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Lie Algebra Cohomology and Holomorphic Continuation of Generalized Jacquet Integrals

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Introduction

In this paper there are two types of theorems. The first are generalizations of vanishing theorems of Kostant [K] and Lynch [L]. The second are the holomorphic continuation of certain integrals. The proofs of the two seemingly unrelated types of results have in common the use of certain operators, Q_j which are non-commutative analogues of the standard Euler operator on the space of polynomials in several variables.

We now describe an important class of examples of the results. Let G be a real reductive group of inner type. Let $\mathfrak{g}$ denote the Lie algebra of G and let Y be a nilpotent element in $\mathfrak{g}$. Then ([Ja, p. 99, Lemma 8]) there exist elements $X, H \in \mathfrak{g}$ such that $[X, Y]=H$ and $[H, X]=2X$, $[H, Y]=-2Y$. Fix a Cartan involution such that $\theta H=-H$. Let $\mathfrak{u}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{u}_2=\{x \in \mathfrak{g} \mid [H, x]=2x\}$. Let $z \in \mathbf{C}-\{0\}$ and let $\psi(x)=zB(Y, x)$ for $x \in \mathfrak{u}$. If V is a $\mathfrak{g}$-module then we define a new action π_ψ of $\mathfrak{u}$ on V by $\pi_\psi(x)v=xv-\psi(x)v$. We denote this $\mathfrak{u}$-module by $V \otimes \mathbf{C}_\psi$. The main theorem on Lie algebra cohomology implies

Theorem. *Let V be a $\mathfrak{g}$-module such that if $v \in V$ then $\pi_\psi(x)^k v=0$ for all $x \in \mathfrak{u}$ for some $k=k(v)$. Then $H^i(\mathfrak{u}, V \otimes \mathbf{C}_\psi)=(0)$ for $i>0$.*

We now describe the other type of results. Let $\mathfrak{p}$ be the sum of the eigenspaces for $\operatorname{ad} H$ with non-negative eigenvalue. Let $P=\{g \in G \mid \operatorname{Ad}(g)\mathfrak{p} \subset \mathfrak{p}\}$ and put $M=P \cap \theta(P)$. Let (σ, H_σ) be a finite dimensional irreducible representation of M. Put $\mathfrak{a}=\{Z \in \mathfrak{m} \mid [Z, \mathfrak{m}]=0,\ \theta Z=-Z\}$. If $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ then let $(\pi_{P,\sigma,\nu}, I_{P,\sigma}^\infty)$ be the corresponding (degenerate) principal series representation (see § 6). Let $\tau=\exp(\pi/2(x-Y))$.

The analytic results involve the study of integrals of the form

$$(*) \qquad J_{P,\sigma,\nu}(f)=\int_{\mathfrak{u}} e^{iB(Y,u)} f_{\sigma,\nu}(\tau \exp(u))\,du.$$

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(See § 6 for unexplained notation.) If Y is a principal nilpotent element of $\mathfrak{g}$ then P is a minimal parabolic subalgebra and $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$. Also $\eta(\exp(x))=e^{iB(Y,x)}$ defines the most general "generic character" of N. In this case our results imply that $J_{P,\sigma,\nu}$ has a weakly holomolorphic (not just meromorphic) continuation in ν to $\mathfrak{a}_{\mathbf{C}}^*$ as an operator on the C^∞ vectors. This result in the case when G has reduced rank 1 is due to Schiffman [S]. In the case when G is split or complex then the result (on K-finite vectors) is due to Jacquet [J] (thus the designation in the title). There are other papers with special cases of this theorem ([HI], [HII]).

If Y is not principal then the results are more difficult to describe. (See § 5, 6, 7). We will instead give an example. Let $G=\mathrm{Sp}(n, \mathbf{R})$ which we realize as the group of all linear transformations of $\mathbf{R}^{2n}$ that preserve the form $\omega(x, y)=\Sigma_{i=1}^n (x_i y_{n+i}-x_{n+i}y_i)$. We take

$$Y=\begin{bmatrix} 0 & 0 \\ I_{pq} & 0 \end{bmatrix},$$

with

$$I_{pq}=\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

with I_j the $j\times j$ identity matrix and $p+q=n$. Then we can take

$$H=\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

with I the $n\times n$ identity matrix and $X=Y^T$. In this case

$$\mathfrak{u}=\left\{\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} : S=S^T,\ S\ n\times n\right\}.$$

The corresponding integrals are of the form

$$\int e^{ix\,\mathrm{tr}\,(I_{pq}S)} f_{\sigma,\nu}\left(J \exp\left(\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}\right)\right) dS$$

the integration over the space of all symmetric $n\times n$ matrices, $x \in \mathbf{R}$-$\{0\}$ and

$$J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

In this example we show that these integrals have holomorphic continuations for smooth f. (For precise results see § 7). If $G=SU(n, n)$,

$SL(2n, \mathbf{R})$, $SO(2n, 2n)$ there are parabolic subgroups with similar descriptions to P above and our results imply the holomorphic continuation of the corresponding integrals (see §4 for a large class of examples). The proofs are given in such a way that if one can prove the conjecture at the end of section 5 then the most general integral $(*)$ has a holomorphic continuation to all of $\mathfrak{a}_{\mathbf{C}}^*$.

As we indicated above the results rest on the algebraic properties of certain generalizations of Euler operators. Lemma 2.1 is the contains the key to these operators. Our proof of this Lemma involves the determination of the minimal polynomial of an element of the group algebra of the symmetric algebra on n letters (see the appendix).

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§ 1. Some observations about unipotent representations

Let $\mathfrak{n}$ be a Lie algebra over $\mathbf{C}$. If M is an $\mathfrak{n}$-module and if $m \in M$ then we set $\mathfrak{n}^0 m = m$ and if $\mathfrak{n}^p m$ has been defined then $\mathfrak{n}^{p+1}m = \{Xv \mid v \in \mathfrak{n}^p m, X \in \mathfrak{n}\}$. Let $\mathcal{N}$ denote the category of all $\mathfrak{n}$-modules M, such that if $m \in M$ then there exists $k = k(m)$ such that $\mathfrak{n}^k m = (0)$. If $M \in \mathcal{N}$ and if $M^{\mathfrak{n}} = \{m \in M \mid \mathfrak{n}m = 0\} = (0)$ then $M = (0)$. Let $M \in \mathcal{N}$. Set $M^k = \{m \in M \mid \mathfrak{n}^k m = (0)\}$. Then $M^0 = (0) \subset M^1 = M^{\mathfrak{n}} \subset M^2 \subset \cdots$ and $\bigcup M^j = M$.

If M is an $\mathfrak{n}$-module then let $H^j(\mathfrak{n}, M)$ denote the usual Lie algebra cohomology space of $\mathfrak{n}$ with coefficients in M (cf. [BW, Chapter I]). For our purposes the most important cohomology spaces are the zeroth which is just $M^{\mathfrak{n}}$ and the first (which we now recall). Let $Z^1(\mathfrak{n}, M)$ denote the space of all $\omega\colon \mathfrak{n} \to M$ such that $\omega([X, Y]) = X\omega(Y) - Y\omega(X)$ for all $X, Y \in \mathfrak{n}$. Let $B^1(\mathfrak{n}, M)$ denote the space of those ω of the form $\omega(X) = Xm$ with $m \in M$ fixed. Then $H^1(\mathfrak{n}, M) = Z^1(\mathfrak{n}, M)/B^1(\mathfrak{n}, M)$.

Lemma 1.1. *Let $M, V \in \mathcal{N}$. Assume that $H^1(\mathfrak{n}, V) = (0)$. If $A \in \mathrm{Hom}_{\mathbf{C}}(M^{\mathfrak{n}}, V^{\mathfrak{n}})$ then there exists $T \in \mathrm{Hom}_{\mathfrak{n}}(M, V)$ such that T agrees with A on $M^{\mathfrak{n}}$. If A is injective then so is T. If A is bijective and if $H^1(\mathfrak{n}, M) = (0)$ then T is surjective.*

Proof. Define T to be A on M^1. Suppose that T has been defined on M^j for $j \geq 1$ as an $\mathfrak{n}$-module homomorphism. If $m \in M^{j+1}$ then $Xm \in M^j$ for $X \in \mathfrak{n}$. Set $\omega_m(X) = T(Xm)$. Then it is easily varified that $\omega_m \in Z^1(\mathfrak{n}, V)$. Hence there exists $v \in V$ such that $T(Xm) = Xv$ for all $X \in \mathfrak{n}$. Let $\{m_\alpha\}$ be a linearly independent set in M^{j+1} that defines a basis modulo

M^j. Let $v_\alpha \in V$ be such that $T(Xm_\alpha)=Xv_\alpha$ for all $X \in \mathfrak{n}$. Define $Tm_\alpha=v_\alpha$ and extend T by linearity to the span of the v_α. This defines a linear map of M^{j+1} into V^{j+1}. If $m \in M^{j+1}$ then $m=\Sigma a_\alpha m_\alpha+u$ with $u \in M^j$. Thus $T(Xm)=\Sigma a_\alpha T(Xm_\alpha)+T(Xu)=\Sigma a_\alpha Xv_\alpha+XT(u)=\Sigma a_\alpha XT(m_\alpha)+XT(u)=XT(m)$. This impliments the "construction" of T.

Assume that A is injective. We show that T is injective on M^j by induction on j. If $j=1$ then T is injective on M^1 by hypothesis. So assume that T is injective on M^j. Suppose that $m \in M^{j+1}$ and $Tm=0$. If $X \in \mathfrak{n}$ then $T(Xm)=XTm=0$. Since $Xm \in M^j$ this implies that $Xm=0$ for all $X \in \mathfrak{n}$. Hence $m \in M^1$ so $m=0$.

Suppose now that $H^1(\mathfrak{n}, M)=(0)$ and that A is bijective. Then the short $\mathfrak{n}$-module exact sequence

$$0 \longrightarrow M \xrightarrow{T} V \longrightarrow V/TM \longrightarrow 0$$

induces the long(er) $\mathfrak{n}$-module exact sequence

$$0 \longrightarrow M^{\mathfrak{n}} \xrightarrow{A} V^{\mathfrak{n}} \longrightarrow (V/TM)^{\mathfrak{n}} \longrightarrow H^1(\mathfrak{n}, M)=(0).$$

Since A is assumed to be bijective, this implies that $(V/TM)^{\mathfrak{n}}=(0)$. Hence $V/TM=(0)$. So $V=TM$. This completes the proof of the Lemma.

If M is an $\mathfrak{n}$-module then put $M[\mathfrak{n}]=\{m \in M \mid \mathfrak{n}^k M=(0) \text{ for some } k\}$. If W is a complex vector space then we put an $\mathfrak{n}$-module structure on $\operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{n}), W)$ by setting $Xf(n)=f(nX)$ or $X \in \mathfrak{n}$, $n \in U(\mathfrak{n})$ and $f \in \operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{n}), W)$. Set $N(w)=\operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{n}), W)[\mathfrak{n}]$.

Lemma 1.2. *If* $\dim \mathfrak{n}<\infty$ *and if* $\mathfrak{n}$ *is nilpotent then* $H^k(\mathfrak{n}, N(W))=(0)$ *for all complex vector spaces* W *and all* $k>0$.

Proof. We first assume that $\dim \mathfrak{n}=1$. Let X be a basis of $\mathfrak{n}$. We must show that $H^1(\mathfrak{n}, N(W))=(0)$. From the definition of the cohomology, this means that we must show that $XN(W)=N(W)$. If $f \in N(W)$ then f is determined by its values $f(X^k)$. Since $f \in N(W)$, there exists r such that $f(X^j)=0$ for $j \geq r$. Define $g(1)=0$ and $g(X^{k+1})=f(X^k)$. Then $Xg=f$. This proves the result in this case.

Suppose that the result has been proved if $\dim \mathfrak{n}=k$. Assume that $\dim \mathfrak{n}=k+1$. Let $X \in \mathfrak{n}$ be such that $\mathfrak{n}=\mathbf{C}X \oplus \mathfrak{n}_1$ with $[X, \mathfrak{n}] \subset \mathfrak{n}_1$ and $\mathfrak{n}_1$ is a Lie subalgebra. Then $U(\mathfrak{n})=\bigoplus_{k \geq 0} X^k U(\mathfrak{n}_1)$. This means that as an $\mathfrak{n}_1$-module, $N(W)$ is a countable direct sum of modules of the form $\operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{n}_1), W)[\mathfrak{n}_1]$. Thus the inductive hypothesis implies that $H^j(\mathfrak{n}_1, N(W))=(0)$ for $j>0$. We can now apply the Hochschild-Serre spectral sequence (cf. [BW, I, 6.5]) to find that $H^j(\mathfrak{n}, N(W))=$

$H^j(\mathfrak{n}/\mathfrak{n}_1, N(W)^{\mathfrak{n}_1})$. Clearly, $N(W)^{\mathfrak{n}_1}=\mathrm{Hom}_{\mathbf{C}}(U(\mathfrak{n}/\mathfrak{n}_1), W)$. Thus the one dimensional case now implies the result.

We assume that $\mathfrak{n}$ is as in the previous Lemma.

Proposition 1.3. *Let $M \in N$. If $H^1(\mathfrak{n}, M)=(0)$ then M is isomorphic with $N(M^{\mathfrak{n}})$. In particular, $H^j(\mathfrak{n}, M)=(0)$ for $j>0$.*

§ 2. The vanishing theorems

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbf{R}$. Let θ be a Cartan involution for $\mathfrak{g}$. We will take this to mean that there exists a symmetric, $\mathfrak{g}$ and θ-invariant, bilinear form B on $\mathfrak{g}$ such that $B(X, \theta X)<0$ for $X \in \mathfrak{g}$, $X \neq 0$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and set $\mathfrak{m}=\mathfrak{p} \cap \theta\mathfrak{p}$. Then $\mathfrak{p}=\mathfrak{m} \oplus \mathfrak{n}$. Let $\mathfrak{z}(\mathfrak{m})$ denote the center of $\mathfrak{m}$ and put $\mathfrak{a}=\{X \in \mathfrak{z}(\mathfrak{m}) \mid \theta X=-X\}$. Then $\mathfrak{m}=\{X \in \mathfrak{g} \mid [X, \mathfrak{a}]=(0)\}$.

Let $\mathfrak{u} \subset \mathfrak{n}$ be an $\mathfrak{m}$-invariant subalgebra and let ψ be a Lie algebra homomorphism of $\mathfrak{u}$ to $\mathbf{C}$. Since B induces a perfect pairing between $\mathfrak{u}_{\mathbf{C}}$ and $\theta(\mathfrak{u}_{\mathbf{C}})$ there exists a unique $Y_\psi \in \theta\mathfrak{u}_{\mathbf{C}}$ such that $B(X, Y_\psi)=\psi(X)$ for $X \in \mathfrak{u}$. We say that ψ is non-degenerate if there exists $H \in \mathfrak{a}$ such that $\mathrm{ad}\, H$ has integral eigenvalues and if we set $\mathfrak{g}_j=\{X \in \mathfrak{g} \mid [H, X]=jX\}$ then the following two conditions are satisfied

(1) $Y_\psi \in (\mathfrak{g}_{\mathbf{C}})_{-2}$, $\mathfrak{g}_0=\mathfrak{m}$, $\mathfrak{g}_2$ is contained in and generates $\mathfrak{u}$.

(2) $\mathrm{ad}\, Y_\psi\colon \mathfrak{u} \to \mathfrak{g}_{\mathbf{C}}$ is injective.

If $\mathfrak{u}=\mathfrak{n}$ then this is the definition of [L] of admissible. We will give some examples in Section 4 (see also [L]). The easiest way to guarantee the crucial condition (2) is to find $X \in (\mathfrak{u}_{\mathbf{C}})_2$ such that $[X, Y_\psi]=H$.

Fix such a non-degenerate ψ. Set $Y=Y_\psi$. If $X \in \mathfrak{g}_{\mathbf{C}}$ then set $\bar{X}$ equal to the complex conjugate of X relative to $\mathfrak{g}$. If $u, v \in \mathfrak{u}_{\mathbf{C}}$ then set $(u, v)=B([[\theta\bar{Y}, \theta\bar{v}], u], Y)$.

(3) $(\ ,\ )$ is Hermitian and positive definite on $\mathfrak{u}_{\mathbf{C}}$ (that is, $(\ ,\ )$ is an inner product).

Indeed, $(u, v)=-B([\theta\bar{Y}, \theta\bar{v}], [Y, u])=-B(\theta[\bar{Y}, \bar{v}], [Y, u])=-B(\overline{[Y, v]}, \theta[Y, u])=\overline{(v, u)}$. If $u\neq 0$ then $(u, u)=-B(\overline{[Y, u]}, \theta[Y, u])>0$ since $[Y, u] \neq 0$.

Let $j>0$ be fixed for the moment. Let $X_1, \cdots, X_d$ be a basis of $(\mathfrak{g}_{\mathbf{C}})_{2j} \cap \mathfrak{u}_{\mathbf{C}}$ such that $(X_i, X_k)=\delta_{jk}$. Set $Z_k=[\theta\bar{Y}, \theta\bar{X}_k] \in (\mathfrak{g}_{\mathbf{C}})_{-2j+2}$. Then $B([Z_i, X_k], Y)=(X_k, X_i)=\delta_{ki}$. Fix $X_0 \in (\mathfrak{g}_{\mathbf{C}})_2$ $(\subset \mathfrak{u}_{\mathbf{C}})$ such that $\psi(X_0)=1$. Then $[Z_i, X_k]=\delta_{ki}X_0+X_{ik}$ with $X_{ik} \in (\mathfrak{g}_{\mathbf{C}})_2$ and $\psi(X_{ik})=0$. Set $Q_j=\Sigma_i Z_i(X_i-\psi(X_i))$. Notice that Q_j is independent of the choice of the X_i.

If V is a $\mathfrak{u}$-module then we can define a new $\mathfrak{u}$-module structure on V by tensoring with the one dimensional $\mathfrak{u}$-module $\mathbf{C}_{-\psi}$ ($\mathfrak{u}$ acts by ψ on $\mathbf{C}$). We write $V_p=(V \otimes \mathbf{C}_{-\psi})^p$ for $p \geq 0$ (see § 1 for notation). The following

Lemma is the key to all of the results in the paper. Since its proof is extremely complicated, we defer the proof to the appendix to this paper.

Lemma 2.1. *Suppose that V is a $\mathfrak{g}$-module such that $V=\bigcup_{p\geq 0}V_p$ (i.e. $(V\otimes\mathbf{C}_{-\psi})=(V\otimes\mathbf{C}_{-\psi})[\mathfrak{u}]$). Fix $j>0$. If $v\in V_k$ then there exist (not necessarily distinct) non-negative integers $n_1,\cdots,n_q$ depending only on k such that*

$$\prod_{i=1}^{q}(Q_j+n_iI)v=0.$$

The next result implies a generalization of a Theorem of Kostant, Lynch. We maintain the above notation.

Theorem 2.2. *Let V be a $\mathfrak{g}$-module such that $V=\bigcup_{p\geq 0}V_p$ as a $\mathfrak{u}$-module. Then $H^j(\mathfrak{u}_{\mathbf{C}}, V\otimes\mathbf{C}_{-\psi})=(0)$ for $j>0$.*

Proof. Proposition 1.4 implies that it is enough to prove that $H^1(\mathfrak{u}, V\otimes\mathbf{C}_{-\varphi})=(0)$. Set $\mathfrak{u}_{2j}=\mathfrak{u}_{\mathbf{C}}\cap(\mathfrak{g}_{\mathbf{C}})_{2j}$. Then $\mathfrak{u}_{\mathbf{C}}=\bigoplus_{j\geq 1}\mathfrak{u}_{2j}$ and $[\mathfrak{u}_{2j},\mathfrak{u}_{2k}]\subset\mathfrak{u}_{2j+2k}$. Thus if $\omega\in Z^1(\mathfrak{u}_{\mathbf{C}}, V_p\otimes\mathbf{C}_{-\varphi})$ then $\omega(\mathfrak{u}_{2i})\subset V_{p-i+1}$ for $i\geq 1$.

Scholium. *Let $\omega\in Z^1(\mathfrak{u}_{\mathbf{C}}, V_p\otimes\mathbf{C}_{-\psi})$. Suppose that $j\geq 1$ and that $\omega(\mathfrak{u}_{2i})\subset V_{p-i}$ for $i>j$. Then there exists $v\in V_{p+1}$ such that $(\omega-dv)(\mathfrak{u}_{2i})\subset V_{p-i}$ for $i\geq j$. Here, as usual, $dv(X)=Xv$.*

We first show how to prove the theorem using the Scholium and then we will prove the Scholium. Since $\mathfrak{u}_{2r+2}=(0)$ for r sufficiently large, $\omega(\mathfrak{u}_{2r+2})\subset V_{p-r-1}$. Hence there exists $v_1\in V_{p+1}$ such that $(\omega-dv_1)(u_{2i})\subset V_{p-i}$ for $i\geq r$. Now, $\omega-dv_1\in Z^1(\mathfrak{u}_{\mathbf{C}}, V_p\otimes\mathbf{C}_{-\psi})$. Hence there exists $v_2\in V_{p+1}$ such that

$$(\omega-dv_1-dv_2)(\mathfrak{u}_{2i})\subset V_{p-i},\qquad i\geq r-1.$$

Continuing in this manner we can find $v\in V_{p+1}$ such that

$$(\omega-dv)(\mathfrak{u})\subset V_{p-1}.$$

Now use the same argument with p replaced by $p-1$ and ω replaced by $\omega-dv$. After p steps we find that $\omega\in B^1(\mathfrak{u}_{\mathbf{C}}, V_p\otimes\mathbf{C}_{-\varphi})$ $(V_0=(0))$. Thus the Scholium implies the theorem.

The proof of the Scholium will take some preparation.

(i) If $Z\in\mathfrak{u}_{2j}$, $j\geq 1$ then $(Z-\psi(Z))V_p\subset V_{p-j}$.

Indeed, $Z-\psi(Z)$ is a linear combination of commutators of elements of the form $Y_1-\psi(Y_1),\cdots,Y_j-\psi(Y_j)$ with $Y_j\in\mathfrak{u}_2$. Since $(X-\varphi(X))V_p\subset V_{p-1}$ for $X\in\mathfrak{u}_2$, (i) follows.

(ii) If $Z\in\mathfrak{g}_{-2j}$, $j\geq 0$ then $ZV_p\subset V_{p+j+1}$.

We prove this by induction on j and for each j by induction on p. If

$j=p=0$ then the result is clear ($V_0=(0)$). Assume that $\mathfrak{m}V_p \subset V_{p+1}$ (don't forget that $\mathfrak{m}=\mathfrak{g}_0$). If $v \in V_{p+1}$, $X \in \mathfrak{u}$, $Z \in \mathfrak{m}$ then $(X-\psi(X))Zv=[X, Z]v + Z(X-\psi(X))v$. Now $[X, Z] \in \mathfrak{u}$ so $[X, Z]V_{p+1} \subset V_{p+1}$. Also, $(X-\psi(X))v \in V_p$. So, $Z(X-\psi(X))v \in V_{p+1}$ by hypothesis. This implies that $(X-\psi(X))ZV_{p+1} \subset V_{p+1}$. Hence, $ZV_{p+1} \subset V_{p+2}$. This proves (ii) for $j=0$ and all p.

Assume (ii) for j (fixed) and all p. We prove it for $j+1$ and all p by induction on p. Let $Z \in (\mathfrak{g}_{\mathbf{C}})_{-2j-2}$. Clearly, $ZV_0=(0)$. Assume that $ZV_p \subset V_{p+j+2}$. If $v \in V_{p+1}$, $X \in \mathfrak{u}$ then $(X-\psi(X))Zv=[X, Z]v+Z(X-\psi(X))v$. Now $[X, Z] \subset \Sigma_{s\geq -j}(\mathfrak{g}_{\mathbf{C}})_{-2s}$. Hence, $[X, Z]V_{p+1} \subset V_{p+j+2}$. $(X-\psi(X))v \in V_p$ so $Z(X-\psi(X))v \in V_{p+j+2}$. We therefore see that $(X-\psi(X))Zv \in V_{p+j+2}$. Hence $Zv \in V_{p+j+3}$. This completes the proof of (ii).

We are (finally) ready to prove the Scholium. Let ω satisfy the hypothesis of the Scholium. Set $v_1=-\Sigma_i Z_i\omega(X_i)$ (Z_i, X_i as above for j). Note that $Z_i \in (\mathfrak{g}_{\mathbf{C}})_{-2j+2}$. We calculate

$$\begin{aligned} dv_1(X_k) &= -\Sigma_i(X_k-\psi(X_k))Z_i\omega(X_i) \\ &= -\Sigma_i\,[X_k, Z_i]\omega(X_i)-\Sigma_i Z_i(X_k-\psi(X_k))\omega(X_i) \\ &= X_0\omega(X_k)-\Sigma_i X_{ik}\omega(X_i)-\Sigma_i Z_i(X_i-\psi(X_i))\omega(X_k) \\ &\quad -\Sigma_i Z_i\omega([X_k, X_i]). \end{aligned}$$

Here we have used the relation $[Z_i, X_k]=\delta_{ik}X_0+X_{ik}$ (don't forget $\psi(X_{ik})=0$) and the cocycle condition on ω. If $X \in \mathfrak{u}_{2i}$ then $\omega(X) \subset V_{p-i+1}$. Hence $(X_0-1)\omega(X_k)$, $X_{ik}\omega(X_i) \in V_{p-j}$. We also note that $\omega([X_k, X_i]) \in V_{p-2j}$ by our hypothesis. Thus $Z_i\omega([X_k, X_i]) \in V_{p-j}$. We have therefore shown that

(iii) $(\omega-dv_1)(X_k)=u_k+Q_j\omega(X_k)$ with $u_k \in V_{p-j}$.

We observe that

(iv) $Q_jV_p \subset V_p$ all p.

Indeed, $(X_i-\psi(X_i))V_p \subset V_{p-j}$ by (i) and $Z_iV_{p-j} \subset V_p$ by (ii).

(v) If $X \in \mathfrak{u}_{2i}$, $i>j$, then $dv_1(X) \in V_{p-i}$.

Indeed, $dv_1(X)=-\Sigma_i XZ_i\omega(X_i)=-\Sigma_i[X, Z_i]\omega(X_i)-\Sigma_i Z_i X\omega(X_i)= -\Sigma_i[X, Z_i]\omega(X_i)-\Sigma_i Z_i(X_i-\psi(X_i))\omega(X)-\Sigma_i Z_i\omega([X, X_i])$. $[X, Z_i] \in \mathfrak{u}_{2i+2-2j}$. $\omega(X_i) \in V_{p-j+1}$ so $[X, Z_i]\omega(X_i) \in V_{p-i}$. $\omega(X) \in V_{p-i}$ and $Q_jV_{p-i} \subset V_{p-i}$. Finally, $[X, X_i] \in \mathfrak{u}_{2i+2j}$ so $\omega([X, X_i]) \in V_{p-i-j}$. Hence $Z_i\omega([X, X_i]) \in V_{p-i}$. (v) now follows.

(v) implies that $\omega-dv_1$ satisfies the same hypothesis as ω. We can thus iterate on (iii) to find

(vi) There exists for each $q=1, 2, \cdots, v_q \in V_{p+1}$ such that $\omega-dv_q$ satisfies the hypothesis of the Scholium and

$$(\omega-dv_q)(X_k)=u_{k,q}+Q_j^q\omega(X_k), \qquad \text{with } u_{k,j} \in V_{p-j}.$$

Lemma 2.1 implies that there exists a polynomial $f(X)=X\Pi_{i=1}^{r}(X+n_i)$ with $n_i\geq 0$ such that $f(Q_j)\omega(X_k)=(0)$ for all k. Thus if we write $f(X)/f(1)=\Sigma_{j\geq 1}b_jX^j$ ($f(1)>0$) then $\Sigma_j b_j=1$. Hence

$$(\omega-d(\Sigma_j b_j v_j))(X_k)=\Sigma_i b_i u_{ki}\in V_{p-j}.$$

This completes the proof of the Scholium hence of the Theorem.

We now show how the above theory implies the theorem in the introduction. Let $Y\in\mathfrak{g}$ be a nilpotent element (i.e. ad $Y^k=0$ for some $k>0$). Then the Jacobson-Morosov theorem (cf. [W, 8. A. 4. 1]) implies that there exist X, $H\in\mathfrak{g}$ such that $[X, Y]=H$, $[H, X]=2X$ and $[H, Y]=-2Y$. In particular, ad H is semi-simple with integral eigenvalues. Thus there exists an inner automorphism g of $\mathfrak{g}$ such that $\theta gH=-H$. So replace θ by $g^{-1}\theta g$. Let $\mathfrak{g}_j$ be (as usual) the eigenspace for ad H with eigenvalue j. Put $\mathfrak{p}=\oplus_{j\geq 0}\mathfrak{g}_j$. $\mathfrak{m}=\mathfrak{g}_0$, $\mathfrak{n}=\oplus_{j>0}\mathfrak{g}_j$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with standard Levi decomposition $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{n}$ (i.e. $\mathfrak{m}=\mathfrak{p}\cap\theta\mathfrak{p}$). Define $\mathfrak{u}$ to be the subalgebra of $\mathfrak{n}$ generated by $\mathfrak{g}_2$. If $z\in\mathbf{C}-\{0\}$ then set $\psi(x)=zB(Y, x)$ for $x\in\mathfrak{u}$. Then ψ is non-degenerate. Now Theorem 2.2 implies the theorem of the introduction.

§ 3. The category W_ψ

Let $\mathfrak{g}$, $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{n}$, $\mathfrak{u}\subset\mathfrak{n}$, ψ, $Y=Y_\psi$, $H\in\mathfrak{a}$ be as in the definition of non-degenerate in the previous section. Let W_ψ be the category of all $\mathfrak{g}$-modules, V, such that $V=\bigcup_{p>0}V_p$ (see § 2 for notation). Note that $V_1=\{v\in V\mid Xv=\psi(X)v \text{ for } X\in\mathfrak{u}\}$. As in Section 2 we set $\mathfrak{u}_{2j}=\{X\in\mathfrak{u}\mid [H, x]=2jX\}$. We assume that $\mathfrak{u}_{2d}\neq(0)$ but $\mathfrak{u}_{2d+2}=(0)$. Let Q_j be defined as in Section 1.

Lemma 3.1. *Let $V\in W_\psi$. If $j\geq 1$ then there exist integers $n_{i,k,j}>0$, $k=1,\cdots,p_i$, $i=2,\cdots,d$ depending only on j such that if*

$$T_j=(Q_1+jI)\Pi_k(Q_2+n_{2,k,j}I)\Pi_k(Q_3+n_{3,k,j})\cdots\Pi_k(Q_d+n_{d,k,j}).$$

Then $T_jV_{j+1}\subset V_j$.

Proof. Let for $r\geq 1$, $V_j^r=\{v\in V_j\mid(X-\psi(X))v\in V_{j-s-1} \text{ for } X\in\mathfrak{u}_{2s}, s\geq r\}$. We note that $V_j^1=V_{j-1}$.

(1) $Q_rV_j^{r+1}\subset V_j^{r+1}$, $r\geq 1$.

Indeed, set $Q_r=\Sigma_iZ_i(X_i=\psi(X_i))$ as in Section 2. If $X\in\mathfrak{u}_{2s}$, $s\geq r+1$ and if $v\in V_j^{r+1}$ then

$$XQ_rv=\Sigma_i[X, Z_i](X_i-\psi(X_i))v+\Sigma_iZ_i[X, X_i]v+Q_rXv.$$

$[X, Z_i]\in\mathfrak{u}_{2(s-r+1)}$. Hence $[X, Z_i](X_i-\psi(X_i))v\in V_{j-s-1}$. $[X, X_i]\in\mathfrak{u}_{2(s+r)}$.

So $[X, X_i]v \in V_{j-s-r-1}$ $(s+r>s)$. Hence $Z_i[X, X_i]v \in V_{j-s-1}$ by (ii) in the proof of Theorem 2.2. Finally, $Xv \in V_{j-s-1}$ so $Q_rXv \in V_{j-s-1}$ by (iv) in the proof of Theorem 2.2.

(2) If $r\geq 2$ then there exist integers $m_i>0$ depending only on j, r such that

$$\Pi_i(Q_r+m_iI)V_j^{r+1}\subset V_j^r.$$

Let $X \in \mathfrak{u}_{2r}$ and $v \in V_j^{r+1}$. Then $XQ_rv=\Sigma_k XZ_kX_kv=Q_rXv+\Sigma_k[X, Z_k]X_k+\Sigma_k Z_k[X, X_k]v$. Now, $[X, X_k] \in \mathfrak{u}_{4r}$ and since $4r>r+1$ this implies that $[X, X_k]v \in V_{j-2r-1}$. Thus $Z_k[X, X_k]v \in V_{j-r-1}$. Also, the definition of the Z_k, X_k implies that $\Sigma_k[X, Z_k]X_k=-X_0X+\Sigma_k u_kX_k$ with $u_k \in \mathfrak{u}_2$ and $\psi(u_k)=0$. Thus $\Sigma_k[X, Z_k]X_kv\equiv -Xv \bmod V_{j-r-1}$. Hence $X(Q_r+I)v\equiv Q_r=v \bmod V_{j-r-1}$. Thus we see that

$$X(Q_r+I)^pv\equiv Q_r^pXv \bmod V_{j-r-1}.$$

Let $n_1, \cdots, n_q$ be such that $n_i\geq 0$ and $\Pi_i\,(Q_r+n_i)V_j=0$ (Lemma 2.1). These integers depend only on j. Then

$$X\Pi_i(Q_r+(n_i+1)I)v \in V_{j-r-1}.$$

This proves (ii).

(iii) If $v \in V_{j+1}^2$ then $(Q_1+jI)v \in V_j$.

Indeed, let $Y_1, \cdots, Y_j \in \mathfrak{u}_2$. Set for $x \in \mathfrak{u}$, $x'=x-\psi(x)$. Then

$$(*)\qquad Y_1'\cdots Y_j'Q_1v=\sum_{i=1}^{j}\sum_k Y_1'\cdots Y_{i-1}'[Y_i, Z_k]X_k'Y_{i+1}'\cdots Y_j'v$$
$$+\sum_{i=1}^{j}\sum_k Y_1'\cdots Y_{i-1}'Z_k[Y_i, X_k]Y_{i+1}'\cdots Y_j'v.$$

We assert that $[Y_i, Z_k]=-(Y_i, X_k)X_0+u_{ik}$ with $u_{i,k} \in \mathfrak{u}_2$ and $\psi(u_{i,k})=0$ thus the first term on the right hand side of $(*)$ contributes

$$-jY_1'\cdots Y_j'v.$$

The second term is zero. To see this we observe as in the proof of (ii) that if $x \in \mathfrak{u}_2$ then $x'V_j^2\subset V_{j-1}^2$. Thus $[Y_i, X_k]Y_{i+1}'\cdots Y_j'v \in V_{j+1-(j-i)-3}=V_{i-2}$. Since $Z_kV_{i-2}\subset V_{i-1}$ the assertion follows. This proves (iii).

The Lemma is a direct consequence of (ii) and (iii).

Corollary 3.2. *If $v \in V_{j+1}$ with $j>0$ then $T_1\cdots T_jv \in V_1$.*

Let F be a finite dimensional $\mathfrak{g}$-module. $\mathfrak{u}$ acts nilpotently on F. Thus if $V \in W_\psi$ then $V\otimes F \in W_\psi$. Let k be the largest eigenvalue of H

on F. We set $F^j=\{f\in F\,|\,Hf=(k-j)f\}$. Let $V\in W_\psi$. Then $V_1\otimes F^j\subset (V\otimes F)_{r+1}$ if $j=2r$ or $j=2r+1$. Fix F and V.

Lemma 2.3. *Let $f\in F^j$, $v\in V_1$. If $r\geq 1$ and $1\leq j_1,\cdots,j_r\leq d$ then*

$$Q_{j_1}\cdots Q_{j_r}(v\otimes f)\in V\otimes\Big(\sum_{i<j}F^i\Big).$$

Proof. As usual, set $Q_r=\Sigma_i Z_{i,r}X'_{i,r}$ $(X'=X-\psi(X))$. Then $Q_{j_1}\cdots Q_{j_r}(v\otimes f)$ is a sum of terms of the form

$$(*)\qquad Y_1\cdots Y_r v\otimes U_1\cdots U_r f$$

with each pair (Y_i, U_i) in one of the following four forms:

$$(1)\qquad (Z_{k,j_i}, X_{k,j_i})$$

$$(2)\qquad (X'_{k,j_i}, Z_{k,j_i})$$

$$(3)\qquad (Z_{k,j_i}X'_{k,j_i}, I)$$

$$(4)\qquad (I, Z_{k,j_i}X_{k,j_i}).$$

We show that each term as in $(*)$ above satisfies the conclusion of the Lemma. Fix such a term. Let for $i=1,2,3,4$, $S_i=\{j\,|\,(Y_j, U_j)$ is as in (i)$\}$. Then $Y_1\cdots Y_r v\in V_p$ with $p=1+\Sigma_{i\in S_1}j_i-\Sigma_{i\in S_2}j_i$. Thus if $Y_1\cdots Y_r v\neq 0$ then $\Sigma_{i\in S_1}j_i\geq\Sigma_{i\in S_2}j_i$. Also, $U_1\cdots U_r f\in F^{j+s}$ with $s=-2\Sigma_{i\in S_1}j_i+2\Sigma_{i\in S_2}(j_i-1)-2|S_4|$. Thus if $Y_1\cdots Y_r v\otimes U_1\cdots U_r f\neq 0$ then $s\leq -2|S_2|-2|S_4|$. Thus if $s\geq 0$ then S_2 and S_4 are empty and $s\leq -2|S_1|$. If S_1 is empty also then $Y_1\cdots Y_r v=0$ (indeed, $Y_r v=0$). Thus $s<0$ if $Y_1\cdots Y_r v\otimes U_i\cdots U_r f\neq 0$. This implies the Lemma.

Let for $v\in V_1$, $f\in F^j$, $j=2r$ or $2r+1$, $\Gamma_j(v\otimes f)=v\otimes f$ for $r=0$. Set $c_r=r!\,|\Pi_{i\geq 2,1\leq j\leq r,k}n_{i,k,j}\neq 0$ (see Lemma 3.1). Set $\Gamma_j(v\otimes f)=c_r^{-1}T_1\cdots T_r(v\otimes f)$, $r>0$. We define a linear map, Γ, from $V_1\otimes F$ to $V\otimes F$ by $\Gamma(v\otimes\Sigma f_j)=\Sigma\Gamma_j(v\otimes f_j)$ $(f_j\in F^j)$.

Theorem 3.4. (1) $\Gamma(V_1\otimes F)\subset(V\otimes F)_1$.
(2) *Γ defines a linear isomorphism of $V_1\otimes F$ onto $(V\otimes F)_1$.*

Proof. (1) is just a restatement of Corollary 3.2.
We now prove that Γ is injective. Let $f_1,\cdots,f_q$ be a basis of F with $f_i\in F^{r_i}$ and $r_1\leq r_2\leq\cdots\leq r_q$. Let $s_1<\cdots<s_m$ be the distinct r_j in order. Let $u\in V_1\otimes F$, $u=\Sigma_i v_i\otimes f_i$, $v_i\in V$. Assume that $u\neq 0$, we show that $\Gamma u\neq 0$. Let i be the largest index such that $v_i\neq 0$. Let $r_i=s_k$. Then

$$u\equiv\sum_{r_j=s_k}v_j\otimes f_j \bmod \sum_{s_n<s_k}v\otimes F^{s_n}.$$

Lemma 3.3 implies that

$$\Gamma u \equiv \sum_{r_j=s_k} v_j \otimes f_j \bmod \sum_{s_n<s_k} V\otimes F^{s_n}.$$

Hence $\Gamma u \neq 0$. To complete the proof we must show that Γ is surjective.

Let $u \in (V\otimes F)_1$. Then $u=\Sigma_i v_i \otimes f_i$ with $v_i \in V$. If $X \in \mathfrak{u}$ then

$$0=(X-\psi(X))u=\sum_i \{(X-\psi(X))v_i\otimes f_i + v_i \otimes X f_i\}.$$

Suppose that $v_i=0$ for $i>p+1$ and that $v_p \neq 0$. Set $r_p=s_k$. Then

$$0=(X-\psi(X))u \equiv \sum_{r_i=s_k} (X-\psi(X))v_i \otimes f_i \bmod (V\otimes \sum_{s_i<s_k} F^{s_i}).$$

Thus $(X-\psi(X))v_i=0$ if $r_i=s_k$. As above,

$$\Gamma \sum_{r_j=s_k} v_j\otimes f_j \equiv \sum_{r_j=s_k} v_j \otimes f_j \bmod \sum_{s_n<s_k} V\otimes F^{s_n}.$$

So

$$u-\Gamma \sum_{r_j=s_k} v_j \otimes f_j \in \sum_{s_n<s_k} V\otimes F^{s_n}.$$

If we iterate this argument we will find that after a finite number of stages that $u=\Gamma w$ for some $w \in V_1 \otimes F$.

Note. The crucial point in this Theorem is that the formula for Γ depends on the structure of F and on ψ but not on V. For the case $\mathfrak{n}=\mathfrak{u}$ and without the explicit formula for Γ this result can be found in [L].

§ 4. Examples

In this section we give some examples of $\mathfrak{g}\supset\mathfrak{p}\supset\mathfrak{u}$ with ψ non-degenerate on $\mathfrak{u}$. We retain the notation of Section 2.

Lemma 4.1. *Let $\mathfrak{p}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ and let ψ be a Lie algebra homomorphism of $\mathfrak{n}$ to $\mathbf{C}$. Let Δ be the set of simple roots of $\mathfrak{a}$ on $\mathfrak{n}$. Then ψ is non-degenerate if and only if $\psi|_{\mathfrak{n}_\lambda}\neq 0$ for $\lambda \in \Delta$.*

Note. This Lemma implies that in the case of minimal parabolic subalgebras the notion of non-degenerate coincides with that of "generic".

Proof. Assume that ψ is non-degenerate. $Y=Y_\psi=\Sigma_{\lambda\in\Delta} Y_\lambda$ with $Y_\lambda \in \theta(\mathfrak{n}_{\mathbf{C}})_\lambda$. Suppose that $H\in\mathfrak{a}$ and that $\mathfrak{m}=\{X\in\mathfrak{g}\,|\,[H,X]=0\}$. We assume that $\operatorname{ad} H$ has integral eigenvalues and that $\mathfrak{g}_2\subset\mathfrak{n}$ generates $\mathfrak{n}$, $[H, Y]=-2Y$ and $\operatorname{ad} Y$ is injective on $\mathfrak{g}_2$. If $Y_\lambda\neq 0$ then $\lambda(H)=2$. Let

$\Phi=\Phi(\mathfrak{p}, \mathfrak{a})$ be the root system of $\mathfrak{p}$ with respect to $\mathfrak{a}$. Then $\mathfrak{g}_2=\bigoplus\mathfrak{z}_\lambda$ the sum over $\lambda \in \Phi$ such that $\lambda(H)=2$. Let $\Sigma=\{\lambda \in \Delta \mid Y_\lambda \neq 0\}$ and let $\Phi(\Sigma)$ denote the set of roots in the span of Σ. If $\lambda \in \Phi-\Phi(\Sigma)$ then $[Y, \mathfrak{g}_\lambda]=0$. Thus if $\Sigma\neq\Delta$ then $\mathfrak{g}_2$ cannot generate $\mathfrak{n}$. Hence if Y is non-degenerate then $Y_\lambda\neq 0$ for all $\lambda \in \Delta$.

Assume now that $Y_\lambda\neq 0$ for all $\lambda \in \Delta$. Set $X_\lambda=-\theta\bar{Y}_\lambda$. Let $H_\lambda \in \mathfrak{a}$ be defined by $B(h, H_\lambda)=\lambda(h)$ for $h \in \mathfrak{a}$. If $a_\lambda \in \mathbf{C}$ for $\lambda \in \Delta$ then

$$[\sum_{\lambda\in\Delta} a_\lambda X_\lambda, Y]=-\sum a_\lambda B(Y_\lambda, \theta\bar{Y}_\lambda)H_\lambda.$$

Now Δ is a basis for $\operatorname{span}_{\mathbf{C}}\{H_\lambda \mid \lambda \in \Delta\}$. Thus we can choose the a_λ such that

$$\mu(-\sum a_\lambda B(Y_\lambda, \theta\bar{Y}_\lambda)H_\lambda)=2, \quad \mu \in \Delta.$$

Set $X=\Sigma a_\lambda X_\lambda$, $H=[X, Y]$. Then $[H, X]=2X$, $[H, X]=-2Y$ and $\mathfrak{g}_2=\bigoplus_{\lambda\in\Delta}\mathfrak{g}_\lambda$. The proposition now follows.

For the rest of this section we describe several techniques for finding examples of non-degenerate ψ's. It is suggested that the reader consult the tables in [H, pp. 532–534 and p. 518].

The next class of examples were the original motivation for this paper. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{R}$. Let $\mathfrak{p}_0$ be a minimal parabolic subalgebra of $\mathfrak{g}$. Let the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_0$ be of type C_r (resp. BC_r). Then there are linear functionals $\varepsilon_1, \cdots, \varepsilon_r$ forming a basis of $(\mathfrak{a}_0)^*$ such that the roots are $\varepsilon_i-\varepsilon_j$, $i\neq j$, $\pm(\varepsilon_i+\varepsilon_j)$, $i\leq j$ (resp. in addition ε_i for $i=1, \cdots, r$). Let $H_i \in \mathfrak{a}_0$ be such that $\varepsilon_i(H_j)=\delta_{ij}$. Let $Y_i \in \mathfrak{g}_{-2\varepsilon_i}$, $i=1, \cdots, r$ and put $Y=\Sigma Y_i$. We may choose $X_i \in \mathfrak{g}_{2\varepsilon_i}$ such that $[X_i, Y_i]=0$. Since $2\varepsilon_i-2\varepsilon_j$ is not a root for $i\neq j$ we see that $[X, Y]=H$. $[H, X]=2X$, $[H, Y]=-2Y$. We choose $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{n}$ with $\mathfrak{m}=\{x \in \mathfrak{g} \mid [H, X]=0\}$, $\mathfrak{n}=\bigoplus_{i\leq j}\mathfrak{g}_{\varepsilon_i+\varepsilon_j}\bigoplus_i\mathfrak{g}_{\varepsilon_i}$. Set $\mathfrak{u}=\bigoplus_{i\leq j}\mathfrak{g}_{\varepsilon_i+\varepsilon_j}$. Then $\mathfrak{u}$ is a subalgebra of $\mathfrak{n}$ and $\mathfrak{u}=\mathfrak{g}_2$. If ψ is defined by $\psi(x)=B(x, Y)$ for $x \in \mathfrak{u}$ then all the conditions for non-degeneracy are satisfied.

We list the examples that are given by this construction.

1. The subclass of examples where $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair. Then $\mathfrak{p}$ is the so called Shilov boundary parabolic. The examples are

(1) $\mathfrak{su}(p, q)$, $p\geq q\geq 1$ which is BC_q if $p>q$, C_q if $p=q$.
(2) $\mathfrak{sp}(n, \mathbf{R})$, C_n.
(3) $\mathfrak{so}(n, 2)$, C_2.
(4) $\mathfrak{so}^*(2n)$, C_n if n is even, BC_n if n is odd.
(5) The Hermitian symmetric real form of E_6, BC_2.
(6) The Hermitian symmetric real form of E_7, C_3.

The other examples from this construction are

2. $\mathfrak{sp}(q, q)$, $p \geq q \geq 1$, BC_q unless $p=q$ in this case C_q.
3. The rank one real form of F_4, BC_1.

Suppose now that the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_0$ is type A_{2r+1}. Then there exists an isometric imbedding of $\mathfrak{a}_0$ into $\mathbf{R}^{2r+2}$ as the hyperplane $\{x \mid \Sigma_i x_i = 0\}$. Set $\varepsilon_i(x) = x_i$. Then the roots are $\varepsilon_i - \varepsilon_j$, $i \neq j$. We take to be the parabolic subalgebra defined as follows. Let $H \in \mathfrak{a}_0$ be the element in $\mathfrak{a}_0$ whose image in $\mathbf{R}^{2r+2}$ is $(1, \cdots, 1, -1, \cdots, -1)$ where the last 1 is in the $r+1^{st}$ position. Set $\mathfrak{m} = \{x \in \mathfrak{g} \mid [H, X] = 0\}$, $\mathfrak{n}$ is the eigenspace for $\operatorname{ad} H$ corresponding to eigenvalue 2. Let $\beta_i = \varepsilon_i - \varepsilon_{r+1+i}$, $i=1, \cdots, r+1$. Then $\beta_i - \beta_j$ is not a root if $i \neq j$. Let $Y_i \in \theta(\mathfrak{n}_C)_{\beta_i} - (0)$ and set $Y = \Sigma_i Y_i$. Put $\psi(x) = B(Y, x)$ for $x \in \mathfrak{n}$. Then we leave it to the reader to check that ψ is non-degenerate. Here are the examples that come from this method.

4. $\mathfrak{sl}(2n, \mathbf{R})$.
5. $\mathfrak{su}^*(2n)$.
6. The real form of E_6 with maximal compactly imbedded subalgebra of type F_4.

There are also two examples corresponding to D_n. Assume that the root system of $(\mathfrak{g}, \mathfrak{a}_0)$ is of type D_{2n}. Then the roots are given by $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \leq i \neq j \leq 2n$. Choose the simple roots to be $\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{2n-1} - \varepsilon_{2n}$, $\varepsilon_{2n-1} + \varepsilon_{2n}$. We take for $\mathfrak{p}$ the parabolic subalgebra corresponding to $\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{2n-1} - \varepsilon_{2n}$. Then the roots of $\mathfrak{a}_0$ on $\mathfrak{n}$ are $\varepsilon_i + \varepsilon_j$ for $i \neq j$. Choose Y_i to be a nonzero element of $\mathfrak{g}_{-\varepsilon_i - \varepsilon_{n+i}}$ for $i=1, \cdots, n$ and $Y = z(\Sigma_i Y_i)$ with $z \neq 0$. Then it is easily checked that there exist for each $i=1, \cdots, n$, $X_i \in \mathfrak{g}_{\varepsilon_i + \varepsilon_{n+1}}$ such that if $X = z^{-1}(\Sigma_i X_i)$ then X, Y, H is a standard basis of a three dimensional simple Lie algebra (TDS) and $\operatorname{ad} H|_{\mathfrak{n}} = 2I$, $\mathfrak{m} = \{x \in \mathfrak{g} \mid [H, x] = 0\}$. In this case we get the following example.

7. $\mathfrak{so}(2n, 2n)$.

For $\mathfrak{sl}(n, \mathbf{R})$ (or $\mathfrak{sl}(n, \mathbf{C})$) examples abound. We now give a method which probably gives all examples for $\mathfrak{sl}(n, \mathbf{R})$. Let H, X, Y be a standard basis of a TDS with $H, X, Y \in M_n(\mathbf{R})$ we assume, as we may, that H is symmetric. We assume that all of the eigenvalues of H are congruent mod 2. We take for $\mathfrak{p}$ the direct sum of the eigenspaces for the non-negative eigenvalues of $\operatorname{ad} H$. Then $\mathfrak{n}$ is generated by the 2-eigenspace for $\operatorname{ad} H$. We take $\psi(x) = z \operatorname{tr} Yx$ for $x \in \mathfrak{n}$ and $z \in \mathbf{C} - \{0\}$.

The last class of examples that we will describe is of a somewhat more sophisticated nature. Let G be a connected semi-simple algebraic group defined over k, a subfield of $\mathbf{R}$. We look upon G as its complex points hence as a Lie group over $\mathbf{C}$. Let P be a minimal parabolic subgroup of G defined over k and let S be a maximal k-split torus of P. Let Φ be the root system of (G, S) and let Φ^+ be the positive roots corresponding to P. We assume that Φ is reduced (i.e. if $\alpha \in \Phi$ then $2\alpha \notin \Phi$).

Let $\Delta \subset \Phi^+$ be the set of simple roots.

Let for $\alpha \in \Delta$, E_α be a one dimensional subgroup of the unipotent subgroup of G corresponding to α that is invariant under conjugation by S and defined over k. Let V be the subgroup of G generated by the E_α for $\alpha \in \Delta$. Then it is shown in [BT, Theorem 7.2] that there is a unique k-split reductive algebraic group $F \subset G$ containing SV with S as its maximal split torus.

Let $G = G_{\mathbf{R}}$, $P = P_{\mathbf{R}}$, $A = (S_{\mathbf{R}})^o$ (upper o indicates identity component). Let N be the unipotent radical of P and $N = N_{\mathbf{R}}$. Let $\bar{P}$ be the opposite parabolic to P and let $\bar{N}$ be its unipotent radical put $\bar{N} = \bar{N}_{\mathbf{R}}$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_{\mathbf{R}} = \mathfrak{g}$. $\mathfrak{g}_k$ is a k-form of $\mathfrak{g}$ and the Killing form of $\mathfrak{g}$ is defined over k. Let ψ be a Lie algebra homomorphism of $\mathfrak{n}$ to $\mathbf{C}$. Then there exists $Y = Y_\psi \in \bar{\mathfrak{n}}_{\mathbf{C}}$ such that $B(Y, x) = \psi(X)$. Write $Y_\psi = \Sigma_{\alpha \in \Delta} Y_\alpha$ with $Y_\alpha \in (\bar{\mathfrak{n}}_{-\alpha})_{\mathbf{C}}$. We assume that $\exp \mathbf{C} Y_\alpha$ is defined over k and that $Y_\alpha \neq 0$ for all $\alpha \in \Delta$. Then (replacing P by $\bar{P}$ in the above mentioned result of Borel and Tits) there exists a unique k-split (hence $\mathbf{R}$-split) reductive subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ containing and such that $\bar{\mathfrak{n}}_{-\alpha} \cap \mathfrak{g}_1 = \mathbf{C} Y_\alpha \cap \mathfrak{g}_1$ is one dimensional over $\mathbf{R}$. We can now argue as in the proof of Lemma 4.1 to find $X \in \oplus_{\alpha \in \Delta} (\mathfrak{n}_\alpha \cap \mathfrak{g}_1)_{\mathbf{C}}$ such that $[X, Y] = H$, $H \in \mathfrak{a}$ and $\operatorname{ad} H|_{\mathfrak{n}_\alpha} = 2I$, $\alpha \in \Delta$.

§ 5. Some structural results

In this section we study the global structure of the non-degenerate functionals in the previous sections in preparation for the results on the holomorphic continuation of the corresponding generalized Jacquet integrals. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{R}$. Fix a Cartan involution θ of $\mathfrak{g}$. Let G be a Lie group of inner type with finite center and Lie algebra $\mathfrak{g}$ (i.e. if $g \in G$ then $\operatorname{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbf{C}}$). We assume that $\mathfrak{g}$ is one of the examples in Section 4 that corresponds to C_n, A_{2r+1} or D_{2n}. Let $\mathfrak{p}$ be the parabolic subalgebra described in those cases in Section 4 and let ψ be a non-degenerate homomorphism of $\mathfrak{n}$ into $\mathbf{R}$ as in that section. Let P be the corresponding parabolic subgroup of G (i.e. $P = \{g \in G \mid \operatorname{Ad}(g)\mathfrak{p} \subset \mathfrak{p}\}$). Let H, Y, X be as in those cases. Put $M = \{g \in G \mid \operatorname{Ad}(g)H = H\}$. Set $M_\psi = \{m \in M \mid \psi(\operatorname{Ad}(m)x) = \psi(x) \text{ for } x \in \mathfrak{n}\}$. Set $N = \exp \mathfrak{n}$. Then $P = MN$ is a Levi decomposition of P.

Proposition 5.1. *There exists $m \in M$ such that if we replace ψ by $\psi \circ \operatorname{Ad}(m)$ then there exists a finite subset $\Sigma \subset K$ such that*

(1) *$G = \bigcup_{w \in \Sigma} M_\psi N w P$ disjoint union.*

(2) *There is a unique element $w_o \in \Sigma$ such that $M_\psi N w_o P$ is open. Furthermore, $M_\psi N w_o P = N w_o P = P w_o P$ is the unique open double coset of P.*

(3) *If* $w \in \Sigma$ *and* $w \neq w_o$ *then* ψ *restricted to the Lie algebra of* $N \cap w^{-1}Pw$ *is non-trivial.*

Proof. In all of these cases $\mathfrak{n} = \mathfrak{g}_2 = \{x \in \mathfrak{g} \mid [H, x] = 2x\}$. Thus $\mathfrak{g}_{-2} = \theta\mathfrak{n}$. So $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{m} \oplus \mathfrak{z}_2$. As a module (under ad) for the Lie algebra $\mathfrak{s}$, with basis X, Y, H, $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{z}^2$ with $\mathfrak{s}$ acting trivially on $\mathfrak{g}^0$ and acting by a multiple of the irreducible three dimensional representation on $\mathfrak{g}^2$. We note that $\mathfrak{m}_\psi = \mathfrak{m} \cap \mathfrak{g}^0$. Set $\sigma = \exp(\pi(X - Y)/2)$. Then a direct calculation in $SL(2, \mathbf{R})$ implies that $\mathrm{Ad}(\sigma)x = x$ for $x \in \mathfrak{m}_\psi$ and $\mathrm{Ad}(\sigma)x = -x$ for $x \in \mathfrak{m} \cap \mathfrak{g}^2$. There exists $m \in M$ such that $\mathrm{Ad}(m\sigma m^{-1})$ commutes with θ. So replace ψ by $\psi \circ \mathrm{Ad}(m)$, Y by $\mathrm{Ad}(m)Y$, X by $\mathrm{Ad}(m)X$. M_ψ contains the identity component of the fixed point set of $m \rightarrow \sigma m \sigma^{-1}$.

We now look at the C_n cases. Fix P_o a minimal parabolic subgroup contained in P. Let A_o be a standard split component of P_o ($\theta h = -h$ for $h \in \mathfrak{a}_o$). Let $\varepsilon_1, \cdots, \varepsilon_n$ be as in Section 4. Then the Weyl group of (G, A_o) is the semi-direct product of the permutations of the ε_i, S_n, and the sign changes on the ε_i, Z_2^n. Let w_i be defined by $w_i\varepsilon_j = -\varepsilon_j$ for $j \leq i$ and $w_i\varepsilon_j = \varepsilon_j$ for $j > i$. The Bruhat lemma implies that

$$G = P \cup \bigcup_{i=1}^n Pw_iP.$$

Set $P_i^* = M \cap w_iP_ow_i^{-1}$. Then P_i^* is a minimal parabolic subgroup of M. Thus Theorem 3 of the introduction of [M] implies that there exist a finite number of $w_{ji} \in K \cap M$ such that $M = \bigcup_j M_\psi w_{ji} P_i^*$. Now

$$P \cup \bigcup_{ij} NM_\psi w_{ji}w_iP = P \cup \bigcup_{ij} NM_\psi w_{ji}w_iw_i^{-1}P_i^*w_iP = P \cup \bigcup_i NMw_iP = G$$

by the above. Suppose that $NM_\psi w_{ji}w_iP$ is open then Pw_iP is open so $i = n$. We observe that $Nw_nP = Pw_nP$. Thus in this case we may take all of the $w_{in} = 1$. This proves (1), (2). The explicit formulas for the w_{ij} in [M, Theorem 3] imply that $\mathrm{Ad}(w_{ij})\mathfrak{g}_{\varepsilon_p + \varepsilon_q} \subset \mathfrak{g}_{\varepsilon_p + \varepsilon_q}$ for all p, q. Hence ψ is non-trivial on $\mathfrak{n} \cap w_{ij}w_j\mathfrak{n}w_j^{-1}w_{ij}^{-1}$ if $j \neq n$. This completes the proof in the C_n case.

We now look at the A_{2r+1} case. Let P_0 be a minimal parabolic subgroup of G. We use analogous notation to the previous case. However, this time the ε_i, $i = 1, \cdots, 2r+2$ satisfy the relation $\Sigma_i\varepsilon_i = 0$. The Weyl group is given by the permutations of the ε_i. Put $w_j = \prod_{i=1}^j (i, r+1+i)$. The Bruhat lemma in this case implies that $G = \bigcup_j Pw_jP$. The rest of the argument is exactly the same as the case of C_n.

The D_{2n} case is completely analogous and we leave it to the reader.

Let $Y \in \mathfrak{g}$ be a nilpotent element and let X, Y, H be as in the end of Section 2. We assume that the eigenvalues of ad H are all even. Let G be a real reductive group of inner type with Lie algebra a $\mathfrak{g}$ and let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ given by the sum of the eigenspaces for the non-negative eigenvalues of ad H. Let P be the corresponding parabolic subgroup of G. We conjecture that a decomposition of the type of Proposition 5.1 is true for G.

§ 6. Some Bruhat theory

Let G be a real reductive group of inner type. Let θ be a Cartan involution of G. Fix a minimal parabolic subgroup, P_o, of G. Let P be a parabolic subgroup of G with unipotent radical N_P and set $M_P=\theta P\cap P$. Let $A_P=\exp(\mathfrak{a}_P)$ with $\mathfrak{a}_p=\{X\in\mathfrak{z}(\mathfrak{m}_P)\,|\,\theta X=-X\}$. Set $\mathfrak{a}_o=\mathfrak{a}_{P_o}$.

Let η be a homomorphism from N_P to S_1 (the circle group). We set $d\eta=i\psi$. Put $M_\psi=M_\eta=\{m\in M_P\,|\,\eta(mnm^{-1})=\eta(n),\ n\in N_p\}=\{m\in M\,|\,\psi(\mathrm{Ad}(m)X)=\psi(X),\ X\in\mathfrak{n}_p\}$. Throughout this section we will make the following assumptions about P and η.

(1) There exists a finite subset Σ of G such that G is the disjoint union $\bigcup_{w\in\Sigma}PwM_\psi N_P$.

(2) There is a unique element $w_o\in\Sigma$ such that $Pw_oM_\psi N$ is open in G and furthermore $Pw_oM_\psi N=Pw_oN$ and $w_oM_\psi w_o^{-1}=M_\psi$.

(3) If $w\in\Sigma$ and if $w\neq w_o$ then η is not identically equal to 1 on $N_P\cap w^{-1}Pw$.

We note that if $P=P_o$ and if ψ is non-degenerate (i.e. generic) then the above conditions are satisfied. Also, all of the C_n, A_{2r+1}, D_{2n} examples of Section 5 satisfy these conditions.

Put ${}^oM_P=\{m\in M_P\,|\,\kappa(m)^2=1$ for all continuous homomorphisms κ of M_P into $\mathbf{R}-(0)\}$. Then $M_P=A_P\,{}^oM_P$ with unique expression. If $\nu\in(\mathfrak{a}_P)^*_{\mathbf{C}}$ then we set $\exp(h)^\nu=\exp(\nu(h))$ for $h\in\mathfrak{a}_P$. Set $\rho_P=1/2(tr(\mathrm{ad}\,h|_{\mathfrak{n}_P})$, $h\in\mathfrak{a}_P$. Let (σ, H_σ) be a finite dimensional representation of ${}^oM_P$. If $\nu\in(\mathfrak{a}_P)^*_{\mathbf{C}}$ then we denote by σ_ν the representation of P given by

$$\sigma_\nu(man)=a^{\nu+\rho_P}\sigma(m),\quad m\in{}^oM_P,\quad a\in A_P,\quad n\in N_P.$$

Let $I^\infty_{P,\sigma,\nu}$ denote the C^∞ induced representation of σ_ν from P to G. That is, $I^\infty_{P,\sigma,\nu}$ is the space of all smooth fumctions f from G to H_σ such that $f(pg)=\sigma_\nu(p)f(g)$ for $p\in P$, $g\in G$. We endow $I^\infty_{P,\sigma,\nu}$ with the C^∞ topology. The action, $\pi_{P,\sigma,\nu}$, is given by $(\pi_{P,\sigma,\nu}(g)f)(x)=f(xg)$.

Let (τ, V_τ) be a finite dimensional representation of $M_\psi N_P$ such that $\tau(n)=\eta(n)I$ for $n\in N_P$. We set $\mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})$ equal to the space of all continuous linear maps, T from $I^\infty_{P,\sigma,\nu}$ to V_τ such that $T(\pi_{P,\sigma,\nu}(m)f)=\tau(m)T(f)$ for $f\in I^\infty_{P,\sigma,\nu}$, $m\in M_\psi N_P$. Let $U_{\sigma,\nu}$ be the space of all $f\in I^\infty_{P,\sigma,\nu}$ such that $\mathrm{supp}f\subset Pw_oN_PM_\psi$.

Theorem 6.1. *Let* $T\in\mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})$.

(1) *If* $T|_{U_{\sigma,\nu}}=0$ *then* $T=0$.

(2) *There exists* $S\colon H_\sigma \to V_\tau$ *such that*

 (i) $S\sigma_\nu(m)=\tau(m)S$ *for* $m \in M_\psi$ *and*

 (ii) $T(f)=S\int_{N_P}\eta(n)^{-1}f(w_0^* n)dn$ (*here* dn *is a fixed choice of invariant measure on* N_P).

Proof. The argument is standard Bruhat theory. If $f \in C_c^\infty(G)$ and if $v \in H_\sigma$ then set

$$S_{\sigma,\nu}(f\otimes v)(g)=\left(\int_P \sigma_\nu(p)^{-1}f(pg)d_r p\right)v$$

where $d_r p$ is a fixed choice of right invariant measure on P.

Then $S_{\sigma,\nu}$ defines a continuous linear map of $C_c^\infty(G)\otimes H_\sigma$ onto $I^\infty_{P,\sigma,\nu}$. If $T \in \mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})$ then set $\hat{T}(f)(v)=T(S_{\sigma,\nu}(f\otimes v))$. Then

$$\hat{T}\colon C_c^\infty(G) \longrightarrow \mathrm{Hom}_C(H_\sigma, V_\tau).$$

Put $L(g)f(x)=f(gx)$ (resp. $R(p)f(x)=f(xg)$). Then

$$\hat{T}(L(p)f)=\hat{T}(f)\cdot\sigma_\nu(p),$$
$$\hat{T}(R(m)f)=\tau(m)\hat{T}(f)$$

for $p \in P$, $m \in M_\psi N$, $f \in C_c^\infty(G)$.

Set ${}^o\mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})=\{T \in \mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})\,|\,\mathrm{supp}\,\hat{T}\subset G-Pw_0^\infty N_P\}$. Then Bruhat theory (cf. [War, 5.3 p. 411]) implies that

$$\dim {}^o\mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu}) \le \sum_{w\in\Sigma-\{w_0\}} \dim \mathrm{Hom}_{w^{-1}Pw\cap M_\psi N_P}(H_{\sigma_\nu}, S(\mathfrak{g}/(\mathrm{Ad}\,(w^*)^{-1}\mathfrak{p}+\mathfrak{m}_\psi+\mathfrak{n}_P)\otimes V_\tau)).$$

Here, $w^* \in K$ is a representative for w, $S(\cdot)$ is the symmetric algebra. The proof uses (1) and (2). (3) implies that if $w\neq w_0$ then $\eta|_{w^{-1}Pw\cap N_D}\neq 1$. Since N_P acts trivially on H_{σ_ν} and unipotently on $S(\mathfrak{g}/\mathfrak{n}_P)$ we conclude that $\dim^o \mathrm{Wh}^\tau(I^\infty_{P,\sigma,\nu})=(0)$.

The second part of the Theorem is also completely standard (cf. [War, Theorem 5.2.2.1]) so we leave it to the reader.

Set $(I^\infty_{P,\sigma,\nu})'$ equal to the space of all continuous linear functionals on $I^\infty_{P,\sigma,\nu}$ put $\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')$ equal the space of all $\lambda \in (I_{P,\sigma,\nu})'$ such that

 (i) $\lambda(\pi_{P,\sigma,\nu}(n)f)=\eta(n)\lambda(f)$, $n \in N_P$, $f \in I^\infty_{P,\sigma,\nu}$,

 (ii) Let π' be the contragradient representation of G on $(I^\infty_{P,\sigma,\nu})'$. Then $\pi'(M_\psi)\lambda$ spans a finite dimensional space on which M_ψ acts semisimply.

Corollary 6.2. *Assume that σ is irreducible. Then* $\dim \mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')\leq \dim H_\sigma$.

Proof. Let $\mathring{M}_\psi$ denote the set of all equivalence classes of irreducible finite dimensional representations of M_ψ. Fix for $\gamma\in\mathring{M}_\psi$, a representative $V_\gamma\in\gamma$. We decompose $\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')$ into M_ψ-isotypic components as

$$\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')=\sum_{\gamma\in\mathring{M}_\psi}\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')(\gamma).$$

Fix $\gamma\in\mathring{M}_\psi$. Let $V\subset\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})')(\tau)$ be a finite dimensional M_ψ-invariant subspace. We define $T\colon I^\infty_{P,\sigma,\nu}\to V^*$ by $T(f)(\lambda)=\lambda(f)$. Let $\mu\in\mathrm{Hom}_{M_\psi}(V^*,V^*_\tau)$. We look upon V^*_γ as a $M_\psi N$-module with N acting by $\eta^{-1}I$. Then $\mu\circ T\in\mathrm{Wh}^{\tau^*_\gamma}(I^\infty_{P,\sigma,\nu})$. Thus Theorem 6.1 implies that there exists $S(\mu)\in\mathrm{Hom}_{M_\psi}(H_\sigma,V^*_\gamma)$ such that

$$\mu(T(f))=S(\mu)\Big(\int_N\eta^{-1}(n)f(w_o n)dn\Big),\qquad f\in U_{\sigma,\nu}.$$

This implies that we have an injective linear mapping

$$S\colon \mathrm{Hom}_{M_\psi}(V^*,V^*_\gamma)\longrightarrow\mathrm{Hom}_{M_\psi}(H_\sigma,V^*_\gamma).$$

Hence $\dim V^*\leq\dim\mathrm{Hom}_{M_\psi}(H_\sigma,V^*_\gamma)\dim V^*_\gamma$. This implies that

$$\dim W(I^\infty_{P,\sigma,\nu})')(\gamma)\leq\dim\mathrm{Hom}_{M_\psi}(H_\sigma,V^*_\gamma)\dim V^*_\gamma.$$

Hence

$$\dim\mathrm{Wh}_\eta((I^*_{P,\sigma,\nu})')\leq\sum_{\gamma\in\mathring{M}_\psi}\dim\mathrm{Hom}_{M_\psi}(H_\sigma,V^*_\gamma)\dim V^*_\gamma=\dim H_\sigma$$

since M_ψ is reductive in M.

§7. The holomorphic continuation of certain integrals

We retain the notation and assumptions of the previous section. We assume in addition that ψ is non-degenerate. We write M, N, A for M_P, N_P, A_P.

Let $I^\infty_{P,\sigma}$ denote the space of all $f\in C^\infty(K;H_\sigma)$ such that $f(pk)=\sigma(p)f(k)$ for $p\in P\cap K$ and $k\in K$. If $f\in K$ then set $f_{\sigma,\nu}(pk)=\sigma_\nu(p)f(k)$ for $p\in P$. Then the correspondence $f\to f_{\sigma,\nu}$ of $I^\infty_{P,\sigma}$ to $I^\infty_{P,\sigma,\nu}$ is a continuous isomorphism. We will thus consider $I^\infty_{P,\sigma,\nu}$ as $I^\infty_{P,\sigma}$ and under this identification $\pi_{P,\sigma,\nu}(g)f(k)=f_{\sigma,\nu}(kg)$.

Proposition 7.1. *There exists a constant $q_\sigma\geq 0$ such that*

$$\int_N \eta(n)^{-1} f_{\sigma,\nu}(w_o n)dn = J_{\sigma,\nu,\eta}(f) = J_\nu(f)$$

converges absolutely and uniformly in compacta of $\{\nu \in \mathfrak{a}_{\mathbf{C}}^* \mid \operatorname{Re}(\nu, \alpha) > q_\sigma$ *for* $\alpha \in \Phi(P, A)\}$ *for all* $f \in I_{P,\sigma}^\infty$. *If* $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, $\operatorname{Re}(\nu, \alpha) > q_\sigma$ *and if* $\lambda \in H_\sigma^*$ *then* $\lambda \circ J_\nu \neq 0$. *Hence in particular*

$$\dim \mathrm{Wh}_\eta ((I_{P,\sigma,\nu}^\infty)') = \dim H_\sigma.$$

Proof. The first assertion follows from (by now) standard inequalities (due to Harish-Chandra) which we now recall. If $g \in G$ then we write $g = p(g)k(g)$. Neither $p(g)$ nor $k(g)$ are well defined. However, $p(g)P \cap K$ and $P \cap Kk(g)$ are well defined. Fix a $K \cap P$-invariant inner product on H_σ. If $f \in I_{P,\sigma}^\infty$ then

$$\|\eta(n)^{-1} f_{\sigma,\nu}(w_o n)\| = \|\sigma_\nu(p(w_o n)) f(k(w_o n))\| \leq \|f\|_\infty \|\sigma_\nu(p(w_o n))\|.$$

We write $p(g) = n(g)m(g)a(g)$ with $n(g) \in N$, $m(g) \in {}^oM$ and $a(g) \in A$. Now the ambiguity is in $m(g)$. We have

$$\|\eta(n)^{-1} f_{\sigma,\nu}(w_o n)\| \leq \|f\|_\infty \|\sigma(m(w_o n))\| \, a(w_o n)^{\operatorname{Re}\nu + \rho}.$$

Let log be the inverse map to exp on $\mathfrak{n}$. Fix on $\mathfrak{g}$ the inner product $\langle x, y\rangle = -B(x, \theta y)$. It is standard that (cf. [W, Proof of 4.5.6])

$$\|\sigma(m(w_o n))\| \leq C(1 + \|\log(n)\|)^d$$

for some positive constants C and d.

It is thus enough to show that there exists $q_\sigma \geq 0$ such that if $\nu \in \mathfrak{a}^*$ and $(\nu, \alpha) > q_\sigma$ then

$$\int_N a(w_o n)^{\nu+\rho}(1 + \|\log(n)\|)^d dn < \infty.$$

But this follows directly from (for example) [W, 4.5.4].

We now prove the second assertion. Let $\varphi \in C_c^\infty(N)$ be such that

$$\int_N \eta(n)^{-1}\varphi(n)dn = 1.$$

If $v \in H_\sigma$ then set $\beta_\nu(v)(pw_o n) = \varphi(n)v$ if $g = pw_o n$, $p \in P$, $n \in N$ and zero otherwise. Then $\beta_\nu(v) \in I_{P,\sigma,\nu}^\infty$ and $J_\nu(\beta_\nu(v)) = v$. This completes the proof.

Let $(I_{P,\sigma}^\infty)'$ denote the space of all continuous functionals on $I_{P,\sigma}^\infty$ (which is endowed with the C^∞ topology). We will use the weak topology on $(I_{P,\sigma}^\infty)'$. We look upon $\mathrm{Wh}_\eta((I_{P,\sigma,\nu}^\infty)')$ as a subspace of $(I_{P,\sigma}^\infty)'$.

Theorem 7.2. *We keep the hypotheses of this section and the previous one. Let $\mu \in H_\sigma^*$ then $\nu \to \mu \circ J_\nu$ extends to a weakly holomorphic map of $\mathfrak{a}_{\mathbf{C}}^*$ into $(I_{P,\sigma}^\infty)'$. Furthermore, if $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ then* $\dim\{\mu \circ J_\nu \mid \mu \in H_\sigma^*\} = \dim H_\sigma$. *In particular,* $\dim \mathrm{Wh}_\eta((I_{P,\sigma,\nu}^\infty)') = \dim H_\sigma$ *for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$.*

Proof. If (π, H) is a smooth Frechet representation of G then we denote by H' the continuous dual of H and we write for $g \in G$ (resp. $X \in \mathfrak{g}$) $\pi'(g)\lambda = \lambda \circ \pi(g)^{-1}$ (resp. $-\lambda \circ \pi(X)$) for $\lambda \in H'$. We set for $X \in \mathfrak{n}$, $\pi'_\psi(X) = \pi'(X) + \psi(X)$ and $H'[\psi] = \{\lambda \in H' \mid \pi'_\psi(\mathfrak{n})^k\lambda = 0$ for some k and $\pi'_\psi(M_\psi)\lambda$ spans a finite dimensional space such that M_ψ acts semi-simply$\}$.

Notice that $H'[\psi] \in W_{-\psi}$ in the notation of Section 3. We set $\mathrm{Wh}_\psi(H') = H^0(\mathfrak{n}, H'[\psi] \otimes \mathbf{C}_\psi) = (H'[\psi])_1$. If F is a finite dimensional representation of G then (since $\mathfrak{n}$ acts nilpotently on F)

$$(H \otimes F)'[\psi] = H'[\psi] \otimes F'.$$

Let (μ, F) be a finite dimensional irreducible representation of G such that

(i) ${}^oM$ acts trivially on $F^{\theta\mathfrak{n}} = \{v \in F \mid (\theta\mathfrak{n})v = (0)\}$.

(ii) If $-\lambda$ is the action of $\mathfrak{a}$ on $F^{\theta\mathfrak{n}}$ (which is one dimensional) then

$$(\lambda, \alpha) > 0, \qquad \alpha \in \Phi(P, A).$$

Such an F always exists (cf. [W, 4.A.2.3]). We note that

$$I_{P,\sigma,\nu}^\infty \otimes F \cong \mathrm{Ind}^{\infty}{}_P^G(\sigma_\nu \otimes \mu|_P)$$

where Ind^∞ indicates C^∞ induction. Let $H_\sigma \otimes F = X_1 \supset X_2 \supset \cdots \supset X_{d+1} = (0)$ be a P-invariant filtration with $X_i/X_{i+1} \cong (\sigma_i)_{\nu_i}$ for appropriate σ_i, ν_i. Here the action of P on H_σ is σ_ν. We assume as we may (see below) that $\sigma_1 = \sigma$ and $\nu_1 = \nu - \lambda$. Writing $I^\infty = I_{P,\sigma,\nu}^\infty$ then

$$I^\infty \otimes F = I_1^\infty \supset I_2^\infty \supset \cdots \supset I_{d+1}^\infty = (0)$$

with each of the spaces closed and invariant and $I_j^\infty / I_{j+1}^\infty \cong I_{P,\sigma_i,\nu_i}^\infty$.

Assume that $\dim \mathrm{Wh}_\eta((I_{P,\sigma,\nu}^\infty)') = \dim H_\sigma$. Then

$$\dim((I_{P,\sigma,\nu}^\infty)'[\psi] \otimes F')_1 = \dim F \dim H_\sigma$$

by Theorem 3.4. Now $((I_{P,\sigma,\nu}^\infty)'[\psi] \otimes F')_1|_{I_d^\infty} \subset \mathrm{Wh}_\eta((I_{P,\sigma_d,\nu_d}^\infty)')$ so

$$\dim((I_{P,\sigma,\nu}^\infty)'[\psi] \otimes F')_1|_{I_d^\infty} \leq \dim H_{\sigma_d}$$

by Corollary 6.2. Set $W^j = \{\lambda \in ((I_{P,\sigma,\nu}^\infty)'[\psi] \otimes F)_1 \mid \lambda|_{I_j^\infty} = 0\}$. Then $\dim W^d \geq \dim F \dim H_\sigma - \dim H_{\sigma_d}$. $W^d|_{I_{d-1}^\infty} \subset \mathrm{Wh}_\eta((I_{P,\sigma_{d-1},\nu_{d-1}}^\infty)')$ so $\dim W^{d-1} \geq \dim F \dim H_\sigma - \dim H_{\sigma_d} - \dim H_{\sigma_{d-1}}$. If we continue in this way we find that $\dim W^2 \geq \dim H_\sigma$. But $W_2 \subset \mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu-\lambda})')$. We have thus proved

(iii) If $\dim \mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu})') = \dim H_\sigma$ then $\dim \mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu-\lambda})') = \dim H_\sigma$.

If $\nu \in \mathfrak{a}^*_{\mathbf{C}}$ then there exists $k>0$ such that $\mathrm{Re}(\nu+k\lambda, \alpha)>q_\sigma$ for all $\alpha \in \Phi(P, A)$. We have thus proved

(iv) If $\nu \in \mathfrak{a}^*_{\mathbf{C}}$ then $\dim \mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu-\lambda})') = \dim H_\sigma$.

We now convert this into an analytic statement. Let

$$\delta\colon I^\infty_{P,\sigma} \otimes F \longrightarrow I^\infty_{P,\sigma\otimes\mu|_{P\cap K}}$$

be defined by

$$\delta(f\otimes v)(k) = f(k)\otimes\mu(k)v.$$

Let q be the natural projection of F onto $F/\mathfrak{n}F$. We define

$$T_\lambda\colon I^\infty_{P,\sigma} \otimes F \longrightarrow I^\infty_{P,\sigma\otimes\mu|_{P\cap K}}$$

by $T_\lambda(f\otimes v) = (I\otimes q)(\delta(f\otimes v))$. We note that T_λ depends only on F and σ and

(iv) $T_\lambda \circ (\pi_{P,\sigma,\nu}(g)\otimes\mu(g)) = \pi_{P,\sigma,\nu-\lambda}(g)T_\lambda$

(v) T_λ is surjective and continuous.

Set $\pi_\nu = {}_{P,\sigma,\nu}$. Suppose that we have proved the theorem for $\mathrm{Re}(\nu, \alpha) > q$ for $\alpha \in \Phi(P, A)$. Let $\lambda_1, \cdots, \lambda_p$ be a basis of H^*_σ. Set $\gamma_i(\nu) = \lambda_i \circ J_\nu$ for $\mathrm{Re}(\nu, \alpha) > q$. Let $\mu_1, \cdots, \mu_r$ be a basis of F^*. We apply the earlier observations and Theorem 3.4 to find $p^k_{ij} \in U(\mathfrak{g}_{\mathbf{C}})$ (depending on F and ψ but *not* on ν) such that

$$\sum_{i,j} (\pi'_\nu \otimes \mu')(p^k_{ij})(\gamma_i(\nu)\otimes\mu_j) = \xi_k(\nu)$$

$k=1, \cdots, pr$ defines a basis of $((I^\infty_{P,\sigma,\nu})'[\psi]\otimes F')_1$. Fix $\nu_o \in \mathfrak{a}^*_{\mathbf{C}}$ such that $\mathrm{Re}(\nu_o, \alpha) > q$ for $\alpha \in \Phi(P, A)$. Let $Z_\lambda = \mathrm{Ker}\, T_\lambda$. Then the above considerations imply that $\dim \sum \mathbf{C}\xi_k(\nu_o)|_{Z_\lambda} = pr - p = t$. By relabeling we may assume that $\xi_i(\nu_o)|_{Z_\lambda}$, $i=1, \cdots, t$ are linearly independent in an open neighborhood, U, of ν_o. Hence there exist holomorphic functions a_{ji}, $1 \leq j \leq t$, $1 \leq i \leq p$ on U such that

$$\xi_{t+j}(\nu)|_{Z_\lambda} = \sum_{j=1}^t a_{ji}(\nu)\xi_j(\nu).$$

Set $\varphi_i(\nu) = \xi_{t+j}(\nu) - \sum_{j=1}^t a_{ji}(\nu)\xi_j(\nu)$ for $\nu \in U$ and $1 \leq i \leq p$. Then $\varphi_i(\nu)|_{Z_\lambda} = 0$, and $\varphi_1(\nu), \cdots, \varphi_p(\nu)$ are linearly independent for $\nu \in U$. Thus $\{\varphi_i(\nu)\}_{i=1}^p$ is a basis of $\mathrm{Wh}_\eta((I^\infty_{P,\sigma,\nu-\lambda})')$ for $\nu \in U$.

We now use the notation of the previous section. Theorem 5.1 implies that if $\nu \in U$ then there exist $\beta_i(\nu) \in H^*_\sigma$ such that if $f \in U_{\sigma,\nu-\lambda}$ then

$$\varphi_i(\nu)(f)=\beta_i(\nu)\left(\int_N \eta(n)^{-1}f(w_o n)dn\right).$$

Let $v_i \in H_\sigma$ be such that $\mu_i(v_j)=\delta_{ij}$. Fix $u \in C_c^\infty(N)$ satisfying

$$\int_N \eta(n)^{-1}u(n)dn=1.$$

Set $f_{\nu,i}(pw_o n)=\sigma_\nu(p)u(n)v_i$ for $p \in P$, $n \in N$ and extend $f_{\nu,i}$ to G by 0. Then $f_{\nu,i} \in U_{\sigma,\nu}$ and $\nu \to f_{\nu,i}$ is holomorphic from $\mathfrak{a}_{\mathbf{C}}^*$ to $I_{P,\sigma}^\infty$. We note that

$$\varphi_i(\nu)(f_{\nu-\lambda,j})=\beta_i(\nu)(v_j).$$

Hence β_i: $U\to H_\sigma^*$ is holomorphic. Since $\beta_1(\nu), \cdots, \beta_r(\nu)$ are linearly independent for $\nu \in U$, we can find b_{ij} holomorphic on $\mathfrak{a}_{\mathbf{C}}^*$ such that if

$$\omega_i(\nu)=\sum b_{ij}(\nu)\varphi_i(\nu)$$

then

$$\omega_i(\nu)(f)=\mu_i\left(\left(\int_N \eta(n)^{-1}f(w_o n)\,dn\right)\right.$$

for $f \in U_{\sigma,\nu-\lambda}$. This implies that if $\mathrm{Re}\,(\nu-\lambda, \alpha)>q$ then $\omega_i(\nu)=\gamma_i(\nu-\lambda)$. This implies that if $q_1=\min_{\alpha\in\Phi(P,A)}(\lambda, \alpha)>0$ then γ_i has a holomorphic continuation to $\mathrm{Re}\,(\nu, \alpha)>q-q_1$, $\alpha \in \Phi(P, A)$. The Theorem now follows.

Corollary 7.3. *Let P and η be given in one of the following ways.*
(1) *P is minimal and η is generic.*
(2) *P and $d\eta$ are as in Proposition* 5.1.
Then the conclusion of Theorem 7.2 *is true.*

Proof. We have already observed that all of the hypotheses of the Theorem are satisfied in these cases.

§ 8. Applications

In this section we give several corollaries to Theorem 7.2 and Corollary 7.3. Let G be a real reductive group of inner type. We fix the notation as in Section 4. We assume that the Lie algebra of $\mathfrak{g}$ is as in the case of the BC_n examples of Section 4. Let P be the parabolic subgroup corresponding to the parabolic subalgebra given as in that section. Let $\mathfrak{g}_1=\bar{\mathfrak{u}}\oplus\mathfrak{m}\oplus\mathfrak{u}$. Then $(\mathfrak{g}_1, \mathfrak{a}_o)$ has a C_n-root system. Let w_o be as in Proposition 5.1 for this case.

Theorem 8.1. *Let $U=\exp\mathfrak{u}$ and let η be a unitary character of U such* *that $d\eta$ is non-degenerate. Let σ be a finite dimensional irreducible representation of ${}^oM$. If $f \in I^\infty_{P,\sigma}$ then set*

$$J_{P,\sigma,\nu}(f) = \int_U \eta(u)^{-1} f_{\sigma,\nu}(w_o u) du.$$

Then there exists $q_\sigma \geq 0$ such that the above integral converges uniformly in compacta of $\{\nu \in \mathfrak{a}^*_{\mathbf{C}} \mid \mathrm{Re}(\nu, \alpha) > q_\sigma, \alpha \in \Phi(m \oplus u, a_o)\}$. $J_{P,\sigma,\nu}$ has a weakly holomorphic extension to $\mathfrak{a}^*_{\mathbf{C}}$.

Proof. Let G^o_1 be the connected subgroup of G with Lie algebra $\mathfrak{g}_1$ MG_1. Set $K_1 = K \cap G_1$. If $f \in I^\infty_{P,\sigma}$ then $S(f) = f|_{K_1} \in I^\infty_{P_1,\sigma}$. $S(\pi_{P,\sigma,\nu}(g)f) = \pi_{P_1,\sigma,\nu}(g) S(f)$. The result therefore follows of Corollary 7.3.

We now give an application to a minimal parabolic subgroup. Let $\bar{P}_o = \theta P_o$, and $\bar{N}_o = \theta(N_o)$.

Theorem 8.2. *Let (σ, H_σ) be an irreducible representation of ${}^oM_o$ an arbitrary unitary character of $\bar{N}_o$. Define for $\nu \in (\mathfrak{a}_0)^*_{\mathbf{C}}$*

$$J_{\eta,\sigma,\nu}(f) = \int_{N_0} \eta(n)^{-1} f_{\sigma,\nu}(w_o n) dn.$$

Then the integral above converges absolutely and uniformly in $\{(\mathfrak{a}_o)^*_{\mathbf{C}} \mid$ $\mathrm{Re}(\nu, \alpha) > 0, \alpha \in \Phi(P_o, A_0)\}$. Furthermore, $J_{\eta,\sigma,\nu}$ has a meromorphic continuation to $(\mathfrak{a}_o)^*_{\mathbf{C}}$.

Note. The proof contains some information on the location of the singularities.

Proof. The proof of the first assertion is essentially the same as (1) in Proposition 7.1.

Let Δ be the set of simple roots of $\Phi(P_o, A_o)$. Let $F = \{\alpha \in \Delta \mid d\eta|_{\mathfrak{n}_\alpha} \neq 0\}$. Let (P_F, A_F) be the corresponding parabolic subgroup (cf. $M_F = P_F \cap \theta(P_F)$ and ${}^*P_F = P_o \cap M_F$, ${}^*N_F = N_o \cap M_F$, ${}^*\bar{N}_F = \theta({}^*N_F)$, $\mathfrak{a}_F = \mathfrak{a}_o \cap {}^o\mathfrak{m}_F$. Then $\mathfrak{a}_o = {}^*\mathfrak{a}_F \oplus \mathfrak{a}_F$. If $\nu \in (\mathfrak{a}_o)^*_{\mathbf{C}}$ then set ${}^*\nu_F = \nu|_{{}^*\mathfrak{a}_F}$. If $m \in M_F$ and if $f \in I^\infty_{P_o,\sigma}$ then put

$$\lambda_\nu(f)(m) = \int_{\bar{N}_F} f_{\sigma,\nu}(m\bar{n}) d\bar{n}.$$

This integral converges absolutely for $\mathrm{Re}(\nu, \alpha) > 0$, $\alpha \in \Phi(P_o, A_0)$. $\lambda_\nu(\pi_{P_o,\sigma,\nu}(m)f) = \pi_{*P_F,\sigma,\nu}(m)\lambda_\nu(f)$ for all $m \in M_F$.

(1) There is a holomorphic function γ on $(\mathfrak{a}_F)^*_{\mathbf{C}}$ such that if $f \in I^\infty_{P,\sigma}$ then

$$\nu \longrightarrow \gamma(\nu_F)\lambda_\nu(f)$$

extends to a holomorphic mapping of $(\mathfrak{a}_o)^*_{\mathbf{C}}$ into $I^\infty_{*P_F,\sigma}$.

This is a direct consequence of the main result on in [S]. Let s_o be the longest element relative to $*P_F$ of the relative to A_o. Let $w_o \in K \cap M_F$ be a representative of w_o. Put $\eta(w_o^{-1}nw_o)$ for $n \in N_F$. Then μ is a generic character of $*N_F$. If then (up to normalization of measures)

$$\int_{*\bar{N}_F} \eta(\bar{n})^{-1}\varphi_{\sigma,\nu}(\bar{n})d\bar{n}$$

$$=\int_{*N_F} \mu(n)^{-1}(\pi_\sigma(w_o^{-1})\varphi)_{\sigma,\nu}(w_o n)dn = J_{*P_F,\sigma,\nu}(\pi_\sigma(w_o^{-1})\varphi).$$

Here $\pi_\sigma(k)\varphi(x)=\varphi(kx)$, $k, x \in K \subset P_F$. Up to normalization of invariant measures, one has

$$J_{\eta,\sigma,\nu}(f)=J_{*P_F,\sigma,\nu}(\pi_\sigma(w_o^{-1})\lambda_\nu(f)).$$

The result now follows from Corollary 7.3.

Appendix. The proof of Lemma 2.1

The proof uses a result on the group algebra of the symmetric group which we do first. Let S_n denote the symmetric group on n letters. Let σ_i denote the cycle $(12\cdots i) \in S_n$ $(\sigma_1=1)$. We use the notation $\mathbf{C}[S_n]$ for the group algebra of S_n over $\mathbf{C}$. Set $\gamma=\sum_{i=1}^n \sigma_i \in \mathbf{C}[S_n]$.

Proposition A.1. $(\gamma-n)\prod_{i=1}^{n-2}(\gamma-j)=0$ *in* $\mathbf{C}[S_n]$.
The proof of this assertion uses several intermediate results.

Lemma A.2. *If* $u \in S_n$ *and if* $u=\sigma_{i_r}\cdots\sigma_{i_1}=\sigma_{j_r}\cdots\sigma_{j_1}$ *with* $i_p \geq p$ *and* $j_p \geq p$, $p=1, \cdots, t$ *then* $i_p=j_p$ *for all* $p=1, \cdots, r$.

Proof. By induction on r. If $r=1$ then the result is clear. Our assumptions on the i_p and the j_p imply that $ui_1=uj_1=r$. We prove this by induction on r. If $r=1$ and if $j \geq 1$ then $\sigma_j j=1$. Assume for $r=k-1$. Then $\sigma_{i_{k-1}}\cdots\sigma_{i_1}i_1=k-1$. Since $i_k \geq k$, $\sigma_{i_k}(k-1)=k$. Thus $i_1=j_1$. Hence $\sigma_{i_r}\cdots\sigma_{i_2}=\sigma_{j_r}\cdots\sigma_{j_2}$ so $i_k=j_k$ for $k=2, \cdots, r$.

Corollary A.3. *If* $u \in S_n$ *then* u *can be written uniquely in the form*

$$u=\sigma_{i_{n-1}}\cdots\sigma_{i_1}$$

with $i_j \geq j$.

Proof. The uniqueness follows from Lemma A.1. The number of elements of the form $\sigma_{i_{n-1}}\cdots\sigma_{i_1}$ with $i_j \geq j$ is $n!$ by Lemma A.1. Thus every element of S_n has such an expression.

Corollary A.4. *Set* $\mu=\Sigma_{w\in S_n}w$ *and* $\gamma_k=\Sigma_{j\leq k}\sigma_j$. *Then*

$$(\gamma-\gamma_{n-2})\cdots(\gamma-\gamma_2)(\gamma-1)\gamma=\mu.$$

Lemma A.5. *Let* $u=\sigma_{i_r}\cdots\sigma_{i_1}$, $i_j\geq j$. *If* $t\leq r$ *then*

$$\sigma_t u=\sigma_{j_r}\cdots\sigma_{j_1},\quad j_p\geq p,\quad p=1,\cdots,s.$$

Proof. The following formulas are easily checked by direct calculation

(i) $\sigma_i\sigma_j=\sigma_j^2\sigma_{j-1}^{-1}\sigma_{i-1}$ for $2\leq i\leq j$.

(ii) $\sigma_2\sigma_r\sigma_j=\sigma_{j+1}\sigma_r$ for $j=1,\cdots,r-1$.

We now prove the Lemma in the case when $t=2$. Suppose that $i_r>i_{r-1}$ then (ii) implies that $\sigma_2\sigma_{i_r}\sigma_{i_{r-1}}=\sigma_{i_{r-1}+1}\sigma_{i_r}$. So

$$\sigma_2 u=\sigma_{i_{r-1}+1}\sigma_{i_r}\sigma_{i_{r-2}}\cdots\sigma_{i_1}$$

which implies the Lemma in this case.

If $i_r=i_{r-1}=p$ then we note that (ii) implies

(iii) $\sigma_2\sigma_p^2=\sigma_p\sigma_{p-1}$.

Hence $\sigma_2 u=\sigma_p\sigma_{p-1}\sigma_{i_{r-2}}\cdots\sigma_{i_1}$.

Suppose that $i_r<i_{r-1}$ then (i) implies that

$$\sigma_{i_r}\sigma_{i_{r-1}}=\sigma_{i_{r-1}}^2\sigma_{i_{r-1}-1}^{-1}\sigma_{i_r-1}.$$

So (iii) implies that

$$\begin{aligned}\sigma_2\sigma_{i_r}\sigma_{i_{r-1}}\cdots\sigma_{i_1}&=\sigma_2\sigma_{i_{r-1}}^2\sigma_{i_{r-1}-1}^{-1}\sigma_{i_r-1}\sigma_{i_{r-2}}\cdots\sigma_{i_1}\\&=\sigma_{i_{r-1}}\sigma_{i_{r-1}-1}\sigma_{i_{r-1}-1}^{-1}\sigma_{i_r-1}\sigma_{i_{r-2}}\cdots\sigma_{i_1}=\sigma_{i_{r-1}}\sigma_{i_r-1}\sigma_{i_{r-1}}\cdots\sigma_{i_1}.\end{aligned}$$

This completes the proof in the case when $t=2$.

Assume the result for $t-1\geq 2$ we now prove it for t. We note that $t\leq r\leq i_r$. Suppose that $t<i_r$. Then (i) implies that

$$\sigma_t u=\sigma_{i_r}^2\sigma_{i_r-1}^{-1}\sigma_{t-1}\sigma_{i_{r-1}}\cdots\sigma_{i_1}.$$

The inductive hypothesis implies that this is equal to

$$\sigma_{i_r}^2\sigma_{i_r-1}^{-1}\sigma_{j_{r-1}}\cdots\sigma_{j_1}\quad\text{with}\quad j_p\geq p.$$

To this we apply (i) with $j=2$ and get

$$\sigma_t u=\sigma_2\sigma_{i_r}\sigma_{j_{r-1}}\cdots\sigma_{j_1}.$$

In this case the inductive step now follows from the case $t=2$.

Suppose now that $i_r=t$. Then $t=i_r=r$. So

$$\sigma_t u = \sigma_r^2 \sigma_{i_{r-1}} \cdots \sigma_{i_1} = \sigma_2 \sigma_r \sigma_{r-1} \sigma_{i_{r-1}} \cdots \sigma_{i_1}$$

by (ii). The inductive hypothesis now implies that

$$\sigma_t u = \sigma_2 \sigma_r \sigma_{j_{r-1}} \cdots \sigma_{j_i} \quad \text{with} \quad j_k \geq k.$$

Thus the inductive step now follows from the case $t=2$.

We are now ready to prove the proposition. Lemma A.5 implies that

$$\gamma_j \prod_{r=0}^{j-1} (\gamma - \gamma_r) = j \prod_{r=0}^{j-1} (\gamma - \gamma_r).$$

Suppose that we have shown that $\prod_{r=0}^{j-1}(\gamma-\gamma_r) = \prod_{r=0}^{j-1}(\gamma - r)$ then the above formula implies $\gamma \prod_{r=0}^{j-1}(\gamma-\gamma_r) = (\gamma-\gamma_j)\prod_{r=0}^{j-1}(\gamma-\gamma_r) + j\prod_{r=0}^{j-1}(\gamma-\gamma_r)$ Since this equation is obvious for $j=2$, Corollary A.4 implies that

$$\prod_{j=0}^{n-2} (\gamma - j) = \mu.$$

The result now follows since $\gamma\mu = n\mu$.

Let now $\nu_i = (i \cdots n)$ for $i = 1, \cdots, n$ and $\tau = \Sigma_i \nu_i$.

Corollary A.6. $(\tau - n) \prod_{j=0}^{n-2} (\tau - j) = 0$.

Proof. Let s_o be the element of S_n such that $s_o i = n+1-i$. Then $\nu_i = s_o \sigma_i^{-1} s_o$. If we set $\psi(\Sigma_{\sigma \in S_n} a_\sigma \sigma) = \Sigma_\sigma a_\sigma s_o \sigma^{-1} s_o$ then ψ defines an anti-automorphism of $\mathbf{C}[S_n]$ ($\psi(xy) = \psi(y)\psi(x)$). Since $\tau = \psi(\gamma)$, the Corollary follows from Proposition A.1.

We are now ready to prove Lemma 2.1. We return to the pertinent notation in Section 2. Fix $j \geq 1$. Define $V_{p,r}$ to be the space of all $v \in V$ satisfying the following two conditions

(1) $(Y_1 - \psi(Y_1)) \cdots (Y_{p+1} - \psi(Y_{p+1}))v = 0$ if $Y_i \in \Sigma_{i \geq j} \mathfrak{u}_{2i}$.

(2) If $Y_i \in \mathfrak{u}_{2r_i}$, $r_i \geq j$ then $(Y_1 - \psi(Y_1)) \cdots (Y_p - \psi(Y_p))v \in V_{r - \Sigma_i(r_i - k)}$.

Here, $V_q = 0$ for $q \leq 0$. Notice that $V_{p,r} \subset V_{p,r+1}$, $V_{p,0} \subset V_{p+1,0}$, $\bigcup_p V_{p,0} = V$ and $V_{p+1,0} = \bigcup_r V_{p,r}$.

We first observe $(Q = Q_j)$

(I) If we can show that $(Q+I)^2 \prod_{i=0}^{p} (Q+iI)^p V_{p,r} \subset V_{p,r-1}$ then Lemma 2.1 follows.

Indeed, set $f_{p,r}(T) = (T+1)^2 \prod_{i=0}^{p} (T+iI)^p$. We note that $V_r \subset V_{r,0}$ and that $V_{s,0} \cap V_r \subset \bigcup_{q \leq r} V_{s,q}$. Thus, $\prod_{q \leq r} f_{r-1,q}(Q) V_r \subset V_r \cap V_{r-1,0}$. Hence $\prod_{q \leq r} f_{r-2,q}(Q) f_{r-1,q}(Q) V_r \subset V_r \cap V_{r-2,0}$, etc.. Since $V_{0,0} = (0)$, (I) implies the Lemma.

We are left with the proof of (I). If $x \in \mathfrak{u}$ then set $x' = x - \psi(x)$. Let $v \in V_{p,r}$ and let $Y_i \in \mathfrak{u}_{2r_i}$ with $r_i \geq j$, $i = 1, \cdots, p$. Then $QY_1' \cdots Y_p' v = 0$. Hence ($\dim \mathfrak{u}_{2j} = m$)

$$
\begin{aligned}
Y_1'\cdots Y_p'Qv &= \sum_{i=1}^{p} Y_1'\cdots Y_{i-1}'[Y_i, Q]Y_{i+1}'\cdots Y_p'v \\
&= \sum_{i=1}^{p}\sum_{k=1}^{m} Y_1'\cdots[Y_i, Z_k]X_k'Y_{i+1}'\cdots Y_p'v \\
&\quad+\sum_{i=1}^{p}\sum_{k=1}^{m} Y_i'\cdots Z_k[Y_i, X_k]Y_{i+1}'\cdots Y_p'v \\
&= \sum_{i=1}^{p}\sum_{k=1}^{m} Y_1'\cdots[Y_i, Z_k]X_k'Y_{i+1}'\cdots Y_p'v \\
&\quad+\sum_{i=1}^{p}\sum_{k=1}^{m} Z_kY_1'\cdots[Y_i, X_k]Y_{i+1}'\cdots Y_p'v \\
&\quad+\sum_{i=1}^{p}\sum_{k=1}^{m}\sum_{u=1}^{i-1} Y_1'\cdots[Y_u,Z_k]Y_{u+1}'\cdots Y_{i-1}'[Y_i, X_k]Y_{i+1}'\cdots Y_p'v.
\end{aligned}
$$

If $r_i>j$ then $[Y_i, Z_k]\in \mathfrak{u}_{2+2(r_i-j)}$ so

$$
\begin{aligned}
&Y_1'\cdots[Y_i, Z_k]X_k'Y_{i+1}'\cdots Y_p'v \\
&\quad=[Y_i, Z_k]Y_1'\cdots X_k'Y_{i+1}'\cdots Y_p'v \\
&\quad\quad+\sum_{u=1}^{i-1} Y_1'\cdots[Y_u, [Y_i, Z_k]]\cdots Y_{i-1}'X_k'Y_{i+1}'\cdots Y_p'v
\end{aligned}
$$

which is in $V_{r-\Sigma_t(r_t-j)-1}$. Also $Y_1'\cdots[Y_i, X_k]Y_{i+1}'\cdots Y_p'v=0$. Furthermore (as above), if $r_u>j$ then

$$Y_1'\cdots[Y_u, [Y_i, Z_k]]\cdots Y_{i-1}'X_k'Y_{i+1}'\cdots Y_p'v \in V_{r-\Sigma_t(r_t-j)-1}.$$

Thus

$$
\begin{aligned}
Y_1'\cdots Y_p'Qv &\equiv \sum_{r_i=j}\sum_{k=1}^{m} Y_1'\cdots[Y_i, Z_k]X_k'Y_{i+1}'\cdots Y_p'v \\
&\quad+\sum_{i=1}^{p}\sum_{k=1}^{m}\sum_{\substack{u\le i-1\\ r_u=k}} Y_1'\cdots[Y_u, Z_k]Y_{u+1}'\cdots Y_{i-1}'[Y_i, X_k]Y_{i+1}'\cdots Y_p'v \\
&\qquad\qquad \operatorname{mod} V_{r-\Sigma_t(r_t-j)-1}.
\end{aligned}
$$

If $r_i=j$ then $[Y_i, Z_k]=-(Y_i, X_k)X_o+u_{i,k}$ with $u_{i,k}\in\mathfrak{u}_2$ and $\psi(u_{i,k})=0$. As above,

$$Y_1'\cdots(u_{i,k}-(Y_i, X_k)(X_o-1))X_k'Y_{i+1}'\cdots Y_p'v\in V_{r+\Sigma_t(r_t-j)-1}$$

as is

$$Y_1'\cdots(u_{i,k}-(Y_i, X_k)(X_o-1))\cdots Y_{i-1}'[Y_i, X_k]Y_{i+1}'\cdots Y_p'v.$$

We therefore have shown that if $s=|\{i\,|\,r_i=j\}|$ then

$$\text{(II)}\quad Y_1'\cdots Y_p'Qv \equiv -sY_1'\cdots Y_p'v$$
$$-\sum_{i=1}^{p}\sum_{\substack{u\leq i-1\\ r_u=k}} Y_1'\cdots Y_{u-1}'Y_{u+1}'\cdots Y_{i-1}'[Y_i, Y_u]Y_{i+1}'\cdots Y_p'v$$
$$\mod V_{r-\Sigma_t(r_t-j)-1}.$$

If $s=1$ and if $r_u=j$ then this says that $Y_1'\cdots Y_p'Qv = -Y_1'\cdots Y_{u-1}'Y_{u+1}'\cdots Y_p'Y_u'v$. Thus $Y_1'\cdots Y_p'Q^2v = -Y_1'\cdots Y_p'Qv$. Hence $Y_1'\cdots Y_p'(Q+1)Qv=0$. We thus assume that $s\geq 2$. If we write $[Y_i, Y_u] = Y_i'Y_u' - Y_u'Y_i'$ we see that (II) "telescopes" to

$$\text{(III)}\quad Y_1'\cdots Y_p'Qv \equiv -Y_1'\cdots Y_p'v$$
$$-\sum_{\substack{r_u=j\\ u<p}} Y_1'\cdots Y_{u-1}'Y_{u+1}'\cdots Y_p'Y_u'v \mod V_{r-\Sigma_t(r_t-j)-1}.$$

(III) implies that, in particular, $Qu \in V_{p,r}$. This implies that

$$Y_1'\cdots Y_p'(Q+I)v$$
$$\equiv -\sum_{\substack{r_u=j\\ u<p}} Y_1'\cdots Y_{u-1}'Y_{u+1}'\cdots Y_p'Y_u'v \mod v_{r-\Sigma_t(r_t-j)-1}.$$

Thus $Y_1'\cdots Y_p'(Q+I)^2v$ is a sum of terms of the form (up to sign) $Z_1'\cdots Z_{p-m}'L_1'\cdots L_m'v$ with $m\geq 2$, the Z_i, L_i a relabeling of the Y_i and $L_i \in \mathfrak{u}_{2j}$. (III) now implies that

$$Z_1'\cdots Z_{p-m}'L_1'\cdots L_m'Qv \equiv -\Sigma_{i=1}^{m} Z_1'\cdots Z_{p-m}'L_{\nu_i 1}'\cdots L_{\nu_i m}'v$$
$$\text{modulo } V_{r-\Sigma_t(r_t-j)-1}$$

and terms of the same form with larger m. Thus Corollary A.6 implies that

$$Z_1'\cdots Z_{p-m}'L_1'\cdots L_m'(Q+mI)\prod_{i=0}^{m-2}(Q+iI)v$$

is congruent modulo $V_{r-\Sigma_t(r_t-j)-1}$ to a sum of terms of the "ZL" form with larger m. If we apply this argument at most $p-1$ times we find that

$$Y_1'\cdots Y_p'(Q+I)^2(Q+pI)^{p-1}\prod_{i=0}^{p-2}(Q+iI)^{p-1}v$$

is congruent modulo $V_{r-\Sigma_t(r_t-j)-1}$ to terms of the form

$$Z_i'Z_2'\cdots Z_{p-s}'L_1'\cdots L_s'v.$$

With $Z_i \in \mathfrak{u}_{2q_i}$, $q_i>j$ for $i=1, \cdots, p-s$ and $L_i \in \mathfrak{u}_{2j}$ for $i=1, \cdots, s$. The Z_i and the L_i constituting (up to sign) a rearrangement of the Y_i. If we now apply (III) again we find that

$$Z_1'Z_2'\cdots Z_{p-s}'L_1'\cdots L_s'Qv$$
$$\equiv -\sum_{i=1}^{s} Z_1'Z_2'\cdots Z_{p-s}'L_{\nu_i 1}'\cdots L_{\nu_i s}'v \bmod V_{r-\Sigma(r_t-j)-1}.$$

We apply Corollary A.6 again to find that

$$Y_1'\cdots Y_p'(Q+I)^2\prod_{u=0}^{s}(Q+uI)^s v \in V_{r-\Sigma(r_t-j)-1}.$$

This implies that $(Q+I)^2\prod_{u=0}^{s}(Q+uI)^s V_{p,r} \in V_{p,r-1}$. This is the assertion in (I). So we are done.

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