known as a result of R. Godement [6]. In [20], we treat the same theorem as above for the case $i(\chi)=1$.

Theorem 3. *Let* χ *be an element in* $\mathfrak{z}^\vee$ *satisfying* $i(\chi)\geqq 1$. *Then each irreducible unitary representation* (π, H) *of* G *realized on* $H(G, \chi)$ *is equivalent to a subrepresentation of principal series of* G *induced from a cuspidal parabolic subgroup* $P=MAN$ *with* $i(\chi)=\dim A$.

Proof. Let ϕ be a fixed K-finite vector in H. We put $f(x)=(\pi(x)\phi, \phi)$. Then we have $f\neq 0$. Define $F=F_f$ as in (4.3). By using Lemma 4.10, we have there exists a cuspidal parabolic subgroup $P=MAN$ such that $F_P\neq 0$. F_P is (τ_{K_M}, U)-spherical function on MA. Bearing in mind Lemma 4.11, F_P is of the form $F_P(am)=\sum_{k=1}^p e^{\lambda_k(\log a)}F_k(m)$ for a in A and

m in M where λ_k is purely imaginary valued linear form on $\mathfrak{a}$ and F_k a square integrable function on M satisfying $(z-\chi_k(z))F_k=0$, $z \in \mathfrak{z}_M$ for a character χ_k of $\mathfrak{z}_M$. We fix a number k. Let V be a closed invariant subspace generated by $\{R_m F_k;\ m \in M\}$ in $U \otimes L^2(M)$. Then the right regular representation σ of M on V is equivalent to a sum of finite number of discrete series. We denote $V=H_\sigma$. Define a $L^2(K)\otimes U \otimes L^2(M)$-valued C^∞-function g on G by $g(kman)=e^{-(\nu+\rho_P)(\log a)}\tau_1(k)F_k(m)$, $\tau=(\tau_1, \tau_2)$. Since F_k is τ_{K_M}-spherical the function g is well defined. Let $(L^2(K)\otimes H_\sigma)_\sigma$ be the same as in (2.3). Then g belongs to $(L^2(K)\otimes H_\sigma)_\sigma$. We define a unitary representation $\pi(\sigma, \nu)$ as in Section 2. We shall prove π is unitary equivalent to a subrepresentation of $\pi(\sigma, \nu)$. Let c be the positive constant determined by $\|f\|^2=c\int_K\int_M |\tau_1(k)g(m)|^2 dm dk$. Using the Schur orthogonality relations of π in Theorem 2, we have $\|f\|^2=(d\pi)^{-1}|\phi|^2$. Let H_0 be the abstract subspace of H generated by $\{\pi(x)\phi;\ x \in G\}$. H_0 is a G-invariant dense subspace of H. Moreover since π is unitary, we have $|\pi(x)\psi|=|\psi|$ for all x in G and ψ in H_0. Let us now define a linear operator η of H_0 to $(L^2(K)\otimes H_\sigma)_\sigma$ by $\eta(\pi(x)\phi)(y)=(cd_\pi)^{1/2}g(x^{-1}y)$, $x, y \in G$. By definition, η is unitary and $\eta\circ\pi(x)=\pi(\sigma, \nu)(x)\circ\eta$ on H_0 for all x in G. Consequently η is extended to a equivalent mapping of H to $(L^2(K)\otimes H_\sigma)_\sigma$. This completes our proof.

Remark 3. Combining Theorem 3 with Theorem 3.1, we see that all irreducible unitary representations realized on $H(G, \chi)$ have the tempered characters. We now correct the error in the proof of Theorem 6.4, [20].

§ 6. Realization of a regular principal series representation

In this section, we shall prove that all regular principal series unitary representation, induced from cuspidal parabolic subgroup, of G is realized on $H(G, \chi)$. Let $P_0=M_0A_0N_0$ be a minimal parabolic subgroup of G with θ-stable split component A_0 and $\Psi(A_0)$ the simple root system of (P_0, A_0). Let f be a K-finite C^∞-function on G. We define a (τ, U)-spherical function $F=F_f$ as in (4.3). Assume that F is tempered. Then F has the constant term F_P of F along a given parabolic subgroup P of G. The function F_P is of the form

(6.1) $F_P(am)=\sum_{k=1}^{s} p_k(\log a)e^{\lambda_k(\log a)}F_k(m)$, $a \in A$, $m \in M$ where p_k is a polynomial function and λ_k a purely imaginary valued linear form on $\mathfrak{a}$, and F_k is a tempered (τ_{K_M}, U)-spherical function on M satisfying $(z-\chi_k(z))F_k=0$ $(z \in \mathfrak{z}_M)$ for some character χ_k of $\mathfrak{z}_M$.

Definition 1. A function F on G belongs to $\mathscr{A}_0(G, \chi)$ (χ is a given character of $\mathfrak{z}$) if F has the following properties:

(1) there exists a finite dimensional double unitary representation (τ, U) of K such that F is τ-spherical,

(2) F is tempered, and satisfies $(z-\chi(z))F=0$ for all z in $\mathfrak{z}$,

(3) for each parabolic subgroup $P=MAN$, if $F_P\neq 0$ then $i(\chi)=\dim A+i(\chi_k)$ and p_k is constant for all $k=1, 2, \cdots, s$ where p_k, χ_k are the same as in (6.1).

A parabolic subgroup P is standard if $P=P_\Theta$ for a suitable subset Θ in $\Psi(A_0)$. All parabolic subgroup P of G is conjugate to a standard parabolic subgroup under an inner automorphism of K. Let F be a τ-spherical $\mathfrak{z}$-finite tempered function on G and $P=MAN$ a parabolic subgroup of G. In view of Lemma 4.6, we have $F_{P^k}(m^k)=\tau_1(k)F_P(m)\tau_2(k)^{-1}$ for all m in MA where $P^k=kPk^{-1}$, $m^k=kmk^{-1}$, k is a fixed element in K. Therefore the above assumption (3) can be restricted to all standard parabolic subgroup of G.

Lemma 1. *Let $P=MAN$ be a standard parabolic subgroup of G and F a function in $\mathscr{A}_0(G, \chi)$ with constant term $F_P=\sum_k e^{\lambda_k}F_k$. Then the τ_{K_M}-spherical function F_k belongs to $\mathscr{A}_0(M, \chi_k)$ where χ_k is the same as in* (6.1).

Proof. Let $P_\Theta=M_\Theta A_\Theta N_\Theta$ be a parabolic subgroup corresponding to a subset Θ in $\Psi(A_0)$ and $(P_\Theta^*)_0=(M_\Theta^*)_0(A_\Theta^*)_0(N_\Theta^*)_0$ the minimal parabolic subgroup of M_Θ. Then we have $\Theta=\Psi((A_\Theta^*)_0)$. Therefore all standard parabolic subgroup of M_Θ are given by $P_{\Theta'}^*=M_\Theta\cap P_{\Theta'}$ for the sets Θ' in Θ. We shall denote the Langlands decomposition of $P_{\Theta'}^*$ by $P_{\Theta'}^*=M_{\Theta'}^*A_{\Theta'}^*N_{\Theta'}^*$. We see that $A_{\Theta'}=A_\Theta A_{\Theta'}^*$. Define for each fixed element a in A_Θ, a τ_{K_M}-spherical function $F_{P_\Theta,a}$ on M_Θ by $(F_{P_\Theta,a})(m)=F_{P_\Theta}(am)$. Then we have $(F_{P_\Theta,a})_{P_{\Theta'}^*}(a^*m)=F_{P_{\Theta'}}(aa^*m)$ for $a\in A_{\Theta'}^*$, $a\in A_\Theta$ and $m\in M_{\Theta'}^*$. Consequently $F_{P_{\Theta'}}(aa^*m)=\sum_k\sum_j p_{k,j}e^{\lambda_k+\lambda_{k,j}}F_{k,j}$, where $(F_k)_{P_{\Theta'}^*}=\sum_j e^{\lambda_{k,j}}F_{k,j}$ and $F_{k,j}$ satisfies $(z-\chi_{k,j}(z))F_{k,j}=0$ for all z in $\mathfrak{z}_{M_{\Theta'}^*}$.

By the assumptions in (3) for F, we have $p_{k,j}=a$ const. and $i(\chi)=\dim A_{\Theta'}+i(\chi_{k,j})=\dim A_\Theta+i(\chi_k)$. Hence $i(\chi_k)=(\dim A_{\Theta'}-\dim A_\Theta)+i(\chi_{k,j})$ $=\dim A_{\Theta'}^*+i(\chi_{k,j})$. Thus the lemma follows.

Let α be a fixed element in $\Psi(A_0)$. For the simplicity of our notations, we denote the parabolic subgroup of G corresponding to $\Theta=\{\alpha\}$ by $P_\alpha=M_\alpha A_\alpha N_\alpha$. Since $\dim A_\alpha=1$, A_α is parametrized by $A_\alpha=\{\exp tH_1;\ t\in \boldsymbol{R}\}$ where H_1 is the element satisfying $\alpha(H_1)=1$. Let $P_\alpha^*=M_\alpha^*A_\alpha^*N_\alpha^*$ be the minimal parabolic subgroup of M_α satisfying $A_0=A_\alpha A_\alpha^*$. We define $D=D_\alpha$ as in Section 1, and extend it by $D_\alpha(aa^*)=D_\alpha(a^*)$ for $a\in A$, $a^*\in A^*$. Let r be a positive real number as in Lemma 4.7. We define a subset

$B_r(t)$ of $\mathrm{cl}((A^*)^+)$ by $B_r(t)=\{a^* \in \mathrm{cl}((A_\alpha^*)^+);\ (1-r\rho(H_1))t \geqq (r\rho-\alpha)(\log a^*)\}$ where $t \geqq 0$ and $\mathrm{cl}((A_\alpha^*)^+)$ is the closure of positive Weyl chamber $(A_\alpha^*)^+$ of A_α^*. Then we have the following; for the set $A(\alpha, r)$ as in Lemma 4.7,

(6.2) $A(\alpha, r)=\bigcup_{t \geqq 0} a_t B_r(t)$, $a_t=\exp tH_1$ (see, for a proof of this fact Lemma 6.4 [20]).

Lemma 2. *Let F be a function in* $\mathscr{A}_0(G, \chi)$. *Then we have*

$$(*) \qquad \lim_{\varepsilon \to +0} \varepsilon^{i(\chi)} \int_{A_0^+} |F(a)|^2 D(a) e^{-\varepsilon\rho(\log a)} da < \infty.$$

(Proof by an induction on $i(\chi)$). If $i(\chi)=0$, our assertion is obvious. Let us assume $i(\chi)=p>0$, and for all linear semisimple linear group G' and the characters χ' of $\mathfrak{z}'$ with property $i(\chi') \leqq p-1$, all functions F' in $\mathscr{A}_0(G', \chi')$ satisfy (*) (where $\mathfrak{z}'$ is the center of universal enveloping algebra of $\mathfrak{g}'_C$). Let F be a function in $\mathscr{A}_0(G, \chi)$. In view of Lemma 4.7, it is sufficient to prove that $I(F)=\lim_{\varepsilon \to +0} \varepsilon^p \int_{A(\alpha, r)} |F(a)|^2 D(a) e^{-\varepsilon\rho(\log a)} da < \infty$ for all α in $\Psi(A_0)$. By using Lemma 4.6, we have

$$I(F) \leq a \text{ const.} \lim_{\varepsilon \to +0} \varepsilon^p \int_{A(\alpha, r)} |F_{P_\alpha}(a)|^2 D_\alpha(a) e^{-\varepsilon\rho(\log a)} da$$

and hence by (6.2)

$$\leqq a \text{ const.} \lim_{\varepsilon \to +0} \varepsilon^p \int_0^\infty \int_{B_r(t)} |F_{P_\alpha}(a_t a^*)|^2 D_\alpha(a^*) e^{-\varepsilon\rho(\log a^*)} e^{-\varepsilon t} da^* dt.$$

In the expression of $F_P=\sum_k e^{\lambda_k} F_k$, $F_k \in \mathscr{A}_0(M, \chi_k)$ and $i(\chi_k)=p-1$ (see Lemma 1). Hence our inductive hypothesis implies that

$$I(F_k)=\lim_{\varepsilon \to +0} \varepsilon^{p-1} \int_{(A^*)_0^+} |F_k(a^*)|^2 D_\alpha(a^*) e^{-\varepsilon\rho(\log a)} da^* \quad \text{is finite.}$$

Consequently we have $I(F) \leqq a \text{ const.} \left(\lim_{\varepsilon \to +0} \varepsilon \int_0 e^{-\varepsilon t} dt\right)\left(\sum_k I(F_k)\right)$. This completes our proof.

Combining Lemma 4.11 with Lemma 2, we have the following.

Theorem 1. *Let* χ *be a character of* $\mathfrak{z}$ *and* $\mathscr{A}_0(G, \chi)$ *the same as in Definition* 1. *Then a K-finite function f belongs to* $H_K(G, \chi)$ *if and only if* $F=F_f \in \mathscr{A}_0(G, \chi)$.

Definition 2. A principal series representation $\pi(\sigma, \nu)$ of G induced from a cuspidal parabolic subgroup $P=MAN$ is regular if ν is regular on $\mathfrak{a}$.

Theorem 2. *Let $\pi(\sigma, \nu)$ be a regular principal P-series representation of G. Then each K-finite matrix coefficient belongs to $H_K(G, \chi)$ where χ is the infinitesimal character of $\pi(\sigma, \nu)$.*

Proof. Let f be a K-finite matrix element of $\pi(\sigma, \nu)$. By using Theorem 1 it is enough to show that $F=F_f$ belongs to $\mathscr{A}_0(G, \chi)$. Let Θ be a subset of $\Psi(A_0)$ and $P_\Theta=M_\Theta A_\Theta N_\Theta$ be the parabolic subgroup of G. Since ν is regular on $\mathfrak{a}$ and σ has the real regular infinitesimal character, we see that $\chi=\chi_{\lambda(\sigma,\nu)}$ is regular. Therefore the constant term of F along P_Θ is of the form $F_{P_\Theta}=\sum_k e^{\nu_k}F_k$, ν_k is regular on $\mathfrak{a}_\Theta$, and F_k satisfies $(z-\chi_{\lambda_k}(z))F_k=0$ for a regular character χ_{λ_k} of $\mathfrak{z}_{M_\Theta}$. Let $\chi_{\lambda(\sigma)}$ be the infinitesimal character of σ. Then there exists w in W such that $w(\nu_k+\lambda_k)=\lambda(\sigma)+\nu$. Hence we have $\dim A_\Theta+i(\alpha_{\lambda_k})=\dim A=i(\chi)$. Therefore F belongs to $\mathscr{A}_0(G, \chi)$ as claimed.

Theorem 5 in the previous section will be proved by using the following lemma.

Lemma 3. *Let ϕ be a K-finite function in $H(G, \chi)$ satisfying $\chi_{\tau'}*\phi=\phi*\chi_\tau=\phi$ for two suitable elements τ, τ' in $\mathscr{E}(K)$. We put*

$$h(x, y)=\int_K \overline{\chi_\tau(k)}\phi(xky)dk, \quad x, y \in G.$$

Then there are $\phi_1, \phi_2, \cdots, \phi_p$ and $\psi_1, \psi_2, \cdots, \psi_p$ in $H_K(G, \chi)$ such that $h(x, y)=\sum_k \phi_k(x)\psi_k(y)$.

Proof. We define two functions f_x and g_x on G by $f_x(y)=h(y, x)$ and $g_x(y)=h(x, y)$. Since $f_x, g_x \in H_K(G, \chi)$ (see Lemma 4.1), there exist $\phi_1, \phi_2, \cdots, \phi_p$ ($\psi_1, \psi_2, \cdots, \psi_q$) in $H_K(G, \chi)$ and $f_1, f_2, \cdots, f_p$ (resp. $g_1, g_2, \cdots, g_q$) in $C^\infty(G)$ such that $f_y(x)=\sum_k f_k(y)\phi_k(x)$ and $g_x(y)=\sum_i g_i(x)\psi_i(y)$. Therefore, since $f_y(x)=g_x(y)$, we have

$$\sum_k f_k(y)\phi_k(x)=\sum_j g_j(x)\psi_j(y). \tag{6.3}$$

We claim all f_k belong to $H_K(G, \chi)$. By (6.3) we have immediately $\sum_k (zf_k)(y)\phi_k(x)=\sum_j g_j(x)(z\psi_j)(y)=\sum_k \chi(z)f_k(y)\phi_k(x)$ for each z in $\mathfrak{z}$, x and y in G. Since $\{\phi_1 \phi_2, \cdots, \phi_p\}$ is linearly independent over $\boldsymbol{C}$, we get $zf_k=\chi(z)f_k$ for all z in $\mathfrak{z}$. Similarly we can prove all f_k's are K-finite.

Define F_{ψ_j} and F_{f_k} as in (4.3). Then we have

$$\sum_k F_{f_k}(y)\phi_i(x)=\sum_j F_{\psi_j}(y)g_j(x). \tag{6.4}$$

Let $d_{P_0}F_{f_k}=\sum\sum_{i,s}p_{k,i,s}e^{\nu_{k,i,s}}F_{k,i,s}$ be the expansion of F_{f_k} on $A_0^+(R)$ as in Lemma 4.4. Bearing in mind $\phi_1, \phi_2, \cdots, \phi_p$ is linearly independent, the temperedness of F_{ψ_j} implies that $\mathrm{Re}\,\nu_{k,i,s}(\omega_t)\leqq 0$ for all k, i, s, t where we use the same notations as in Section 4. Consequently by a result of Casselman and Miličić (Theorem 8.4.7, [16]), all F_{f_k}'s are tempered. Let P be a standard parabolic subgroup of G and $(F_{\psi_j})_P$, $(F_{f_k})_P$ the constant term of F_{ψ_j}, F_{f_k} along P. By (6.4), we have

$$\sum_k (F_{f_k})_P\phi_k(x)=\sum_j (F_{\psi_j})_P g_j(x) \qquad \text{for all } x \text{ in } G. \tag{6.5}$$

Since $F_{\psi_j}\in\mathscr{A}_0(G, \chi)$ (see Theorem 1) and $\phi_1, \phi_2, \cdots, \phi_p$ is linearly independent, we conclude that all F_{f_k} belong to $\mathscr{A}_0(G, \chi)$. Hence again by Theorem 1, we have $f_k\in H_K(G, \chi)$. Thus we can prove the lemma.

Proof of Theorem 5.1. Bearing in mind Lemma 5.2, it is sufficient to show that if (π, H) is realized on $H(G, \chi)$ then $(\pi(x)v, v)$ belongs to $H(G, \chi)$ for a suitable K-finite vector in H. We put $E_l(\tau)f=\chi_\tau * f$ and $E_r(\tau)f=\chi_\tau * f$ for each fixed $[\tau]$ in $\mathscr{E}(K)$. Let η be the equivalent mapping of H into $H(G, \chi)$, and denote $H'=\eta(H)$, $\pi'=$the restriction of R to H'. Then we have $\pi'(x)\circ\eta=\eta\circ\pi(x)$ for x in G. Let $[\tau]$ be an element in $\mathscr{E}(K)$. Since $\pi'(x)$ and $E_l(\tau)$ are commutative, $\pi'(x)\circ(E_l(\tau)\circ\eta)=(E_l(\tau)\circ\eta)\circ\pi(x)$. Consequently, it follows from the irreducibilities of π and π' that $E_l(\tau)\circ\eta$ $=0$ or $E_l(\tau)\circ\eta$ is bijective. On the other hand since $H'=\oplus_{\tau\in\mathscr{E}(K)}E_l(\tau)H'$, there exists a unique $[\tau']$ in $\mathscr{E}(K)$ such that $E_l(\tau')H'=H'$. Let us now choose $[\tau]$ in $\mathscr{E}(K)$ satisfying $[\pi|_K: \tau]>0$. Then there exists v in H such that $E(\tau)v=v$. We put $\phi(x)=((E_l(\tau')\circ\eta)(v))(x)$. Then ϕ is K-finite and $(\pi'(x)\phi, \phi)=(\pi(x)v, v)$. We shall prove that $f(x)=(\pi(x)v, v)\in H(G, \chi)$. Since $E_r(\tau)\phi=\phi$,

$$f(x)=\lim_{\varepsilon\to+0}\varepsilon^{i(\chi)}\int_G\left(\int_K\overline{\chi_\tau(k)}\phi(ykx)dk\right)\phi(y)e^{-\varepsilon d(y)}dy.$$

We now apply Lemma 3. Then we have $\int_K\overline{\chi_\tau(k)}\phi(ykx)dk=\sum_{i=1}^p\phi_i(y)\psi_i(x)$ for a finite number of elements $\phi_1, \phi_2, \cdots, \phi_p$ and $\psi_1, \psi_2, \cdots, \psi_p$ in $H_K(G, \chi)$. This implies that $f(x)=\sum_i\psi_i(x)(\phi_i, \phi)\in H_K(G, \chi)$. Hence we can prove Theorem 5.1 completely.

§ 7. Irreducibilities for regular principal series representations

First of all in this section, we shall state a minimal K-type theorem of principal P-series representation of G. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{b}$ a Cartan subalgebra of $\mathfrak{k}$. Φ_K is the root system of $(\mathfrak{k}_C, \mathfrak{b}_C)$ and ρ_K one half the sum of all positive roots in Φ_K. All irreducible unitary representations of K are parametrized by the dominant integral forms on $\mathfrak{b}_C$ which is the highest weight. We shall denote by $\tau=\tau_\mu$ the irreducible unitary representation with highest weight μ. Let $\pi(\sigma, \nu)$ be a fixed principal series representation of G induced from a cuspidal parabolic subgroup $P=MAN$. Then we have $\pi(\sigma, \nu)|_K = \oplus_{\mu\in\mathfrak{b}^*}[\pi(\sigma, \nu)|_K : \tau_\mu]\tau_\mu$ where $\mathfrak{b}^*$ is the set of all dominant integral forms on $\mathfrak{b}_C$, $\pi(\sigma, \nu)|_K$ is the restriction of $\pi(\sigma, \nu)$ to K and $[\pi(\sigma, \nu)|_K : \tau_\mu]$ the multiplicity of τ_μ appearing in $\pi(\sigma, \nu)|_K$.

Definition 1. An irreducible unitary representation τ of K is a minimal (lowest) K-type of $\pi(\sigma, \nu)$ if $[\pi(\sigma, \nu)|_K : \tau_\mu]>0$ and $|\mu+\rho_K|\leqq |\mu'+\rho_K|$ for all $\tau_{\mu'}$ in $\mathscr{E}(K)$ satisfying $[\pi(\sigma, \nu)|_K : \tau_{\mu'}]>0$.

The follownig theorem is due to D. Vogan [24].

Lemma 1. *Each principal P-series representation $\pi(\sigma, \nu)$ has a minimal K-type with multiplicity one.*

For a proof of the lemma, see Theorem 15.1, [16] ([24] and [3]).

Remark 1. The proof of Theorem 15.1 in [16] is given by using the minimal K-type theorem of the discrete series representation σ. For the minimal K-type theorem of discrete series, see [10].

Let $\pi(\sigma, \nu)$ be a regular principal P-series unitary representation of G with infinitesimal character $\chi=\chi_{\lambda(\sigma,\nu)}$. Consider an irreducible component π of $\pi(\sigma, \nu)$. Then the characters Θ_π and $\Theta_{\pi(\sigma,\nu)}$ satisfy the following properties (see Lemma 2.2 and Theorem 3.1);

(1) Θ_π and $\Theta_{\pi(\sigma,\nu)}$ are the solutions of differential equation $(z-\chi(z))\Theta=0$, $z\in\mathfrak{z}$ where χ is the same as above,

(2) Θ_π and $\Theta_{\pi(\sigma,\nu)}$ are tempered.

Therefore by using the uniqueness theorem for tempered invariant eigendistributions on G (see Theorem 13, [13]), there exists a constant c_π such that

$$\Theta_\pi=c_\pi\Theta_{\pi(\sigma)}. \tag{7.1}$$

We now give a proof of the irreducibility of regular principal P-series unitary representation $\pi(\sigma, \nu)$ of G.

Theorem 1. *All regular principal P-series unitary representation $\pi(\sigma, \nu)$ are irreducible.*

Proof. Let $P_0 = M_0A_0N_0$ be a minimal parabolic subgroup of G with θ-stable split component A_0. We put $G_1 = KA_0^+K$, $A_0^+ =$ the positive Weyl chamber of (P_0, A_0). Then G_1 is K-invariant open dense subset of G. Let ϕ be a K-finite element in $H(\sigma, \nu)$. We define a function $f_\varepsilon(x) = (\pi(\sigma, \nu)(x)\phi, \phi)e^{-\varepsilon d(x)}$ for a fixed positive real number ε. We see that f_ε is a tempered C^∞-function on G_1 (see Lemma 5.4, [20]). Let (π, H) be an irreducible component of $(\pi(\sigma, \nu), H(\sigma, \nu))$ and $\phi_1, \phi_2, \cdots$ be orthonormal basis of H satisfying $E(\tau_i)\phi_i = \phi_i$ for some $[\tau_i]$ in $\mathscr{E}(K)$. We denote $\phi = \phi_1$ and $\tau = \tau_1$, and define $f_\varepsilon = (f_\varepsilon)_\phi$ as above. Bearing in mind f_ε is K-finite, we have immediately

$$\Theta_\pi(\overline{f_\varepsilon}) = \sum_{i=1}^{\infty} \int_G \overline{f_\varepsilon(x)}(\pi(x)\phi_i, \phi_i)dx = \sum_{i=1}^{n} \int_G \overline{f_\varepsilon(x)}(\pi(x)\phi_i, \phi_i)dx$$

for a suitable number n.

On the other hand since $\pi(\sigma, \nu)$ is a regular principal series, it follows from Theorem 5.2 that $\lim_{\varepsilon\to+0} \varepsilon^{i(\chi)}\Theta(f_\varepsilon) = d_\pi^{-1}$ where d_π is the formal degree of π. Similarly we have $\mathrm{im}_{\varepsilon\to+0}\, \varepsilon^{i(\chi)}\Theta_{\pi(\sigma,\nu)} = [\pi(\sigma, \nu): \pi](d_\pi)^{-1}$. Hence by (7.1), we have

$$[\pi(\sigma, \nu): \pi] = c_\pi. \tag{7.2}$$

Let us now consider a following special subrepresentation π of $\pi(\sigma, \nu)$. By using Lemma 1, we can choose a minimal K-type τ of $\pi(\sigma, \nu)$ with multiplicity one. Let (π, H) be an irreducible component of $\pi(\sigma, \nu)$ satisfying $[\pi|_K: \tau] \neq 0$. Then $[\pi(\sigma, \nu): \pi] = 1$, and therefore by (7.2) $c_\pi = 1$. This implies that $\Theta = \Theta_{\pi(\sigma\ \nu)}$. Thus $\pi(\sigma, \nu)$ is irreducible.

Remark 1. The irreducibility of regular principal series $\pi(\sigma, \nu)$ induced from minimal parabolic subgroup of G is proved by F. Bruhat [2]. In general Harish-Chandra proves the irreducibilities of all regular principal P-series representations ([9]).

Remark 2. B. Kostant [18] gives an criterion for the irreducibility of spherical principal series (not necessary unitary) of G in an algebraic situation.

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Department of Mathematics
Tsuda College
Tsuda-machi, Kodaira, Tokyo 187
Japan