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Multiplicity One Theorems for Generalized Gelfand-Graev Representations of Semisimple Lie Groups and Whittaker Models for the Discrete Series

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Contents

§ 0. Introduction

The idea of induced representation goes way back to G. Frobenius and I. Schur, and it has been always playing central and indispensable

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roles in the theory of group representations. This idea contributes greatly to construct irreducible representations. For example, every irreducible unitary representation of a simply connected nilpotent Lie group is obtained as an induced representation, and the Kirillov orbit method [16] tells us how to construct it explicitly using polarization. Further, for semisimple Lie groups, Langlands' classification of irreducible admissible representations relies largely (although not completely) upon the induction from parabolic subgroups (see e.g., [17, Chap. VIII]).

On the other hand, *generalized Gelfand-Graev representations* (abbreviated as GGGRs for short) we treat in this article are very interesting induced representations, though far from being irreducible. These GGGRs were constructed by Kawanaka ([13], [14]) for reductive algebraic groups over a finite field, through the Dynkin-Kostant theory on nilpotent classes. They form a series of representations (induced from certain kinds of unipotent subgroups), parametrized by nilpotent orbits in the corresponding Lie algebras. Among GGGRs, the (original) Gelfand-Graev representations (= GGRs; cf. [29], see also Remark 1.2) correspond to regular nilpotent classes.

Kawanaka's results on GGGRs, enumerated below, show the importance of these representations.

(1) The characters of GGGRs form a $\boldsymbol{Z}$-basis of the space of unipotently supported virtual characters [15, 2.4.3].

(2) These characters can be utilized to determine explicitly the values of irreducible characters at the unipotent elements [14].

(3) Multiplicities in GGGRs give very nice informations through which one could classify irreducibles intrinsically. Among irreducible constituents of GGGRs, those occurring with "finite" multiplicity are especially important toward this direction. Here, multiplicity is understood to be "finite" if it is independent of the cardinal number q of a finite field (when q is considered as a variable). See [15, 2.4.1 and 2.5.2] for more detail.

As pointed out in [13, 1.3.5], GGGRs can be constructed analogously for reductive algebraic groups over an archimedean or non-archimedean local field. So we define GGGRs of semisimple *Lie* groups (see Section 1 for the precise definition), and study them detailedly. By the above (3), multiplicities in GGGRs are of particular interest.

From this point of view, the original GGRs have a very typical character: GGRs of quasi-split, linear semisimple Lie groups are of multiplicity free (Shalika [29], see also [35, Th. 4.5]). Then it is quite natural to ask if GGGRs as well as non-generalized GGRs have multiplicity free property or not. Here is a more direct motivation of Part I of this paper, and also of our earlier works [35] and [36] on finite multiplicity property. But, as shown in [34, 4.1], the GGGRs are far from being of multiplicity finite except the extreme case, case of GGRs. So, in order to recover the multiplicity free (or finite multiplicity) property, we need to consider *reduced* GGGRs, variants of GGGRs (see 1.2).

In this direction, we have shown in [36] that finite multiplicity property is actually regained for some important types of reduced GGGRs. We now recall the construction of such (reduced) GGGRs and explain our finite multiplicity theorems. (For all of the facts mentioned below, consult [36] or its short summary given in Section 5 and 9.3 of this paper.)

Let G be a connected simple Lie group with finite center, and $\mathfrak{g}$ its Lie algebra. Denote by K a maximal compact subgroup of G. Assume that G/K carries a structure of hermitian symmetric space. We put $l=\mathrm{rank}(G/K)$. Let $G=KA_pN_m$ be an Iwasawa decomposition of G. Then, there exists an at most two step nilpotent Lie subgroup $N\subseteqq N_m$, canonically diffeomorphic to the Šilov boundary of Siegel domain which realizes G/K. Let P be the normalizer of N in G. Then P is a maximal parabolic subgroup of G containing N as its unipotent radical, and so P admits a Levi decomposition $P=LN=L\ltimes N$. The Levi subgroup L acts on $\mathfrak{n}\equiv\mathrm{Lie}\,N$ through the adjoint action, whence it acts also on the center $\mathfrak{z}_\mathfrak{n}$ of $\mathfrak{n}$. Under this action, $\mathfrak{z}_\mathfrak{n}$ has exactly $(l+1)$-number of open orbits ω_i $(0\leqq i\leqq l)$. Putting $\omega_i=\mathrm{Ad}(G)\tilde{\omega}_i$, we thus get nilpotent $\mathrm{Ad}(G)$-orbits $\tilde{\omega}_i\subseteqq\mathfrak{g}$, $0\leqq i\leqq l$, which are all contained in the same nilpotent class o of $\mathfrak{g}_C\equiv\mathfrak{g}\otimes_R C$. Further, we obtain $o\cap\mathfrak{g}=\coprod_{0\leqq i\leqq l}\omega_i$ (disjoint union).

The GGGR Γ_i associated with ω_i is an induced representation $\Gamma_i=\mathrm{Ind}_N^G(\rho_i)$, where ρ_i is an irreducible unitary representation of N on a Fock space of $\mathfrak{n}/\mathfrak{z}_\mathfrak{n}$ with respect to a certain complex structure J_i''. (If G/K is of tube type, then $\mathfrak{n}$ is abelian. So, ρ_i is a one-dimensional representation on the Fock space C of $\mathfrak{n}/\mathfrak{z}_\mathfrak{n}=(0)$.) Throughout, we will mean by Ind either C^∞-induction ($=C^\infty$-Ind) or unitary induction ($=L^2$-Ind). Unitarily induced GGGRs L^2-Γ_i have the following important property:

$$\lambda_G\simeq\bigoplus_{0\leqq i\leqq l}[\infty]\cdot L^2\text{-}\Gamma_i, \tag{0.1}$$

where λ_G denotes the regular representation of G on $L^2(G)$, and $[\infty]\cdot\pi$ means the infinite multiple of a representation π. In particular, the orthogonal direct sum $\bigoplus_i L^2$-Γ_i is quasi-equivalent to λ_G.

The reduced GGGRs coming from Γ_i are defined in accordance with the Mackey machine for representations of semidirect product groups, as follows. Since L acts on N, it acts also on the unitary dual $\hat{N}$ of N in the canonical way. Let H^i denote the stabilizer of the unitary equivalence class $[\rho_i]\in\hat{N}$ of ρ_i in L. We can see that every ρ_i is extended canonically to an actual (not just projective) unitary representation $\tilde{\rho}_i$ of the semidirect product subgroup $H^iN = H^i \ltimes N \subseteqq P$ acting on the same Hilbert space. For an irreducible (unitary, in case of L^2-Ind) representation c of H^i, the induced representation

$$(0.2) \qquad \Gamma_i(c) \equiv \mathrm{Ind}_{H^iN}^G(\tilde{c} \otimes \tilde{\rho}_i) \qquad \text{with } \tilde{c} \equiv c \otimes 1_N,$$

is said to be a reduced GGGR coming from Γ_i. By construction, unitary GGGR is decomposed into a direct integral of the corresponding reduced GGGRs.

It should be remarked that, for any $0 \leqq i \leqq l$, (L, H^i) has a structure of reductive symmetric pair (at least on Lie algebra level) associated with a signature of the restricted root system of $\mathfrak{l} \equiv \mathrm{Lie}\, L$. Moreover, among (L, H^i)'s there exist precisely two cases: (say) $i=0$ and $i=l$, for which L/H^i is riemannian, or equivalently H^i is a maximal compact subgroup of L.

Our principal results in [36] say that a fairly large number of reduced GGGRs $\Gamma_i(c)$ have finite multiplicity property. Especially, in case $i=0$ or l, the corresponding reduced GGGRs are all of multiplicity finite.

In the present paper, we continue the study of above types of (reduced) GGGRs. The contents are divided into two parts: Part I and Part II. We prove in Part I that, when $i=0$ or l, reduced GGGRs $\Gamma_i(c)$ have further multiplicity free property under some additional (but reasonable) assumptions on (G, c). Part II is devoted to describing embeddings of certain irreducible representations of G into (reduced) GGGRs. Such an embedding is called *Whittaker model.*

Now let us explain in more detail the contents of each part respectively.

0.1. Multiplicity one theorems for reduced GGGRs.

The beginning four sections proceed in more general setting. First of all, we shall define GGGRs and reduced GGGRs of semisimple Lie groups in full generality (Section 1).

After that, we give, following the formulation of Benoist [2] and Kostant [18, Section 6], nice sufficient conditions for a given monomial induced representation (C^∞- or L^2-) $\mathrm{Ind}_Q^G(\zeta)$ of a Lie group G to have multiplicity free property (Section 2, see e.g., Theorem 2.12), where ζ is a unitary character of a closed subgroup Q of G. These criterions require the existence of an automorphism σ of G with the following property: the anti-automorphism $\sigma^\vee$ of G defined by $\sigma^\vee(g) = \sigma(g^{-1})$ ($g \in G$), induces the trivial action on the space of $(\zeta, \bar{\zeta})$-quasi-invariant (eigen) distributions on G (see Definition 2.5) in the canonical way. In case of L^2-induction, the existence of such a σ implies that the commuting algebra $\mathrm{End}_G(L^2\text{-}\mathrm{Ind}_Q^G(\zeta))$ is abelian, or equivalently $L^2\text{-}\mathrm{Ind}_Q^G(\zeta)$ is of multiplicity free.

Benoist [2] further gave a sufficient condition for a given automorphism of G to have the above property, in connection with a certain geometric structure on G (cf. Proposition 2.13). There, he bears in mind the case of quasi-regular representation $\mathrm{Ind}_Q^G(1_Q)$ associated with a symmetric space G/Q, with the involution σ of G canonically attached to G/Q. So, the criterion of Benoist works very well for such a case ([2], [19]; cf. [6]). However, his result can not be applied directly to the case of reduced GGGRs $\Gamma_i(c)$ which lies outside the scope of [2]. For our multiplicity one theorems, we need to investigate matters more detailedly. This is the theme of Sections 3 and 4.

Now let G be a connected semisimple Lie group with finite center. In Section 3, we introduce spaces of Whittaker distribution on G which realize G-modules induced (in the category of distributions) from one-dimensional representations of unipotent subgroups. By differentiating the group action, Whittaker distributions on any open subset of G can be defined analogously.

Section 4 is devoted to examining the supports of Whittaker distributions in connection with various kinds of Bruhat decompositions of G. More precisely, generalizing the technique of [29] and [34], we estimate supports of quasi-elementary Whittaker distributions on certain open dense subsets $C_s \subseteqq G$ with supports contained in Bruhat cells $G_s \subseteqq C_s$ which are closed in C_s and of lower dimension in general. (See 3.2 for the precise definition of C_s and G_s.) Here, an eigendistribution of the Casimir operator L_Ω is called quasi-elementary. To get the estimation, we proceed as follows. Let T be such a Whittaker distribution and take $z \in G_s$ from its support. In a sufficiently small neighbourhood of z, T can be expressed uniquely as a finite linear combination of transversal derivatives of distributions on G_s. Calculating $L_\Omega T$, we get the same kind of expression of $L_\Omega T$. Since T is quasi-elementary, these two expressions coincide up to scalar multiples. This produces our estimation for the support of T (Theorem 4.2).

Returning to our original objects, we apply in 6.1 this estimation to the case of GGGRs Γ_i. Then we can show that, if $i=0$ or l, any quasi-elementary Whittaker distribution on the whole G corresponding to Γ_i is uniquely determined by its restriction to an open Bruhat cell (Theorem 6.5). This means the non-existence of such type of distributions with "singular supports". This is somewhat similar to the following well-known fact due to Harish-Chandra (cf. [10, Th. 4]): there do not exist nilpotently supported non-zero invariant eigendistributions on semisimple Lie groups. But, if $i \neq 0$, l, then there may exist non-zero quasi-elementary

Whittaker distributions with singular supports.

Combining the above non-existence theorem with criterions for multiplicity free property in Section 2, we can show that our reduced GGGRs (C^∞- or L^2-) $\Gamma_i(c)$ with $i=0$ or l, have multiplicity free property under the assumptions: (i) G/K is of tube type (or N is abelian), and (ii) c is a real valued character of the (compact) stabilizer H^i. (See Theorems 6.9 and 6.10). These are our main results of Part I.

Our results generalize, in a certain sense, the uniqueness theorem of generalized Bessel models for the symplectic group of rank 2 over a non-archimedian local field (Novodvorskii and Piatetskii-Shapiro; [24], [25]).

0.2. Whittaker models for the discrete series.

By virtue of (0.1), any irreducible constituent of the regular representation of G appears in our GGGR L^2-Γ_i for some i, $0 \leq i \leq l$. So in particular, each member of the discrete series of G can be embedded into some L^2-Γ_i. Thus there naturally arises an interesting problem: Describe Whittaker models ($=$ embeddings) for the discrete series into GGGRs Γ_i.

In Part II, we settle this problem for the holomorphic discrete series by the *method of highest weight vectors* (explained below). This method enables us to obtain a nice description of Whittaker models, more generally for any irreducible highest weight representation of G. We deal with two types of Whittaker models. One is embeddings into C^∞-induced GGGR (C^∞-Whittaker model), the other is into unitarily induced GGGR (L^2-Whittaker model).

Since a highest weight module is characterized by its highest weight vector, one may obtain a fairly good description of Whittaker models by determining highest weight vectors in GGGRs. Such a vector must be K-finite, for we are concerned with admissible representations of G.

Suggested by this idea, Hashizume [11] treated a subject similar to the above C^∞-Whittaker model. But there exist some mistakes in that paper. (For example, he asserts uniqueness of Whittaker models, which is, however, not always true. See 8.1 and Remark 10.2). Nevertheless, his method of highest weight vectors is, we feel, very nice and gives the most direct and elementary way to describe Whittaker models. Furthermore, this method is applicable also for embeddings into other types of representations. For instance, we can describe completely embeddings of irreducible highest weight respresentations of G into the (non-unitary) principal series induced from a minimal parabolic subgroup of G (although such a description has been obtained by other methods, cf. [5], [32]). In this case, embedding into the principal series is unique up to scalar multiples (Theorem 8.1). Furthermore, we can specify the parameter of the principal series into which a given highest weight module can be embedded.

Our method of highest weight vectors proceeds, in the present case, as follows.

First, we determine explicitly all the K-finite highest weight vectors in C^∞-induced GGGRs by solving a system of differential equations on G characterizing such vectors (Theorem 10.6). Second, among these highest weight vectors, those contained in the representation spaces of L^2-induced GGGRs are specified through evaluation of L^2-norm (Theorem 11.3). In the latter step, we utilize the classical technique of Harish-Chandra [9, VI], which was used to study the non-vanishing condition for the holomorphic discrete series. These two results enable us to produce descriptions of C^∞- and L^2-Whittaker models: Theorems 12.6 and 12.10, respectively. Furthermore, we can describe Whittaker models in reduced GGGRs (Theorem 12.13). These three theorems are our principal results of Part II. Our results are complete for the (limit of) holomorphic discrete series.

It should be remarked that irreducible highest weight representations occur in GGGRs with at most finite multiplicity (although non-reduced GGGRs Γ_i are not of multiplicity finite). So, such a Whittaker model is important from the viewpoint of Kawanaka (3) (see supra).

Our result on L^2-Whittaker model (Theorem 12.9) has a certain connection with Rossi-Vergne's result [28, Cor. 5.23] that describes the restriction of the holomorphic discrete series to an Iwasawa subgroup $S \equiv A_p N_m \subsetneqq G$. Precisely, one of them can be derived from the other. However, to get our Theorem 12.10 from the result in [28], one needs many things: for instance, (a) Anh reciprocity [1] (cf. [27]), (b) detailed informations for representations of the solvable Lie group S ([4], [7], [23]), and so on. (See 12.5 for more detail.) For this reason, we wanted to take a short cut and give a more direct and more elementary proof of Theorem 12.10. Our method of highest weight vectors realizes this hope satisfactorily. Moreover, our proof is independent not only of their result but also of the technique in [28].

Our result in the present paper, as well as those in the earlier article [36], would be useful to get irreducible decomposition of unitary GGGRs L^2-Γ_i explicitly, which we hope to treat in near future.

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Notation. Let G be a Lie group, and H a closed subgroup of G. For a continuous representation ζ of H acting on a Fréchet space, C^∞-$\mathrm{Ind}_H^G(\zeta)$ denotes the smooth representation of G induced from ζ in C^∞-context, defined in [35, 2.1]. If ζ is a unitary representation of H, we consider the unitarily induced representation also, which will be denoted by L^2-$\mathrm{Ind}_H^G(\zeta)$ (see e.g., [35, 3.6]). Throughout this paper, we will deal with these two types of induced representations. The notation Ind (without C^∞- or L^2-) will be used to mean any one of C^∞-Ind or L^2-Ind.

Part I. Multiplicity one theorems for generalized Gelfand-Graev representations

§ 1. Generalized Gelfand-Graev representations (GGGRs)

To begin with, we redefine here Kawanaka's generalized Gelfand-Graev representations for semisimple Lie groups, after our previous paper [36] (referred as [II] later on). These representations are main subjects of the present article. We employ the abbreviation GGGR to denote such a representation.

1.1. Definition of GGGRs ([13], [14], [II]).

Let G be a connected semisimple Lie group with finite center, and $\mathfrak{g}$ the Lie algebra of G. $G=KA_pN_m$ denotes an Iwasawa decomposition of G, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}_p\oplus\mathfrak{n}_m$ the corresponding decomposition of $\mathfrak{g}$. We take a Cartan involution θ of G compatible with these decompositions: $K=\{g\in G; \theta(g)=g\}$ and $\theta(a)=a^{-1}$ for all $a\in A_p$. Let Λ be the root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Choose a positive system Λ^+ of Λ such that $\mathfrak{n}_m=\sum_{\lambda\in\Lambda^+}\mathfrak{g}(\mathfrak{a}_p; \lambda)$, where $\mathfrak{g}(\mathfrak{a}_p; \lambda)$ is the root space of a root λ.

Let ω be a non-trivial nilpotent $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$. Using the Dynkin-Kostant theory on nilpotent classes and the Kirillov orbit method for nilpotent Lie groups, we associate to ω an induced representation Γ_ω of G, called GGGR, in the following way. (For more detail, one should consult [II, § 1].) For each nilpotent class $\omega\neq(0)$, there exists a unique element $H\in\mathfrak{a}_p$ dominant with respect to Λ^+, such that H is the semisimple element of an $\mathfrak{sl}_2$-triplet (X, H, Y) in $\mathfrak{g}$ containing an $X\in\omega$:

$$(1.1)\qquad [H, X]=2X, \qquad [H, Y]=-2Y, \qquad [X, Y]=H.$$

Moreover, H is uniquely determined by the nilpotent G_C-orbit $o\equiv G_C\cdot\omega$ in $\mathfrak{g}_C$ containing ω. Here G_C is the adjoint group of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. So we express H as $H(o)$.

Let $\mathfrak{g}=\bigoplus_{j\in Z}\mathfrak{g}(j)_o$ denote the gradation of $\mathfrak{g}$ determined by ad $H(o)$.

Put $\mathfrak{p}_o=\bigoplus_{j\geq 0}\mathfrak{g}(j)_o$, $\mathfrak{l}_o=\mathfrak{g}(0)_o$ and $\mathfrak{n}_o=\bigoplus_{j\geq 1}\mathfrak{g}(j)_o$. Then $\mathfrak{p}_o$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{p}_o=\mathfrak{l}_o\oplus\mathfrak{n}_o$ gives its Levi decomposition. Let $P_o=L_oN_o$ be the corresponding decomposition on the group level. Here $P_o=N_G(\mathfrak{n}_o)$, the normalizer of $\mathfrak{n}_o$ in G; $L_o=Z_G(H(o))$, the centralizer of $H(o)$ in G; and $N_o=\exp\mathfrak{n}_o$, the analytic subgroup of G with Lie algebra $\mathfrak{n}_o$.

Take an $\mathfrak{sl}_2$-triplet $(X, H(o), Y)$ with $X\in\omega$ as above. We define a linear from X^* on $\mathfrak{n}_o$ by

$$\langle X^*, Z\rangle=B(Z, \theta X) \qquad \text{for } Z\in\mathfrak{n}_o, \tag{1.2}$$

where B is the Killing form of $\mathfrak{g}$. Let Ad* denote the coadjoint representation of N_o on the dual space $\mathfrak{n}_o^*$ of $\mathfrak{n}_o$. Since N_o is a simply connected nilpotent Lie group, the coadjoint orbit space $\mathfrak{n}_o^*/\mathrm{Ad}^*(N_o)$ corresponds bijectively to the unitary dual $\hat{N}_o$ of N_o through the Kirillov map (see [II, 1.3]). Let ξ_X be the irreducible unitary representation of N_o corresponding to the orbit $[X^*]=\mathrm{Ad}^*(N_o)X^*$. One can realize ξ_X explicitly as a monomial representation of N_o, using a real polarization at $X^*\in\mathfrak{n}_o^*$. Actually, it is easily seen that the ideal $\mathfrak{n}(2)_o=\bigoplus_{j\geq 2}\mathfrak{g}(j)_o$ of $\mathfrak{n}_o$ coincides with the radical of X^*:

$$\mathfrak{n}(2)_o=\{Y\in\mathfrak{n}_o;\ \mathrm{ad}^*(Y)X^*=0\},$$

where $\mathrm{ad}^*(Y)\equiv(d/dt)\mathrm{Ad}^*(\exp tY)|_{t=0}$ for $Y\in\mathfrak{n}_o$. So there exists a subspace $\mathfrak{v}(X)$ of $\mathfrak{g}(1)_o$ such that $\mathfrak{n}(X)\equiv\mathfrak{v}(X)\oplus\mathfrak{n}(2)_o$ is a real polarization at X^*:

$$X^*([\mathfrak{n}(X), \mathfrak{n}(X)])=(0), \qquad 2\dim\mathfrak{v}(X)=\dim\mathfrak{g}(1)_o.$$

Let η_X be a unitary character of $N(X)\equiv\exp\mathfrak{n}(X)$ given as

$$\eta_X(\exp Z)=\exp\sqrt{-1}\langle X^*, Z\rangle \qquad \text{for } Z\in\mathfrak{n}(X). \tag{1.3}$$

Then ξ_X can be realized as the unitarily induced representation:

$$\xi_X=L^2\text{-}\mathrm{Ind}_{N(X)}^{N_o}(\eta_X). \tag{1.4}$$

From this construction, ξ_X is either one-dimensional or infinite-dimensional according as $\mathfrak{g}(1)_o=(0)$ or not.

Now consider the induced representation $\Gamma_X=\mathrm{Ind}_{N_o}^G(\xi_X)$. Here, Ind means, as in our previous paper [35] (referred as [I] later on), either L^2-Ind (=the unitary induction) or C^∞-Ind (the C^∞-induction). The equivalence class $[\Gamma_X]$ of Γ_X does not depend on a choice of X, since any of such X belongs to an $\mathrm{Ad}(L_o)$-orbit in $\mathfrak{g}(2)_o$ in common (see [II, Prop.

1.9]). So we may express Γ_X as Γ_ω without any confusion.

Definition 1.1. The induced representation $\Gamma_\omega = \mathrm{Ind}_{N_o}^G(\xi_X)$ is called the *generalized Gelfand-Graev representation* (=GGGR) associated with the nilpotent class $\omega \subseteq \mathfrak{g}$.

Remark 1.2 [II, 1.5]. A real semisimple Lie algebra $\mathfrak{g}$ contains a regular nilpotent Ad(G)-orbit if and only if $\mathfrak{g}$ is quasi-split, that is, the centralizer $\mathfrak{m}$ of $\mathfrak{a}_p$ in $\mathfrak{k}$ is abelian. In such a case, the GGGRs associated with regular nilpotent classes coincide with the original Gelfand-Graev representations (= GGRs for short), or the representations of G induced from non-degenerate unitary characters of the maximal unipotent subgroup N_m. These (unitary) GGRs are known to be of multiplicity free if G is a linear group ([29], see also [I, Th. 4.5]).

1.2. Reduced GGGRs ([15, 2.5], [II, Section 2]).

Contrary to the case of original GGRs, the generalized GGGRs fail to be of multiplicity finite in general. So, in order to reduce the infinite multiplicities in GGGRs to be finite or to be one if possible, we introduced in [II] a variant of GGGR, called reduced GGGR (= RGGGR for short). We actually gave there finite multiplicity theorems for some important classes of RGGGRs closely related to the regular representation (see Section 5 below).

In the Part I of this paper, we shall prove that the above important types of RGGGRs, already known to be of multiplicity finite, have further multiplicity free property under some reasonable assumptions.

Now let us recall the construction of RGGGRs. Let $\Gamma_\omega = \mathrm{Ind}_{N_o}^G(\xi_X)$ ($o = G_C \cdot \omega$, $X \in \omega$) be the GGGR associated with a nilpotent class ω of $\mathfrak{g}$. The Levi subgroup $L_o = Z_G(H(o))$ acts on N_o through the conjugation. Hence it acts also on the unitary dual $\hat{N}_o$ through

$$(1.5) \qquad l \cdot [\xi] = [l \cdot \xi], \qquad (l \cdot \xi)(n) = \xi(l^{-1} n l) \qquad (n \in N_o)$$

for $l \in L_o$ and $[\xi] \in \hat{N}_o$, where $[\xi]$ is the unitary equivalence class of a unitary representation ξ of N_o. Let $H_o(X)$ be the stabilizer of $[\xi_X]$ in L_o. It coincides with $Z_{L_o}(\theta X)$, which is known to be reductive. Then, ξ_X can be extended canonically to a unitary (projective, in general) representation $\tilde{\xi}_X$ of the semidirect product subgroup $S_o(X) \equiv H_o(X) N_o \subseteq P_o$ acting on the same Hilbert space. Moreover, for any case treated in later sections, such an extension $\tilde{\xi}_X$ can be chosen to be genuine (not just projective). So, we suppose here such a property for $\tilde{\xi}_X$ for the sake of simplicity.

Definition 1.3. For an irreducible admissible (unitary, in case of L^2-Ind) representation c of the reductive subgroup $H_o(X) \subseteqq L_o$, the induced representation

$$(1.6) \qquad \Gamma_\omega(c) = \mathrm{Ind}_{S_0(X)}^G(\tilde{c} \otimes \tilde{\xi}_X) \qquad (\tilde{c} \equiv c \otimes 1_{N_o})$$

is called the *reduced* generalized Gelfand-Graev representation (= RGG GR) associated with (ω, c). Here 1_{N_o} is the trivial character of N_o.

§ 2. Sufficient conditions for multiplicity free property

In this section, we give, following Benoist [2] and Kostant [18], sufficient conditions for a monomial representation of a Lie group to have multiplicity free property. These conditions will be utilized in later sections to prove multiplicity one theorems for reduced generalized Gelfand-Graev representations of semisimple Lie groups.

2.1. Distributions on C^∞-manifolds.

First, we prepare here the notation and elementary facts on distributions on C^∞-manifolds, and then in 2.2 those on Lie groups, which will be used throughout the Part I of this paper.

Let M be a σ-compact C^∞-manifold. Denote by $\mathscr{E}(M) = C^\infty(M)$ (resp. $\mathscr{D}(M) = C_0^\infty(M)$) the space of C^∞-functions on M (resp. such functions with compact supports) equipped with the usual Schwartz topology (see [33, p. 479]). Let $\mathscr{D}'(M)$ (resp. $\mathscr{E}'(M)$) be the topological dual space of $\mathscr{D}(M)$ (resp. $\mathscr{E}(M)$). Each element $\mathscr{D}'(M) \ni T: \mathscr{D}(M) \to \boldsymbol{C}$ is said to be a distribution on M. Notice that the identical map $\mathscr{D}(M) \hookrightarrow \mathscr{E}(M)$ gives a continuous embedding of $\mathscr{D}(M)$ into $\mathscr{E}(M)$ with a dense image. Hence the space $\mathscr{E}'(M)$ can be identified with a subspace of $\mathscr{D}'(M)$ via

$$\mathscr{E}'(M) \ni T \longmapsto T | \mathscr{D}(M) \in \mathscr{D}'(M).$$

Moreover, $\mathscr{E}'(M)$ coincides with the space of distributions on M with compact supports.

Now suppose that M be orientable. Then M admits a volume form Ω_M. Let ν denote the Borel measure on M canonically associated with Ω_M. Then, any locally integrable function f on M (with respect to ν) is viewed as a distribution T_f on M via

$$\mathscr{D}(M) \ni \phi \longmapsto \langle T_f, \phi \rangle = \int_M f \cdot \phi d\nu.$$

Let $\mathrm{Diff}(M)$ be the algebra of C^∞-differential operators on M. For any $D \in \mathrm{Diff}(M)$, there exists a unique $D^* \in \mathrm{Diff}(M)$ (which depends on the measure ν) such that

$$\int_M (D\phi)\cdot\psi\, d\nu = \int_M \phi\cdot(D^*\psi)\, d\nu \qquad \text{for all } \phi,\ \psi \in \mathscr{D}(M).$$

D^* is called the adjoint of D with respect to ν. The assignment $D \mapsto D^*$ gives an involutive anti-automorphism of $\mathrm{Diff}(M)$. If $T \in \mathscr{D}'(M)$ and $D \in \mathrm{Diff}(M)$, then the functional $\phi \mapsto \langle T, D^*\phi\rangle$ is also a distribution on M, which will be denoted by DT. In this fashion, the algebra $\mathrm{Diff}(M)$ acts on the spaces $\mathscr{D}'(M)$ and $\mathscr{E}'(M)$. Furthermore, this action, restricted to $\mathscr{E}(M) \subseteq \mathscr{D}'(M)$, coincides with the usual differentiation of functions: $DT_f = T_{Df}$ for $D \in \mathrm{Diff}(M)$ and $f \in \mathscr{E}(M)$.

2.2. Distributions on Lie groups.

Now let G be a Lie group, and $\mathfrak{g}$ the Lie algebra of G. For simplicity, we always assume that G be unimodular. Denote by $d_G(x)$ a Haar measure on G. Using this measure, we regard "functions" on G as distributions, and consider the adjoints of differential operators.

Let $g \in G$. For a function f on G, we put

$$(2.1) \qquad (L_g f)(x) = f(g^{-1}x), \quad (R_g f)(x) = f(xg), \quad \check{f}(x) = f(x^{-1}),$$

for $x \in G$. Then the maps $f \mapsto L_g f$, $f \mapsto R_g f$ and $f \mapsto \check{f}$ induce topological isomorphisms of spaces $\mathscr{D}(G)$ and $\mathscr{E}(G)$. Furthermore, $g \mapsto R_g$ and $g \mapsto L_g$ define representations of G on spaces of functions, and $\vee$ gives an intertwining operator between them. These operations L_g, R_g and $\vee$ are extended to to those on spaces $\mathscr{E}'(G) \subseteq \mathscr{D}'(G)$ of distributions on G respectively by

$$(2.2) \qquad \langle L_g T, \psi\rangle = \langle T, L_{g^{-1}}\psi\rangle, \qquad \langle R_g T, \psi\rangle = \langle T, R_{g^{-1}}\psi\rangle,$$

$$(2.3) \qquad \langle \check{T}, \psi\rangle = \langle T, \check{\psi}\rangle,$$

where T is a distribution and ψ a test function.

By differentiating representations R and L respectively, one can associate to each $X \in \mathfrak{g}$, differential operators R_X and L_X on G:

$$(2.4) \qquad \begin{cases} (R_X f)(g) = \dfrac{d}{dt} f(g \exp(tX))|_{t=0}, \\[2ex] (L_X f)(g) = \dfrac{d}{dt} f(\exp(-tX)g)|_{t=0}, \end{cases}$$

for $f \in \mathscr{E}(M)$ and $g \in G$. Let $U(\mathfrak{g}_C)$ denote the enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Then $X \mapsto L_X$ (resp. $X \mapsto R_X$) extends uniquely to an isomorphism, denoted again by L (resp. R), from $U(\mathfrak{g}_C)$ onto the algebra of right (resp. left) G-invariant differential operators on G. The adjoints of L_D and R_D ($D \in U(\mathfrak{g}_C)$) are given respectively as

$$(L_D)^* = L_{\check{D}}, \qquad (R_D)^* = R_{\check{D}},$$

where $\vee$ denotes the principal anti-automorphism of $U(\mathfrak{g}_C)$, or the anti-automorphism of $U(\mathfrak{g}_C)$ such that $\check{X} = -X$ for $X \in \mathfrak{g}_C$. This notation $\vee$ is consistent with that in (2.1): $(L_D\check{f})(1) = L_{\check{D}}f(1)$, $1 =$ the unit element of G.

Let T_1 and T_2 be two distributions on G. Suppose that either T_1 or T_2 has compact support. Then the convolution $T_1 * T_2 \in \mathscr{D}'(G)$ of T_1 with T_2 can be defined by

$$\text{(2.5)} \qquad \langle T_1 * T_2, \psi \rangle = \langle T_2(g), \langle T_1, R_g\psi \rangle\rangle \qquad (\psi \in \mathscr{D}(G)),$$

where $T_2(g)$ denotes the distribution T_2 applied on functions in $g \in G$. By definition, we see easily

$$\text{(2.6)} \qquad \operatorname{supp}(T_1 * T_2) \subseteqq \operatorname{supp}(T_1)\operatorname{supp}(T_2),$$

where $\operatorname{supp}(T)$ is the support of a distribution T. So in particular, $\mathscr{E}'(G)$ has a structure of algebra under convolution, and $\mathscr{D}'(G)$ is a two-sided $\mathscr{E}'(G)$-module. If both T_1 and T_2 are functions: $T_i = T_{f_i}$ ($i = 1, 2$), then the convolution is given by $T_1 * T_2 = T_{f_1 * f_2}$ with

$$\text{(2.7)} \qquad (f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, d_G(x) \qquad (g \in G).$$

We define for $g \in G$ and $D \in U(\mathfrak{g}_C)$, distributions δ_g and δ_D in $\mathscr{E}'(G)$ respectively by

$$\text{(2.8)} \qquad \langle \delta_g, \psi \rangle = \psi(g), \qquad \langle \delta_D, \psi \rangle = (L_{\check{D}})(1) \qquad \text{for } \psi \in \mathscr{E}(G).$$

Through the map $g \mapsto \delta_g$ (resp. $D \mapsto \delta_D$), one can identify G (resp. $U(\mathfrak{g}_C)$) with the group of all dirac measures (resp. the algebra of all distributions with supports at the identity). Moreover, one sees easily by definition the following relations.

$$\text{(2.9)} \qquad \begin{cases} L_g T = \delta_g * T, & R_g T = T * \delta_{g^{-1}}, \\ L_D T = \delta_D * T, & R_D T = T * \delta_{\check{D}}, \\ (T_1 * T_2)^\vee = \check{T}_2 * \check{T}_1. \end{cases}$$

2.3. Generalized matrix coefficients for representations [18, 6.1].

Let $(\pi, \mathscr{H})$ be a contiouous representation of G on a Hilbert space $\mathscr{H}$. A vector $v \in \mathscr{H}$ is called smooth if the map $G \ni g \mapsto \pi(g)v \in \mathscr{H}$ is of class C^∞. The totality $\mathscr{H}^\infty$ of smooth vectors for π is a $\pi(G)$-stable, dense subspace of $\mathscr{H}$. Moreover, $\mathscr{H}^\infty$ has a structure of $\mathscr{E}'(G)$-module which extends the action $G \ni g \mapsto \pi(g)|\mathscr{H}^\infty$ of $G \subseteqq \mathscr{E}'(G)$: for $T \in \mathscr{E}'(G)$ and $v \in \mathscr{H}^\infty$, the vector $\pi(T)v \in \mathscr{H}^\infty$ is characterized by

$$(2.10) \qquad (\pi(T)v, w) = \langle T(g), (\pi(g)v, w)\rangle \qquad \text{for } w \in \mathscr{H},$$

where (,) is the inner product on $\mathscr{H}$. Notice that, for $X \in \mathfrak{g}$, $\pi(X) \equiv \pi(\delta_X)$ is given by differentiating the G-action:

$$(2.11) \qquad \pi(X)v = \frac{d}{dt}\pi(\exp tX)v|_{t=0}.$$

We equip $\mathscr{H}^\infty$ with the usual Fréchet space topology, defined by the family of seminorms $\{\|\cdot\|_D;\, D \in U(\mathfrak{g}_C)\}$, where $\|v\|_D = (\pi(D)v, \pi(D)v)^{1/2}$ $(v \in \mathscr{H}^\infty)$. Then, for every $T \in \mathscr{E}'(G)$, $\pi(T)$ gives a linear map on $\mathscr{H}^\infty$, continuous with respect to this topology.

Let $\mathscr{H}^*$ be the continuous dual of $\mathscr{H}$, which is also a Hilbert space. Denote by π^* the representation of G on $\mathscr{H}^*$ contragredient to π: $\pi^*(g) = {}^t\pi(g^{-1})$ $(g \in G)$, where, for a bounded operator A on $\mathscr{H}$, ${}^tA: \mathscr{H}^* \to \mathscr{H}^*$ is its transpose. Let $(\mathscr{H}^*)^{-\infty}$ denote the continuous dual space of $\mathscr{H}^\infty$. By restricting each $v^* \in \mathscr{H}^*$, a linear functional on $\mathscr{H}$, to the dense subspace $\mathscr{H}^\infty$, we can regard $\mathscr{H}^* \subseteqq (\mathscr{H}^*)^{-\infty}$. Equip $(\mathscr{H}^*)^{-\infty}$ with the $\mathscr{E}'(G)$-module structure contragredient to that on $\mathscr{H}^\infty$:

$$(2.12) \qquad \langle \pi^*(T)a^*, w\rangle = \langle a^*, \pi(\check{T})w\rangle \qquad (w \in \mathscr{H}^\infty)$$

for $T \in \mathscr{E}'(G)$ and $a^* \in (\mathscr{H}^*)^{-\infty}$. On the other hand, the subspace $(\mathscr{H}^*)^\infty \subseteqq \mathscr{H}^*$ of smooth vectors for π^* has a structure of $\mathscr{E}'(G)$-module consistent with that on $(\mathscr{H}^*)^{-\infty}$.

Starting from $(\pi^*, \mathscr{H}^*)$ instead of $(\pi, \mathscr{H})$ and identifying $((\pi^*)^*, (\mathscr{H}^*)^*)$ with $(\pi, \mathscr{H})$ in the canonical way, one obtains similarly $\mathscr{E}'(G)$-modules $\mathscr{H}^\infty \subseteqq \mathscr{H}^{-\infty} = ((\mathscr{H}^*)^*)^{-\infty}$. Consequently, we have the picture of G-modules:

$$(2.13) \qquad \begin{tikzcd}[column sep=small] \mathscr{H}^\infty \subseteqq & \mathscr{H} = (\mathscr{H}^*)^* \arrow[d, leftrightarrow] & \subseteqq \mathscr{H}^{-\infty}, \\ (\mathscr{H}^*)^\infty \subseteqq \arrow[urr] & \mathscr{H}^* & \subseteqq (\mathscr{H}^*)^{-\infty}, \arrow[from=1-1, to=2-3, crossing over] \end{tikzcd}$$

where $A \to B$ means that B is the dual space of A. In addition, the right and left ends are $\mathscr{E}'(G)$-modules. Suggested by (2.13), a vector in $\mathscr{H}^{-\infty}$ will be called a *generalized vector* for π.

Now, for $v \in \mathscr{H}$ and $w^* \in \mathscr{H}^*$, define a continuous function T^{π}_{v,w^*} on G by

$$T^{\pi}_{v,w^*}(g) = \langle \pi(g)v, w^* \rangle \qquad (g \in G). \tag{2.14}$$

Such a function is said to be the matrix coefficient of π associated to (v, w^*). The notion of matrix coefficient can be generalized as follows.

Proposition 2.1 [18, Prop. 6.1.1.]. *Let π be any continuous representation of a Lie group G on a Hilbert space $\mathscr{H}$, and keep to the above notations. Then one has*

(1) *$\pi(\psi)a \in \mathscr{H}^{\infty}$ for any $a \in \mathscr{H}^{-\infty}$ and any $\psi \in \mathscr{D}(G)$.*

(2) *For $a \in \mathscr{H}^{-\infty}$ and $b^* \in (\mathscr{H}^*)^{-\infty}$, define a linear functional T^{π}_{a,b^*} on $\mathscr{D}(G)$ by*

$$\langle T^{\pi}_{a,b^*}, \psi \rangle = \langle \pi(\psi)a, b^* \rangle \qquad (\psi \in \mathscr{D}(G)). \tag{2.15}$$

Then T^{π}_{a,b^*} gives a distribution on G, which is said to be the generalized matrix coefficient (=*GMC for short*) *of π associated with the pair (a, b^*).*

We can see immediately that the GMCs have the following properties.

Lemma 2.2 [2, Lemma 4.1.1]. *Let $a \in \mathscr{H}^{-\infty}$ and $b^* \in (\mathscr{H}^*)^{-\infty}$. Then, for a $T \in \mathscr{E}'(G)$, the GMC T^{π}_{a,b^*} satisfies the relations*

$$T * T^{\pi}_{a,b^*} = T^{\pi}_{a,\pi^*(T)b^*}, \tag{2.16}$$

$$T^{\pi}_{a,b^*} * T = T^{\pi}_{\pi(\check{T})a,b^*}, \tag{2.17}$$

$$(T^{\pi^*}_{a,b^*})^{\vee} = T^{\pi}_{b^*,a}. \tag{2.18}$$

The more important property for GMC is that the equivalence class of an irreducible unitary representation is determined by the corresponding GMCs.

Proposition 2.3 [2, Lemma 4.1.2]. *Let $(\pi, \mathscr{H})$ and $(\rho, \mathscr{F})$ be two irreducible unitary representations of G, and take non-zero vectors $a \in \mathscr{H}^{-\infty}$, $b^* \in (\mathscr{H}^*)^{-\infty}$, $c \in \mathscr{F}^{-\infty}$ and $d^* \in (\mathscr{F}^*)^{-\infty}$.*

(1) *If $T^{\pi}_{a,b^*} = T^{\rho}_{c,d^*}$, then π is unitarily equivalent to ρ.*

(2) *In such a case, let $A: \mathscr{H} \xrightarrow{\sim} \mathscr{F}$ be a unitary intertwining operator, and $A^*: \mathscr{F}^* \to \mathscr{H}^*$ the adjoint operator of A. Then, $A(\mathscr{H}^{\infty}) = \mathscr{F}^{\infty}$ and $A^*((\mathscr{F}^*)^{\infty}) = (\mathscr{H}^*)^{\infty}$. So, by taking the transpose, A (resp. A^*) can be*

extended canonically to an isomorphism of $\mathscr{E}'(G)$-modules, denoted again by A (resp. A^*), from $\mathscr{H}^{-\infty}$ onto $\mathscr{F}^{-\infty}$ (resp. from $(\mathscr{F}^*)^{-\infty}$ onto $(\mathscr{H}^*)^{-\infty}$). For these extended operators A and A^*, one has

$$A(a)=\lambda c \quad \text{and} \quad A^*(d^*)=\lambda b^* \quad \text{with } \lambda\neq 0, \ \in \boldsymbol{C}.$$

Now assume that G be a connected semisimple Lie group with finite center. K denotes a maximal compact subgroup of G. For such a group G, there is an important category of irreducible representations, including all the irreducible unitary representations. It consists of irreducible admissible representations. Here, a continuous Hilbert space representation π of G is said to be admissible if the restriction of π to K is unitary and has finite multiplicity property.

We wish give here a variant of Proposition 2.3 which works not only for unitary representations but also for irreducible admissible representations of G. For this purpose, we need some notation.

If $(\pi, \mathscr{H})$ is an admissible representation, then the space $\mathscr{H}_K$ of K-finite vectors for π consists of analytic vectors, so in particular, one has $\mathscr{H}_K \subseteqq \mathscr{H}^\infty$. This subspace $\mathscr{H}_K$ is stable under $\pi(K)$ and $\pi(\mathfrak{g}_C)$. We thus get a $(\mathfrak{g}_C, K)$-module $\mathscr{H}_K$. The representation $(\pi, \mathscr{H})$ is irreducible if and only if $\mathscr{H}_K$ is an (algebraically) irreducible $(\mathfrak{g}_C, K)$-module. Two admissible representations $(\pi_i, \mathscr{H}_i)$ $(i=1, 2)$ are said to be infinitesimally equivalent if the corresponding $(\mathfrak{g}_C, K)$-modules are equivalent. If π_1 and π_2 are irreducible and unitary, then π_1 is unitarily equivalent to π_2 if and only if they are infinitesimally equivalent. (For these facts, one can consult [31, Chap. 8]. See also [I, Section 2].)

For an admissible $(\pi, \mathscr{H})$, $\mathscr{H}'_K$ denotes the algebraic dual space of $\mathscr{H}_K$. Equip $\mathscr{H}'_K$ with the $(\mathfrak{g}_C, K)$-module structure contragredient to that on $\mathscr{H}_K$. Since $\mathscr{H}_K$ is dense in $\mathscr{H}^\infty$, we have natural inclusions of $(\mathfrak{g}_C, K)$-modules:

$$\tag{2.19} \begin{cases} (\mathscr{H}^*)_K \subseteqq (\mathscr{H}^*)^\infty \subseteqq (\mathscr{H}^*)^{-\infty} \subseteqq \mathscr{H}'_K, \\ \mathscr{H}_K \subseteqq \mathscr{H}^\infty \subseteqq \mathscr{H}^{-\infty} \subseteqq (\mathscr{H}^*)'_K. \end{cases}$$

Let A: $\mathscr{H}_K \xrightarrow{\sim} \mathscr{F}_K$ be an isomorphism of $(\mathfrak{g}_C, K)$-modules which gives an infinitesimal equivalence between admissible representations $(\pi, \mathscr{H})$ and $(\rho, \mathscr{F})$. Then, making use of the admissibility of π and ρ, we can define an operator A^*: $(\mathscr{F}^*)_K \rightarrow (\mathscr{H}^*)_K$ via

$$\tag{2.20} \langle Av, w^* \rangle = \langle v, A^*w^* \rangle \qquad (v \in \mathscr{H}_K, \ w^* \in (\mathscr{F}^*)_K).$$

It is easy to see that $(\mathscr{F}^*)_K$ is, as a $(\mathfrak{g}_C, K)$-module, equivalent to $(\mathscr{H}^*)_K$ through A^*. By taking the transposes of A and A^*, one thus gets $(\mathfrak{g}_C, K)$-module isomorphisms

$$(2.21) \qquad {}^tA: \mathscr{F}'_K \xrightarrow{\sim} \mathscr{H}'_K, \quad {}^t(A^*): (\mathscr{H}^*)'_K \xrightarrow{\sim} (\mathscr{F}^*)'_K.$$

We can now strengthen Proposition 2.3 for semisimple case to the following

Proposition 2.4. *Let G be a connected semisimple Lie group with finite center, and keep to the above notation. Let $(\pi, \mathscr{H})$ and $(\rho, \mathscr{F})$ be two irreducible admissible representations.*

(1) *If $T^{\pi}_{a,b^*} = T^{\rho}_{c,d^*}$ for some non-zero vectors $a \in \mathscr{H}^{-\infty}$, $b^* \in (\mathscr{H}^*)^{-\infty}$, $c \in \mathscr{F}^{-\infty}$ and $d^* \in (\mathscr{F}^*)^{-\infty}$, then π is infinitesimally equivalent to ρ.*

(2) *In such a case, let A be an isomorphism of $(\mathfrak{g}_C, K)$-modules from $\mathscr{H}_K$ onto $\mathscr{F}_K$. Then, there exists a non-zero complex number λ such that ${}^t(A^*)(a) = \lambda c$ and ${}^tA(d^*) = \lambda b^*$.*

Proof. Suppose that $T^{\pi}_{a,b^*} = T^{\rho}_{c,d^*}$, which means that

$$(2.22) \qquad \langle \pi(\phi)a, b^* \rangle = \langle \rho(\phi)c, d^* \rangle \qquad (\phi \in \mathscr{D}(G)).$$

Replacing the above ϕ by $\check{\psi} * \phi$, we get

$$(2.23) \qquad \langle \pi(\phi)a, \pi^*(\psi)b^* \rangle = \langle \rho(\phi)c, \rho^*(\psi)d^* \rangle$$

for all ϕ, $\psi \in \mathscr{D}(G)$. Notice that $\pi^*(\mathscr{D}(G))b^*$ (resp. $\rho^*(\mathscr{D}(G))d^*$) is a non-zero $\mathscr{E}'(G)$-stable (so in particular, G-stable) subspace of $(\mathscr{H}^*)^\infty$ (resp. $(\mathscr{F}^*)^\infty$). By the irreducibility of π^* (resp. ρ^*), it must be dense in $\mathscr{H}^*$ (resp. in $\mathscr{F}^*$). Keeping this fact in mind, we deduce from (2.23) that the assignment $\tilde{A}$; $\pi(\phi)a \mapsto \rho(\phi)c$ ($\phi \in \mathscr{D}(G)$), is well-defined and it gives an $\mathscr{E}'(G)$-module isomorphism from $\pi(\mathscr{D}(G))a$ onto $\rho(\mathscr{D}(G))c$.

Let $\mathscr{D}_K(G) \subseteq \mathscr{D}(G)$ be the space of left K-finite C^∞-functions with compact supports. We can see easily that $\pi(\mathscr{D}_K(G))a$ and $\rho(\mathscr{D}_K(G))c$ are respectively $(\mathfrak{g}_C, K)$-submodules of $\mathscr{H}_K$ and $\mathscr{F}_K$, which are non-zero since $\mathscr{D}_K(G)$ is dense in $\mathscr{D}(G)$. Irreducibility of $\mathscr{H}_K$ and $\mathscr{F}_K$ implies that $\mathscr{H}_K = \pi(\mathscr{D}_K(G))a$ and $\mathscr{F}_K = \rho(\mathscr{D}_K(G))c$ respectively. Put $A = \tilde{A} | \mathscr{H}_K$. Then, A gives an isomorphism of $(\mathfrak{g}_C, K)$-modules: $\mathscr{H}_K \xrightarrow{\sim} \mathscr{F}_K$. This shows the assertion (1), the infinitesimal equivalence between π and ρ. Furthermore, this A clearly satisfies ${}^tA^*(a) = c$ and ${}^tA(d^*) = b^*$. Notice that any $(\mathfrak{g}_C, K)$-module isomorphism V: $\mathscr{H}_K \xrightarrow{\sim} \mathscr{F}_K$ is a scalar multiple of A. This proves (2). Q.E.D.

2.4. Sufficient conditions for multiplicity free property.

Let G be a (unimodular, for simplicity) Lie group again, and ζ a one-dimensional representation (= character) of a closed subgroup Q of G.

Consider the induced representation (L^2- or C^∞-) $\mathrm{Ind}_Q^G(\zeta)$. An induced representation of this form is said to be monomial. In this subsection, we give, mainly after [2], sufficient conditions for a monomial representation to have multiplicity free property in the form of four theorems. For this purpose, Propositions 2.3 and 2.4 play an essential role.

2.4.1. *Case of C^∞-induced representations.*
First, let us consider the representation $(\pi_\zeta, C^\infty(G;\zeta)) = C^\infty\text{-}\mathrm{Ind}_Q^G(\zeta)$ induced in C^∞-context:

$$C^\infty(G;\zeta)=\{f\in\mathscr{E}(G);\, f(xq)=\delta_Q(q)^{1/2}\zeta(q)^{-1}f(x)\ (x\in G,\ q\in Q)\},$$
$$\pi_\zeta(g)f(x)=f(g^{-1}x) \qquad (f\in C^\infty(G;\zeta),\ g,\ x\in G),$$

where δ_Q is the modular function on Q with respect to a left Haar measure $d_Q(q)$ on Q: $\delta_Q(q)=d_Q(xq)/d_Q(x)$.

For a continuous Hilbert space representation $(\pi, \mathscr{H})$ of G, let $(\mathscr{H}^*)_\zeta^{-\infty}$ denote the space of generalized vectors $b^*\in(\mathscr{H}^*)^{-\infty}$ satisfying

$$\pi^*(q)b^*=\delta_Q(q)^{1/2}\zeta(q)^{-1}b^* \qquad \text{for all } q\in Q. \tag{2.24}$$

Then, in view of [I, Lemma 2.2], we have

$$\mathrm{Hom}_G(\mathscr{H}^\infty, C^\infty(G;\zeta))\simeq(\mathscr{H}^*)_\zeta^{-\infty} \qquad \text{(as vector spaces)}, \tag{2.25}$$

where the left hand side is the space of continuous G-module homomorphisms from $\mathscr{H}^\infty$ into $C^\infty(G;\zeta)$. This correspondence is given as

$$\begin{aligned}&\mathrm{Hom}_G(\mathscr{H}^\infty, C^\infty(G;\zeta))\ni B^*\longmapsto b^*\in(\mathscr{H}^*)_\zeta^{-\infty},\\ &b^*(v)=(B^*v)(1) \qquad (v\in\mathscr{H}^\infty,\ 1=\text{the unit element of } G).\end{aligned} \tag{2.26}$$

So, for irreducible π, we can call $\dim(\mathscr{H}^*)_\zeta^{-\infty}$ the multiplicity of $(\pi, \mathscr{H}^\infty)$ in $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_Q^G(\zeta)$.

Now let ζ' be another character of a closed subgroup Q' of G. For $b^*\in(\mathscr{H}^*)_\zeta^{-\infty}$ and $a\in\mathscr{H}_{\zeta'}^{-\infty}$, the corresponding GMC $T=T_{a,b^*}^\pi$ satisfies, in view of Lemma 2.2, the following relation.

$$L_qR_{q'}T=\delta_Q(q)^{1/2}\zeta(q)^{-1}\delta_{Q'}(q')^{1/2}\zeta'(q')^{-1}T \qquad (q\in Q,\ q'\in Q'). \tag{2.27}$$

Definition 2.5. An element $T\in\mathscr{D}'(G)$ satisfying (2.27) is said to be a (ζ,ζ')-*quasi-invariant distribution* ($=(\zeta,\zeta')$-QID for short). If such a T is, in addition, a joint eigendistribution of Laplace operators on G: $L_DT=\chi(D)T$ ($D\in Z(\mathfrak{g}_C)$) for some algebra homomorphism χ from the center $Z(\mathfrak{g}_C)$ of $U(\mathfrak{g}_C)$ to the complex number field $\boldsymbol{C}$, then it will be called a quasi-invariant *eigen*distribution ($=$QIED).

Remark 2.6. If G is connected, any irreducible unitary representation $(\pi, \mathscr{H})$ of G is quasi-simple, i.e., $Z(\mathfrak{g}_C)$ acts on $\mathscr{H}^\infty$ by scalars (see e.g., [33, 4.4.1.6]). So, in such a case, any QID on G obtained as a GMC of π is necessarily a QIED. The same holds also for irreducible admissible representations of semisimple Lie groups.

Example 2.7. Consider the case $G=G_1\times G_1$ (a direct product of a Lie group G_1 with itself), $Q=Q'=\mathrm{diag}(G_1\times G_1)=\{d(g_1)=(g_1, g_1); g_1 \in G_1\}$, and $\zeta=\zeta'=1_Q$, the trivial character of Q. Then, Q coincides with the fixed subgroup of the involution $(x, y)\mapsto(y, x)$ of G (so G/Q is a symmetric space), and G is diffeomorphic to the product $G_1\times Q$ through

$$G_1\times Q \ni (g_1, d(g_2))\longmapsto(g_1g_2, g_2)=(g_1, 1)d(g_2) \in G.$$

Hence, each right Q-invariant distribution T on G (so is any $(1_Q, 1_Q)$-QID) is identified with a $T_1 \in \mathscr{D}'(G_1)$ via

$$\langle T_1, \psi_1\rangle=\langle T, \psi\rangle \qquad \text{for } \psi \in \mathscr{D}(G),$$

where we define $\psi_1 \in \mathscr{D}(G_1)$ by

$$\psi_1(g_1)=\int_{G_1} \psi(g_1g_2, g_2)d_{G_1}(g_2) \qquad (g_1 \in G_1).$$

Through this identification, $(1_Q, 1_Q)$-QIEDs T on G coincide with so-called invariant eigendistributions on the Lie group G_1:

$$L_{g_1}R_{g_1}T_1=T_1 \qquad \text{for all } g_1 \in G_1, \tag{2.28}$$

$$L_DT_1=\chi(D)T_1 \qquad \text{for all } D \in Z((\mathfrak{g}_1)_C), \tag{2.29}$$

where $\mathfrak{g}_1=\mathrm{Lie}\, G_1$ and $\chi\colon Z((\mathfrak{g}_1)_C)\to \boldsymbol{C}$ is an algebra homomorphism. These distributions on G_1 have been playing an important role in the theory of group representations (e.g., the characters of irreducible admissible representations of semisimple Lie groups are QIEDs).

Using the notion of QIDs, one can give sufficient conditions for a monomial representation $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_Q^G(\zeta)$ to be of multiplicity free.

Let Aut (G) denote the group of automorphisms of G. First,

Theorem 2.8 (cf. [2, 4.4]). *Let G be any (unimodular) Lie group, and ζ a character of a closed subgroup $Q\subseteqq G$. Assume that there exists a $\sigma \in$ Aut (G) with following two properties:*

$$\sigma(Q)=Q, \tag{2.30}$$

(2.31) $\qquad T^{\sigma}=\check{T}$ *holds for any* $(\zeta, \zeta\circ\sigma)$-QID T *on* G,

where $\langle T^{\sigma}, \psi\rangle=\langle T, \psi\circ\sigma\rangle$ $(\psi\in\mathscr{D}(G))$. *Then, one has for any irreducible unitary representation* $(\pi, \mathscr{H})$ *of* G,

(1) $(\dim \mathscr{H}^{-\infty}_{\zeta\circ\sigma})\cdot(\dim(\mathscr{H}^*)^{-\infty}_{\zeta})\leqq 1$, *where* $0\cdot\infty$ *should be understood as* 0. *The equality holds only if* $\pi^{\sigma}\equiv\pi\circ\sigma$ *and* π^* *are unitarily equivalent.*

(2) *If* π^* *is unitarily equivalent to* π^{σ}, *the multiplicity of* $\dim(\mathscr{H}^*)^{-\infty}_{\zeta}$ *of* $(\pi, \mathscr{H}^{\infty})$ *in* $\pi_{\zeta}=C^{\infty}\text{-}\mathrm{Ind}_Q^G(\zeta)$ *is at most one.*

Proof. The proof of this theorem is same as that of [2, Prop. 4.4.1]. But we need to give it here to clarify the succeeding discussion. Suppose that $(\mathscr{H}^*)^{-\infty}_{\zeta}\neq(0)$ and $\mathscr{H}^{-\infty}_{\zeta\circ\sigma}\neq(0)$. Take non-zero vectors $b^*\in(\mathscr{H}^*)^{-\infty}_{\zeta}$ and $a\in\mathscr{H}^{-\infty}_{\zeta\circ\sigma}$. Then the GMC T^{π}_{a,b^*} gives a non-zero $(\zeta, \zeta\circ\sigma)$-QID on G. By (2.31) together with (2.18), we have

$$T^{\pi^{\sigma}}_{a,b^*}=(T^{\pi}_{a,b^*})^{\sigma}=(T^{\pi}_{a,b^*})^{\vee}=T^{\pi^*}_{b^*,a}.$$

Thanks to Proposition 2.3, π^{σ} and π^* are unitarily equivalent. Let A: $\mathscr{H}\xrightarrow{\sim}\mathscr{H}^*$ be a unitary intertwining operator from π^{σ} to π^*. Extend A, as in Proposition 2.3 (2), to an $\mathscr{E}'(G)$-module isomorphism $(\pi^{\sigma}, \mathscr{H}^{-\infty})\xrightarrow{\sim}(\pi^*,(\mathscr{H}^*)^{-\infty})$. Then, we have $C(Aa)\ni b^*$. Once a non-zero vector a is fixed, any b^* lies in the one-dimensional subspace $C(Aa)\subseteqq(\mathscr{H}^*)^{-\infty}$. This proves $\dim(\mathscr{H}^*)^{-\infty}_{\zeta}=1$. Similarly one can show $\dim\mathscr{H}^{-\infty}_{\zeta\circ\sigma}=1$. We have thus proved (1).

The assertion (2) follows from (1), by keeping in mind the fact that, if $\pi^*\simeq\pi^{\sigma}$, then the above intertwining operator A naturally gives rise to an isomorphism of vector spaces: $\mathscr{H}^{-\infty}_{\zeta\circ\sigma}\simeq(\mathscr{H}^*)^{-\infty}_{\zeta}$. Q.E.D.

Remark 2.9. If G is connected, the assumption (2.31) for σ can be weakened, in view of Remark 2.6 and the above proof, to the following

(2.31′) $\qquad T^{\sigma}=\check{T}$ for any $(\zeta, \zeta\circ\sigma)$-QIED T on G.

In semisimple case, we can yield, using Proposition 2.4 instead of Proposition 2.3, a variant of Theorem 2.8 as follows.

Theorem 2.10. *Suppose that G be a connected semisimple Lie group with finite center. Consider a monomial representation* $\pi_{\zeta}=C^{\infty}\text{-}\mathrm{Ind}_Q^G(\zeta)$ *induced in* C^{∞}*-context. Assume the existence of* $\sigma\in\mathrm{Aut}(G)$ *with properties* (2.30) *and* (2.31′). *Then, one has for any irreducible admissible representation* $(\pi, \mathscr{H})$ *of* G,

(1) $(\dim\mathscr{H}^{-\infty}_{\zeta\circ\sigma})\cdot(\dim(\mathscr{H}^*)^{-\infty}_{\zeta})\leqq 1$. *The equailty holds only if* π^{σ} *and* π^* *are infinitesimally equivalent.*

(2) *If π^σ is equivalent to π^* through a bicontinuous linear operator, then the multiplcity* $\dim(\mathscr{H}^*)_\zeta^{-\infty}$ *of* $(\pi, \mathscr{H}^\infty)$ *in* π_ζ *does not exceed one.*

The proof of this theorem is analogous to that of Theorem 2.8, so we omit it.

Let G be any Lie group again. If one can find out, for a given character ζ of Q, an automorphism σ of G with one more nice property ((2.32) below) in addition to (2.30) and (2.31), one can deduce the unique embedding property for irreducible representations into $\pi_\zeta = C^\infty\text{-Ind}_Q^G(\zeta)$.

Theorem 2.11. *Let G, Q and ζ be as in Theorem* 2.8. *Assume that there exists a $\sigma \in \mathrm{Aut}(G)$ with properties* (2.30), (2.31) (*one can replace* (2.31) *by a weaker condition* (2.31′) *if G is connected*), *and*

$$\zeta \circ \sigma = \bar{\zeta}, \tag{2.32}$$

where the bar means the complex conjugation. Then one has

$$\dim \mathrm{Hom}_G(\mathscr{H}^\infty, C^\infty(G;\zeta)) = \dim(\mathscr{H}^*)_\zeta^{-\infty} \leqq 1 \tag{2.33}$$

for any irreducible unitary representation $(\pi, \mathscr{H})$ of G.

Note. Shalika utilized this criterion in the proof of multiplicity one theorem for Gelfand-Graev representations of quasi-split semisimple Lie groups [29, Th. 3.1], although he did not specify it in such full generality as above.

Proof of Theorem 2.11. Let $(\pi, \mathscr{H})$ be a unitary representation of G. Denote by A the conjugate linear isomorphism of Hilbert spaces from $\mathscr{H}$ onto $\mathscr{H}^*$ such that $(v, w)_{\mathscr{H}} = \langle v, Aw\rangle_{\mathscr{H}\times\mathscr{H}^*}$ $(v, w \in \mathscr{H})$. This operator A commutes with the G-actions, whence it naturally gives rise to an isomorphism $\mathscr{H}_{\bar\zeta}^{-\infty} \simeq (\mathscr{H}^*)_\zeta^{-\infty}$. By the assumption (2.32), one obtains $\mathscr{H}_{\zeta\circ\sigma}^{-\infty} \simeq (\mathscr{H}^*)_\zeta^{-\infty}$. In view of (2.25) and Theorem 2.8 (1), we get (2.33) as desired. Q.E.D.

2.4.2. *Case of unitarily induced representations.*

Now let ζ be a unitary character of Q, and consider the unitarily induced representation $(\mathscr{U}_\zeta, L^2(G;\zeta)) = L^2\text{-Ind}_Q^G(\zeta)$. Assume throughout that $\mathscr{U}_\zeta$ is of type I. Let

$$\mathscr{U}_\zeta \simeq \int_{\hat{G}} [m_\zeta(\pi)] \cdot \pi \, d\mu_\zeta(\pi) \tag{2.34}$$

denote the factor decomposition of $\mathscr{U}_\zeta$ (see [I, 3.4]). Here, μ_ζ is a Borel measure on the unitary dual $\hat{G}$ of G equipped with the Mackey Borel structure, and m_ζ is the multiplicity function for $\mathcal{U}_\zeta$. Then, in view of [I, Lemma 3.10], the function m_ζ admits an upper bound as

$$m_\zeta(\pi) \leqq \dim (\mathcal{H}^*)_\zeta^{-\infty} \tag{2.35}$$

for almost all $\pi \in \hat{G}$ with respect to μ_ζ. Theorem 2.11 together with this inequality produces a criterion for multiplicity free property as follows.

Theorem 2.12. *Let the notation be as above. Let* $\mathcal{U}_\zeta = L^2\text{-}\mathrm{Ind}_Q^G(\zeta)$ *be a monomial unitary representation of a Lie qroup* G. *Assume the existence of a* $\sigma \in \mathrm{Aut}(G)$ *as in Theorem* 2.11. *Then,* $\mathcal{U}_\zeta$ *has multiplicity free property, that is, the multiplicity function for* $\mathcal{U}_\zeta$ *takes value* 1 *or* 0 *almost everywhere on the unitary dual* $\hat{G}$ *of* G.

2.5. Case of finite groups.

We now explain the meaning of sufficient conditions for multiplicity free property given in 2.4. To avoid technical complexity, let G assume to be a finite group, and $C(G)$ denotes the algebra of all functions on G under convolution (cf. (2.7)). Consider a monomial representation $\pi_\zeta = \mathrm{Ind}_Q^G(\zeta)$. The space $I(\zeta)$ of $(\zeta, \bar{\zeta})$-QIDs consists of $T \in C(G)$ such that

$$T(qxq') = \zeta(q)T(x)\zeta(q') \qquad (q, q' \in Q,\ x \in G),$$

and it is a subalgebra of $C(G)$. We can identify $I(\zeta)$ with the endomorphism algebra $\mathrm{End}_G(\pi_\zeta)$ of π_ζ in the following way. For a $T \in I(\zeta)$, put

$$A_T(f) = f * \check{T} \qquad (f \in C(G; \zeta)).$$

Here, $C(G; \zeta) \equiv \{f \in C(G); f(xq) = \zeta(q)^{-1}f(x)\ (x \in G,\ q \in Q)\}$ is the representation space of π_ζ. Then we get $A_T \in \mathrm{End}_G(\pi_\zeta)$, and the map $T \mapsto A_T$ gives the desired isomorphism $I(\zeta) \simeq \mathrm{End}_G(\pi_\zeta)$ as algebras.

A finite-dimensional representation ρ of G is of multiplicity free if and only if its emdomorphism algebra $\mathrm{End}_G(\rho)$ is commutative. So, in order to prove multiplicity free property for π_ζ, it is enough to find out an anti-automorphism of $C(G)$ which acts on the subalgebra $I(\zeta)$ trivially. For this purpose, let σ be any automorphism of G. It induces an algebra automorphism $T \mapsto T^\sigma = T \circ \sigma$ on $C(G)$, whence $T \mapsto T^{\sigma^\vee} = (T^\sigma)^\vee$, $\sigma^\vee \equiv \sigma \circ {}^\vee$, gives an anti-automorphism.

This σ leaves $I(\zeta)$ stable if and only if $\sigma(Q) = Q$ and $\zeta \circ \sigma = \bar{\zeta}$, which correspond to (2.30) and (2.32) respectively. Under these assumptions on σ, (2.31) means that σ acts on $I(\zeta)$ trivially. We have thus proved also for finite group case that three conditions (2.30), (2.31) and (2.32) assure multiplicity free property for $\pi_\zeta = \mathrm{Ind}_Q^G(\zeta)$.

2.6. Benoist's condition for multiplicity free property.

Let $L^2\text{-Ind}_Q^G(\zeta)$ be, as in 2.4, a monomial representation of a Lie group G. Let σ denote an involution of G such that $\sigma(Q)=Q$. Benoist gave a sufficient condition for the property (2.31).

Proposition 2.13 [2, Prop. 3.1]. *Assume the following property for* σ (*referred as* ($\mathscr{P}$) *later on*): *there exists a submanifold* P *of* G *such that*

(i) *the multiplication* $m: Q\times P\to G$ *is a surjective submersion.*

(ii) *If* $p\in P$, *then* $p^{-1}\in P$ *and* $\sigma(p)p\in Q$,

(iii) $qPq^{-1}=P$ *for all* $q\in Q$,

(iv) $\zeta(\sigma(p)p)>0$ *for all* $p\in P$.

Then, one gets (2.31): *any* $(\zeta, \zeta\circ\sigma)$-QID T *on* G *satisfies* $T^\sigma=\check{T}$.

This proposition together with Theorem 2.12 yields the following

Proposition 2.14 [2, Th. 5.1]. *Let* $\mathscr{U}_\zeta=L^2\text{-Ind}_Q^G(\zeta)$ *be a monomial unitary representation of a Lie group* G. *Assume that there exists an involution* σ *of* G *with property* ($\mathscr{P}$). *If* σ *satisfies, in addition,* $\bar{\zeta}=\zeta\circ\sigma$, *then* $\mathscr{U}_\zeta$ *is of multiplicity one.*

Example 2.15. Let G be a connected semisimple Lie group with finite center, and ζ a real-valued character of a maximal compact subgroup $K\subseteqq G$. Then, the induced representation $\text{Ind}_K^G(\zeta)$ is of multiplicity free. In fact, take as σ a Cartan involution of G such that K is its fixed subgroup. Then σ has property ($\mathscr{P}$) by putting $P=\exp\{X\in\mathfrak{g};\ \sigma X=-X\}$ and $Q=K$. Moreover, we have $\bar{\zeta}=\zeta=\zeta\circ\sigma$, so one can apply Proposition 2.14 successfully.

Benoist's coindition works very well in case where G/Q is a symmetric space. (Notice that his emphasis in [2] is placed on the property ($\mathscr{P}$) and on the case of exponential symmetric spaces.) But his result can not be applied directly to the case of generalized Gelfand-Graev representations. For our multiplicity one theorems, we need to investigate more closely quasi-invariant eigendistributions on semisimple Lie groups, attached to GGGRs. This is the main theme of the succeeding two sections.

§ 3. Spaces of Whittaker distributions

Hereafter, let G be a connected semisimple Lie group with finite center. In this section, we define spaces of Whittaker distributions on open subsets of G in connection with monomial representations of G induced from characters of unipotent subgroups (cf. [12], [20]). We also introduce the notion of Whittaker distributions with singular supports with respect to the Bruhat decompositions of G.

If there do not exist non-zero Whittaker distributions with singular supports, then the study of the above monomial representations is simplified to a certain extent. This can be seen from Proposition 3.1. We will utilize it in later sections to prove our multiplicity one theorems.

3.1. Whittaker vectors and Whittaker distributions.

Let $G=KA_pN_m$ be an Iwasawa decomposition of G, and keep to the notation in the previous sections. Denote by η a character of a connected Lie subgroup N of the maximal unipotent subgroup N_m. For an irreducible admissible representation $(\pi, \mathscr{H})$ of G, a generalized vector in $(\mathscr{H}^*)_\eta^{-\infty}$ (see (2.24)) is said to be a *Whittaker vector* of type (N, η) (see [34, Def. 1.1]). Let $b^* \in (\mathscr{H}^*)_\eta^{-\infty}$, $\neq 0$. In view of (2.25), b^* characterizes an embedding B^* of the smooth representation $(\pi, \mathscr{H}^\infty)$ into $(\pi_\eta, C^\infty(G; \eta)) = C^\infty\text{-Ind}_Q^G(\eta)$:

$$B^*: \mathscr{H}^\infty \ni v \longmapsto (T^\pi_{v,b^*})^\vee \in C^\infty(G; \eta), \tag{3.1}$$

where $T^\pi_{v,b^*}(g)=\langle \pi(g)v, b^*\rangle$ $(g \in G)$ is a GMC of π. Furthermore, this map B^* can be extended naturally to an $\mathscr{E}'(G)$-module embedding $\mathscr{H}^{-\infty} \hookrightarrow \mathscr{D}'(G)$, $a \mapsto (T^\pi_{a,b^*})^\vee$. We thus meet with the distributions $T=T^\pi_{a,b^*}$ $(a \in \mathscr{H}^{-\infty})$ on G which satisfy in view of Lemma 2.2, the following conditions:

$$L_nT=\eta(n)^{-1}T \qquad (n \in N), \tag{3.2}$$

$$L_DT=\chi(D)T \qquad (D \in Z(\mathfrak{g}_C)) \tag{3.3}$$

for some algebra homomorphism χ: $Z(\mathfrak{g}_C)\to \boldsymbol{C}$. This means that T is a left η-QIED on G. Since N is simply connected, (3.2) is equivalent to

$$L_ZT=-\eta'(Z)T \qquad \text{for } Z \in \mathfrak{n}\equiv \mathrm{Lie}\, N, \tag{3.4}$$

where η': $\mathfrak{n}\to \boldsymbol{C}$ is a Lie algebra homomorphism such that $\eta'(Z)=(d/dt)\eta(\exp tZ)|_{t=0}$.

Bearing in mind that the conditions (3.3) and (3.4) have a meaning also for distributions T on any open subset $O\subseteq G$, we put

$$\begin{cases} \mathscr{D}'(O; \eta')=\{T \in \mathscr{D}'(O);\ T \text{ satisfies (3.4)}\}, \\ \mathscr{D}'(O; \eta': \chi)=\{T \in \mathscr{D}'(O);\ T \text{ satisfies (3.3) and (3.4)}\}. \end{cases} \tag{3.5}$$

Each element of $\mathscr{D}'(O; \eta')$ is called a η'-*Whittaker distribution* on O. We call such a distribution χ-*elementary*, if it lies further in the subspace $\mathscr{D}'(O; \eta': \chi)\subseteq \mathscr{D}'(O; \eta')$.

Among Laplace operators L_D $(D \in Z(\mathfrak{g}_C))$ on G, the Casimir operator

L_Ω, Ω = the Casimir element of $U(\mathfrak{g}_C)$, plays an important role. Replacing (3.3) by a weaker condition

$$L_\Omega T = \kappa T \qquad \text{for } \kappa \in \boldsymbol{C}, \tag{3.6}$$

we can define similarly $\mathscr{D}'(O; \eta', \kappa)$, which will be called the space of κ-*quasi*-elementary η'-Whittaker distributions on O. Clearly one has

$$\mathscr{D}'(O; \eta' : \chi) \subseteqq \mathscr{D}'(O; \eta', \kappa) \subseteqq \mathscr{D}'(O; \eta') \quad \text{with} \quad \kappa = \chi(\Omega).$$

Each of these spaces has a right $\mathscr{E}'(G)$-module structure under convolution.

3.2. Whittaker distributions on Bruhat cells.

We now recall after [34, 2.1] the Bruhat decompositions of G. Let $P_m = MA_pN_m$ with $M = Z_K(A_p)$, the centralizer of A_p in K, be a minimal parabolic subgroup of G. Denote by P a parabolic subgroup of G containing P_m. Then, the unipotent radical N_P of P is a normal subgroup of P contained in N_m. Putting $L_P = \theta P \cap P$ (θ = the Cartan involution), we have a Levi decomposition $P = L_P \ltimes N_P$. Let $\bar{P} \equiv \theta P$ denote the parabolic subgroup opposite to P, then $\bar{P} = L_P \ltimes U_P$ with $U_P \equiv \theta N_P$.

We denote by W the Weyl group of (G, A_p): $W = M^*/M$ with $M^* = N_K(A_p)$, the normalizer of A_p in K. Then, the Weyl group $W(P)$ of (L_P, A_p) is canonically identified with the subgroup $(M^* \cap L_P)/M \subseteqq W$. Take a complete system W_P of representatives of the coset space $W/W(P)$. We may assume $W_P \ni 1$ = the unit element of W. Then, the Bruhat decomposition of G with respect to $(P_m, \bar{P})$ is given as

$$G = \coprod_{s \in W_P} G_s \quad \text{with} \quad G_s \equiv P_m s^* \bar{P}, \tag{3.7}$$

where s^* is any representative of $s \in W$ in M^*. Moreover, for each $s \in W$, the double coset G_s is expressed as

$$G_s = N_m s^* \bar{P} = (N_m \cap s^* N_P s^{*-1}) s^* \bar{P}, \tag{3.8}$$

because $P_m = (N_m \cap s^* N_P s^{*-1})(P_m \cap s^* \bar{P} s^{*-1})$. Every element of G_s is expressed uniquely as a product of elements of $N_m \cap s^* N_P s^{*-1}$ and $s^* \bar{P}$. G_s has a structure of a normal submanifold of G diffeomorphic to $(N_m \cap s^* N_P s^{*-1}) \times \bar{P}$ in the canonical way.

We put $C_s = s^* N_P \bar{P} = s^* G_1$ ($s \in W$). Then C_s is an open dense subset of G, and it admits a decomposition

$$C_s = (s^* N_P s^{*-1}) s^* \bar{P} = (U_m \cap s^* N_P s^{*-1})(N_m \cap s^* N_P s^{*-1}) s^* \bar{P}, \tag{3.9}$$

where $U_m = \theta N_m$. The right end is canonically diffeomorphic to the direct product $(U_m \cap s^* N_P s^{*-1}) \times G_s$. So in particular, C_s contains G_s as a closed submanifold.

Now we indroduce the spaces of Whittaker distributions with supports contained in Bruhat double cosets. Let N, $\eta = \exp \eta'$ be as in 3.1, $\chi \in \mathrm{Hom}_{alg}(Z(\mathfrak{g}_C), \boldsymbol{C})$ and $\kappa \in \boldsymbol{C}$. For an $s \in W_P$, we set

$$(3.10)\qquad \begin{cases} W(s;\eta':\chi)=\{T\in\mathscr{D}'(C_s;\eta':\chi);\ \mathrm{supp}(T)\subseteqq G_s\}, \\ W(s;\eta',\kappa)=\{T\in\mathscr{D}'(C_s;\eta',\kappa);\ \mathrm{supp}(T)\subseteqq G_s\}. \end{cases}$$

Namely, $W(s;\eta':\chi)$ (resp. $W(s;\eta',\kappa)$) is the space of χ-elementary (resp. κ-quasi-elementary) Whittaker distributions on C_s with supports contained in G_s.

If $s=1$, then $G_1=C_1$, whence $W(1;\eta':\chi)=\mathscr{D}'(C_1;\eta':\chi)$ and $W(1;\eta',\kappa)=\mathscr{D}'(C_1;\eta',\kappa)$. For any $T\in\mathscr{D}'(G;\eta':\chi)$, the restriction $T|G_1$ gives an element of $W(1;\eta':\chi)$. On the other hand, if $s\neq 1$, then $\dim G_s < \dim C_s = \dim G$. In such a case, we say that the distributions in $W(s;\eta',\kappa)$ ($\kappa\in\boldsymbol{C}$) have singular supports (with respect to $(P_m, \bar{P})$).

If there does not exist any non-zero (*quasi-*) elementary Whittaker distribution with singular support, the study of Whittaker distributions on the whole G becomes easy to some extent.

Proposition 3.1. *Under the above notation, assume that* $W(s;\eta':\chi)=(0)$ *for all* $s\in W_P$, $\neq 1$. *Then the restriction mapping* $\mathscr{D}'(G;\eta':\chi)\ni T\mapsto T|G_1\in W(1;\eta':\chi)$ *is injective. The same holds also for* $\mathscr{D}'(G;\eta',\kappa)$ *and* $W(1;\eta',\kappa)$.

Proof. For an $s\in W_P$, let $\tilde{C}_s$ be any open subset of G containing G_s as a closed submanifold. Replacing C_s in (3.10) by $\tilde{C}_s$, we can define the space $\tilde{W}(s;\eta':\chi)$ of χ-elementary Whittaker distributions on $\tilde{C}_s$ with supports in G_s. Take T from $W(s;\eta':\chi)$, and put $T'=T|(C_s\cap\tilde{C}_s)$. Since $\mathrm{supp}(T)\subseteqq G_s\subseteqq C_s\cap\tilde{C}_s$, T' can be extended uniquely to an element $\tilde{T}\in\tilde{W}(s;\eta':\chi)$. Through this assignment $T\mapsto\tilde{T}$, we get

$$(3.11)\qquad W(s;\eta':\chi)\simeq\tilde{W}(s;\eta':\chi)\qquad\text{(as vector spaces).}$$

Now put $\tilde{C}_s=C_s\cup(\bigcup_{s'}G_{s'})$ for $s\in W_P$, where s' runs through the elements of W_P such that $\dim G_{s'}>\dim G_s$. Then, $\tilde{C}_s$ is an open dense subset of G containing G_s as a closed submanifold (see [34, p. 272]), and $\tilde{C}_1=C_1$. In view of (3.11), it suffices to prove the assertion for $\tilde{W}(s;\eta':\chi)$. This is carried out exactly as in the proof of [34, Prop. 2.3].

The second assertion is proved analogously, thus we complete th proof. Q.E.D

Bearing this proposition in mind, we shall study in the succeeding section the spaces $W(s; \eta', \kappa)$ $(s \in W_{P'} \neq 1, \kappa \in C)$ of quasi-elementary Whittaker distributions with singular supports.

§ 4. Quasi-elementary Whittaker distributions on Bruhat cells

We estimate in this section the supports of quasi-elementary Whittaker distributions $T \in W(s; \eta', \kappa)$ (see 3.2) on Bruhat cells by a technique analogous to the one employed in our previous paper [34]. The main result here is Theorem 4.2. This extends, to a much more general setting, the result of Shalika [29, Prop. 2.10], which was crucial to get multiplicity one theorem for the original Gelfand-Graev representations (=GGRs) of quasi-split semisimple Lie groups. Theorem 4.2 enables us to prove multiplicity free property not only for the GGRs but also for the generalized ones (=GGGRs). Later, we shall utilize Theorem 4.2 to show the non-existence of non-zero quasi-elementary Whittaker distributions with singular supports, which is the key step toward our multiplicity one theorems.

4.1. Supports of distributions in $W(s; \eta', \kappa)$.
Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}=\mathrm{Lie}\, G$. We assume throughout this section the following two properties for $\mathfrak{n}$:

$$(4.1) \qquad \begin{cases} \text{(i)} & \mathfrak{n} \text{ is an ideal of the maximal nilpotent subalgebra} \\ & \mathfrak{n}_m=\mathrm{Lie}\, N_m, \\ \text{(ii)} & \mathfrak{n} \text{ is stable under } \mathrm{Ad}\,(A_p). \end{cases}$$

The condition (ii) means that $\mathfrak{n}$ is compatible with the root space decomposition of $\mathfrak{n}_m$: $\mathfrak{n}=\sum_{\lambda \in \Delta^+} (\mathfrak{n} \cap \mathfrak{g}(\mathfrak{a}_p; \lambda))$, where $\mathfrak{g}(\mathfrak{a}_p; \lambda)$ is the root space of a root λ.

Example 4.1. For a nilpotent class o of $\mathfrak{g}_C$ which intersects with $\mathfrak{g}$, the subalgebras $\mathfrak{n}(i)_o=\oplus_{j \geqq i} \mathfrak{g}(j)_o \subseteqq \mathfrak{n}_m$ $(i \geqq 1)$ satisfy (4.1), where $\mathfrak{g}=\oplus_{j \in Z} \mathfrak{g}(j)_o$ is the gradation of $\mathfrak{g}$ in 1.1.

Let η': $\mathfrak{n} \rightarrow C$ be a Lie algebra homomorphism. Fix a parabolic subgroup $P=L_P N_P$ containing P_m, and keep to the notation in 3.2. For an $s \in W_P$ and $\kappa \in C$, we consider the space $W(s; \eta', \kappa)$ of κ-quasi-elementary η'-Whittaker distributions on $C_s=s^* N_P \bar{P}$ $(\bar{P}=\theta P)$ with supports contained in the Bruhat coset $G_s=P_m s^* \bar{P} \subseteqq C_s$.

Let $D^s_{\eta'}$ denote the closed submanifold of $N_m \cap s^* N_P s^{*-1}$ consisting of $n \in N_m \cap s^* N_P s^{*-1}$ such that

(4.2) $\eta'(\mathrm{Ad}(n)Z)=0$ for all $Z \in \mathfrak{n} \cap \mathrm{Ad}(s^*)\mathfrak{u}_P$,

where $\mathfrak{u}_P \equiv \theta\mathfrak{n}_P$ with $\mathfrak{n}_P = \mathrm{Lie}\, N_P$.

Using an argument similar to the one in [34, Section 2], we can estimate the supports of Whittaker distributions on Bruhat cells as follows. Its proof will be given in 4.3–4.7.

Theorem 4.2. *For any $s \in W_P$, a Whittaker distribution in $W(s; \eta', \kappa)$ ($\kappa \in C$) has always its support contained in $D^s_{\eta'}s^*\bar{P}$.*

This is the main result of this section, which generalizes, in a certain sense, Theorem 2.11 of [34]. There, we gave a good estimation of supports of Whittaker distributions associated with Whittaker vectors for the (degenerate) principal series. This generalization enables us to deal with Whittaker distributions coming from *any* irreducible representation of G.

4.2. An application of Theorem 4.2.

Before proving this theorem, we give here an application.

Theorem 4.3. *Let $\eta': \mathfrak{n}_m \to C$ be a non-degenerate Lie algebra homomorphism of $\mathfrak{n}_m$, i.e., $\eta' | \mathfrak{g}(\mathfrak{a}_p; \lambda) \neq 0$ for any simple root λ. Consider the spaces $W(s; \eta', \kappa)$ ($s \in W_{P_m} = W$, $\kappa \in C$) of quasi-elementary Whittaker distributions on Bruhat cells with respect to the pair $(P_m, \bar{P}_m = \theta P_m)$ of minimal parabolics. Then,*

(1) *$W(s; \eta', \kappa) = (0)$ for any $s \neq 1$, $\in W$ and any $\kappa \in C$.*

(2) *The restriction map $\mathscr{D}'(G; \eta', \kappa) \ni T \to T | G_1 \in W(1; \eta', \kappa)$ is injective, where $G_1 = N_m\bar{P}_m$ is open dense in G.*

Proof. We see easily from the non-degeneracy of η' that $D^s_{\eta'}$ is empty for any $s \neq 1$, $\in W$. Then, the assertions follow from Proposition 3.1 and Theorem 4.2. Q.E.D.

The above theorem includes, as a special case $G = GL_n$, the result of Shalika [29, Prop. 2.10]. Although we state here our result for connected semisimple Lie groups for simplicity, Theorems 4.2 and 4.3 remain true for reductive Lie groups, including GL_n as a special case.

Moreover, Theorem 4.3 can be used to get another proof of Corollary 2.12 of [34], in which we gave a nice (maybe best possible) upper bound for multiplicities of the most continuous principal series in GGRs.

The rest of this section is devoted to proving Theorem 4.2.

4.3. The Casimir operator L_Ω on C_s.

Until the end of this section, we fix, $\mathfrak{n}$, η', $P = L_P N_P$ and $s = s^*M \in W_P \subseteq W$. For $x, y \in G$ and $D \in U(\mathfrak{g}_C)$, we put ${}^y x = yxy^{-1}$, ${}^y D = \mathrm{Ad}(y)D$ for short.

For later use, we prepare here an explicit expression of the Casimir operator of G on the open dense subset C_s. Let us now define vector fields $\partial(X)$ $(X \in {}^{s*}\mathfrak{n}_P)$ and $\partial'(Y)$ $(Y \in {}^{s*}\bar{\mathfrak{p}})$ on $C_s = ({}^{s*}N_P)s^*\bar{P}$, which is canonically diffeomorphic to ${}^{s*}N_P \times \bar{P}$, respectively by

$$(4.3) \qquad \begin{cases} \partial(X)_z \phi = \dfrac{d}{dt} \phi(n' \exp(tX)s^*\bar{p})\,|_{t=0}, \\ \partial'(Y)_z \phi = \dfrac{d}{dt} \phi(n' \exp(-tY)s^*\bar{p})\,|_{t=0}, \end{cases}$$

for $z = n's^*\bar{p} \in ({}^{s*}N_P)s^*\bar{P} = C_s$ and $\phi \in \mathscr{E}(C_s) = C^\infty(C_s)$. Here $\bar{\mathfrak{p}}$ denotes the Lie algebra of $\bar{P}$. Then, $X \mapsto \partial(X)$ (resp. $Y \mapsto \partial'(Y)$) extends uniquely to an isomorphism from the enveloping algebra $U(({}^{s*}\mathfrak{n}_P)_C)$ (resp. $U(({}^{s*}\bar{\mathfrak{p}})_C)$) into the algebra $\mathrm{Diff}(C_s)$ of differential operators on C_s. This extension will be denoted still by ∂ (resp. ∂'). By definition, $\partial(X)$ commutes with $\partial'(Y)$. The tangent space $T_z(C_s)$ of C_s at the point z is decomposed as

$$(4.4) \qquad T_z(C_s) = \partial({}^{s*}\mathfrak{n}_P)_z \oplus \partial'({}^{s*}\bar{\mathfrak{p}})_z.$$

Now assume that $z = n's^*\bar{p}$ lies in the closed submanifold $G_s = (N_m \cap {}^{s*}N_P)s^*\bar{P}$, or equivalently $n' \in N_m \cap {}^{s*}N_P$. We put

$$(4.5) \qquad \mathfrak{e} = \mathfrak{n}_m \cap {}^{s*}\mathfrak{n}_P, \qquad \mathfrak{f} = \mathfrak{u}_m \cap {}^{s*}\mathfrak{n}_P,$$

where $\mathfrak{u}_m = \theta\mathfrak{n}_m$. Then, in view of (3.9) one gets

$$(4.6) \qquad T_z(C_s) = \partial(\mathfrak{f})_z \oplus T_z(G_s), \qquad T_z(G_s) = \partial(\mathfrak{e})_z \oplus \partial'({}^{s*}\bar{\mathfrak{p}})_z,$$

where $T_z(G_s) \subseteq T_z(C_s)$ is the tangent space of G_s at z. The first equality in (4.6) means by definition that $\partial(\mathfrak{f})$ is transversal to G_s in the sense of [29, p. 180].

We express here the Casimir operator L_Ω on C_s by means of the differential operators in $\partial(U(({}^{s*}\mathfrak{n}_P)_C))$ and $\partial'(U(({}^{s*}\bar{\mathfrak{p}})_C))$. For this purpose, construct the Casimir element $\Omega \in U(\mathfrak{g}_C)$ explicitly as follows. Consider the positive definite inner product $\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto -B(X, \theta Y)$, with the Killing form B of $\mathfrak{g}$. For each $\lambda \in \Lambda^+$, take an orthonormal basis $\{E_\lambda^p;\ 1 \leqq p \leqq d(\lambda) \equiv \dim \mathfrak{g}(\mathfrak{a}_p; \lambda)\}$ of $\mathfrak{g}(\mathfrak{a}_p; \lambda)$ with respect to this inner product. Furthermore, $\{H_j';\ 1 \leqq j \leqq \dim \mathfrak{m}\}$ (resp. $\{H_k;\ 1 \leqq k \leqq \dim \mathfrak{a}_p\}$) denotes an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{a}_p$). Put $F_\lambda^p = -\theta E_\lambda^p$. Then, $\Omega \in Z(\mathfrak{g}_C)$ is expressed as

$$(4.7) \qquad \Omega = 2\Sigma_{\lambda \in \Lambda^+, 1 \leqq p \leqq d(\lambda)} F_\lambda^p E_\lambda^p - \sum_j H_j'^2 + \sum_k H_k^2 + H_{2\delta},$$

where $\delta = 2^{-1} \sum_{\lambda \in \Lambda^+} d(\lambda) \cdot \lambda$ and, for a $\tau \in \mathfrak{a}_p^*$, $H_\tau \in \mathfrak{a}_p$ is such that $B(H, H_\tau) = \tau(H)$ $(H \in \mathfrak{a}_p)$.

We put

$$I_1=\{\lambda\in\Lambda^+;\ \mathfrak{g}(\mathfrak{a}_p;\ -\lambda)\subseteqq\mathfrak{f}\},\qquad I_2=\{\lambda\in\Lambda^+;\ \mathfrak{g}(\mathfrak{a}_p;\ \lambda)\subseteqq\mathfrak{e}\}$$

and

$$I_3=\Lambda^+\setminus(I_1\cup I_2).$$

Then one gets the following

Lemma 4.4. *The Casimir operator L_Ω is expressed on the open dense subset C_s as*

$$\begin{aligned}L_\Omega= & -2\textstyle\sum_{\lambda\in I_1,1\leqq p\leqq d(\lambda)}\partial(F_\lambda^p)\partial'(E_\lambda^p)-2\sum_{\lambda\in I_2,1\leqq p\leqq d(\lambda)}\partial(E_\lambda^p)\partial'(F_\lambda^p)\\ & +2\textstyle\sum_{\lambda\in I_3,1\leqq p\leqq d(\lambda)}\partial'(F_\lambda^p)\partial'(E_\lambda^p)\\ & -\textstyle\sum_j\partial'(H_j')^2\\ & +\textstyle\sum_k\partial'(H_k)^2+\partial'(H_\tau),\end{aligned}\tag{4.8}$$

where $\tau=2\delta-\sum_{\lambda\in I_1}d(\lambda)\cdot\lambda$.

Proof. For a differential operator D defined on a neighbourhood of the unit element 1 of G, we define $\tilde{D}\in\mathrm{Diff}(C_s)$ by

$$(\tilde{D}\phi)(n's^*\bar{p})=((D\circ L_{n'^{-1}}\circ R_{s^*\bar{p}})\phi)(1)\tag{4.9}$$

for $\phi\in\mathscr{E}(C_s)$ and $n's^*\bar{p}\in({}^{s*}N_P)s^*\bar{P}$. (Note that $n'\in{}^{s*}N_P$ is an analytic function in $z=n's^*\bar{p}\in C_s$.) Since $\Omega\in Z(\mathfrak{g}_C)$, one gets $(L_\Omega)^\sim=L_\Omega$ on C_s. In view of (4.7), let us calculate $(L_{F_\lambda^pE_\lambda^p})^\sim$, $((L_{H_j'})^2)^\sim$, etc.

First, assume that $\lambda\in I_1$. By the definition of I_1, we have $F_\lambda^p\in\mathfrak{f}\subseteqq{}^{s*}\mathfrak{n}_P$ $(1\leqq p\leqq d(\lambda))$, whence $E_\lambda^p\in{}^{s*}\bar{\mathfrak{p}}$. Notice that $F_\lambda^pE_\lambda^p=E_\lambda^pF_\lambda^p-H_\lambda$. Then keeping (4.3) in mind, we see easily

$$(L_{F_\lambda^pE_\lambda^p})^\sim=-\partial(F_\lambda^p)\partial'(E_\lambda^p)-\partial'(H_\lambda)\qquad(\lambda\in I_1).\tag{4.10}$$

Similarly one gets

$$(L_{F_\lambda^pE_\lambda^p})^\sim=\begin{cases}-\partial(E_\lambda^p)\partial'(F_\lambda^p) & (\lambda\in I_2),\\ \partial'(F_\lambda^p)\partial'(E_\lambda^p) & (\lambda\in I_3),\end{cases}\tag{4.11}$$

$$((L_{H_j'})^2)^\sim=\partial'(H_j')^2,\qquad((L_{H_k})^2)^\sim=\partial'(H_k)^2.\tag{4.12}$$

The equalities (4.10)–(4.12) give the expression (4.8). Q.E.D.

4.4. Local expression of distributions with singular supports.

Let T be a distribution on C_s such that $\mathrm{supp}(T)\subseteqq G_s$. Then, T can be expressed locally as a finite linear combination of transversal derivatives of distributions on G_s. We give here such an expression. For this purpose, fix a linear order $>$ on $\mathfrak{a}_p^*$ such that $\Lambda^+=\{\lambda\in\Lambda;\ \lambda>0\}$. For a sequence $\gamma=(\gamma(\lambda, p))$ of non-negative integers $\gamma(\lambda, p)$ ($\lambda \in I_1$, $1 \leqq p \leqq d(\lambda)$), called a multi-index later on, we put

$$F^{\gamma}=\prod{}'(F_{\lambda}^{p})^{\gamma(\lambda,p)} \in U(\mathfrak{k}_C). \tag{4.13}$$

Here, the product $\prod'$ is taken in such a way that the term $(F_{\lambda'}^{p'})^{\gamma(\lambda',p')}$ always sits on the left of $(F_{\lambda}^{p})^{\gamma(\lambda,p)}$ if either $\lambda'>\lambda$ or $\lambda'=\lambda$, $p'>p$. By the Poincaré-Birkhoff-Witt theorem, these F^{γ}'s form a basis of $U(\mathfrak{k}_C)$.

Now take $z_0=n_0s^*\bar{p}_0 \in G_s$ with $n_0 \in N_m \cap {}^{s*}N_P$, $p_0 \in \bar{P}$, from the support of T. Then, according to Schwartz (cf. [29, Lemma 2.4]), there exists an open nieghbourhood $O_{z_0} \subseteqq C_s$ of z_0 on which $T_0=T \mid O_{z_0}$ is expressed uniquely as

$$T_0=\sum_{\gamma} \partial(F^{\gamma})\bar{T}_{\gamma} \qquad \text{(finite sum)} \tag{4.14}$$

with $T_{\gamma} \in \mathscr{D}'(G_s[z_0])$, $G_s[z_0] \equiv G_s \cap O_{z_0}$. Here, for an $S \in \mathscr{D}'(G_s[z_0])$, $\bar{S}$ denotes its trivial extension to O_{z_0}:

$$\langle \bar{S}, \phi \rangle=\langle S, \phi \mid G_s[z_0] \rangle \qquad (\phi \in \mathscr{D}(O_{z_0})). \tag{4.15}$$

Replacing O_{z_0} by a smaller neighbourhood of z_0 if necessary, we may (and do) assume:

(i) $O_{z_0}=\mathscr{N}_0\mathscr{U}s^*\bar{\mathscr{P}}_0$, where $\mathscr{N}_0$ (resp. $\mathscr{U}$, $\bar{\mathscr{P}}_0$) is an open neighbourhood of n_0 (resp. 1, $\bar{p}_0$) in $N_m \cap {}^{s*}N_P$ (resp. in $U_m \cap {}^{s*}N_P$, in $\bar{P}$), whence $G_s[z_0]=\mathscr{N}_0s^*\bar{\mathscr{P}}_0$,

(ii) $z_0 \in \operatorname{supp}(T_{\gamma})$ for all non-zero T_{γ}'s.

Let a denote the largest integer attained by $|\gamma| \equiv \sum\gamma(\lambda, p)=a$ for $T_{\gamma} \neq 0$. Then, a is independent of the choice of a neighbourhood O_{z_0} with the above properties, which is said to be the transversal order of T at z_0.

Now let $D \in U(\mathfrak{e}_C)$ and $D' \in U({}^{s*}\bar{\mathfrak{p}}_C)$. Since $\partial(D)$ and $\partial'(D')$ are tangential to G_s from the second equality of (4.6), the restrictions of $\partial(D)$ and $\partial'(D')$ onto the open subset $G_s[z_0] \subseteqq G_s$ naturally give rise to differential operators on $G_s[z_0]$, which are denoted still by $\partial(D')$ and $\partial'(D')$ respectively. Note that $G_s=(N_m \cap {}^{s*}N_P)s^*\bar{P}$ is canonically diffeomorphic to the direct product $(N_m \cap {}^{s*}N^P) \times \bar{P}$. We will consider the action of Diff$(G_s[z_0])$ on $\mathscr{D}'(G_s[z_0])$ as in 2.1, through the measure $dnd\bar{p}$ restricted to $G_s[z_0]$. Here dn and $d\bar{p}$ denote respectively right Haar measures on $N_m \cap {}^{s*} N_P$ and on $\bar{P}$.

Now suppose that $T \in W(s; \eta', \kappa)$ for $\kappa \in \boldsymbol{C}$. We henceforth go to the local situation around $z_0 \in \operatorname{supp}(T)$, and study $T_0=T \mid O_{z_0}$ instead of the global T itself.

4.5. The condition $L_{\Omega}T_0=\kappa T_0$.

First, we rewrite this condition for T_0 on O_{z_0} to that for T_{γ}'s on $G_s[z_0]$.

Proposition 4.5. *The distribution T_γ on $G_s[z_0]$ coming from $T \in W(s;\eta',\kappa)$, satisfies the relation*

$$\sum_{\lambda\in I_1, 1\leqq p\leqq d(\lambda)} \partial'(E_\lambda^p) T_{\gamma-\delta[\lambda,p]} = 0 \tag{4.16}$$

for any multi-index γ such that $|\gamma| = a+1$ (a = the transversal order of T at z_0). Here we put

$$\begin{cases} \delta[\lambda, p] = (\delta^{\lambda p}_{\lambda' p'})_{\lambda', p'}, \\ \delta^{\lambda p}_{\lambda' p'} = 1 \quad \text{if } (\lambda', p') = (\lambda, p), \quad \delta^{\lambda p}_{\lambda' p'} = 0 \quad \text{otherwise}. \end{cases} \tag{4.17}$$

Proof. Let us calculate $L_\Omega T_0$. In view of Lemma 4.4 and (4.14), one gets easily

$$L_\Omega T_0 = -2\sum_{|\gamma|=a}\sum_{\lambda\in I_1, p} \partial(F^{\gamma+\delta[\lambda,p]})\overline{\partial'(E_\lambda^p)}T_\gamma + T', \tag{4.18}$$

where $T' \in \mathscr{D}'(O_{z_0})$ is such that $\operatorname{supp}(T') \subseteqq G_s[z_0]$ and that its transversal order at z_0 is less than $a+1$. Here we used the following fact: *Let $D \in$* Diff(O_{z_0}) *be tangential to $G_s[z_0]$. If $S \in \mathscr{D}'(O_{z_0})$ such that* supp$(S) \subseteqq G_s[z_0]$, *then the transversal order of DS at z_0 does not exceed that of S* (see [29, p. 181]).

Notice the uniqueness of the local expression (4.14). Then, the condition $L_\Omega T_0 = \kappa T_0$ together with (4.18) yields the desired property (4.16).

Q.E.D.

4.6. The condition $L_Z T_0 = -\eta'(Z)T_0$ ($Z \in \mathfrak{n}$).

Secondly, we wish to rewrite the condition of left η'-quasi-invariancy for T_0. By virtue of the assumption (4.1, ii), $\mathfrak{n}$ admits a direct sum decomposition as vector space:

$$\mathfrak{n} = (\mathfrak{n} \cap {}^{s*}\mathfrak{n}_P) \oplus (\mathfrak{n} \cap {}^{s*}\bar{\mathfrak{p}}). \tag{4.19}$$

So, we consider the following two cases separately.

4.6.1. *Case of $Z \in \mathfrak{n} \cap {}^{s*}\mathfrak{n}_P$.* Then, L_Z is tangential to G_s, and it commutes with $\partial(F^\gamma)$'s since $(\mathfrak{n} \cap {}^{s*}\mathfrak{n}_P) \cap \mathfrak{f} = (0)$. Thus, $L_Z T_0 = -\eta'(Z)T_0$ is equivalent to

$$L_Z T_\gamma = -\eta'(Z)T_\gamma \qquad \text{for all } \gamma.$$

4.6.2. *Case of $Z \in \mathfrak{n} \cap {}^{s*}\bar{\mathfrak{p}}$.* In this case, L_Z does not in general commute with the transversal differential operators $\partial(F^\gamma)$, so we need to calculate brackets of these two types of differential operators. For this purpose, we return to the global situation on C_s for a while.

For a differential operator D on a neighbourhood of $1 \in G$, we put

(4.20) $$(D^\dagger\phi)(z)=((D\circ L_{n^{-1}}\circ R_{us^*\bar{p}})\phi)(1) \qquad (\phi\in\mathscr{E}(C_s)),$$

where $z=n\cdot u\cdot s^*\cdot\bar{p}=nus^*\bar{p}$ with $n\in N_m\cap{}^{s^*}N_P$, $u\in U_m\cap{}^{s^*}N_P$ and $\bar{p}\in\bar{P}$. Then $D^\dagger$ gives a differential operator on C_s. (Compare with $\tilde{D}$ in (4.9).) Bearing the assumption (4.1, i) in mind, we see easily that the condition (3.4) for T is equivalent to

(4.21) $$(L_Z)^\dagger T=-\psi_Z T \qquad (Z\in\mathfrak{n}),$$

where $\psi_Z\in\mathscr{E}(C_s)$ is defined by

(4.22) $$\psi_Z(z)=\eta'({}^nZ) \qquad \text{for } z=nus^*\bar{p}\in C_s.$$

Now let Q denote the ring of polynomial functions on $\mathfrak{f}$, and Q_+ the maximal ideal of Q consisting of $h\in Q$ without constant term. Identifying $\mathfrak{f}$ with $U_m\cap{}^{s^*}N_P$ through the exponential mapping, we regard each $h\in Q$ as a function on the nilpotent Lie group $U_m\cap{}^{s^*}N_P$. Further, we extend each h to a C^∞-function on $C_s=(N_m\cap{}^{s^*}N_P)(U_m\cap{}^{s^*}N_P)s^*\bar{P}$ via $C_s\ni nus^*\bar{p}\mapsto h(u)$, which will be denoted still by h for simplicity.

Under these notations, one gets

Lemma 4.6. *Let $Z\in{}^{s^*}\bar{\mathfrak{p}}$ and $F\in\mathfrak{f}$. Then, the bracket of differential operators $\partial(F)$ and $(L_Z)^\dagger$ is given as*

(4.23) $$[\partial(F),(L_Z)^\dagger]\equiv-(L_{[F,Z]})^\sim$$

modulo $Q_+\partial({}^{s^*}\mathfrak{n}_P)\oplus Q_+\partial'({}^{s^*}\bar{\mathfrak{p}})\subseteq\mathrm{Diff}(C_s)$, where $(L_{[F,Z]})^\sim$ is as in (4.9).

Proof. Fix a basis (Z_n) of $\mathfrak{g}$. For each integer $j\geqq 0$, put $((-1)^j/j!)\cdot(\mathrm{ad}\,V)^jZ=\sum_n a_j^n(V)Z_n$ $(V\in\mathfrak{f})$. Clearly, a_j^n is a homogeneous polynomial on $\mathfrak{f}$ of degree j, especially $a_j^n\in Q$. By definitions of $\tilde{D}$ and $D^\dagger$ $(D\in\mathrm{Diff}(G))$, we see easily

$$(L_Z)^\dagger=\sum_{j\geqq 0}\sum_n a_j^n\cdot(L_{Z_n})^\sim, \qquad \partial(F)=-(L_F)^\sim.$$

Hence the bracket in question is calculated as

$$\begin{aligned}[\partial(F),(L_Z)^\dagger]&=-\sum_{j,n}[(L_F)^\sim,\ a_j^n\cdot(L_{Z_n})^\sim]\\&=-\sum_{j,n}\{((L_F)^\sim a_j^n)\cdot(L_{Z_n})^\sim+a_j^n\cdot[(L_F)^\sim,(L_{Z_n})^\sim]\}\\&\quad\cdots\cdots\\&\equiv-\sum_n((L_F)^\sim a_1^n)\cdot(L_{Z_n})^\sim\ (\mathrm{mod.}\ Q_+\partial({}^{s^*}\mathfrak{n}_P)\oplus Q_+\partial'({}^{s^*}\bar{\mathfrak{p}})).\end{aligned}$$

For the last equality, we used the facts

$$a_j^n\in Q_+ \qquad \text{if } j\geqq 1,$$
$$(L_F)^\sim a_j^n\in Q_+\cup(0) \qquad \text{if } j\neq 1,$$
$$\sum_n a_0^n\cdot[(L_F)^\sim,(L_{Z_n})^\sim]=[\partial'(Z),\partial(F)]=0 \qquad \text{(see 4.2).}$$

From the definition of a_1^n, $(L_F)^{\sim}a_1^n$ is the constant determined by

$$[F, Z]=\sum_n ((L_F)^{\sim}a_1^n)\cdot Z_n.$$

Consequently, we deduce

$$[\partial(F), (L_Z)^{\dagger}]=-\sum_n ((L_F)^{\sim}a_1^n)\cdot(L_{Z_n})^{\sim}=-(L_{[F,Z]})^{\sim}$$

as desired. Q.E.D.

We need one more lemma.

Lemma 4.7. *Let $D \in Q_+\partial({}^{s*}\mathfrak{n}_P)\oplus Q_+\partial'({}^{s*}\bar{\mathfrak{p}})$. Then, for each $S \in \mathscr{D}'(G_s)$, there exists a constant $c_{D,S} \in \boldsymbol{C}$ such that $D\bar{S}=c_{D,S}\bar{S}$.*

Proof. Let $h \in Q_+$ and $Y \in \mathfrak{g}={}^{s*}\mathfrak{n}_P\oplus{}^{s*}\bar{\mathfrak{p}}$.

Case 1. If $Y \in {}^{s*}\bar{\mathfrak{p}}$, then $\partial'(Y)h=0$. Hence we get $h\cdot\partial'(Y)\bar{S}=\partial'(Y)\cdot h\cdot\bar{S}=0$ since $h \in Q_+$ is identically zero on G_s.

Case 2. If $Y \in \mathfrak{e}=\mathfrak{n}_m\cap{}^{s*}\mathfrak{n}_P$, then $\partial(Y)$ is tangential to G_s. Hence, $h\cdot\partial(Y)\bar{S}=h\cdot\overline{\partial(Y)S}=0$.

Case 3. If $Y \in \mathfrak{f}=\mathfrak{u}_m\cap{}^{s*}\mathfrak{n}_P$, then $h\cdot\partial(Y)\bar{S}=\partial(Y)\cdot h\cdot\bar{S}+\partial(Y)h\cdot\bar{S}=(\partial(Y)h)(1)\bar{S}$.

With the decomposition ${}^{s*}\mathfrak{n}_P=\mathfrak{e}\oplus\mathfrak{f}$ in mind, we obtain the lemma. Q.E.D.

Now let us consider the local expression (4.14) of T on O_{z_0}. From (4.21), we get the following condition on $T_\gamma \in \mathscr{D}'(G_s[z_0])$.

Proposition 4.8. *Let $Z \in \mathfrak{n}\cap{}^{s*}\bar{\mathfrak{p}}$. For any multi-index $\gamma=(\gamma(\lambda, p))_{\lambda\in I_1, 1\leqq p\leqq d(\lambda)}$ such that $|\gamma|=a$ (=the transversal order of T at z_0), T_γ satisfies*

$$\partial'(Z)T_\gamma=-\psi_Z T_\gamma+\sum C_{\lambda'p'}^{\lambda p}(\gamma(\lambda, p)+1)T_{\gamma+\delta[\lambda,p]-\delta[\lambda',p']}, \tag{4.24}$$

where the sum is over $(\lambda, p, \lambda', p')$ with $\lambda, \lambda' \in I_1$, $1\leqq p\leqq d(\lambda)$, $1\leqq p'\leqq d(\lambda')$. Here the constants $C_{\lambda'p'}^{\lambda p}$ are determined by

$$[Z, F_\lambda^p]=\sum_{\lambda',p'}C_{\lambda'p'}^{\lambda p}F_{\lambda'}^{p'} \qquad (\text{mod. } \mathfrak{e}\oplus{}^{s*}\bar{\mathfrak{p}}), \tag{4.25}$$

and ψ_Z is as in (4.21).

Proof. For a multi-index γ, let us calculate $(L_Z)^{\dagger}\partial(F^\gamma)\bar{T}_\gamma$. By definitions of $\tilde{D}$ and $D^{\dagger}$ ($D \in \mathrm{Diff}(G)$), it is easy to see that $(L_Z)^{\dagger}\bar{T}_\gamma=\overline{\partial'(Z)T_\gamma}$. Lemmas 4.6 and 4.7 together with this equality imply that

$$(4.26)\qquad \begin{aligned}&(L_Z)^\dagger\partial(F^\gamma)\overline{T_\gamma}\\&=\partial(F^\gamma)\overline{\partial'(Z)T_\gamma}-\textstyle\sum_{\lambda,p}\gamma(\lambda,p)\partial(F^{\gamma-\delta[\lambda,p]})(L_{[Z,F^p_\lambda]})^{\sim}\overline{T_\gamma}+\overline{D'T_\gamma},\end{aligned}$$

where $D'\in\mathrm{Diff}(O_{z_0})$ has the order less than $|\gamma|$. By using the expansion (4.25), the right hand side of (4.26) turns to be

$$\partial(F^\gamma)\overline{\partial'(Z)T_\gamma}-\textstyle\sum_{\lambda,\lambda',p,p'}C^{\lambda p}_{\lambda' p'}\gamma(\lambda,p)\partial(F^{\gamma-\delta[\lambda,p]+\delta[\lambda',p']})\overline{T_\gamma}+\sum_{|\gamma'|<|\gamma|}\partial(F^{\gamma'})\overline{S_{\gamma'}}$$

for some $S_{\gamma'}\in\mathscr{D}'(G_s[z_0])$. Consequently, we have

$$(4.27)\qquad (L_Z)^\dagger T_0=\textstyle\sum_{|\gamma|\leq a}\partial(F^\gamma)T'_\gamma$$

with $T'_\gamma\in\mathscr{D}'(G_s[z_0])$ given as

$$(4.28)\qquad T'_\gamma=\partial'(Z)T_\gamma-\textstyle\sum C^{\lambda p}_{\lambda' p'}(\gamma(\lambda,p)+1)T_{\gamma+\delta[\lambda,p]-\delta[\lambda',p']}$$

for any γ such that $|\gamma|=a$. Taking into the uniqueness of the local expression (4.14), we thus obtain the desired equality (4.24), from (4.27), (4.28) and the condition (4.21). Q.E.D.

4.7. Proof of Theorem 4.2.

Making use of the results of the subsections 4.3–4.6, we can now prove Theorem 4.2, the main result of this section. Let $T\in W(s;\eta',\kappa)$ and $z_0\in\mathrm{supp}(T)\subseteq G_s$. Keep to the notation in the previous subsections. We will show that z_0 necessarily lies in the subset $D^s_{\eta'}\cdot s^*P$ with $D^s_{\eta'}$ as in (4.2). In terms of the function ψ_Z on C_s introduced in (4.22), this condition means that

$$\psi_Z(z_0)=0\qquad\text{for all } Z\in\mathfrak{n}\cap{}^{s*}\mathfrak{u}_P\subseteq\mathfrak{n}\cap{}^{s*}\mathfrak{p}.$$

Fix $\lambda_0\in\Lambda^+$ such that $\mathfrak{g}(\mathfrak{a}_p;\lambda_0)\subseteq\mathfrak{n}\cap{}^{s*}\mathfrak{u}_P$, and p_0, $1\leq p_0\leq d(\lambda_0)$. We put $E=E^{p_0}_{\lambda_0}$. Then it suffices to show that $\psi_E(z_0)=0$. This is done in the following way.

First, let us introduce a lexicographic order on the set of multi-indices γ as follows: $\gamma_1>\gamma_2$ if and only if there exists a pair (λ,p) such that

(i) $\gamma_1(\lambda',p')=\gamma_2(\lambda',p')$ if either $\lambda'>\lambda$ or $\lambda=\lambda'$, $p'>p$,

(ii) $\gamma_1(\lambda,p)>\gamma_2(\lambda,p)$.

We put

$$b=\max\{\gamma(\lambda_0,p_0);\ |\gamma|=a,\ z_0\in\mathrm{supp}(T_\gamma)\}.$$

Let $\mathscr{J}$ denote the totality of multi-indices $\gamma=(\gamma(\lambda,p))$ such that $\gamma(\lambda_0,p_0)=b$, $|\gamma|=a$ and $z_0\in\mathrm{supp}(T_\gamma)$. From now on, γ denotes the largest element of $\mathscr{J}$.

Applying Proposition 4.5 to $\gamma+\delta[\lambda_0, p_0]$, we get

$$\sum_{\lambda \in I_1, 1 \leqq p \leqq d(\lambda)} \partial'(E_\lambda^p) T_{\gamma+\delta[\lambda_0, p_0]-\delta[\lambda, p]} = 0.$$

From the definition of $\mathscr{J}$, one has $z_0 \notin \operatorname{supp}(T_{\gamma+\delta[\lambda_0, p_0]-\delta[\lambda, p]})$ if $(\lambda, p) \neq (\lambda_0, p_0)$. So, in a sufficiently small neighbourhood of z_0 in G_s, there holds that

$$\partial'(E) T_\gamma = 0. \tag{4.29}$$

Second, we apply Proposition 4.8 putting $Z=E$. Then,

$$\partial'(E) T_\gamma = -\psi_E T_\gamma + \sum C_{p'}^{\lambda p} (\gamma(\lambda, p)+1) T_{\gamma+\delta[\lambda, p]-\delta[\lambda-\lambda_0, p']}. \tag{4.30}$$

Here, the sum is over (λ, p, p') with $\lambda \in I_1+\lambda_0$, $1 \leqq p \leqq d(\lambda)$, $1 \leqq p' \leqq d(\lambda-\lambda_0)$, and the coefficients $C_{p'}^{\lambda p}$ are defined by

$$[E, F_\lambda^p] = \sum_{1 \leqq p' \leqq d(\lambda-\lambda_0)} C_{p'}^{\lambda p} F_{\lambda-\lambda_0}^{p'}. \tag{4.31}$$

Let us investigate each term in the summand of (4.31) separately.

Case 1. If $(\lambda-\lambda_0, p') \neq (\lambda_0, p_0)$, then $(\gamma+\delta[\lambda, p]-\delta[\lambda-\lambda_0, p'])(\lambda_0, p_0) = \gamma(\lambda_0, p_0) = b$ and $\gamma+\delta[\lambda, p]-\delta[\lambda-\lambda_0, p'] > \gamma$. By the maximality of $\gamma \in \mathscr{J}$, we obtain

$$T_{\gamma+\delta[\lambda, p]-\delta[\lambda-\lambda_0, p']} = 0 \qquad \text{near } z_0 \in \operatorname{supp}(T_\gamma).$$

Case 2. Assume $(\lambda-\lambda_0, p') = (\lambda_0, p_0)$, or equivalently $\lambda = 2\lambda_0$ and $p' = p_0$. In this case we have for any $1 \leqq p \leqq d(\lambda)$,

$$\begin{aligned} 0 &= B([E, E], F_\lambda^p) = B(E, [E, F_\lambda^p]) \\ &= \sum_{p''} C_{p''}^{\lambda p} B(E, F_{\lambda_0}^{p''}) = C_{p_0}^{\lambda p}, \end{aligned}$$

because $B(E, F_{\lambda_0}^{p''}) = B(E_{\lambda_0}^{p_0}, F_{\lambda_0}^{p''}) = \delta_{p'', p_0}$ (Kronecker's δ).

(4.29) and (4.30) together with the results of above two cases imply that $\psi_E T_\gamma = 0$ near $z_0 \in \operatorname{supp}(T_\gamma)$. Consequently, we deduce $\psi_E(z_0) = 0$, which completes the proof of Theorem 4.2. Q.E.D.

§ 5. Important types of generalized Gelfand-Graev representations and their finite multiplicity property

In this section, we first construct after [II] important classes of generalized Gelfand-Graev representations (=GGGRs) closely related with the regular representation. Then we quote from that paper finite multiplicity theorems for reduced GGGRs coming from these important GGGRs under the names of Theorems 5.4 and 5.5. These theorems are, in a sense, analogous to Harish-Chandra's finite multiplicity theorem for induced representations $\mathrm{Ind}_K^G(\tau)$ ($\tau \in \hat{K}$). We shall clarify in 5.4 the similarity between them.

In the succeeding sections, we shall concentrate on these (reduced) GGGRs. To be more precise, we give in Section 6 multiplicity one theorems for reduced GGGRs, and Part II (Sections 7–12) is devoted to describing embeddings of irreducible highest weight representations into GGGRs.

5.1. Simple Lie groups of hermitian type.

Let G be a connected non-compact simple Lie group with finite center. Hereafter, we always assume that G/K carries a structure of hermitian symmetric space, where K is a maximal compact subgroup of G. We recall here after [II, 3.1] refined structure theorems due to Moore, for such a simple Lie group G and its Lie algebra $\mathfrak{g}$, which will be heavily utilized from Section 6 onward.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, $\mathfrak{k}\equiv\mathrm{Lie}\,K$, be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of G. The given G-invariant complex structure on G/K naturally gives rise to an $\mathrm{Ad}(K)$-invariant complex structure J on $\mathfrak{p}$, which can be expressed as $J=\mathrm{ad}(Z_0)|\mathfrak{p}$ for a uniquely determined central element Z_0 of $\mathfrak{k}$. (Note that the center of $\mathfrak{k}$ is one-dimensional under the above assumption on G.) Extending J to a linear map on $\mathfrak{p}_C$ by complex linearity, we put $\mathfrak{p}_\pm=\{X \in \mathfrak{p}_C;\ JX=\pm\sqrt{-1}X\}$. Then, $\mathfrak{g}_C$ admits the decomposition

$$\mathfrak{g}_C=\mathfrak{p}_-\oplus\mathfrak{k}_C\oplus\mathfrak{p}_+ \tag{5.1}$$

such that $[\mathfrak{k}_C, \mathfrak{p}_\pm]\subseteqq\mathfrak{p}_\pm$, $[\mathfrak{p}_+, \mathfrak{p}_+]=[\mathfrak{p}_-, \mathfrak{p}_-]=(0)$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then it is also a Cartan subalgebra of $\mathfrak{g}$. Denote by Σ the root system of $(\mathfrak{g}_C, \mathfrak{t}_C)$. Let $\Sigma_\mathfrak{k}$ (resp. $\Sigma_\mathfrak{p}$) be the subset of compact (resp. non-compact) roots in Σ:

$$\Sigma_\mathfrak{k}=\{\gamma \in \Sigma;\ \mathfrak{g}_C(\mathfrak{t}_C;\gamma)\subseteqq\mathfrak{k}_C\}, \qquad \Sigma_\mathfrak{p}=\{\gamma \in \Sigma;\ \mathfrak{g}_C(\mathfrak{t}_C;\gamma)\subseteqq\mathfrak{p}_C\},$$

where $\mathfrak{g}_C(\mathfrak{t}_C;\gamma)$ is the root space of a $\gamma \in \Sigma$. Then we have $\Sigma=\Sigma_\mathfrak{k}\cup\Sigma_\mathfrak{p}$ (disjoint union). We can take a positive system Σ^+ of Σ consistent with the decomposition (5.1) in the following sense:

$$\mathfrak{p}_\pm=\oplus_{\gamma\in\Sigma_\mathfrak{p}^+}\mathfrak{g}_C(\mathfrak{t}_C;\pm\gamma) \quad \text{with} \quad \Sigma_\mathfrak{p}^+\equiv\Sigma^+\cap\Sigma_\mathfrak{p}. \tag{5.2}$$

For each $\gamma \in \Sigma$, select a non-zero vector $X_\gamma \in \mathfrak{g}_C(\mathfrak{t}_C;\gamma)$ such as

$$X_\gamma-X_{-\gamma},\quad \sqrt{-1}(X_\gamma+X_{-\gamma}) \in \mathfrak{k}\oplus\sqrt{-1}\,\mathfrak{p},\quad [X_\gamma, X_{-\gamma}]=H'_\gamma, \tag{5.3}$$

where $H'_\gamma=H_\gamma/2\gamma(H_\gamma)$ with $H_\gamma \in \sqrt{-1}\,\mathfrak{t}$ determined by $B(H, H_\gamma)=\gamma(H)$ for

$H \in \mathfrak{t}_C$ (B = the Killing form of $\mathfrak{g}_C$).

A root γ is said to be strongly orthogonal to $\beta \in \Sigma$ if $\gamma \pm \beta \notin \Sigma \cup (0)$. Construct a maximal family $(\gamma_1, \gamma_2, \cdots, \gamma_l)$ of mutually strongly orthogonal roots in $\Sigma_{\mathfrak{p}}^+$ inductively as follows: for each k, γ_k is the largest root in $\Sigma_{\mathfrak{p}}^+$ strongly orthogonal to $\gamma_{k+1}, \cdots, \gamma_l$, where we consider a linear order on $\sqrt{-1}\mathfrak{t}^*$ which determines the positive system Σ^+. Now we set

$$\mathfrak{t}^- = \sum_{1 \leqq k \leqq l} \boldsymbol{R} H'_{\gamma_k} \subseteqq \sqrt{-1}\mathfrak{t}, \tag{5.4}$$

and $\mathfrak{t}^+$ will denote the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to B.

For a $\gamma \in \mathfrak{t}_C^*$, put $\pi(\gamma) = \gamma \,|\, \mathfrak{t}_C^-$, the restriction of γ to $\mathfrak{t}_C^-$. If γ is identically zero on $\mathfrak{t}_C^+$, then one may express $\pi(\gamma)$ still as γ without any confusion. So are the cases $\gamma = \gamma_k$ $(1 \leqq k \leqq l)$. Let us describe the restrictions $\pi(\gamma)$ $(\gamma \in \Sigma^+)$ by means of the orthogonal basis $(\gamma_1, \gamma_2, \cdots, \gamma_l)$ of $\mathfrak{t}_C^{-*}$. We put for $1 \leqq m < k \leqq l$,

$$C_k = \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = \gamma_k/2\}, \qquad P_k = \{\gamma \in \Sigma_{\mathfrak{p}}^+ ;\ \pi(\gamma) = \gamma_k/2\}, \tag{5.5}$$

$$\begin{aligned} C_{km} &= \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = (\gamma_k - \gamma_m)/2\}, \\ P_{km} &= \{\gamma \in \Sigma_{\mathfrak{p}}^+ ;\ \pi(\gamma) = (\gamma_k + \gamma_m)/2\}, \end{aligned} \tag{5.6}$$

$$C_0 = \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = 0\}, \qquad P_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}. \tag{5.7}$$

According to Harish-Chandra (cf. [II, Prop. 3.1]), the subsets $\Sigma_{\mathfrak{k}}^+$ and $\Sigma_{\mathfrak{p}}^+$ of Σ^+ are expressed respectively as

$$\Sigma_{\mathfrak{k}}^+ = C_0 \cup \left(\bigcup_{1 \leqq k \leqq l} C_k\right) \cup \left(\bigcup_{1 \leqq m < k \leqq l} C_{km}\right), \tag{5.8}$$

$$\Sigma_{\mathfrak{p}}^+ = \left(\bigcup_{1 \leqq k \leqq l} P_k\right) \cup P_0 \cup \left(\bigcup_{1 \leqq m < k \leqq l} P_{km}\right), \tag{5.9}$$

where the unions are disjoint.

We put $H_k = X_{\gamma_k} + X_{-\gamma_k} \in \mathfrak{p}$ for $1 \leqq k \leqq l$. Then $\sum_{1 \leqq k \leqq l} \boldsymbol{R} H_k$ is a maximal abelian subspace of $\mathfrak{p}$, so we denote it by $\mathfrak{a}_p$. Let μ be the inner automorphism of $\mathfrak{g}_C$ defined by

$$\mu = \exp\left(\frac{\pi}{4} \cdot \sum_{1 \leqq k \leqq l} \mathrm{ad}\,(X_{\gamma_k} - X_{-\gamma_k})\right) \qquad \text{(Cayley transform)}. \tag{5.10}$$

A simple calculation yields $\mu(H_k) = H'_{\gamma_k}$, whence $\mu(\mathfrak{a}_p) = \mathfrak{t}^-$. Moreover, the restriction $\mu \,|\, \mathfrak{t}^+$ is identity on $\mathfrak{t}^+$. So, the invesrse μ^{-1} gives a transform on $\mathfrak{g}_C$ carrying the complexification of the compact Cartan subalgebra $\mathfrak{t}$ onto that of a maximally splitting one $\mathfrak{t}^+ \oplus \mathfrak{a}_p$.

Let Λ be the root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Select a positive system $\Lambda^+ \subseteqq \Lambda$ consistent with $\Sigma^+ \subseteqq \Sigma$ through the Cayley transform μ: $\Lambda^+ \cup (0) = \Sigma^+ \circ (\mu \,|\, \mathfrak{a}_p) \cup (0)$. We set $\lambda_k = \gamma_k \circ (\mu \,|\, \mathfrak{a}_p)$ for $1 \leqq k \leqq l$. Then $(\lambda_k)_{1 \leqq k \leqq l}$ is an orthogonal basis of $\mathfrak{a}_p^*$. Moore's restricted root theorem tells us the following description of Λ^+, the set Π of simple roots in Λ^+, and the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_p)$ by means of the basis (λ_k).

Proposition 5.1 [22, Th. 3]. (1) *There are two possibilities for the positive system* Λ^+:

(CASE I) $\Lambda^+=\{(\lambda_k-\lambda_m)/2;\ 1\leqq m<k\leqq l\}\cup\{(\lambda_k+\lambda_m)/2;\ 1\leqq m\leqq k\leqq l\}$,
(CASE II) $\Lambda^+=\{(\lambda_k-\lambda_m)/2;\ 1\leqq m<k\leqq l\}\cup\{\lambda_k/2;\ 1\leqq k\leqq l\}$
$\cup\{(\lambda_k+\lambda_m)/2; 1\leqq m\leqq k\leqq l\}$.

(2) *According as* (CASE I) *or* (CASE II) *above,* Π *is expressed as*

(CASE I) $\Pi=\{\lambda_1, (\lambda_2-\lambda_1)/2, \cdots, (\lambda_l-\lambda_{l-1})/2\}$,
(CASE II) $\Pi=\{\lambda_1/2, (\lambda_2-\lambda_1)/2, \cdots, (\lambda_l-\lambda_{l-1})/2\}$.

(3) W *consists of all transforms of the form* $\lambda_k\mapsto\varepsilon_k\lambda_{\sigma(k)}$, *where* σ *is an arbitrary permutation of* $1, 2, \cdots, l$, *and* $\varepsilon_k=\pm 1$. *So,* W *is identified canonically with the semidirect product group* $\mathfrak{S}_l\ltimes(\boldsymbol{Z}/2\boldsymbol{Z})^l$. *Here, the action of the symmetric group* $\mathfrak{S}_l$ *of degree* l *on* $(\boldsymbol{Z}/2\boldsymbol{Z})^l\simeq\{1, -1\}^l$ *is given as* $\sigma\cdot(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)=(\varepsilon_{\sigma^{-1}(1)}, \varepsilon_{\sigma^{-1}(2)}, \cdots, \varepsilon_{\sigma^{-1}(l)})$.

The (CASE I) happens if and only if G/K is holomorphically equivalent to a tube domain. So are exactly the cases $\mathfrak{g}\simeq\mathfrak{su}(l, l)$, $\mathfrak{sp}(l, \boldsymbol{R})$, $\mathfrak{so}^*(4l)$, $\mathfrak{so}(2, q)$ $(l=2)$ and $\mathfrak{g}$ of type EVII in the sense of Cartan. On the other hand, simple Lie algebras $\mathfrak{g}$ corresponding to non-tube type hermitian symmetric spaces are listed up as $\mathfrak{g}\simeq\mathfrak{su}(p, l)$ $(p>l)$, $\mathfrak{so}^*(4l+2)$ and the exceptional Lie algebra of type EVIII.

5.2. Important types of GGGRs $\boldsymbol{\Gamma}_i$.

Let o denote the nilpotent G_C-orbit in $\mathfrak{g}_C$ through the point $\sum_{1\leqq k\leqq l} X_{\gamma_k} \in \mathfrak{p}_+$, where G_C denotes the adjoint group of $\mathfrak{g}_C$. Then o intersects with $\mathfrak{g}$ [II, Lemma 3.4]. We now explain the description of $\mathrm{Ad}(G)$-orbits in $o\cap\mathfrak{g}$ given in [II]. For this purpose, set

$$E_k=\sqrt{-1}(H'_{\gamma_k}-X_{\gamma_k}+X_{-\gamma_k})/2 \qquad (1\leqq k\leqq l). \tag{5.11}$$

It is easily verified that $E_k\in\mathfrak{g}(\mathfrak{a}_p; \lambda_k)$, the root space of λ_k, and that $\mu^{-1}(X_{\gamma_k}) =\sqrt{-1}E_k$. For $0\leqq i\leqq l$, define a nilpotent element $A[i]\in\mathfrak{n}_m= \sum_{\lambda\in\Lambda^+}\mathfrak{g}(\mathfrak{a}_p; \lambda)$ by

$$A[i]=-\sum_{1\leqq k\leqq i} E_k+\sum_{i<m\leqq l} E_m. \tag{5.12}$$

Theorem 5.2 [II, Th. 3.13]. *The intersection* $o\cap\mathfrak{g}$ *splits into* $(l+1)$-

number of nilpotent $\mathrm{Ad}(G)$*-orbits* $\omega_i \equiv \mathrm{Ad}(G)A[i]$ $(0 \leqq i \leqq l)$: $o \cap \mathfrak{g} = \coprod_{0 \leqq i \leqq l} \omega_i$.

The generalized Gelfand-Graev representations of G associated to nilpotent classes ω_i $(0 \leqq i \leqq l)$ are constructed after Definition 1.1 in the following manner. Keep to the notation in Section 1. The dominant element $H(o) \in \mathfrak{a}_p$ is given as

$$H(o) = \sum_{1 \leqq k \leqq l} H_k, \tag{5.13}$$

since $(A[i], \sum_{1 \leqq k \leqq l} H_k, \theta A[i])$ is an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}$ for each i. One gets a gradation $\mathfrak{g} = \oplus_{j \in Z} \mathfrak{g}(j)$, $\mathfrak{g}(j) \equiv \mathfrak{g}(j)_o$, by ad $H(o)$ such as

$$\begin{cases} \mathfrak{g}(0) = \mathfrak{m} \oplus \mathfrak{a}_p \oplus (\oplus_{1 \leqq m \neq k \leqq l} \mathfrak{g}(\mathfrak{a}_p;\, (\lambda_k - \lambda_m)/2)), \\ \mathfrak{g}(1) = \sum_{1 \leqq k \leqq l} \mathfrak{g}(\mathfrak{a}_p;\, \lambda_k/2) \qquad \text{(possibly (0))}, \\ \mathfrak{g}(2) = \sum_{1 \leqq m \leqq k \leqq l} \mathfrak{g}(\mathfrak{a}_p;\, (\lambda_k + \lambda_m)/2), \\ \mathfrak{g}(j) = (0) \quad \text{if } |j| > 3, \quad \mathfrak{g}(j) = \theta \mathfrak{g}(-j) \quad (j \in Z), \end{cases} \tag{5.14}$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_p$ in $\mathfrak{k}$. The parabolic subalgebra $\mathfrak{p}_o = \oplus_{j \geqq 0} \mathfrak{g}(j)$ is maximal, and $\mathfrak{p}_o = \mathfrak{l} \oplus \mathfrak{n}$ with $\mathfrak{l} \equiv \mathfrak{g}(0)$, $\mathfrak{n} \equiv \mathfrak{n}_o = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, gives its Levi decomposition. Let $P = LN$ with $L = P \cap \theta P$ and $N = \exp \mathfrak{n}$, be the corresponding decomposition of the parabolic subgroup $P \equiv P_o$ with Lie algebra $\mathfrak{p}_o$. Then, N is an at most two-step nilpotent Lie group, and it is canonically diffeomorphic to the Šilov boundary of Siegel domain which realizes G/K. Moreover, N is abelian if and only if G/K is of tube type. This is exactly the (CASE I) of Proposition 5.1.

For each i, let $\xi_i \equiv \xi_{A[i]}$ denote the irreducible unitary representation of N corresponding to the coadjoint N-orbit through $A[i]^* \in \mathfrak{n}^*$, where $\langle A[i]^*, Z \rangle = B(Z, \theta A[i])$ $(Z \in \mathfrak{n})$. Then, the GGGR $\Gamma_i \equiv \Gamma_{\omega_i}$ associated with ω_i is defined to be

$$\Gamma_i = (L^2\text{- or } C^\infty\text{-})\ \mathrm{Ind}_N^G(\xi_i). \tag{5.15}$$

These GGGRs have the following important property.

Theorem 5.3 [II, Th. 4.2]. *The left regular representation* $(\lambda_G, L^2(G))$ *of* G *splits into an orthogonal direct sum of the infinite multiples of unitary GGGRs* $L^2\text{-}\Gamma_i$: $\lambda_G \simeq \oplus_{0 \leqq i \leqq l} [\infty] \cdot L^2\text{-}\Gamma_i$. *In particular, the direct sum* $\oplus_i L^2\text{-}\Gamma_i$ *is quasi-equivalent to* λ_G.

Now let us construct after Definition 1.3, reduced GGGRs (= RGGGRs) coming from these GGGRs. We denote by H^i the stabilizer of the unitary equivalence class $[\xi_i] \in \hat{N}$ in the Levi subgroup L. Thanks to [II, Th. 4.6], each ξ_i can be extended canonically to an actual (not just projective) unitary representation $\tilde{\xi}_i$ of the semidirect product group $H^iN = H^i \ltimes N \subseteqq P$, acting on the same Hilbert space. For an irreducible (unitary, in case of unitary induction) representation c of H^i, the RGGGR $\Gamma_i(c) \equiv \Gamma_{\omega_i}(c)$ associated to (ω_i, c) is defined to be

$$\Gamma_i(c) = \mathrm{Ind}_{H^iN}^{G} (\tilde{c} \otimes \tilde{\xi}_i) \quad \text{with} \quad \tilde{c} = c \otimes 1_N. \tag{5.16}$$

We now recall reductive symmetric pair structure of (L, H^i) (at least on the Lie algebra level), which plays an important role for our study on multiplicities in RGGGRs $\Gamma_i(c)$. For any $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l) \in \{1, -1\}^l$, define a map $\tilde{\varepsilon}$ from the root system $\Delta(\mathfrak{l}) = \{(\lambda_k - \lambda_m)/2;\ 1 \leqq m \neq k \leqq l\}$ of $(\mathfrak{l}, \mathfrak{a}_p)$ to the set $\{1, -1\}$ by

$$\tilde{\varepsilon}((\lambda_k - \lambda_m)/2) = \varepsilon_k \varepsilon_m.$$

Then $\tilde{\varepsilon}$ gives a signature of roots in the sense of [26, Def. 1.1]. So one can construct a reductive symmetric pair $(\mathfrak{l}, \mathfrak{h}^\varepsilon)$ as follows. Let θ_ε be a linear map on $\mathfrak{l}$ such that

$$\theta_\varepsilon(X) = \begin{cases} \tilde{\varepsilon}(\lambda)\theta X & (X \in \mathfrak{g}(\mathfrak{a}_p; \lambda), \quad \lambda \in \Delta(\mathfrak{l})), \\ \theta X & (X \in \mathfrak{m} \oplus \mathfrak{a}_p). \end{cases} \tag{5.17}$$

Then θ_ε gives an involutive automorphism of $\mathfrak{l}$. We take as $\mathfrak{h}^\varepsilon$ the Lie algebra of fixed points of θ_ε.

Put $\chi(i) = (\overbrace{-1, \cdots, -1}^{i}, \overbrace{1, \cdots, 1}^{(l-i)})$ for each $0 \leqq i \leqq l$. By virtue of [II, Lemmas 2.1 and 3.5], the Lie algebra $\mathfrak{h}^i$ of H^i coincides with $\mathfrak{h}^{\chi(i)}$. Furthermore, if $i = 0$ or l, then $H^i = K \cap L$, a maximal compact subgroup of L. So, in this case, the reductive symmetric pair (L, H^i) is riemannian.

5.3. Finite multiplicity theorems for RGGGRs $\Gamma_i(c)$ [II].

Thanks to the above symmetric pair structure, one can apply to RGGGRs $\Gamma_i(c)$ our results obtained in [I]. There, we have presented good sufficient conditions for induced representations of semisimple Lie groups to have finite multiplicity property. As a consequence, we find out, for instance, the following hereditary character of finite multiplicity property:

For an irreducible admissible representation c of H^i, the RGGGR C^∞-$\Gamma_i(c) = C^\infty$-$\mathrm{Ind}_{H^iN}^{G} (\tilde{c} \otimes \tilde{\xi}_i)$ is of multiplicity finite if so is the restriction $(\tilde{c} \otimes \tilde{\xi}_i) \mid M$, where $M = Z_K(A_p)$. (Note that M is contained in every H^i.)

Investigating closely the multiplicities in $(\tilde{c} \otimes \tilde{\xi}_i) \mid M$, we have obtained in [II] finite multiplicity theorems for RGGGRs $\Gamma_i(c)$ as follows. First, for C^∞-induced RGGGRs, we deduce

Theorem 5.4 [II, Th. 6.5]. *If c is a finite-dimensional representation of H^i, then the RGGGR C^∞-$\Gamma_i(c)$ has finite multiplicity property, that is, every irreducible admissible $(\mathfrak{g}_C, K)$-module occurs as a submodule of C^∞-$\Gamma_i(c)$ with at most finite multiplicity.*

Estimating the multiplicity function of a unitary RGGGR L^2-$\Gamma_i(c)$ by the multiplicities of irreducible admissible $(\mathfrak{g}_C, K)$-submodules of C^∞-$\Gamma_i(c)$, we have the following finite multiplicity theorem for unitary RGGGRs.

Theorem 5.5 [II, Th. 6.6]. (1) *If $i=0$ or l, then all the RGGGRs L^2-$\Gamma_i(c)$ $(c \in (K\cap L)^\wedge)$ associated with the nilpotent orbit ω_i have finite multiplicity property, where $(K\cap L)^\wedge$ is the unitary dual of the compact group $K\cap L=H^0=H^l$. Namely, if $L^2\text{-}\Gamma_i(c)\simeq\int_{\hat{G}}^{\oplus}[m_{i,c}(\pi)]\cdot\pi d\mu_{i,c}(\pi)$ denotes the factor decomposition* (see [I, 3.4]), *then the multiplicity function $m_{i,c}$ on $\hat{G}$ for L^2-$\Gamma_i(c)$ takes finite values almost everywhere with respect to the Borel measure $d\mu_{i,c}$.*

(2) *Assume that G/K is of tube type. Then, the representation L^2-$\Gamma_i(c)$ is of multiplicity finite for any $0\leqq i\leqq l$ and any finite-dimensional unitary representation c of H^i.*

5.4. Comparison with Harish-Chandra's finite multiplicity theorem.

We now compare the above theorems for $i=0$ or l, with Harish-Chandra's finite multiplicity theorem for $\mathrm{Ind}_K^G(\tau)$ $(\tau \in \hat{K})$. Let $i=0$ or l. The original GGGR L^2-Γ_i is far from being of multiplicity finite except the unique case $\mathfrak{g}\simeq\mathfrak{sl}(2, \boldsymbol{R})\simeq\mathfrak{su}(1, 1)\simeq\mathfrak{sp}(1, \boldsymbol{R})$. (In this exceptional case, our GGGR becomes non-generalized GGRs, which is of multiplicity free). In view of [II, (4.13)], it is decomposed into a direct sum of RGGGRs as

$$L^2\text{-}\Gamma_i\simeq\oplus_{c\in(K\cap L)^\wedge}[\dim c]\cdot L^2\text{-}\Gamma_i(c). \tag{5.18}$$

Theorem 5.5(1) assures that each constituent L^2-$\Gamma_i(c)$ has finite multiplicity property.

One can find out a same kind of phenomenon in Harish-Chandra's case. Now let G be any connected semisimple Lie group with finite center, and $K\subseteqq G$ a maximal compact subgroup. Thanks to the Peter-Weyl theorem for compact groups, the regular representation λ_K of K is decomposed into irreducibles as

$$\lambda_K\simeq\oplus_{\tau\in\hat{K}}[\dim\tau]\cdot\tau,$$

which implies immediately

$$\lambda_G \simeq \oplus_{\tau \in \hat{K}} [\dim \tau] \cdot L^2\text{-}\mathrm{Ind}_K^G(\tau). \tag{5.19}$$

According to Harish-Chandra, the induced representations L^2-$\mathrm{Ind}_K^G(\tau)$ have finite multiplicity property for all $\tau \in \hat{K}$ (cf. [I, Th. 3.1]), although the regular representation λ_G is quite far from being of multiplicity finite.

In view of Example 2.15, the representation L^2-$\mathrm{Ind}_K^G(\tau)$ is furthermore of multiplicity free if τ is a real-valued character of K. By analogy, we can expect that our RGGGRs L^2-$\Gamma_i(c)$ also have multiplicity free property under some additional assumptions. We shall realize this expectation in the next section, giving multiplicity one theorems for RGGGRs.

§ 6. Multiplicity one theorems for reduced GGGRs

We give in this section multiplicity one theorems for reduced generalized Gelfand-Graev representations (=RGGGRs) $\Gamma_i(c)$ associated with the nilpotent classes $\omega_i = \mathrm{Ad}(G)A[i]$ with $i=0$ or l (Theorems 6.9 and 6.10). These are our main results of Part I of this paper. To prove these theorems, we first give non-existence theorem of non-zero quasi-elementary Whittaker distributions with singular supports (Theorem 6.5). This non-existence theorem enables us to apply successfully criterions for multiplicity free property given in Section 2.

6.1. Non-existence of quasi-elementary Whittaker distributions with singular supports.

For each integer $0 \leqq i \leqq l$, let $\Gamma_i = \mathrm{Ind}_N^G(\xi_i)$ be the GGGR in (5.15). The irreducible unitary representation ξ_i of N is realized as

$$\xi_i = L^2\text{-}\mathrm{Ind}_{N_i}^N(\tilde{\eta}_i). \tag{6.1}$$

Here, $\mathfrak{n}_i = \mathrm{Lie}\, N_i$ is a real polarization at $A[i]^* \in \mathfrak{n}^*$ containing the center $\mathfrak{g}(2)$ of $\mathfrak{n}$, and $\tilde{\eta}_i$ denotes the unitary character of N_i: $\tilde{\eta}_i(\exp Z) = \exp\sqrt{-1}\langle A[i]^*, Z\rangle$ $(Z \in \mathfrak{n}_i)$. We put

$$\eta'_i = \sqrt{-1}A[i]^* \,|\, \mathfrak{g}(2). \tag{6.2}$$

For $\kappa \in C$, let us consider the space $W(s; \eta'_i, \kappa)$ (see (3.10)) of κ-quasi-elementary η'_i-Whittaker distributions on Bruhat cells with respect to $(P_m, \bar{P} = \theta P)$, where $P_m = MA_pN_m$ is a minimal parabolic subgroup of G, and $P = LN$ is a maximal one in 5.2.

We now give, using Proposition 5.1(3), an explicit parametrization of the double cosets in $P_m \backslash G / \bar{P}$. Identify the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_p)$ with

$(Z/2Z)^l \rtimes \mathfrak{S}_l$. Then the subgroup $W(P)$ corresponding to the Levi subgroup of P coincides with $\mathfrak{S}_l$. Thus we can take a complete system of representatives W_P in $W/W(P)$ as $W_P=(Z/2Z)^l=\{1, -1\}^l$. In view of (3.7), the Bruhat decomposition of G with respect to $(P_m, \bar{P})$ is given as

$$G=\coprod_{\varepsilon \in \{1,-1\}^l} G_\varepsilon \quad \text{with} \quad G_\varepsilon=P_m \varepsilon^* \bar{P}, \tag{6.3}$$

where ε^* denotes a representative of $\varepsilon \in W$ in $M^*=N_K(A_p)$.

We wish to apply Theorem 4.2 to $W(\varepsilon; \eta'_i, \kappa)$. First, let us quote a lemma due to Rossi and Vergne which plays an important role for our purpose. Take a pair (k, m) of integers such that $1 \leqq m<k \leqq l$. Then we see immediately

$$[X, [Y, E_m]] \in \mathfrak{g}(\mathfrak{a}_p; \lambda_k) \qquad \text{for all } X, Y \in \mathfrak{g}(\mathfrak{a}_p; (\lambda_k-\lambda_m)/2),$$

where E_m is the root vector of λ_m in (5.11). Note that the root space $\mathfrak{g}(\mathfrak{a}_p; \lambda_k)$ is one-dimensional and is spanned by E_k. So, by putting

$$(\operatorname{ad} X)^2 E_m/2=\zeta_{km}(X) E_k \quad \text{with} \quad \zeta_{km}(X) \in \boldsymbol{R}, \tag{6.4}$$

we obtain a quadratic form ζ_{km} on $\mathfrak{g}(\mathfrak{a}_p; (\lambda_k-\lambda_m)/2)$.

Lemma 6.1 [28, Th. 4.10]. *Under the above notation, the quadratic form ζ_{km} is positive definite for every $1 \leqq m<k \leqq l$.*

Making use of this lemma, let us describe, for each fixed $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l) \in \{1, -1\}^l$, the closed subset $D(\varepsilon, i) \equiv D^\varepsilon_{\eta'_i} \subseteqq N_m \cap {}^{\varepsilon^*}N$ (${}^{\varepsilon^*}N=\varepsilon^* N \varepsilon^{*-1}$) defined by (4.2). For this purpose, we put

$$V_0=(N_m \cap {}^{\varepsilon^*}N) \cap L, \qquad V_1=N \cap {}^{\varepsilon^*}N.$$

Then $N_m \cap {}^{\varepsilon^*}N$ admits a semidirect product decomposition

$$N_m \cap {}^{\varepsilon^*}N=V_0 \ltimes V_1. \tag{6.5}$$

Moreover we have

$$V_0=\exp \mathfrak{v}_0 \quad \text{with} \quad \mathfrak{v}_0=\textstyle\sum_{(k,m) \in S} \mathfrak{g}(\mathfrak{a}_p; (\lambda_k-\lambda_m)/2), \tag{6.6}$$

where S is the totality of pairs (k, m) such that $1 \leqq m<k \leqq l$ and $(\varepsilon_k, \varepsilon_m)=(1, -1)$. V_0 is an abelian group isomorphic $\mathfrak{v}_0$ through the exponential mapping, since $(\lambda_k-\lambda_m)/2+(\lambda_{k'}-\lambda_{m'})/2 \notin \Lambda$ for $(k, m), (k', m') \in S$.

On the other hand, the abelian subalgebra $\mathfrak{g}(2) \cap {}^{\varepsilon^*}\mathfrak{u}$ ($\mathfrak{u} \equiv \theta \mathfrak{n}$) has an expression

$$\mathfrak{g}(2) \cap {}^{\varepsilon^*}\mathfrak{u}=\textstyle\sum_{m \in S'} \mathfrak{g}(\mathfrak{a}_p; \lambda_k) \oplus \sum_{(k,m) \in S''} \mathfrak{g}(\mathfrak{a}_p; (\lambda_k+\lambda_m)/2), \tag{6.7}$$

where

$$S' \equiv \{k;\ 1 \leqq k \leqq l,\ \varepsilon_k = -1\},$$
$$S'' \equiv \{(k, m);\ 1 \leqq m < k \leqq l,\ (\varepsilon_k, \varepsilon_m) = (-1, -1)\}.$$

Let $n = v_0 v_1 \in N_m \cap {}^{\varepsilon^*}N$ with $v_j \in V_j$ $(j=0, 1)$. Since $\mathfrak{g}(2)$ is the center of $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, n lies in $D(\varepsilon, i)$ if and only if so does the V_0-component v_0. Put

$$v_0 = \exp\left(\textstyle\sum_{(k,m) \in S} X_{km}\right) \quad \text{with} \quad X_{km} \in \mathfrak{g}(\mathfrak{a}_p;\ (\lambda_k - \lambda_m)/2).$$

Keeping (4.2) in mind, we calculate $\eta_i'(\mathrm{Ad}\,(v_0)Z)$ for each $Z \in \mathfrak{g}(2) \cap {}^{\varepsilon^*}\mathfrak{u}$. From (6.7) and the fact $\mathfrak{g}(\mathfrak{a}_p;\ \lambda_k) = \boldsymbol{R}E_k$, it suffices to treat the following two cases.

Case 1. Let $Z = E_m$ for an $m \in S'$. Then it follows from (6.4) that

$$\mathrm{Ad}\,(v_0)E_m \equiv E_m + \textstyle\sum_{k>m,\, \varepsilon_k = 1} \zeta_{km}(X_{km})E_k$$

modulo $\sum_{k>m} \mathfrak{g}(\mathfrak{a}_p;\ (\lambda_k + \lambda_m)/2)$, on which η_i' is identically zero. Hence we get

$$\eta_i'(\mathrm{Ad}\,(v_0)E_m) = \sqrt{-1}\, b\{\chi_m^i + \textstyle\sum_{k>m,\, \varepsilon_k=1} \chi_k^i \cdot \zeta_{km}(X_{km})\}, \tag{6.8}$$

where $b \equiv -B(E_k, \theta E_k) = \|\gamma_k\|^2/2$ is a positive constant independent of k, and we put

$$\chi_k^i = 1 \quad \text{if} \quad k \leqq i, \quad \text{and} \quad \chi_k^i = -1 \quad \text{if} \quad k > i.$$

Case 2. Let $Z \in \mathfrak{g}(\mathfrak{a}_p;\ (\lambda_k + \lambda_m)/2)$ for $(k, m) \in S''$. Define a bilinear form $\zeta_{k',Z}$ on $\mathfrak{g}(\mathfrak{a}_p;\ (\lambda_{k'} + \lambda_k)/2) \times \mathfrak{g}(\mathfrak{a}_p;\ (\lambda_{k'} + \lambda_m)/2)$ by $[X_{k'k}[X_{k'm}, Z]] = \zeta_{k',Z}(X_{k'k}, X_{k'm})E_{k'}$, where $k' > k, m$. Then,

$$\eta_i'(\mathrm{Ad}\,(v_0)Z) = \sqrt{-1}\, b \textstyle\sum_{k',\, \varepsilon_{k'}=1} \zeta_{k',Z}(X_{k'k}, X_{k'm})\chi_{k'}^i. \tag{6.9}$$

By virtue of (6.8) and (6.9), one concludes a complete description of $D(\varepsilon, i)$, and in particular,

Proposition 6.2. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l) \in \{1, -1\}^l$ and i be an integer such that $0 \leqq i \leqq l$. Keep to the above notation. The closed submanifold $D(\varepsilon, i) = D^{\varepsilon}_{\eta_i'} \subseteq N_m \cap {}^{\varepsilon^*}N$ consists of $n = v_0 v_1 \in V_0 V_1 = N_m \cap {}^{\varepsilon^*}N$ with $v_0 = \exp(\sum_{(k,m) \in S} X_{km})$, satisfying the condition*

$$\chi_m^i + \textstyle\sum_{k>m,\, \varepsilon_k = 1} \chi_k^i \cdot \zeta_{km}(X_{km}) = 0$$

for all $1 \leqq m \leqq l$ such that $\varepsilon_m = -1$. Hence ζ_{km} is the positive definite quadratic form in Lemma 6.1.

We deduce immediately from this proposition a sufficient condition for the set $D(\varepsilon, i)$ to be empty as follows.

Proposition 6.3. *Let ε and i be an in Proposition* 6.2. *Assume that $\varepsilon \neq \mathbf{1} \equiv (1, 1, \cdots, 1)$, and denote by m_ε the largest integer m such that $\varepsilon_m = -1$. If either $i < m_\varepsilon$ or $i = l$, then the set $D(\varepsilon, i)$ is empty.*

This proposition combined with Theorem 4.2 produces the following theorem on the non-existence of quasi-elementary Whittaker distributions on Bruhat cells.

Theorem 6.4. *Let $\varepsilon \in \{1, -1\}^l = W_P$ and i be an integer such as $0 \leqq i \leqq l$. For a $\kappa \in \boldsymbol{C}$, consider the space $W(\varepsilon; \eta'_i, \kappa)$ of κ-quasi-elementary η'_i-Whittaker distributions on $C_\varepsilon = \varepsilon^* N\bar{P}$ with supports contained in the Bruhat cell $G_\varepsilon = P_m \varepsilon^* \bar{P} \subseteqq C_\varepsilon$. If the pair (ε, i) satisfies the assumption in Proposition* 6.3, *then $W(\varepsilon; \eta'_i, \kappa)$ reduces to* (0). *So in particular, if $i = 0$ or l, then $W(\varepsilon; \eta'_i, \kappa) = (0)$ for all $\varepsilon \in \{1, -1\}^l$, $\neq \mathbf{1}$.*

Note. The second assertion turns to be false for $i \neq 0$, l. Actually, in case of $G = \mathrm{Sp}(2, \boldsymbol{R})$, the real symplectic group of rank $l = 2$, the spaces $W(\varepsilon; \eta'_i, \kappa)$ $(\kappa \in \boldsymbol{C})$ with $\varepsilon = (-1, 1)$ and $i = 1$, do not reduce to (0) in general. In this case, a non-zero distribution in $W(\varepsilon; \eta'_i, \kappa)$ can be found, for instance, as a trivial extension (to C_ε) of a certain $S \in \mathscr{D}'(G_\varepsilon)$ which is quasi-invariant under the action $N \times \bar{P}_m \ni (n, \bar{p}) \mapsto L_n R_{\bar{p}}$ on $\mathscr{D}'(G_\varepsilon)$, where $\bar{P}_m = \theta P_m$.

In view of Proposition 3.1, we deduce immediately from Theorem 6.4 the following

Theorem 6.5. *Assume that $i = 0$ or l. For a $\kappa \in \boldsymbol{C}$, let $\mathscr{D}'(G; \eta'_i, \kappa)$ denote the space of κ-quasi-elementary η'_i-Whittaker distributions on the whole G. Then, the restriction map*

$$\mathscr{D}'(G; \eta'_i, \kappa) \ni T \longmapsto T | N\bar{P} \in W(\mathbf{1}; \eta'_i, \kappa)$$

is injective, where $N\bar{P}$ is open dense in G. In other words, there does not exist any non-zero quasi-elementary η'_i-Whittaker distribution with singular support with respect to $(P_m, \bar{P})$: if $T \in \mathscr{D}'(G; \eta'_i, \kappa)$ is such that $\mathrm{supp}(T) \subseteqq G \backslash N\bar{P}$, *then $T = 0$.*

6.2. An involution β and its relation to QIEDs on G.

Making use of this theorem and criterions for multiplicity free property given in Section 2, we wish to give multiplicity one theorems for RGGGRs $\Gamma_i(c)$. For this purpose, we assume until the end of this section that hermitian symmetric space G/K is of tube type. Under this assumption, one has the following

Lemma 6.6. (1) *The element* $Y \equiv (\pi\sqrt{-1}/2) \cdot \sum_{1 \leq k \leq l} H'_{\gamma_k}$ *lies in the center of* $\mathfrak{k}$, *where* H'_{γ_k} *is the element of* $\mathfrak{t}^-$ *in* (5.3).

(2) *Put* $\varepsilon^*_- = \exp Y \in K$. *Then, the Cartan involution* θ *is realized as the inner automorphism* $g \mapsto \varepsilon^*_- g \varepsilon^{*-1}_-$ ($g \in G$). *In particular,* $\varepsilon^*_- \in K$ *is a representative of the longest element of the Weyl group* $W = \mathfrak{S}_l \ltimes \{1, -1\}^l$.

Proof. By virtue of [II, Lemma 5.1], we get immediately $\mathrm{ad}(Y)|\mathfrak{k}=0$ and $\mathrm{ad}(Y)|\mathfrak{p}_\pm = \pm\pi\sqrt{-1} I_\pm$ with $I_\pm =$ the identity map on $\mathfrak{p}_\pm$, where $\mathfrak{p}_\pm$ is as in (5.2). This proves the assertions (1) and (2) at the same time.
Q.E.D.

Now assume that G be a linear group. We denote by $G^C \supseteqq G$ the complexification of G. Then the Cayley transform μ extends naturally to an inner automorphism of G^C, which will be denoted again by μ. In view of the above lemma, we can define an involutive inner automorphism β of G^C by

$$\beta(x) = \mu^{-1}(\varepsilon^*_- \mu(x) \varepsilon^{*-1}_-) \qquad (x \in G^C). \tag{6.10}$$

Lemma 6.7. *The subgroups* G, N *and* $K \cap L$ *are stable under* β. *Moreover, the differential of* β *(denoted still by* β *for simplicity) acts on the Lie algebra* $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(2)$ *in such a way as*

$$\beta|\mathfrak{g}(j) = (-1)^{j/2} I_j \qquad (j = 0, \pm 2), \tag{6.11}$$

where I_j *is the identity map on* $\mathfrak{g}(j)$.

Proof. First, let us prove (6.11). Recall that each $\lambda_k \in \mathfrak{a}^*_p$ corresponds to $\gamma_k|\mathfrak{t}^-$ through the Cayley transform μ. In view of the expressions (5.8), (5.9) and (5.14), one gets

$$\mu(\mathfrak{g}(\pm 2)_C) = \mathfrak{p}_\pm \quad \text{and} \quad \mu(\mathfrak{g}(0)_C) = \mathfrak{k}_C.$$

Combining this with Lemma 6.6(2), we obtain (6.11) from the definition of β. So in particular, the real Lie subalgebras $\mathfrak{g}$, $\mathfrak{n} = \mathfrak{g}(2)$ of $\mathfrak{g}_C$ are stable under β. Since both G and N are connected, these subgroups of G_C are β-stable, too. Furthermore, we can show $\mu(k) = k$ for all $k \in K \cap L$, exactly as in the proof of [II, Lemma 4.15]. This completes the proof. Q.E.D.

Let us consider RGGGRs $\Gamma_i(c)$ with $i = 0$ or l:

$$\Gamma_i(c) = \mathrm{Ind}^G_{(K \cap L)N}(\zeta_{c,i}), \qquad \zeta_{c,i} \equiv \tilde{c} \otimes \tilde{\xi}_i,$$

for irreducible unitary representations c of $K\cap L$. Since N is abelian, the representation ξ_i is one-dimensional. So, we can take as an extension $\tilde{\xi}_i$ a unitary character of $(K\cap L)N$ trivial on $K\cap L$.

Thanks to Theorem 6.5, we can deduce the following theorem which enables us to prove multiplicity one theorems for $\Gamma_i(c)$'s.

Theorem 6.8. *Assume that c is a unitary character of $K\cap L$. Then, any $(\zeta_{c,i}, \zeta_{c,i}\circ\beta)$-QIED T on G (see Definition 2.5) satisfies the relation $T^\beta=\check{T}$, where, $T^\beta=T\circ\beta$ and $\check{T}$ is defined by* (2.3).

Proof. For a distribution T on G, put $S=R_{\varepsilon^*_-}T$, the right translation of T by ε^*_-. In view of Lemma 6.6(2), T is a $(\zeta_{c,i}, \zeta_{c,i}\circ\beta)$-QIED if and only if S satisfies

$$L_{z_1}R_{\theta z_2}S=\zeta_{c,i}(z_1)^{-1}\zeta_{c,i}(\beta(z_2))^{-1}S \qquad (z_1, z_2\in(K\cap L)N), \tag{6.12}$$

$$L_D S=\chi(D)S \quad (D\in Z(\mathfrak{g}_C)) \quad \text{for some } \chi\in \mathrm{Hom}_{alg}(Z(\mathfrak{g}_C), C). \tag{6.13}$$

On the other hand, the condition $T^\beta=\check{T}$ on T is transferred to that on S as follows. Let ι denote the involutive anti-automorphism of G such as

$$\iota(g)=(\theta\circ\beta)(g^{-1})=\varepsilon^*_-\beta(g^{-1})\varepsilon^{*-1}_- \qquad (g\in G).$$

(In view of (6.11), θ commutes with β, whence ι is actually involutive.) Then, a simple calculation yields

$$S^\iota=R_{\beta(\varepsilon^*_-)^{-1}}((T^\beta)^\vee).$$

With the definition of ε^*_- in mind, we get from (5.11)

$$\varepsilon^*_-=\exp(-\pi\cdot\textstyle\sum_{1\leq k\leq l}(E_k+\theta E_k)/2).$$

Notice that $E_k\in\mathfrak{g}(2)$ and $\theta E_k\in\mathfrak{g}(-2)$. Then (6.11) implies that $\beta(\varepsilon^*_-)^{-1}=\varepsilon^*_-$. We thus obtain

$$S^\iota=R_{\varepsilon^*_-}((T^\beta)^\vee). \tag{6.14}$$

Consequently, the condition $T^\beta=\check{T}$ is equivalent to $S^\iota=S$.

By the above consideration, the proof of theorem is now reduced to showing that any $S\in\mathscr{D}'(G)$ with properties (6.12) and (6.13) necessarily satisfies $S^\iota=S$. This is achieved in the following manner.

Assume that $S\in\mathscr{D}'(G)$ satisfies (6.12) and (6.13). Then, also S^ι satisfies (6.12) and

(6.13′) $$L_D(S^\iota)=\chi(\iota'(D))S \qquad \text{for } D\in Z(\mathfrak{g}_C).$$

Here ι' denotes the automorphism of $Z(\mathfrak{g}_C)$ induced from $\theta\circ\beta$ on $\mathfrak{g}$ in the canonical way. Notice that the Casimir element Ω is, by definition, fixed under any automorphism of $U(\mathfrak{g}_C)$ coming from that of $\mathfrak{g}$. Thus, both S and S^ι are eigendistributions of L_Ω with the common eigenvalue $\kappa\equiv\chi(\Omega)$. In particular, these two distributions are in the same space $\mathscr{D}'(G;\eta_i',\kappa)$ of Whittaker distributions.

In view of Theorem 6.5, we have only to show that $S^\iota=S$ on the open dense subset $G_1=N\bar{P}$ of G. Put $T_0=T|G_1$ for a distribution T on G. By the definition of ι, G_1 is ι-stable. This implies that $(S^\iota)_0=(S_0)^\iota$. Notice that S_0 still satisfies (6.12). Hence, there exists a unique distribution X on L such that

(6.15) $$S_0=\xi_i\otimes X\otimes(\xi_i\circ(\theta\,|\,U)) \text{ with } U=\theta N,\ \xi_i=\exp\eta_i'.$$

Here, for distributions $T_1\in\mathscr{D}'(N)$, $T_2\in\mathscr{D}'(L)$ and $T_3\in\mathscr{D}'(U)$, $T=T_1\otimes T_2\otimes T_3$ denotes the distribution on $G_1=NLU\simeq N\times L\times U$ defined by $\langle T,\psi\rangle=\langle T_1,\psi_1\rangle\cdot\langle T_2,\psi_2\rangle\cdot\langle T_3,\psi_3\rangle$ for $\psi\in\mathscr{D}(G_1)$ of the form $\psi(nlu)=\psi_1(n)\times\psi_2(l)\psi_3(n)$, $(n,l,u)\in N\times L\times U$. The uniqueness of X together with (6.12) implies

(6.16) $$L_kR_{k'}X=c(k)^{-1}c(k')^{-1}X \qquad \text{for all } k,k'\in K\cap L.$$

Now we apply Proposition 2.13 by putting $G=L$, $\sigma=\theta\,|\,L$, $Q=K\cap L$, $\zeta=c$ and $P=\exp(\mathfrak{p}\cap\mathfrak{l})$. We thus conclude

(6.17) $$X^\theta=X^\vee.$$

Notice that $\iota(l)=\theta(l^{-1})$ for $l\in L$. Then (6.15) and (6.17) imply $(S_0)^\iota=S_0$, which completes the proof. Q.E.D.

6.3. Multiplicity one theorems.

We can now give multiplicity one theorems for RGGGRs. First, we deduce for the smooth RGGGRs $C^\infty\text{-}\Gamma_i(c)$

Theorem 6.9. *Let G be a connected, linear simple Lie group of hermitian type. Assume that the corresponding hermitian symmetric space G/K is of tube type. For a character (necessarily unitary) c of $K\cap L$, consider the RGGGR $C^\infty\text{-}\Gamma_i(c)=C^\infty\text{-}\mathrm{Ind}_{(K\cap L)N}^G(\tilde{c}\otimes\tilde{\xi}_i)$ with $i=0$ or l, induced in C^∞-context. Then one has for each irreducible admissible representation π of G,*

(1) $\dim\mathrm{Hom}_G((\pi^*)_\infty,\ C^\infty\text{-}\Gamma_{l-i}(c))\cdot\dim\mathrm{Hom}_G(\pi_\infty,\ C^\infty\text{-}\Gamma_i(c))\leqq 1$, *where π^* denotes the representation of G contragredient to π, and π_∞ (resp.*

$(\pi^*)_\infty)$ *the smooth representation of* G *on the space of* C^∞*-vectors for* π *(resp. for* π^**). The equality holds only if* $\pi^\beta=\pi\circ\beta$ *and* π^* *are infinitesimally equivalent.*

(2) *If* π^β *is equivalent to* π^* *through a bicontinuous linear operator, the multiplicity* $\dim \mathrm{Hom}_G(\pi_\infty, C_\infty\text{-}\Gamma_i(c))$ *does not exceed one.*

(3) *Assume that c be further real-valued. Then there holds*

$$\dim \mathrm{Hom}_G(\pi_\infty, C^\infty\text{-}\Gamma_i(c))\leqq 1$$

for all irreducible unitary representations π *of* G.

Proof. Assertions follow immediately from Theorems 2.10, 2.11 and 6.8. Q.E.D.

Secondly, we obtain the following multiplicity one theorem for unitary RGGGRs.

Theorem 6.10. *Let* G *be as in Theorem* 6.9. *If c is a real-valued character of* $K\cap L$, *then the unitarily induced RGGGR* $L^2\text{-}\Gamma_i(c)$ *with* $i=0$ *or* l, *has multiplicity free property.*

Proof. One gets the statement immediately by combining Theorem 6.8 with Theorem 2.12. Q.E.D.

These Theorems 6.9 and 6.10 are our main results of this Part I. They extend, in a certain sense, the uniqueness theorem of generalized Bessel models for the symplectic group of rank 2 over a non-archimedian local field, due to Novodvorskii and Piatetskii-Shapiro ([24], [25]).

Part II. Whittaker models for the discrete series

Let G be, as in Sections 5 and 6, a connected simple Lie group of hermitian type. We still assume that G is a matrix group. But, we now treat, contrary to 6.2 and 6.3, such a Lie group G associated to a non-tube type hermitian symmetric space, too. Let l be the real rank of G. For an integer i, $0\leqq i\leqq l$, let $\Gamma_i=\mathrm{Ind}_N^G(\xi_i)$ be the generalized Gelfand-Graev representation (=GGGR) associated with the nilpotent $\mathrm{Ad}(G)$-orbit $\omega_i=\mathrm{Ad}(G)A[i]\subseteqq\mathfrak{g}$ (see 5.2).

In Part II, we deal with embeddings of irreducible representations π of G into GGGRs Γ_i. Such an embedding is called a *Whittaker model* for π in Γ_i.

Since $\mathfrak{g} = \mathrm{Lie}\, G$ has a compact Cartan subalgebra, our simple Lie group G admits the discrete series, consisting of irreducible subrepresentations of the regular representation of G on $L^2(G)$. In view of Theorem 5.3, one finds that any given discrete series representation does occur in a unitary GGGR L^2-Γ_i for some i, as a subrepresentation. Suggested by this fact, we have addressed in [II, 4.1] the following *Embeddings of Discrete Series problem.*

Problem EDS. Describe explicitly Whittaker models for the discrete series in GGGRs Γ_i.

We shall settle this problem for holomorphic (or anti-holomorphic) discrete series, determining by the method of Hashizume [11] all the K-finite highest weight vectors in GGGRs. This method of highest weight vectors is applicable more generally for irreducible admissible highest weight representations of G (which include both the holomorphic discrete series and irreducible finite-dimensional representations).

§ 7. Irreducible highest weight representations and the holomorphic discrete series

In this section, we construct after Harish-Chandra, irreducible admissible representations of G with highest weights. Our main reference here is Varadarajan's excellent survey [30, Section 2].

7.1. Highest weight modules of complex semisimple Lie algebras. In this subsection, let $\mathfrak{g}_C$ be any complex semisimple Lie algebra, and $\mathfrak{t}_C$ a Cartan subalgebra of $\mathfrak{g}_C$. Σ denotes the root system of $(\mathfrak{g}_C, \mathfrak{t}_C)$. Choose a positive system Σ^+ of Σ, and put $\mathfrak{n}_\pm = \sum_{\gamma \in \Sigma^+} \mathfrak{g}_C(\mathfrak{t}_C; \pm\gamma)$, where $\mathfrak{g}_C(\mathfrak{t}_C; \gamma)$ is the root space of a root γ.

Let $\lambda \in \mathfrak{t}_C^*$, a linear form on $\mathfrak{t}_C$. A $U(\mathfrak{g}_C)$-module V is said to be a *highest weight module* with Σ^+-extreme highest weight λ, or a λ-highest weight module, if there exists a non-zero vector $v \in V$ satisfying the following three conditions:

$$(7.1) \qquad U(\mathfrak{g}_C) \cdot v = V,$$

$$(7.2) \qquad \mathfrak{n}_+ \cdot v = (0),$$

$$(7.3) \qquad H \cdot v = \lambda(H) v \qquad \text{for all } H \in \mathfrak{t}_C.$$

Such a vector v is called a highest weight vector. It is unique up to scalar multiples.

Now extend $\lambda \in \mathfrak{t}_C^*$ to a one-dimensional representation $\tilde{\lambda}$ of the Borel subalgebra $\mathfrak{b}_+ \equiv \mathfrak{t}_C \oplus \mathfrak{n}_+$ trivially on $\mathfrak{n}_+$. Consider the induced $U(\mathfrak{g}_C)$-module

$$M[\lambda] \equiv U(\mathfrak{g}_C) \otimes_{U(\mathfrak{b}_+)} \tilde{\lambda}, \tag{7.4}$$

where we regard $U(\mathfrak{g}_C)$ as a right $U(\mathfrak{b}_+)$-module in the canonical way. Then $M[\lambda]$ is a λ-highest weight module with highest weight vector $v_\lambda = 1 \otimes 1$, and it is a free $U(\mathfrak{n}_-)$-module with generator v_λ. Moreover, it has, and is characterized by, the following universal property:

(7.5) If V is a λ-highest weight module with highest weight vector $v \in V$, then there exists a unique $U(\mathfrak{g}_C)$-module homomorphism A: $M[\lambda] \to V$ such that $A(v_\lambda) = v$.

$M[\lambda]$ is called the *Verma module* with highest weight λ.

One can see easily that $M[\lambda]$ has the unique largest proper $U(\mathfrak{g}_C)$-submodule. Denote it by K_λ. Then the quotient $L_\lambda \equiv M[\lambda]/K_\lambda$ is the unique (up to equivalence) irreducible λ-highest weight module. L_λ is finite-dimensional if and only if λ is Σ^+-dominant and integral: $\lambda(H'_\gamma) \in \{0, 1, 2, \cdots\}$ for all $\gamma \in \Sigma^+$, where H'_γ denotes, as in (5.3), the element of $\mathfrak{t}_C$ corresponding to the co-root $\gamma^\vee \equiv 2\gamma/\langle \gamma, \gamma \rangle$ through the Killing form of $\mathfrak{g}_C$. Furthermore, such L_λ's exhaust all the irreducible finite-dimensional $U(\mathfrak{g}_C)$-modules.

7.2. Admissible highest weight modules.

Now we return to original objects, and let $\mathfrak{g}$ be the Lie algebra of our simple linear Lie group G of hermitian type, and $\mathfrak{t}$, Σ, Σ^+, $\Sigma_\mathfrak{k}^+$, $\Sigma_\mathfrak{p}^+$, $\cdots$ be as in 5.1. For these $\mathfrak{g}_C = \mathfrak{g} \otimes_R C$, $\mathfrak{t}_C = \mathfrak{t} \otimes_R C$ and Σ^+, we consider λ-highest weight modules in 7.1. [Throughout Part II, we will keep the symbol λ for highest weights (not for restricted roots in Λ as in Part I). But, we still employ the notation λ_k for such a specified root in Λ.] In the present case, L_λ has further a structure of admissible $(\mathfrak{g}_C, K)$-module under a certain assumption on $\lambda \in \mathfrak{t}_C^*$. We now specify this condition on λ.

Let Ξ_K^+ denote the set of $\lambda \in \mathfrak{t}_C^*$ satisfying

$$\lambda(H'_\gamma) \geqq 0 \text{ for all } \gamma \in \Sigma_\mathfrak{k}^+, \tag{7.6}$$

(7.7) the map $\exp H \mapsto \exp \lambda(H)$ $(H \in \mathfrak{t})$ gives a unitary character of the compact Cartan subgroup $T \equiv \exp \mathfrak{t}$ of G.

An element of Ξ_K^+ is said to be *$\mathfrak{k}$-dominant* and *K-integral.*

For a $\lambda \in \mathfrak{t}_C^*$, if L_λ is further admissible, then necessarily $\lambda \in \Xi_K^+$, because the highest weight vector must be K-finite.

Conversely, assume that $\lambda \in \Xi_K^+$. Denote by $(\tau_\lambda, V_\lambda)$ an irreducible finite-dimensional $\mathfrak{k}_C$-module with $\Sigma_{\mathfrak{t}}^+$-extreme highest weight λ. We put $\mathfrak{q}_+ = \mathfrak{k}_C \oplus \mathfrak{p}_+$ with $\mathfrak{p}_+$ as in 5.1. Extend τ_λ to a representation $(\tilde{\tau}_\lambda, V_\lambda)$ of the maximal parabolic subalgebra $\mathfrak{q}_+$ trivially on $\mathfrak{p}_+$. Let $M[\lambda]'$ denote the induced $U(\mathfrak{g}_C)$-module given as

$$M[\lambda]' \equiv U(\mathfrak{g}_C) \otimes_{U(\mathfrak{q}_+)} V_\lambda. \tag{7.8}$$

Clearly, $M[\lambda]'$ is a λ-highest weight module.

By the assumption (7.7), τ_λ extends to an irreducible representation of K in the canonical way. $U(\mathfrak{g}_C)$ and $U(\mathfrak{q}_+)$ are K-modules through the adjoint action of K on $\mathfrak{g}_C$ and $\mathfrak{q}_+$ respectively. Thus, also $M[\lambda]'$ has a structure of K-module (consistent with its $\mathfrak{k}_C$-module structure) via

$$k \cdot (D \otimes v) = (\mathrm{Ad}\,(k)D) \otimes (\tau_\lambda(k)v) \quad \text{for } k \in K, \quad D \in U(\mathfrak{g}_C), \quad v \in V_\lambda.$$

Let $\mathfrak{p}_-$ be, as in 5.1, the $\mathrm{Ad}\,(K)$-stable abelian subalgebra of $\mathfrak{g}_C$ opposite to $\mathfrak{p}_+$. Then $\mathfrak{g}_C = \mathfrak{p}_- \oplus \mathfrak{q}_+$, which implies that $M[\lambda]'$ is a free $U(\mathfrak{p}_-)$-module of rank $d_\lambda \equiv \dim V_\lambda$ with generators $1 \otimes v_j$ $(1 \leqq j \leqq d_\lambda)$. Here $\{v_j\}$ is a basis of V_λ. In addition, one has a canonical isomorphism of K-modules:

$$M[\lambda]' = U(\mathfrak{p}_-) \cdot (1 \otimes V_\lambda) \simeq U(\mathfrak{p}_-) \otimes V_\lambda. \tag{7.9}$$

Notice that $U(\mathfrak{p}_-)$ is of K-multiplicity finite under $\mathrm{Ad}\,(K) \,|\, U(\mathfrak{p}_-)$. Therefore, $M[\lambda]'$ also has finite K-multiplicity property, that is, it is admissible.

Consequently, one obtains for each $\lambda \in \Xi_K^+$ an admissible infinite-dimensional $(\mathfrak{g}_C, K)$-module $M[\lambda]'$ with highest weight λ. Since the irreducible object L_λ is isomorphic to a quotient of any λ-highest weight module, we thus find that L_λ is admissible if $\lambda \in \Xi_K^+$.

From the above discussion, one gets

Proposition 7.1. *Let $\lambda \in \mathfrak{t}_C^*$. Then, the irreducible highest weight $U(\mathfrak{g}_C)$-module L_λ has further a structure of admissible $(\mathfrak{g}_C, K)$-module if and only if λ is $\mathfrak{k}$-dominant and K-integral, or $\lambda \in \Xi_K^+$.*

Let $\lambda \in \Xi_K^+$. Then, $M[\lambda]'$ has, just as in the case of Verma module, the universal property:

Proposition 7.2 (cf. [30, 2.3]). *Any admissible $(\mathfrak{g}_C, K)$-module with highest weight λ is isomorphic to a quotient of $M[\lambda]'$.*

On the irreducibility of the universal admissible highest weight modules $M[\lambda]'$, one has the following

Proposition 7.3 ([30, Prop. 2.3.3]). *Assume that $\lambda \in \Xi_K^+$ satisfies*

$(\lambda+\rho)(H'_\gamma) \notin \{1, 2, 3, \cdots\}$ *for every non-compact positive root* $\gamma \in \Sigma_\mathfrak{p}^+$, *where* $\rho \equiv 2^{-1} \sum_{\gamma \in \Sigma^+} \gamma$. *Then,* $M[\lambda]'$ *is irreducible, or equivalently* $M[\lambda]' \simeq L_\lambda$.

Later on, we shall say that $\lambda \in \Xi_K^+$ has the property $(\mathscr{I})$ when $M[\lambda]' \simeq L_\lambda$.

7.3. Highest weight G-modules and the holomorphic discrete series. Thanks to Harish-Chandra's subquotient theorem [8, Th. 4], every irreducible admissible $(\mathfrak{g}_C, K)$-module V can be extended to a representation of G in the following sense: there exists a continuous representation π of G on a Hilbert space $\mathscr{H}$ such that the corresponding $(\mathfrak{g}_C, K)$-module $\mathscr{H}_K$ of K-finite vectors is isomorphic to V. Such an extension $(\pi, \mathscr{H})$ is said to be a globarization of V. An irreducible admissible $(\mathfrak{g}_C, K)$-module is called unitarizable if it admits a unitary globarization. In such a case, its unitary globarization is unique up to unitary equivalence.

For $\lambda \in \Xi_K^+$, let $(\pi_\lambda, \mathscr{H}_\lambda)$ be a globarization of L_λ, where we choose π_λ to be unitary whenever L_λ is unitarizable. One thus gets a series of irreducible admissible highest weight representations π_λ of G.

Harish-Chandra gave a sufficient condition for π_λ to be unitary as follows.

Proposition 7.4 ([9, V], cf. [30, Prop. 2.3.5]). *Assume that* $\lambda \in \Xi_K^+$ *satisfies*

$$(7.10) \qquad (\lambda+\rho)(H'_\gamma) \leqq 0 \qquad \textit{for every } \gamma \in \Sigma_\mathfrak{p}^+.$$

Then, the irreducible admissible highest weight module $L_\lambda \simeq M[\lambda]'$ (*by Proposition* 7.3) *is unitarizable, that is,* π_λ *is a unitary representation of* G. *If, in addition,* λ *fulfills*

$$(7.11) \qquad (\lambda+\rho)(H'_\gamma) < 0 \qquad \textit{for every } \gamma \in \Sigma_\mathfrak{p}^+,$$

then the matrix coefficients of π_λ *are square-integrable on* G (*with respect to a Haar measure*). *This means that* π_λ *belongs to the discrete series.*

Remark 7.5. The condition (7.11) on λ is equivalent to the single condition $(\lambda+\rho)(H'_{\gamma_l}) < 0$, where γ_l is the largest non-compact positive root. The proof of this fact is analogous to that of Lemma 11.10 given later.

These square-integrable representations π_λ will be denoted by D_λ in the sequel. D_λ can be realized explicitly as the representation $\text{Hol-Ind}_K^G(\tau_\lambda)$ induced holomorphically from $\tau_\lambda \in \hat{K}$ (cf. [28, Def. 5.8]). For this reason, (D_λ) is called the holomorphic discrete series. Moreover, (7.11) gives a necessary and sufficient condition (on $\lambda \in \Xi_K^+$) for Hol-Ind$_K^G(\tau_\lambda) \neq (0)$, and so it is said to be the non-vanishing condition for the holomorphic discrete series.

§ 8. Method of highest weight vectors

Let Y be any $(\mathfrak{g}_C, K)$-module. Consider the problem of describing embeddings of irreducible admissible highest weight $(\mathfrak{g}_C, K)$-modules L_λ $(\lambda \in \Xi_K^+)$ into the given Y. Since any highest weight module is characterized by its highest weight vector, this problem is reduced, to a large extent, to the problem of determining K-finite highest weight vectors in Y (see 12.1 for more detail). If Y is, in addition, the $(\mathfrak{g}_C, K)$-module associated with a (smoothly) induced representation of G, then the latter problem amounts to solving a system of differential equations on G characterizing highest weight vectors.

8.1. Our present work in connection with Hashizume [11].

Suggested by the above idea of highest weight vectors, Hashizume treated in [11] the embedding problem into smooth representations $Y = C^\infty$-Ind$_{N_m}^G(\zeta)$ of G induced from the following two types of unitary representations ζ of the maximal unipotent subgroup N_m:

(1) unitary characters ζ of N_m,

(2) representations $\zeta = L^2$-Ind$_{Z_N}^{N_m}(\eta)$ induced from characters η of the center Z_N of the subgroup N in 5.2.

The second case is closely related to our GGGRs $\Gamma_i = \mathrm{Ind}_N^G(\xi_i)$. But unfortunately, in that paper there exist some mistakes in determination of highest weight vectors (see Remark 10.2), and so some of his results, especially those on the uniqueness of embeddings (e.g., [11, Cor. 4.5]), are not correct in general.

So, we are going to modify his calculation to determine completely K-finite highest weight vectors for GGGRs C^∞-Γ_i induced in C^∞-context (§ 10). After that, we shall specify, among such vectors, those contained in the unitary GGGRs L^2-Γ_i, by evaluating L^2-norms of highest weight vectors (§ 11). The determination of highest weight vectors in C^∞-context and in L^2-context enables us to get a nice description of Whittaker models for irreducible admissible highest weight representations (§ 12). Our description is complete at least for L_λ's with the property $(\mathscr{S})$: $M[\lambda]' \simeq L_\lambda$, in 7.2. Furthermore, we find that every L_λ is contained in the GGGR C^∞-Γ_i with $i=0$, as a subquotient.

8.2. Another application: embeddings into the principal series.

Now let $P_m = MA_pN_m$ be a Langlands decomposition of the minimal parabolic subgroup P_m of G (see 3.2). For an irreducible unitary representation σ of M and a continuous character ψ: $A_p \to C^\times$, consider the smoothly induced representation $\pi(\sigma, \psi) \equiv C^\infty\text{-Ind}_{P_m}^G (\sigma \otimes \psi \otimes 1_{N_m})$, with 1_{N_m} = the trivial character of N_m. The family $\{\pi(\sigma, \psi)\}_{\sigma,\psi}$ of representations of G is called the principal series along P_m. $\mathscr{H}(\sigma, \psi)_K$ denotes the corresponding $(\mathfrak{g}_C, K)$-module consisting of K-finite vectors for $\pi(\sigma, \psi)$.

The method of highest weight vectors can be applied successfully also to the present case $Y = \mathscr{H}(\sigma, \psi)_K$, which is related to Hashizume's case (1) (with $\zeta = 1_{N_m}$) in 8.1. We can describe completely embeddings of irreducible admissible highest weight modules L_λ into the principal series $\mathscr{H}(\sigma, \psi)_K$. Especially one gets the following

Theorem 8.1. *All the irreducible admissible highest weight modules have the unique embedding property into the principal series. In other words, for each $\lambda \in \Xi_K^+$, there exist a unique principal series $(\mathfrak{g}_C, K)$-module $\mathscr{H}(\sigma, \psi)_K$ into which L_λ can be embedded. Furthermore, such an embedding $L_\lambda \hookrightarrow \mathscr{H}(\sigma, \psi)_K$ is unique up to scalar multiples.*

The proof is carried out through a calculation of highest weight vectors in $\mathscr{H}(\sigma, \psi)_K$. This calculation is somewhat similar to the one which will be given in Section 10 for GGGRs. Its details are omitted here. Other aspects of embeddings into the principal series are also omitted here. We will treat this subject in another paper.

Remark 8.2. Collingwood proved the above unique embedding property for L_λ's with regular infinitesimal characters [5, Prop. 5.15]. He makes use of the formal characters of the universal admissible highest weight modules $M[\lambda]'$ in (7.8) and those of their asymptotic modules. But, our proof of Theorem 8.1 is more elementary than his, in the sense that one need not to use neither characters nor asymptotic modules.

§ 9. Preliminaries for determination of highest weight vectors

This section is devoted to making some preparations for the succeeding sections, Sections 10 and 11, in which we shall determine all the K-finite highest weight vectors for GGGRs Γ_λ in C^∞-context and in L^2-context respectively.

9.1. Iwasawa decomposition of root vectors ([11, § 1], cf. [3, 4.6]). Keep to the notation in Section 5 for the simple Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ of hermitian type. For each non-compact positive root γ, we express here the root vector X_γ in (5.3) explicitly according as the complexified Iwasawa decomposition $\mathfrak{g}_C = \mathfrak{k}_C \oplus (\mathfrak{a}_p)_C \oplus (\mathfrak{n}_m)_C$, where $\mathfrak{n}_m = \sum_{\psi \in \Delta^+} \mathfrak{g}(\mathfrak{a}_p; \psi)$ with the positive restricted root system Λ^+ in Proposition 5.1.

Now let us construct explicitly bases of root spaces $\mathfrak{g}(\mathfrak{a}_p; \psi)_C \equiv \mathfrak{g}(\mathfrak{a}_p; \psi)\otimes_R C$ ($\psi \in \Lambda^+$). First, let k and m be integers such that $1 \leqq m < k \leqq l = \dim \mathfrak{a}_p$. For $\gamma \in P_{km}$ (see (5.6)), put

$$E_\gamma^\pm = (X_\gamma + [X_{-\gamma_k}, X_\gamma] \pm [X_{-\gamma_m}, X_\gamma] \pm [X_{-\gamma_m}, [X_{-\gamma_k}, X_\gamma]])/2. \tag{9.1}$$

Bearing Proposition 5.1 in mind, we obtain by a simple computation

$$E_\gamma^+ = \mu^{-1}(X_\gamma), \qquad E_\gamma^- = \mu^{-1}([X_\gamma, X_{-\gamma_m}]), \tag{9.2}$$

where μ is the Cayley transform in (5.10). This implies that $E_\gamma^\pm \in \mathfrak{g}(\mathfrak{a}_p; (\lambda_k \pm \lambda_m)/2)_C$. Notice that $\gamma \mapsto \gamma - \gamma_m$ induces a bijection from P_{km} to C_{km} (cf. [II, Prop. 3.1]). Then one gets immediately the following

Lemma 9.1. *For $1 \leqq m < k \leqq l$, $\{E_\gamma^\pm; \gamma \in P_{km}\}$ is a basis of the complexified root space $\mathfrak{g}(\mathfrak{a}_p; (\lambda_k \pm \lambda_m)/2)_C$.*

Secondly, for $1 \leqq k \leqq l$, we construct a basis of $\mathfrak{g}(\mathfrak{a}_p; \lambda_k/2)_C$ convenient for later calculation. For this purpose, recall the complex structure J'' on $\mathfrak{g}(1) = \sum_{1 \leqq k \leqq l} \mathfrak{g}(\mathfrak{a}_p; \lambda_k/2)$ in [II, Lemma 4.4]. It is defined as follows. Let $r: \mathfrak{s} \equiv \mathfrak{a}_p \oplus \mathfrak{n}_m \to \mathfrak{p} = \{X \in \mathfrak{g}; \theta X = -X\}$, be the isomorphism of vector spaces given as $r(X) = (X - \theta X)/2$ ($X \in \mathfrak{s}$). Through this r, the given $\mathrm{Ad}(K)$-invariant complex structure J on $\mathfrak{p}$ (see 5.1) is transferred to the complex structure J' on $\mathfrak{s}$:

$$J' = r^{-1} \circ J \circ r. \tag{9.3}$$

Then, the root space $\mathfrak{g}(\mathfrak{a}_p; \lambda_k/2)$ is stable under J' for every k, whence so is $\mathfrak{g}(1)$, too. We put $J'' = J' | \mathfrak{g}(1)$. Extend J'' to a map on $\mathfrak{g}(1)_C$ by complex linearity, and denote by $V^\pm$ the $(\pm\sqrt{-1})$-eigenspace for J'' on $\mathfrak{g}(1)_C$ respectively. Then we find out by [II, Lemma 4.4] that $V^\pm$ is an abelian subalgebra of $\mathfrak{g}_C$. It is decomposed as

$$V^\pm = \oplus_{1 \leqq k \leqq l} V^\pm(k) \quad \text{with} \quad V^\pm(k) = V \cap \mathfrak{g}(\mathfrak{a}_p; \lambda_k/2)_C. \tag{9.4}$$

Under the above notation, we can prove

Lemma 9 2. *For a $\gamma \in C_k \cup P_k$ (see (5.5)), put*

$$E_\gamma^1 = X_\gamma + [X_{-\gamma_k}, X_\gamma]. \tag{9.5}$$

Then, $\{E_\gamma^1; \gamma \in P_k\}$ (resp. $\{E_\gamma^1; \gamma \in C_k\}$) forms a basis of $V^+(k)$ (resp. $V^-(k)$).

Proof. By [II, (4.9)], we have $E_\gamma^1 = \sqrt{2}\,\mu^{-1}(X_\gamma)$ for $\gamma \in C_k \cup P_k$. Then, the assertion follows from [II, Lemma 4.4(3)]. Q.E.D.

Thirdly, the element $E_k=\sqrt{-1}(H'_{\gamma_k}-X_{\gamma_k}+X_{-\gamma_k})/2$ in (5.11) lies in the root space $\mathfrak{g}(\mathfrak{a}_p;\lambda_k)$ (which is one-dimensional) for $1\leqq k\leqq l$.

Consequently, the elements $E^{\pm}_\gamma$ ($\gamma\in P_{km}$, $1\leqq m<k\leqq l$), E^1_γ ($\gamma\in C_k\cup P_k$, $1\leqq k\leqq l$) and E_k ($1\leqq k\leqq l$) form a basis of $(\mathfrak{n}_m)_C$. Using this basis, we get the following expressions of root vectors X_γ ($\gamma\in\Sigma^+_p$) along the Iwasawa decomposition $\mathfrak{g}_C=\mathfrak{k}_C\oplus(\mathfrak{a}_p)_C\oplus(\mathfrak{n}_m)_C$.

Proposition 9.3 (cf. [11, Lemma 1.1]). (1) *If $\gamma=\gamma_k$, then*

$$X_{\gamma_k}=(H'_{\gamma_k}+H_k+2\sqrt{-1}E_k)/2 \tag{9.6}$$

with $H'_{\gamma_k}\in\sqrt{-1}\mathfrak{t}\subseteqq\mathfrak{k}_C$, $H_k=X_{\gamma_k}+X_{-\gamma_k}\in\mathfrak{a}_p$ and $E_k\in\mathfrak{g}(\mathfrak{a}_p;\lambda_k)\subseteqq\mathfrak{n}_m$.

(2) *Let $\gamma\in P_{km}$. Then one gets*

$$X_\gamma=[X_\gamma,X_{-\gamma_k}]+E^+_\gamma+E^-_\gamma, \tag{9.7}$$

where $[X_\gamma,X_{-\gamma_k}]\in[\mathfrak{p}_C,\mathfrak{p}_C]\subseteqq\mathfrak{k}_C$, $E^{\pm}_\gamma\in\mathfrak{g}(\mathfrak{a}_p;(\lambda_k\pm\lambda_m)/2)_C\subseteqq(\mathfrak{n}_m)_C$. Furthermore, there holds that

$$[A,E^-_\gamma]=\sqrt{-1}E^+_\gamma \quad \text{with} \quad A=A[0]=\textstyle\sum_{1\leqq k\leqq l}E_k. \tag{9.8}$$

(3) *For $\gamma\in P_k$, the root vector X_γ is expressed as*

$$X_\gamma=[X_\gamma,X_{-\gamma_k}]+E^1_\gamma, \tag{9.9}$$

where $[X_\gamma,X_{-\gamma_k}]\in\mathfrak{k}_C$ and $E^1_\gamma\in V^+(k)\subseteqq(\mathfrak{n}_m)_C$.

Proof. The assertion (1) follows from the definition of E_k. We get (3) and the first half of (2) respectively from Lemmas 9.2 and 9.1. The equality (9.8) is derived by a simple calculation, by taking into account Proposition 5.1 and definitions of $E^{\pm}_\gamma$ and E_k. Q.E.D.

9.2. An embedding α: $S^C_0\hookrightarrow K^C$.

Let G^C be the complexification of our linear group G, that is, G^C is the connected linear Lie group with Lie algebra $\mathfrak{g}_C$, containing G. For any analytic subgroup R of G with Lie algebra $\mathfrak{r}$, R^C will denote the connected Lie subgroup of G^C corresponding to the complexification $\mathfrak{r}_C$ of $\mathfrak{r}$.

We now introduce an embedding of a solvable subgroup of G^C into K^C, which will be utilized in our expression of highest weight vectors for GGGRs. Consider the subgroups $A^C_p\equiv(A_p)^C$, $(N_m\cap L)^C$ and $\exp V^+$ of G^C, where L is the Levi subgroup in 5.2. The corresponding Lie algebras are expressed respectively as

$$(\mathfrak{a}_p)_C=\mu^{-1}(\mathfrak{t}^-_C) \qquad \text{(see 5.1)},$$
$$(\mathfrak{n}_m\cap\mathfrak{l})_C=\mu^{-1}(\textstyle\sum_{k>m}\sum_{\gamma\in C_{km}}\mathfrak{g}_C(\mathfrak{t}_C;\gamma)),$$

$$V^+=\mu^{-1}(\sum_{1\leqq k\leqq l}\sum_{\gamma\in P_k}\mathfrak{g}_C(\mathfrak{t}_C;\gamma)).$$

Here, the second equality follows from (5.14), and the third one from [II, Lemma 4.4]. These expressions imply that $(\mathfrak{s}_0)_C\equiv(\mathfrak{a}_p)_C\oplus(\mathfrak{n}_m\cap\mathfrak{l})_C\oplus V^+$ is a solvable Lie subalgebra of $\mathfrak{g}_C$, and the corresponding analytic subgroup S_0^C is decomposed as

$$S_0^C=A_p^C\ltimes((N_m\cap L)^C\ltimes(\exp V^+))\qquad\text{(semidirect product)}.$$

Let α be the restriction of the automorphism $\mu\circ\theta$ of $\mathfrak{g}_C$ onto the subalgebra $(\mathfrak{s}_0)_C$. Then, α can be extended canonically to a group isomorphism from S_0^C into G^C, which will be denoted still by α.

We now describe the image $\alpha((\mathfrak{s}_0)_C)$. The elements H_k $(1\leqq k\leqq l)$ form a basis of $\mathfrak{a}_p$. Recall that $\mu(H_k)=H'_{\gamma_k}\in\mathfrak{t}^-$. We thus obtain

$$\alpha(H_k)=-H'_{\gamma_k}. \tag{9.10}$$

Moreover, the basis E_γ^- $(\gamma\in P_{km},\ m<k)$, E_γ^1 $(\gamma\in P_k,\ 1\leqq k\leqq l)$ of $(\mathfrak{n}_m\cap\mathfrak{l})_C\oplus V^+$ is transferred by α as follows.

Lemma 9.4. (1) *The elements $\alpha(E_\gamma^1)$ $(\gamma\in P_k)$ are expressed as*

$$\alpha(E_\gamma^1)=-\sqrt{2}\,[X_\gamma,X_{-\gamma_k}]\in[\mathfrak{p}_C,\mathfrak{p}_C]\subseteqq\mathfrak{k}_C. \tag{9.11}$$

(2) *For $\gamma\in P_{km}$, one has*

$$\alpha(E_\gamma^-)=-[X_\gamma,X_{-\gamma_k}]\in\mathfrak{k}_C. \tag{9.12}$$

Proof. (1) For $\gamma\in P_k$, we can calculate $\mu^{-1}([X_\gamma,X_{-\gamma_k}])$ exactly as in [II, (4.9)], and get

$$\mu^{-1}([X_\gamma,X_{-\gamma_k}])=(X_\gamma-[X_{-\gamma_k},X_\gamma])/\sqrt{2}.$$

Since $X_\gamma\in\mathfrak{p}_C$ and $[X_{-\gamma_k},X_\gamma]\in\mathfrak{k}_C$, it holds that

$$\begin{aligned}\alpha^{-1}([X_\gamma,X_{-\gamma_k}])&=\theta(\mu^{-1}([X_\gamma,X_{-\gamma_k}]))\\&=-(X_\gamma+[X_{-\gamma_k},X_\gamma])/\sqrt{2}=-E_\gamma^1/\sqrt{2},\end{aligned}$$

which proves (9.11).

(9.12) is proved in the same line as above. Q.E.D.

Thanks to this lemma, we now find out

$$\begin{aligned}\alpha((\mathfrak{s}_0)_C)&=\mathfrak{t}_C^-\oplus\sum_{\gamma\in C_{km},k>m}\mathfrak{g}_C(\mathfrak{t}_C;-\gamma)\oplus\sum_{\gamma\in C_k,1\leqq k\leqq l}\mathfrak{g}_C(\mathfrak{t}_C;-\gamma)\\&=\mathfrak{t}_C^-\oplus\sum_{\gamma\in\Sigma_{\mathfrak{t}}^+\setminus C_0}\mathfrak{g}_C(\mathfrak{t}_C;-\gamma)\subseteqq\mathfrak{k}_C,\end{aligned} \tag{9.13}$$

where C_0 is, as in (5.7), the set of compact positive roots corresponding to imaginary roots through the Cayley transform μ. In fact, the inclusion $\subseteqq$ of the first equality is a direct consequence of (9.10)–(9.12). In view of Moore's restricted root theorem (Proposition 5.1), the dimensions of spaces in the both hand sides must be the same. This proves the first equality. The second one follows from (5.8).

So in particular, the isomorphism α on the group level carries S_0^C into K^C.

By using α, the expressions (9.6), (9.7) and (9.9) along the Iwasawa decomposition are rewritten respectively as follows.

Proposition 9.5. *Let $A=\sum_{1\leqq k\leqq l} E_k$. Then one gets the expressions of root vectors as*

$$(9.6') \qquad X_{\gamma_k}=-(\alpha(H_k)-H_k+\sqrt{-1}[A, H_k])/2 \qquad (1\leqq k\leqq l),$$

$$(9.7') \qquad X_\gamma=-(\alpha(E_\gamma^-)-E_\gamma^-+\sqrt{-1}[A, E_\gamma^-]) \qquad (\gamma \in P_{km}),$$

$$(9.9') \qquad X_\gamma=-(\alpha(E_\gamma^1)/\sqrt{2})+E_\gamma^1 \qquad (\gamma \in P_k).$$

9.3. The Fock model ρ_i.

For $0\leqq i\leqq l$, let $\Gamma_i=\mathrm{Ind}_N^G(\xi_i)$ be the GGGR in (5.15), associated with the nilpotent class $\omega_i=\mathrm{Ad}(G)A[i]$. We have constructed in (6.1) the irreducible unitary representation ξ_i of $N=\exp(\mathfrak{g}(1)\oplus\mathfrak{g}(2))$ through a real polarization at $A[i]^* \in \mathfrak{n}^*$. But, for later calculation, it is more convenient to adopt another type of realization ρ_i $(\simeq\xi_i)$, so-called Fock model, which is constructed through a positive polarization.

Now we recall the construction of ρ_i after [II, 4.3]. Let J_i'' denote the complex structure on $\mathfrak{g}(1)$ obtained by twisting J'' in 9.1 in the following fashion:

$$(9.14) \qquad J_i''(X)=\begin{cases} -J''(X) & \text{if } X \in \sum_{k\leqq i} \mathfrak{g}(\mathfrak{a}_p;\ \lambda_k/2), \\ J''(X) & \text{if } X \in \sum_{m>i} \mathfrak{g}(\mathfrak{a}_p;\ \lambda_m/2). \end{cases}$$

By $(\mathfrak{g}(1), J_i'')$, we mean the complex vector space $\mathfrak{g}(1)$ equipped with this complex structure J_i''. Define a bilinear form $(\,,)_i$ on $\mathfrak{g}(1)$ by

$$(9.15) \qquad (X, X')_i=-\{A[i]^*([J_i''X, X'])+\sqrt{-1}A[i]^*([X, X'])\}/4$$

for $X, X' \in \mathfrak{g}(1)$. By [II, Lemma 4.11], $(\,,)_i$ gives a positive definite hermitian form on $(\mathfrak{g}(1), J_i'')$.

Let $\mathscr{F}_i$ be the Fock space of the finite-dimensional Hilbert space $(\mathfrak{g}(1), J_i'', (\,,)_i)$ over $\boldsymbol{C}$. Namely, $\mathscr{F}_i$ consists of all holomorphic functions ϕ on $(\mathfrak{g}(1), J_i'')$ satisfying

$$(9.16)\qquad \|\phi\|_{\mathscr{F}_i}^2 \equiv \int_{\mathfrak{g}(1)} |\phi(X)|^2 \exp(-2\|X\|_i^2) dX < +\infty,$$

where $\|X\|_i^2 \equiv (X, X)_i$ and dX is a Lebesgue measure on $\mathfrak{g}(1)$. Then, $\mathscr{F}_i$ has a structure of Hilbert space induced from this norm. Further, it contains the space of all polynomial functions on $(\mathfrak{g}(1), J_i'')$ as a dense subspace.

Let $n = \exp(X_0 + Y_0)$ with $X_0 \in \mathfrak{g}(1)$ and $Y_0 \in \mathfrak{g}(2)$. We put

$$(9.17)\quad (\rho_i(n)\phi)(X) = \exp\{2(X, X_0)_i - \|X_0\|_i^2 + \sqrt{-1}A[i]^*(Y_0)\} \cdot \phi(-X_0 + X)$$

for $X \in \mathfrak{g}(1)$ and $\phi \in \mathscr{F}_i$. Then, ρ_i gives an irreducible unitary representation of N on the Fock space $\mathscr{F}_i$, which is unitarily equivalent to ξ_i. We call $(\rho_i, \mathscr{F}_i)$ the *Fock model* of the unitary equivalence class $[\xi_i] \in \hat{N}$.

§ 10. Determination of highest weight vectors (Step I): Case of C^∞-induced GGGRs

In this section, using the preparatory results in Section 9, we determine explicitly all the K-finite λ-highest weight vectors for GGGRs C^∞-Γ_i in C^∞-context, for any $\mathfrak{k}$-dominant, K-integral linear form $\lambda \in \mathfrak{t}_C^*$. The main result here is given as Theorem 10.6.

10.1. Spaces of highest weight vectors.

Let us realize our GGGR C^∞-Γ_i as C^∞-$\mathrm{Ind}_N^G(\rho_i) = (\pi_i, C^\infty(G; \rho_i))$ by making use of the Fock model $(\rho_i, \mathscr{F}_i)$ in 9.3. Here, the representation space $C^\infty(G; \rho_i)$ consists of all $\mathscr{F}_i$-valued C^∞-functions F on G such that $F(gn) = \rho_i(n)^{-1}F(g)$ ($g \in G$, $n \in N$), and the action π_i of G is given by left translation. By differentiating this G-action, we equip $C^\infty(G; \rho_i)$ with a $\mathfrak{g}_C$-module structure denoted again by π_i. Let $C^\infty(G; \rho_i)_K$ denote the $(\mathfrak{g}_C, K)$-module of K-finite vectors for π_i. As seen in [I, 2.2], it admits a direct sum decomposition such as

$$(10.1)\qquad C^\infty(G; \rho_i)_K = \oplus_{\tau \in \hat{K}} C^\infty(G; \rho_i)_\tau \qquad (\text{as } K\text{-modules}),$$

where, for any $\tau \in \hat{K}$, $C^\infty(G; \rho_i)_\tau$ denotes the τ-isotypic component of $C^\infty(G; \rho_i)$.

Further, each constituent $C^\infty(G; \rho_i)_\tau$ is decomposed as a K-module into irreducibles in the following way. Take an irreducible unitary representation of K realizing $\tau \in \hat{K}$, and denote it still by τ. Let V_τ be its representation space. By (τ^*, V_τ^*), we mean the representation of K contragredient to (τ, V_τ). Let $C_\tau^\infty(G; \rho_i)$ denote the space of $(V_\tau^* \otimes \mathscr{F}_i)$-valued C^∞-functions $\tilde{F}$ on G satisfying

$$(10.2)\qquad \tilde{F}(kgn) = (\tau^*(k) \otimes \rho_i(n)^{-1})\tilde{F}(g) \qquad (k \in K, g \in G, n \in N).$$

Equip this space with the trivial K-module structure. Then one obtains an isomorphism of K-modules:

(10.3) $$C^\infty(G;\rho_i)_\tau \simeq C_\tau^\infty(G;\rho_i)\otimes V_\tau.$$

The isomorpihsm is given as

(10.4) $$C_\tau^\infty(G;\rho_i)\otimes V_\tau \ni \tilde{F}\otimes v \longmapsto \tilde{F}_v \in C^\infty(G;\rho_i)_\tau,$$

where $\tilde{F}_v$ is defined by

(10.5) $$\tilde{F}_v(g)=\langle\!\langle v, \tilde{F}(g)\rangle\!\rangle \qquad (g \in G).$$

Here, $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes an $\mathscr{F}_i$-valued bilinear form on $V_\tau\times(V_\tau^*\otimes\mathscr{F}_i)$ such that

$$\langle\!\langle v, v^*\otimes\phi\rangle\!\rangle=\langle v, v^*\rangle\cdot\phi \qquad (v\in V_\tau,\ v^*\in V_\tau^*,\ \phi\in\mathscr{F}_i).$$

Now let $\lambda\in\Xi_K^+$, a $\mathfrak{k}$-dominant, K-integral linear form on $\mathfrak{t}_C$. We denote by $C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger$ the space of Σ^+-extreme λ-highest weight vectors in $C^\infty(G;\rho_i)_K$. This is the main object of this section. Necessarily, we have

$$C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger \subseteqq C^\infty(G;\rho_i)_{\tau_\lambda},$$

where $(\tau_\lambda, V_\lambda)$ is an irreducible representation of K with $\Sigma_{\mathfrak{t}}^+$-extreme highest weight λ. Furthermore, if $v_\lambda\in V_\lambda$, $\neq 0$, be a highest weight vector, then one gets through the isomorphism (10.3) the following embedding of vector space:

(10.6) $$C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger \hookrightarrow C_{\tau_\lambda}^\infty(G;\rho_i)\otimes v_\lambda.$$

The image of this embedding is characterized as follows.

Proposition 10.1. *Let $C_{\tau_\lambda}^\infty(G;\rho_i\|\Sigma_{\mathfrak{p}}^+;\lambda)^\dagger$ be the space of all $\tilde{F}\in C_{\tau_\lambda}^\infty(G;\rho_i)$ satisfying*

(10.7) $$L_X\tilde{F}=0 \qquad \textit{for all } X\in\mathfrak{p}_+,$$

where L_X is as in (2.4). *Then one obtains an isomorphism of vector spaces as*

(10.8) $$C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger \simeq C_{\tau_\lambda}^\infty(G;\rho_i\|\Sigma_{\mathfrak{p}}^+;\lambda)^\dagger\otimes v_\lambda$$

through the correspondence (10.4).

Proof. If $\tilde{F}\in C_{\tau_\lambda}^\infty(G;\rho_i\|\Sigma_{\mathfrak{p}}^+;\lambda)^\dagger$, then $\tilde{F}_v$ is clearly a Σ^+-extreme λ-highest weight vector, or $\tilde{F}_v\in C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger$.

Conversely, take an F from $C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$. By (10.6), there exists a unique $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i)$ such that

$$F(g) = \langle\langle v_\lambda, \tilde{F}(g) \rangle\rangle. \tag{10.9}$$

Let $X \in \mathfrak{p}_+$, $g \in G$ and $k \in K$. Then, a simple calculation yields

$$(\pi_i(\mathrm{Ad}(k)X)F)(k^{-1}g) = \langle\langle \tau_\lambda(k)v_\lambda, (L_X\tilde{F})(g) \rangle\rangle.$$

Notice that $\mathfrak{p}_+$ is $\mathrm{Ad}(K)$-stable. Then we see that the left hand side vanishes, because $\pi_i(\mathfrak{p}_+)F = (0)$. We thus obtain

$$\langle\langle \tau_\lambda(k)v_\lambda, (L_X\tilde{F})(g) \rangle\rangle = 0 \qquad \text{for all } g \in G,\ k \in K \text{ and } X \in \mathfrak{p}_+,$$

which is equivalent to (10.7) since V_λ is an irreducible K-module. This completes the proof. Q.E.D.

Thanks to this proposition, the problem of determining K-finite highest weight vectors for C^∞-Γ_i is now reduced to describing the spaces $C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$ ($\lambda \in \Xi^+_K$). Hereafter, we shall settle the latter problem in the following procedure.

Note that G admits the decomposition $G = KA_p(N_m \cap L)N = K \cdot A_p(N_m \cap L) \cdot N$, where L is the Levi subgroup in 5.2. In view of (10.2), each $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$ is uniquely determined by its restriction $\Phi \equiv \tilde{F} | A_p(N_m \cap L)$. We shall rewrite the system of differential equations (10.7) for $\tilde{F}$ on G to that for Φ on $A_p(N_m \cap L)$, using Iwasawa decomposition of non-compact root vectors given in 9.2. Then, solving it, we construct all $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$ from solutions Φ.

Remark 10.2. Hashizume asserts in [11, p. 65] that any element $F \in C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ (but not $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$) is uniquely determined by its restriction onto $A_p(N_m \cap L)$. But this is not true if $\dim \tau_\lambda > 1$. Correcting this error, we shall determine $\tilde{F} | A_p(N_m \cap L)$ in the succeeding subsections. Because of this modification, our calculation becomes more complicated than that in [11]. Further, we will modify the statements on uniqueness of embeddings, in Theorem 3.1 and Corollary 4.5 of [11]. In our case of GGGR Γ_i, the holomorphic discrete series D_λ occurs in Γ_i with $i = 0$, exactly $\dim \tau_\lambda$ times (not with multiplicity one).

10.2. Description of spaces $C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$.

Let $\lambda \in \Xi^+_K$ and $0 \leqq i \leqq l$. Take an $\tilde{F}$ from $C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma^+_{\mathfrak{p}}; \lambda)^\dagger$. Since the representation space $\mathscr{F}_i$ of ρ_i consists of holomorphic functions on $(\mathfrak{g}(1), J_i'')$, the $(V_\lambda^* \otimes \mathscr{F}_i)$-valued function $\tilde{F}$ is regarded canonically as a V_λ^*-valued function $(g, U) \mapsto \tilde{F}(g: U)$ on $G \times \mathfrak{g}(1)$, where J_i'' is the complex structure on $\mathfrak{g}(1)$ given in (9.14). For a differential operator D_1 on G (resp. D_2 on $\mathfrak{g}(1)$), we mean by $\tilde{F}(g; D_1: U)$ (resp. $\tilde{F}(g: U; D_2)$) the differential $[D_1\tilde{F}(\cdot : U)](g)$ (resp. $[D_2\tilde{F}(g: \cdot)](U)$) with respect to $g \in G$ (resp. $U \in \mathfrak{g}(1)$).

Put $\Phi=\tilde{F}|(A_p(N_m \cap L)\times\mathfrak{g}(1))$. Then the differentiability of $\tilde{F}$ together with (10.2) implies that

(10.10) $A_p(N_m \cap L) \ni an_0 \mapsto \Phi(an_0: \cdot)$ gives a $(V_\lambda^* \otimes \mathscr{F}_i^\infty)$-valued C^∞-function on $A_p(N_m \cap L)$, where $\mathscr{F}_i^\infty \subseteqq \mathscr{F}_i$ is the space of C^∞-vectors for ρ_i.

In particular, $\Phi(an_0: U)$ is smooth in $(an_0, U) \in A_p(N_m \cap L)\times\mathfrak{g}(1)$, and is holomorphic in $U \in (\mathfrak{g}(1), J_i'')$ for any fixed an_0.

Let us now rewrite the condition (10.7) for $\tilde{F}$ to that for Φ. First, thanks to Lemma 9.1 and Proposition 9.5, the map

$$X \longmapsto X^\flat \equiv \alpha(X) - X - \sqrt{-1}[X, A] \qquad \text{with } A=\textstyle\sum_{1\leqq k\leqq l} E_k,$$

induces an isomorphism of vector spaces:

$$(\mathfrak{a}_p \oplus \mathfrak{n}_m \cap \mathfrak{l})_C \xrightarrow{\sim} \textstyle\sum_{1\leqq k\leqq l} \mathfrak{g}_C(\mathfrak{t}_C; \gamma_k) \oplus \sum_{\gamma \in P_{km}, k>m} \mathfrak{g}_C(\mathfrak{t}_C; \gamma) \subseteqq \mathfrak{p}_+,$$

where α is the embedding $(\mathfrak{z}_0)_C \hookrightarrow \mathfrak{k}_C$ in 9.2. Its image coincides with the whole $\mathfrak{p}_+$ if and only if G/K is of tube type. Furthermore, $\alpha(X) \in \mathfrak{k}_C$ and $[X, A] \in \mathfrak{g}(2)_C$, the center of $\mathfrak{n}_C$, for all X. Taking (9.17) and (10.2) into account, one gets

$$\begin{aligned}(10.11)\qquad \tilde{F}(an_0; L_{X^\flat}: U) &= -\Phi(an_0; L_X: U)\\ &\quad +(-(\tau_\lambda^* \circ \alpha)(X)+\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}[X, A]\rangle I)\cdot\Phi(an_0: U),\end{aligned}$$

where I is the identity map on V_λ^*. Since $X^\flat \in \mathfrak{p}_+$, (10.7) implies that

$$\begin{aligned}(10.12)\qquad &\Phi(an_0; L_X: U)\\ &=(-(\tau_\lambda^* \circ \alpha)(X)+\langle A[i]^* \ \mathrm{Ad}(an_0)^{-1}[X, A]\rangle I)\cdot\Phi(an_0; U)\end{aligned}$$

for all $X \in (\mathfrak{a}_p \oplus \mathfrak{n}_m \cap \mathfrak{l})_C$.

Secondly, in view of Lemma 9.2 and (9.9′), the assignment

$$V^+ \ni Z \longmapsto Z^\# = Z - \alpha(Z)/\sqrt{2} = \mu(Z)/\sqrt{2}$$

gives an isomorphism: $V^+ \xrightarrow{\sim} \sum_{\gamma\in P_{k}, 1\leqq k\leqq l} \mathfrak{g}_C(\mathfrak{t}_C; \gamma) \subseteqq \mathfrak{p}_+$, where V^+ is the subspace of $\mathfrak{g}(1)_C$ in (9.4). For $Z \in \mathfrak{g}(1)_C$ and any i, $0 \leqq i \leqq l$, let $X_i^\pm[Z]$ denote the elements of $\mathfrak{g}(1)$ characterized by

$$(10.13)\qquad Z=(X_i^-[Z]+\sqrt{-1}\,J_i''X_i^-[Z])+(X_i^+[Z]-\sqrt{-1}\,J_i''X_i^+[Z]).$$

Then, $Z \mapsto X_i^{\mp}[Z] \mp \sqrt{-1}\, J_i'' X_i^{\mp}[Z]$ gives the projection of $\mathfrak{g}(1)_C$ onto $V_i^{\pm}$ along the decomposition $\mathfrak{g}(1)_C = V_i^+ \oplus V_i^-$. We put for $X \in \mathfrak{g}(1)$, $\phi \in C^\infty(\mathfrak{g}(1))$,

$$\partial(X)\phi(U) = \frac{d}{dt}\phi(U+tX)|_{t=0} \qquad (U \in \mathfrak{g}(1)), \tag{10.14}$$

and $\partial(Z) = \partial(X) + \sqrt{-1}\,\partial(Y)$ for $Z = X + \sqrt{-1}\, Y$ with $X, Y \in \mathfrak{g}(1)$. Then one gets easily for $Z \in V^+$,

$$\begin{aligned} &\tilde{F}(an_0; L_{(\mathrm{Ad}(an_0)Z)^{\#}}: U) = -\Phi(an_0: U; \partial(Z)) \\ &\qquad + \{(\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z)/\sqrt{2} + 4(U, X_i^-[Z])_i \cdot I\} \cdot \Phi(an_0: U), \end{aligned} \tag{10.15}$$

where $(\ ,\ )_i$ is the inner product on $(\mathfrak{g}(1), J_i'')$ in (9.15). Since $Z^{\#} \in \mathfrak{p}_+$, the left hand side vanishes by (10.7), whence the function Φ on $A_p(N_m \cap L) \times \mathfrak{g}(1)$ satisfies

$$\begin{aligned} &\Phi(an_0: U; \partial(Z)) \\ &\qquad = \{(\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z)/\sqrt{2} + 4(U, X_i^-[Z])_i \cdot I\} \cdot \Phi(an_0: U) \end{aligned} \tag{10.16}$$

for every $Z \in V^+$. Note that (10.16) has no contribution if G/K is of tube type.

From (5.2) and (5.9), we see that $\mathfrak{p}_+$ is spanned by $X^{\flat}$'s and $Z^{\#}$'s, so we get the following

Proposition 10.3. *For $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_{\mathfrak{p}}^+; \lambda)^{\dagger}$, its restriction Φ onto $A_p(N_m \cap L) \times \mathfrak{g}(1)$ satisfies* (10.10), (10.12) *and* (10.16). *Conversely, any V_λ^*-valued function Φ on $A_p(N_m \cap L) \times \mathfrak{g}(1)$ with properties* (10.10), (10.12) *and* (10.16) *can be extended uniquely to an element $\tilde{F} = \tilde{F}[\Phi]$ in $C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_{\mathfrak{p}}^+; \lambda)^{\dagger}$.*

Proof. We have already proved the former statement, so it sufficies to show the latter one. It is obvious that any Φ satisfying (10.10), (10.12) and (10.16) can be extended uniquely to an element $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i)$ through the relation (10.2). Let $X \in \mathfrak{p}_+$. By the construction of $\tilde{F}$, these $\tilde{F}$ and Φ satisfy (10.11) and (10.15). This together with (10.12) and (10.16) implies $\tilde{F}(an_0; L_X: U) = 0$. Express a given $g \in G$ as $g = kan_0n$ with $(k, a, n_0, n) \in K \times A_p \times (N_m \cap L) \times N$. Then, a simple computation yields

$$\tilde{F}(g; L_X: U) = \{(\tau_\lambda^*(k) \otimes \rho_i(n)^{-1}) \cdot \tilde{F}(an_0; L_{\mathrm{Ad}(k)^{-1}X}: \cdot)\}(U),$$

where $\tilde{F}(an_0; L_{\mathrm{Ad}(k)^{-1}X}: \cdot)$ is the element of $V_\lambda^* \otimes \mathscr{F}_i$ such that $\mathfrak{g}(1) \ni U \mapsto \tilde{F}(an_0; L_{\mathrm{Ad}(k)^{-1}X}: U) \in V_\lambda^*$. Noting that $\mathrm{Ad}(k)\mathfrak{p}_+ = \mathfrak{p}_+$, we thus get $L_X\tilde{F} = 0$, that is, $\tilde{F} \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_{\mathfrak{p}}^+; \lambda)^{\dagger}$ as desired. Q.E.D.

We now solve the system of differential equations (10.12) and (10.16), and describe the spaces $C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger$. Define a $GL(V_\lambda^*)$-valued C^∞-function G_λ^i on $A_p(N_m \cap L)$ by

(10.17) $$G_\lambda^i(an_0) = (\exp \langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A \rangle) \cdot (\tau_\lambda^* \circ \alpha)(an_0),$$

where α denotes the embedding $S_0^C \hookrightarrow K^C$ in 9.2, and we extend τ_λ^* from K to a holomorphic representation of K^C in the canonical way. It is easy to see that this operator valued function G_λ^i satisfies (10.12). This implies immediately that any solution Φ of (10.12) is of the form

(10.18) $$\Phi(an_0 \colon U) = G_\lambda^i(an_0)\tilde{\Phi}(U)$$

with a V_λ^*-valued function $\tilde{\Phi}$ on $\mathfrak{g}(1)$.

From now on, we treat two cases: (Case I) and (Case II) in Proposition 6.1, separately.

10.2.1. (Case I): *Tube Case.* Assume that G/K is holomorphically equivalent to a tube domain. Then, $\mathfrak{g}(1) = (0)$, whence ρ_i is one-dimensional. Each solution Φ of (10.12) is regarded as a V_λ^*-valued function on $A_p(N_m \cap L)$. (10.18) means that

(10.19) $$\Phi_{v^*}^{\lambda i}(an_0) \equiv G_\lambda^i(an_0)v^*$$

satisfies (10.12) for every $v^* \in V_\lambda^*$, and any solution is of this from. Clearly, $\Phi_{v^*}^{\lambda i}$ satisfies (10.10), too. By virtue of Proposition 10.3, we deduce immediately

Proposition 10.4. *If G/K is of tube type, one has an isomorphism $V_\lambda^* \simeq C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger$ (as vector spaces) for each $0 \leqq i \leqq l$. The isomorphism is given as*

$$V_\lambda^* \ni v^* \longmapsto \tilde{F}_{v^*}^{\lambda i} \equiv \tilde{F}[\Phi_{v^*}^{\lambda i}] \in C^\infty_{\tau_\lambda}(G; \rho_i \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger,$$

where $\tilde{F}[\cdot]$ is as in Proposition 10.3.

This is a complete result for tube case.

10.2.2. (Case II): *Non-Tube Case.* We now proceed to the case where G/K does not reduce to a tube domain, or the (Case II) in Proposition 5.1. Then $\mathfrak{g}(1) \neq (0)$, and so the Fock representation ρ_i of N is actually infinite-dimensional for evrey i. Therefore, we need to (and do) solve the differential equation (10.16), too.

First, suppose that $i \neq 0$. Take a non-zero element Z from $V^+(1) = V^+ \cap \mathfrak{g}(\mathfrak{a}_p; \lambda_1/2)_C$. Then Z belongs to $V_i^- = \{W \in \mathfrak{g}(1)_C; J_i'' W = -\sqrt{-1}\, W\}$

by the definition of J_i'', which implies that

$$Z = X_i^-[Z] + \sqrt{-1}\, J_i'' X_i^-[Z] \qquad \text{with } X_i^-[Z] \in \mathfrak{g}(1), \ \neq 0.$$

If Φ is a solution of (10.16), then it holds that

$$\{(\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z)/\sqrt{2} + 4(U, X_i^-[Z])_i \cdot I\} \cdot \Phi(an_0;\ U) = 0.$$

Notice that $(\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z)$ is a nilpotent operator on V_λ^*. Then we find out that, if $(U, X_i^-[Z])_i \neq 0$, then the linear map $(\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z)/\sqrt{2} + (U, X_i^-[Z])_i \cdot I$ on V_λ^* is invertible. This implies that $\Phi(an_0 \colon U) = 0$ whenever $(U, X_i^-[Z])_i \neq 0$. Hence one gets $\Phi = 0$ because Φ is holomorphic in $U \in (\mathfrak{g}(1), J_i'')$. We thus conclude

$$C_{\tau_\lambda}^\infty(G; \rho_i \| \Sigma_{\mathfrak{p}}^+; \lambda)^\dagger = (0) \qquad \text{if } i \neq 0. \tag{10.20}$$

So, let us consider the remaining case $i = 0$. In this case, $X_i^-[Z] = X_0^-[Z] = 0$ for all $Z \in V^+$. Hence the equation (10.16) becomes

$$\Phi(an_0 \colon U;\ \partial(Z)) = \frac{1}{\sqrt{2}} (\tau_\lambda^* \circ \alpha)(\mathrm{Ad}(an_0)Z) \cdot \Phi(an_0 \colon U) \qquad (Z \in V^+). \tag{10.16$'$}$$

In view of (10.18), this equation for Φ is rewritten immediately to that for $\tilde{\Phi} \colon \mathfrak{g}(1) \to V_\lambda^*$, as

$$\tilde{\Phi}(U;\ \partial(Z)) = \frac{1}{\sqrt{2}} (\tau_\lambda^* \circ \alpha)(Z) \cdot \tilde{\Phi}(U) \qquad (Z \in V^+). \tag{10.21}$$

Notice that $\tilde{\Phi}$ is holomorphic with respect to U in $(\mathfrak{g}(1), J'')$, $J'' = J_0''$:

$$\tilde{\Phi}(U;\ \partial(\bar{Z})) = 0 \qquad \text{for all } \bar{Z} \in V^-. \tag{10.22}$$

We can solve these two equations for $\tilde{\Phi}$ explicitly as follows. Let $v^* \in V_\lambda^*$. We put

$$\begin{aligned} &\tilde{\Phi}_{v^*}^\lambda(U) = (\tau_\lambda^* \circ \alpha)(\exp p(U)) v^*, \\ &p(U) = (U - \sqrt{-1} J'' U)/2\sqrt{2} \in V^+ \end{aligned} \tag{10.23}$$

for $U \in \mathfrak{g}(1)$. Then it is easily checked that these $\tilde{\Phi}_{v^*}^\lambda$ ($v^* \in V_\lambda^*$) exhaust all the solutions of (10.21) and (10.22). Furthermore, $v^* \longmapsto \tilde{\Phi}_{v^*}^\lambda$ sets up an isomorphism of vector space from V_λ^* onto the space of solutions.

We define a V_λ^*-valued function $\Phi_{v^*}^\lambda$ on $A_p(N_m \cap L) \times \mathfrak{g}(1)$ by

$$\Phi_{v^*}^\lambda(an_0 \colon U) = G_\lambda^0(an_0) \tilde{\Phi}_{v^*}^\lambda(U) = \tilde{G}_\lambda(an_0 \colon U) v^*, \tag{10.24}$$

where

$$(10.25)\qquad \tilde{G}_\lambda(an_0 : U) \equiv (\exp \langle A^*, \mathrm{Ad}(an_0)^{-1}A\rangle)\cdot(\tau_\lambda^* \circ \alpha)(an_0 \exp p(U)).$$

Then we can see that $\{\Phi_{v*}^\lambda;\ v^* \in V_\lambda^*\}$ coincides with the space of C^∞-functions on $A_p(N_m \cap L) \times \mathfrak{g}(1)$ satisfying (10.10), (10.12) and (10.16). To see this, we have only to notice the following two facts: (1) each $\tilde{\Phi}_{v*}^\lambda$ is a polynomial mapping from $(\mathfrak{g}(1), J'')$ into V_λ^*, and (2) any polynomial on $(\mathfrak{g}(1), J'')$ belongs to the space $\mathscr{F}_0^\infty$ of C^∞-vectors for the Fock representation ρ_0.

Consequently, we can describe spaces $C_{\tau_\lambda}^\infty(G; \rho_i \| \Sigma_\rho^+; \lambda)^\dagger$ completely as follows.

Proposition 10.5. *Assume that G/K is not eqiuvalent to a tube domain. Then one has for each $\mathfrak{k}$-dominant, K-integral linear form $\lambda \in \Xi_K^+$,*
(1) $C_{\tau_\lambda}^\infty(G; \rho_i \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger = (0)$ *if* $i \neq 0$,
(2) $C_{\tau_\lambda}^\infty(G; \rho_i \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger \simeq V_\lambda^*$ *(as vector spaces) if* $i=0$.
The isomorphism is given as

$$(10.26)\qquad V_\lambda^* \ni v^* \longmapsto \tilde{F}_{v*}^\lambda = \tilde{F}[\Phi_{v*}^\lambda] \in C_{\tau_\lambda}^\infty(G; \rho_0 \| \Sigma_\mathfrak{p}^+; \lambda)^\dagger,$$

where $\tilde{F}[\cdot]$ is as in Proposition 10.3.

10.3. Determination of highest weight vectors for GGGRs C^∞-Γ_i.
Summing up Propositions 10.1, 10.4 and 10.5, we deduce immediately the following theorem, which gives a complete description of spaces $C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ ($\lambda \in \Xi_K^+$) of K-finite highest weight vectors for GGGRs C^∞-Γ_i ($0 \leq i \leq l$).

Theorem 10.6. (1) *Assume that G/K is of tube type. For each $v^* \in V_\lambda^*$ and each $0 \leq i \leq l$, put*

$$(10.27)\qquad F_{v*}^{\lambda i}(g) = \rho_i(n)^{-1}\langle \tau_\lambda(k)^{-1}v_\lambda,\ G_\lambda^i(an_0)v^*\rangle,$$

where $g = kan_0n \in KA_p(N_m \cap L)N$, and v_λ is a highest weight vector for the irreducible K-module $(\tau_\lambda, V_\lambda)$ with highest weight λ. Then the map $v^* \mapsto F_{v*}^{\lambda i}$ gives an isomorphism of vector spaces:

$$(10.28)\qquad V_\lambda^* \simeq C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger \qquad \text{for any } i.$$

(2) *Assume that G/K is of non-tube type. Then one has*

$$(10.29)\qquad C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger \begin{cases} \simeq V_\lambda^* & \text{if } i=0, \\ =(0) & \text{if } i \neq 0. \end{cases}$$

For $i=0$, then isomorphism is given as $V_\lambda^* \ni v^* \mapsto F_{v*}^{\lambda 0}$, where the highest weight vector $F_{v*}^{\lambda 0}$ is defined by

(10.30) $\qquad F^{\lambda 0}_{v^*}(g: U)=\{\rho_0(n)^{-1}(\langle\tau_\lambda(k)^{-1}v_\lambda,\ \tilde{G}_\lambda(an_0:\ \cdot)v^*\rangle)\}(U)$

for $g=kan_0n\in G$ *and* $U\in\mathfrak{g}(1)$. *Here,* $\langle\tau_\lambda(k)^{-1}v_\lambda,\ \tilde{G}_\lambda(an_0:\ \cdot)v^*\rangle$ *denotes the element of the Fock space* $\mathcal{F}_0$ *such that*

$$\mathfrak{g}(1)\ni U\longmapsto\langle\tau_\lambda(k)^{-1}v_\lambda,\ \tilde{G}_\lambda(an_0:\ U)v^*\rangle\in \boldsymbol{C},$$

for any fixed (k, a, n_0).

This is the main result of this section, which enables us to describe C^∞-Whittaker models for highest weight modules.

Remark 10.7. The notation $F^{\lambda 0}_{v^*}$ in (10.30) is consistent with $F^{\lambda i}_{v^*}$ in (10.27) with $i=0$, in the following sense. Even if G/K is of tube type, the right hand side of (10.30) has a meaning by identifying the Fock space of $\mathfrak{g}(1)=(0)$ with the one-dimensional vector space $\boldsymbol{C}$ in the canonical way. Then, $F^{\lambda 0}_{v^*}(g: 0)$ coincides with $F^{\lambda 0}_{v^*}(g)$ in (10.27).

Remark 10.8. We can determine K-finite lowest weight vectors in GGGRs C^∞-Γ_i in an analogous way. It should be remarked that, in non-tube case, the representation C^∞-Γ_i admits a non-zero K-finite lowest weight vector if and only if $i=l$.

Remark 10.9. We have thus modified Hashizume's calculation of highest weight vectors ([11, § 4]) in our setting of C^∞-GGGRs.

From the above theorem, we can express dimensions of spaces of highest weight vectors in C^∞-Γ_i with $i=0$, such as

(10.31) $\qquad \dim C^\infty(G;\rho_0\|\Sigma^+;\lambda)^\dagger=\dim\tau_\lambda \qquad$ for every $\lambda\in\Xi^+_K$.

So in particular, K-finite λ-highest weight vectors in C^∞-Γ_0 are not always unique (up to scalar multiples). This differs from [11].

§ 11. Determination of highest weight vectors (Step II): Case of unitarily induced GGGRs

Let L^2-$\Gamma_i=L^2$-$\mathrm{Ind}^G_N(\rho_i)$ be the unitarily induced GGGR associated with the nilpotent Ad(G)-orbit $\omega_i=\mathrm{Ad}(G)A[i]$ in Theorem 5.2. In this section, we determine all the K-finite highest weight vectors for L^2-Γ_i, by evaluating L^2-norms of highest weight vectors for C^∞-Γ_i described in Section 10.

11.1. Spaces $L^2(G;\rho_i\|\Sigma^+;\lambda)^\dagger$ of highest weight vectors.

To begin with, we introduce spaces of highest weight vectors for unitary rerpesentations L^2-Γ_i $(0\leqq i\leqq l)$ in connection with such spaces for C^∞-Γ_i. For this purpose, realize our unitary GGGRs explicitly as follows. The representation space $L^2(G;\rho_i)$ of L^2-Γ_i consists of all $\mathscr{F}_i$-valued Borel functions F on G satisfying

$$(11.1)\qquad F(gn)=\rho_i(n)^{-1}F(g) \qquad (g\in G,\, n\in N),$$

$$(11.2)\qquad \|F\|^2=\int_{G/N}\|F(\dot{g})\|^2_{\mathscr{F}_i}d\dot{g}<+\infty \qquad (\dot{g}=gN),$$

where $d\dot{g}$ denotes a G-invariant measure on the factor space G/N. $L^2(G;\rho_i)$ becomes a Hilbert space with the innder product induced from the above norm. The group G acts on $L^2(G;\rho_i)$ unitarily by left translation:

$$(11.3)\qquad \mathscr{U}_i(g)F(x)=F(g^{-1}x) \qquad (g, x\in G).$$

Let $\lambda\in\Xi_K^+$, a $\mathfrak{k}$-dominant, K-integral linear form on $\mathfrak{t}_C$. We denote by $L^2(G;\rho_i\|\Sigma^+;\lambda)^\dagger$ the space of K-finite, λ-highest vectors for $\mathscr{U}_i$, which belong to the space $L^2(G;\rho_i)^\omega$ of analytic vectors for $\mathscr{U}_i$:

$$(11.4)\qquad L^2(G;\rho_i\|\Sigma^+;\lambda)^\dagger=L^2(G;\rho_i)^\omega\cap C^\infty(G;\rho_i\|\Sigma^+;\lambda)^+.$$

As shown later (Proposition 12.5), embeddings of unitary highest weight G-modules π_λ into $L^2(G;\rho_i)$ correspond bijectively to elements of $L^2(G;\rho_i\|\Sigma^+;\lambda)^\dagger$. For this reason, we treat this space of highest weight vectors.

Nelson's characterization of C^∞-vectors and the regularity theorem for elliptic operators enable us to deduce the following

Proposition 11.1. *If a highest weight vector $F\in C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger$ lies in $L^2(G;\rho_i)$, then F is an analytic vector for $\mathscr{U}_i$. Therefore one has*

$$(11.5)\qquad \begin{aligned} L^2(G;\rho_i\|\Sigma^+;\lambda)^\dagger&=L^2(G;\rho_i)^\infty\cap C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger\\ &=L^2(G;\rho_i)\cap C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger,\end{aligned}$$

where $L^2(G;\rho_i)^\infty$ denotes the space of C^∞-vectors for $\mathscr{U}_i$.

Proof. Take an F from $L^2(G;\rho_i)\cap C^\infty(G;\rho_i\|\Sigma^+;\lambda)^\dagger$. First we show that $F\in L^2(G;\rho_i)^\infty$. For this purpose, let $X_1,\cdots,X_r,X_{r+1},\cdots,X_p$ be an orthogonal basis of $\mathfrak{g}$ with respect to the positive definite inner product $\mathfrak{g}\times\mathfrak{g}\ni(X,Y)\mapsto -B(X,\theta Y)$, such that $X_k\in\mathfrak{k}$ $(1\leqq k\leqq r)$ and $X_j\in\mathfrak{p}$ $(r<j\leqq p)$. Put $\Delta=\sum_{1\leqq m\leqq p}X_m^2\in U(\mathfrak{g}_C)$. Then, Nelson's theorem (cf. [33, Th. 4.4.4.5]) tells us the following characterization of the subspace $L^2(G;\rho_i)^\infty$. Let $\mathscr{U}_{i\infty}(\Delta)$ denote the operator $\mathscr{U}_i(\Delta)$ on $L^2(G;\rho_i)^\infty$, and $\mathscr{U}_{i\infty}(\Delta)^*$ its adjoint operator in the Hilbert space $L^2(G;\rho_i)$. Then,

$$(11.6) \qquad L^2(G; \rho_i)^\infty = \bigcap_{n=1}^{\infty} \mathrm{Dom}\,[(\mathcal{U}_{i\infty}(\Delta)^*)^n],$$

where Dom$[A]$ denotes the domain of a linear operator A.

So, in order to show that F is a C^∞-vector, it suffices to prove: $F \in \mathrm{Dom}[(\mathcal{U}_{i\infty}(\Delta)^*)^n]$ for all $n \geqq 1$. This is done as follows. We put

$$\Omega = \Omega_K + \textstyle\sum_{j>r} X_j^2 \qquad \text{with } \Omega_K = -\textstyle\sum_{k \leqq r} X_k^2.$$

Then Ω and Ω_K lie in the centers of enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Actually, Ω is the Casimir element of $\mathfrak{g}$. By the definition of highest weight vectors, we see easily that F is an eigenfunction of the differential operators L_Ω and L_{Ω_K} (see (2.4)). This implies that

$$(11.7) \qquad L_\Delta F = cF \qquad \text{for some } c \in \boldsymbol{C},$$

because $\Delta = \Omega - 2\Omega_K$.

Now let $\Phi \in C^\infty(G; \rho_i)$ be such that supp(Φ) is compact modulo N. Then one has $\Phi \in L^2(G; \rho_i)^\infty$ since $L_D\Phi \in L^2(G; \rho_i)$ for all $D \in U(\mathfrak{g}_C)$. Moreover, such Φ's form a dense subspace of $L^2(G; \rho_i)$. Using (11.7), we get through a simple computation

$$(\mathcal{U}_{i\infty}(\Delta)\Phi, F) = (L_\Delta\Phi, F) = (\Phi, L_\Delta F) = (\Phi, cF),$$

where $(\ , \ )$ denotes the inner product on $L^2(G; \rho_i)$. This means that $F \in \mathrm{Dom}[\mathcal{U}_{i\infty}(\Delta)^*]$ and $\mathcal{U}_{i\infty}(\Delta)^* F = cF$. One thus obtains $F \in L^2(G; \rho_i)^\infty$.

Secondly, let us prove the analyticity of F for $\mathcal{U}_i$, which means by definition that $L^2(G; \rho_i)$-valued function $\hat{F}$: $g \mapsto \hat{F}(g) \equiv \mathcal{U}_i(g)^{-1}F$ on G is real analytic. As seen above, this function $\hat{F}$ is of class C^∞, and so (11.7) implies that $L_\Delta \hat{F} = c\hat{F}$. Since the differential operator L_Δ on G is elliptic, $\hat{F}$ must be real analytic by virtue of the regularity theorem for elliptic differential operators. Q.E.D.

Remark 11.2. From the above proof, we see further that any K-finite highest weight vector in $L^2(G; \rho_i)$ of C^2-class is necessarily analytic for $\mathcal{U}_i$.

By virtue of the above proposition, in order to determine K-finite highest weight vectors analytic for $\mathcal{U}_i$, we have only to specify, among highest weight vectors $F^{\lambda i}_{v*} \in C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ in Theorem 10.6, those contained in $L^2(G; \rho_i)$. We shall do this in the succeeding subsections.

11.2. Highest weight vectors for GGGRs L^2-Γ_i.

We now present the main result of this section, which gives a complete description of spaces $L^2(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ of highest weight vectors.

Theorem 11.3. *Let $0 \leqq i \leqq l$, $\lambda \in \Xi_K^+$, the set of $\mathfrak{k}$-dominant, K-integral linear forms on $\mathfrak{t}_C$.*

(1) *If $i \neq 0$, then $L^2(G; \rho_i \| \Sigma^+; \lambda) = (0)$ for every λ.*

(2) *Assume that $i = 0$. Then $L^2(G; \rho_i \| \Sigma^+; \lambda) \neq (0)$ if and only if λ satisfies the non-vanishing condition* (7.11) *for the holomorphic discrete series: $(\lambda+\rho)(H_\gamma') < 0$ for all $\gamma \in \Sigma_{\mathfrak{p}}^+$. For such a λ, one has*

$$L^2(G; \rho_i \| \Sigma^+; \lambda)^\dagger = C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger \simeq V_\lambda^*, \tag{11.8}$$

where V_λ^* is the dual space of the irreducible K-module V_λ with highest weight λ.

The rest of this section is devoted to the proof of this theorem.

11.3. L^2-norms $N^i(v^*, \lambda)$ of highest weight vectors.

Keeping Proposition 11.1 in mind, let us calculate L^2-norms of highest weight vectors $F_{v^*}^{\lambda i}$. For this purpose, recall that $G = KA_p(N_m \cap L)N$ is diffeomorphic to the direct product $K \times A_p \times (N_m \cap L) \times N$ in the canonical way. Through this diffeomorphic isomorphism, the factor space G/N is identified with $K \times A_p \times (N_m \cap L)$. Then the G-invariant measure $d\dot{g}$ on G/N is expressed as

$$d\dot{g} = a^{2\delta} dk\, da\, dn_0 \qquad (\dot{g} = kan_0N).$$

Here, dk (resp. da, dn_0) denotes a Haar measure on K (resp. on A_p, on $(N_m \cap L)$), and we put $(\exp H)^{2\delta} = \exp 2\delta(H)$ $(H \in \mathfrak{a}_p)$ with $\delta(H) \equiv (1/2) \cdot \mathrm{tr}(\mathrm{ad}(H) | \mathfrak{n}_m)$. Further we normalize dk so that $\int_K dk = 1$.

For each highest weight vector $F_{v^*}^{\lambda i}$ for C^∞-Γ_i given in Theorem 10.6, we denote by $N^i(v^*, \lambda)$ its L^2-norm: $N^i(v^*, \lambda) = \| F_{v^*}^{\lambda i} \|^2$. Then we obtain

$$\begin{aligned} N^i(v^*, \lambda) = \int_{KA_p(N_m \cap L) \times \mathfrak{g}(1)} & (\exp 2\langle A[i]^*, \mathrm{Ad}(an_0)^{-1} A \rangle) \\ & \times |\langle v_\lambda, \tau_\lambda^*(k)(\tau_\lambda^* \circ \alpha)(an_0 \exp p(U)) v^* \rangle|^2 a^{2\delta} e^{-2\|U\|_0^2} dk\, da\, dn_0\, dU, \end{aligned}$$

where $\|\cdot\|_0$ is the norm on $\mathfrak{g}(1)$ in (9.15), and dU a Lebesgue measure on $\mathfrak{g}(1)$. For each fixed (a, n_0, U), the above integral with respect to $k \in K$ is calculated as

$$\begin{aligned} & \int_K |\langle v_\lambda, \tau_\lambda^*(k)(\tau_\lambda^* \circ \alpha)(an_0 \exp p(U)) v^* \rangle|^2 dk \\ & = d_\lambda^{-1} \| (\tau_\lambda^* \circ \alpha)(an_0 \exp p(U)) v^* \|^2 \qquad \text{with } d_\lambda \equiv \dim V_\lambda. \end{aligned} \tag{11.9}$$

Here we normalized v_λ in $F^{\lambda i}_{v^*}$ as $\|v_\lambda\|=1$, and used the orthogonality relation for matrix coefficients of τ^*_λ. Thus, $N^i(v^*, \lambda)$ is expressed as

$$(11.10)\qquad N^i(v^*, \lambda)=d_\lambda^{-1}\int_{A_p(N_m\cap L)\times\mathfrak{g}(1)}(\exp 2\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A\rangle)$$
$$\times\|(\tau^*_\lambda\circ\alpha)(an_0\exp p(U))v^*\|^2 a^{2\delta}e^{-2\|U\|_0^2}\,da\,dn_0\,dU.$$

Remark 11.4. Highest weight vectors $F^{\lambda i}_{v^*}$ are not necessarily defined for all $(i, v^*, \lambda)\in\bigcup_{\nu\in\Xi_K^+}(\{0, 1, 2, \cdots, l\}\times V^*_\nu\times\{\nu\})$. Nevertheless, the right hand side of (11.10) has a meaning for every (i, v^*, λ). So we now redefine $N^i(v^*, \lambda)$ for any triplet (i, v^*, λ), through (11.10).

In view of Theorem 10.6, the proof of Theorem 11.3 is reduced to showing:

Theorem 11.5. *Let* $(i, v^*, \lambda)\in\bigcup_{\nu\in\Xi_K^+}(\{0, 1, 2, \cdots, l\}\times V^*_\nu\times\{\nu\})$. *Then the integral* $N^i(v^*, \lambda)$ *is finite if and only if* $i=0$ *and* λ *satisfies the non-vanishing condition* (7.11) *for the holomorphic discrete series.*

11.4. Proof of Theorem 11.5, Step I: the "only if" part.

First let us prove the "only if" part. Let $0\leqq i\leqq l$ and $\lambda\in\Xi_K^+$. For any fixed $(n_0, U)\in(N_m\cap L)\times\mathfrak{g}(1)$ and any non-zero vector $v^*\in V^*_\lambda$, consider the integral over the maximal split subgroup $A_p\subseteqq G$, coming from (11.10):

$$(11.11)\qquad \int_{A_p}(\exp 2\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A\rangle)\cdot\|(\tau^*_\lambda\circ\alpha)(an_0\exp p(U))v^*\|^2 a^{2\delta}da.$$

Let $V^*_\lambda=\bigoplus_\nu(V^*_\lambda)_\nu$ (orthogonal direct sum) be the weight space decomposition of V^*_λ, where $(V^*_\lambda)_\nu\equiv\{w^*\in V^*_\lambda;\ \tau^*_\lambda(H)w^*=-\nu(H)w^*\ (H\in\mathfrak{t}_C)\}$ for $\nu\in\mathfrak{t}^*_C$. We put

$$(\tau^*_\lambda\circ\alpha)(n_0\exp p(U))v^*=\textstyle\sum_\nu v^*(n_0, U;\nu)$$

with $v^*(n_0, U; v)\in(V^*_\lambda)_\nu$. Notice that

$$(\tau^*_\lambda\circ\alpha)(a)v^*_\nu=\tau^*_\lambda(\mu(a)^{-1})v^*_\nu\qquad(\text{by }\alpha(a)=\mu(a)^{-1})$$
$$=\mu(a)^\nu v^*_\nu\qquad\text{for each } v^*_\nu\in(V^*_\lambda)_\nu.$$

Here $\mu(a)^\nu\equiv a^{\nu'}$ $(a\in A_p)$ with $\nu'\equiv\nu\circ(\mu\,|\,\mathfrak{a}_p)\in\mathfrak{a}^*_p$. Then the integral (11.11) becomes

$$(11.12)\qquad \textstyle\sum_\nu\displaystyle\int_{A_p}(\exp 2\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A\rangle)\mu(a)^{2(\nu+\rho)}da\,\|v^*(n_0, U;\nu)\|^2.$$

Let us rewrite the integrals in (11.12) in terms of the system of coordinates $(t_1, t_2, \cdots, t_l)$ of $\mathfrak{a}_p=\{\sum_{1\leq m\leq l} t_m H_m;\ -\infty<t_m<+\infty\}$. For this purpose, put

$$a=\exp(\textstyle\sum_{1\leq m\leq l} t_m H_m), \qquad n_0=\exp X \text{ with } X=\sum_{k>m} X_{km},$$

where $X_{km} \in \mathfrak{g}(\mathfrak{a}_p; (\lambda_k-\lambda_m)/2)$. Then a simple computation yields

$$\mu(a)^{2(\nu+\rho)}=\exp(\textstyle\sum_{1\leq m\leq l} 2t_m(\nu+\rho)(H'_{\gamma m})), \tag{11.13}$$

$$\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A\rangle=b\textstyle\sum_{1\leq m\leq l} e^{-2t_m}(\chi^i_m+\sum_{m<k\leq l}\zeta_{km}(X_{km})\chi^i_k). \tag{11.14}$$

Here $b=-B(E_m, \theta E_m)=\|\gamma_m\|^2/2$ (independent of m), ζ_{km} is the (positive definite) quadratic form on $\mathfrak{g}(\mathfrak{a}_p; (\lambda_k-\lambda_m)/2)$ in (6.4), and χ^i_m is defined as in 6.1:

$$\chi^i_m=1 \quad \text{if } m\leq i, \quad \chi^i_m=-1 \text{ if } m>i. \tag{11.15}$$

Identifying the Haar measure da on A_p with the product $\prod_{1\leq m\leq l} dt_m$ of Lebesgue measures dt_m $(1\leq m\leq l)$ on $\boldsymbol{R}$ through the exponential mapping, we thus obtain

$$\begin{aligned}&\int_{A_p}(\exp 2\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A\rangle)\mu(a)^{2(\nu+\rho)}da \\ &=\prod_{m=1}^{l}\int_{-\infty}^{\infty}\exp\{2be^{-2t}(\chi^i_m+\textstyle\sum_{k>m}\zeta_{km}(X_{km})\chi^i_k)+2t(\nu+\rho)(H'_{\gamma m})\}dt.\end{aligned} \tag{11.16}$$

This expression together with (11.12) implies the following

Lemma 11.6. *Keep to the above notation. Let* $(n_0, U) \in (N_m\cap L)\times \mathfrak{g}(1)$ *and* $v^*_\lambda \in V^*_\lambda$, $\neq 0$. *Then the integral* (11.11) *converges if and only if the following two conditions are fulfilled.*

$$\chi^i_m+\textstyle\sum_{k>m}\zeta_{km}(X_{km})\chi^i_k<0 \qquad (1\leq m\leq l), \tag{11.17}$$

$$(\nu+\rho)(H'_{\gamma m})<0 \qquad (1\leq m\leq l) \tag{11.18}$$

for all $\nu \in \mathfrak{t}^*_C$ *such that* $v^*(n_0, U; \nu)\neq 0$.

Proof. Let c_1 and c_2 be two real numbers. Then one finds out easily that the integral $\int \exp(c_1e^{-2t}+c_2t)dt$ converges if and only if $c_1<0$ and $c_2<0$. This combined with (11.12) and (11.16) proves the lemma.
Q.E.D.

From this lemma we get a necessary condition for the integral $N^i(v^*, \lambda)$ in (11.10) to be finite as follows.

Proposition 11.7. *Let $v^* \in V_\lambda^*$ be a non-zero vector. If $N^i(v^*, \lambda) < \infty$, then necessarily $i=0$ and*

$$(\nu+\rho)(H'_{\gamma_m})<0 \qquad (1 \leqq m \leqq l) \tag{11.19}$$

for any $\nu \in \mathfrak{t}_C^*$ such that the $(V_\lambda^*)_\nu$-valued function $(n_0, U) \mapsto v^*(n_0, U; \nu)$ on $(N_m \cap L) \times \mathfrak{g}(1)$ is not identically zero.

Proof. If $v^* \in V_\lambda^*$, $\neq 0$ such that $N^i(v, \lambda) < \infty$, then the integral (11.11) converges for almost all ($=a.a.$) $(n_0, U) \in (N_m \cap L) \times \mathfrak{g}(1)$ with respect to the measure $dn_0 \times dU$. By virtue of Lemma 11.6, (11.17) and (11.18) hold for a.a. (n_0, U)'s. Further, (11.18) implies (11.19) because the $(V_\lambda^*)_\nu$-valued function $v^*(n_0, U; \nu)$ is continuous in (n_0, U) for any fixed ν. We can take an $n_0 = \exp(\sum_{k>m} X_{km})$ satisfying the condition (11.17) from a sufficiently small neighbourhood of the unit element of $N_m \cap L$. Then we see that there should be $\chi_m^i < 0$ for all $1 \leqq m \leqq l$, because $\sum_{k>m} \zeta_{km}(X_{km})\chi_k^i \to 0$ as $n_0 \to 1$. This means that $i=0$. Q.E.D.

Remark 11.8. We utilized the quadratic form ζ_{km} in Lemma 6.1, only to get an exact expression of $\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A \rangle$ in (11.14). We need not to use the positive-definiteness of ζ_{km} (Lemma 6.1) due to Rossi and Vergne, anywhere in this subsection.

Using Proposition 11.7, we can now prove the "only if" part of Theorem 11.5. This is done as follows. First notice that

$$N^i(\tau_\lambda^*(m)v^*, \lambda) = N^i(v^*, \lambda) \qquad \text{for } v^* \in V_\lambda^*,\ m \in M = Z_K(A_p).$$

Indeed, each $m \in M$ is fixed under the Cartan involution θ and the Cayley transform μ: $\theta(m) = \mu(m) = m$. The compact group M normalizes subgroups $N_m \cap L$, $\exp V^+ = \{\exp p(U);\ U \in \mathfrak{g}(1)\}$ of G^C, and subspace $\mathfrak{g}(1)$ of $\mathfrak{g}$. Moreover, the isomorphism $p\colon \mathfrak{g}(1) \rightleftarrows V^+$ commutes with $\mathrm{Ad}(M)$-action (see [II, Lemma 4.11]). So we obtain

$$\begin{aligned}
&\|(\tau_\lambda^* \circ \alpha)(an_0 \exp p(U))\tau_\lambda^*(m)v^*\| \\
&\quad = \|\tau_\lambda^*(m)(\tau_\lambda^* \circ \alpha)(a \cdot m^{-1}n_0 m \cdot \exp p(\mathrm{Ad}(m)^{-1}U))v^*\| \\
&\quad = \|(\tau_\lambda^* \circ \alpha)(a \cdot m^{-1}n_0 m \cdot \exp p(\mathrm{Ad}(m)^{-1}U))v^*\|.
\end{aligned}$$

Keeping in mind the fact that M acts on the inner product space $(\mathfrak{g}(1), J'', (\ ,\ )_0)$ (see 9.3) as a group of unitary operators, we exchange the variables n_0 and U in (11.10) by $m^{-1}n_0 m$ and $\mathrm{Ad}(m)^{-1}U$ respectively. One thus gets $N^i(\tau_\lambda^*(m)v^*, \lambda) = N^i(v^*, \lambda)$.

The complexification $\mathfrak{m}_C$ of $\mathfrak{m} = \mathrm{Lie}\, M$ is expressed as

$$\mathfrak{m}_C = \mathfrak{t}_C^+ \oplus \textstyle\sum_{\gamma \in C_0 \cup (-C_0)} \mathfrak{g}_C(\mathfrak{t}_C; \gamma), \tag{11.20}$$

where $C_0=\{\gamma \in \Sigma_{\mathfrak{k}}^+;\ \gamma \mid \mathfrak{t}_C^- =0\}$, and $\mathfrak{t}_C^+$ is the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}_C$. Let $\mathfrak{b}^\pm$ be the Borel subalgebra of $\mathfrak{k}_C$ defined by

$$\text{(11.21)} \qquad \mathfrak{b}^\pm = \mathfrak{t}_C \oplus \mathfrak{k}^\pm \qquad \text{with } \mathfrak{k}^\pm = \textstyle\sum_{\gamma \in \Sigma_{\mathfrak{k}}^+} \mathfrak{g}_C(\mathfrak{t}_C;\ \pm\gamma).$$

Then (11.20) together with (9.13) implies that

$$\text{(11.22)} \qquad \mathfrak{m}_C \oplus \alpha((\mathfrak{n}_m \cap \mathfrak{l})_C \oplus V^+) \supseteqq \mathfrak{k}^-.$$

Let $v^* \in V_\lambda^*$, $\neq 0$, and consider the subspace $\tau_\lambda^*(U(\mathfrak{k}^-))v^* \subseteqq V_\lambda^*$. Since this space is $\mathfrak{t}_C$-stable, it is decomposed as

$$\tau_\lambda^*(U(\mathfrak{k}^-))v^* = \textstyle\sum_\nu [\tau_\lambda^*(U(\mathfrak{k}^-))v^* \cap (V_\lambda^*)_\nu].$$

By the irreducibility of τ_λ^*, we have $V_\lambda^* = \tau_\lambda^*(U(\mathfrak{k}_C))v^*$, and so

$$\text{(11.23)} \qquad V_\lambda^* = \tau_\lambda^*(U(\mathfrak{b}^+)) \cdot \tau_\lambda^*(U(\mathfrak{k}^-))v^*$$

since $\mathfrak{k}_C = \mathfrak{b}^+ \oplus \mathfrak{k}^-$. This implies immediately that $\tau_\lambda^*(U(\mathfrak{k}^-))v^*$ should contain the $(-\lambda)$-lowest weight space $(V_\lambda^*)_\lambda$. Thus we get

$$\text{(11.24)} \qquad \langle (\tau_\lambda^* \circ \alpha)((N_m \cap L) \exp V^+) \tau_\lambda^*(M) v^* \rangle \supseteqq (V_\lambda^*)_\lambda.$$

Here, $\langle S \rangle$ denotes the vector subspace of V_λ^* spanned by a subset $S \subseteqq V_\lambda^*$. (11.24) means that there exists an $m \in M$ for which the mapping

$$(N_m \cap L) \times \mathfrak{g}(1) \ni (n_0, U) \longmapsto [\tau_\lambda^*(m)v^*](n_0, U; \lambda) \in (V_\lambda^*)_\lambda$$

is not identically zero.

Now assume that $N^i(v^*, \lambda) < \infty$ for some non-zero vector $v^* \in V_\lambda^*$. Then, Proposition 11.7 combined with above consideration implies that $i=0$ and

$$\text{(11.25)} \qquad (\lambda+\rho)(H'_{\gamma_m}) < 0 \qquad \text{for } 1 \leqq m \leqq l.$$

In view of Remark 7.5, this is equivalent to (7.11): $(\lambda+\rho)(H'_\gamma)<0$ for all $\gamma \in \Sigma_{\mathfrak{p}}^+$. We have now completed the proof of the "only if" part of Theorem 11.5.

11.5. Proof of Theorem 11.5, Step II: the "if" part.

Now assume that $i=0$ and $\lambda \in \Xi_K^+$ satisfies $(\lambda+\rho)(H'_\gamma)<0$ for $\gamma \in \Sigma_{\mathfrak{p}}^+$. Then we are going to prove that $N^i(v^*, \lambda) = N^0(v^*, \lambda) < \infty$ for every $v^* \in V_\lambda^*$. In view of (11.10), for this purpose it suffices to show that the integral

$$\text{(11.26)} \quad I[\lambda] \equiv \int_{A_p(N_m \cap L)} (\exp 2\langle A^*, \operatorname{Ad}(an_0)^{-1}A\rangle) \cdot \|(\tau_\lambda^* \circ \alpha)(an_0)\|^2 a^{2\delta} da\, dn_0$$

converges, where $A=A[0]=\sum_{1\leq m\leq l} E_m \in \mathfrak{g}(2)$. Here we denote by $\|Y\|$ the operator norm of a linear operator Y on V_λ^*.

First, we rewrite this integral on $A_p(N_m \cap L)$ to that on the whole Levi subgroup $L=(K\cap L)A_p(N_m\cap L)=(K\cap L)\cdot A_p(N_m\cap L)$. As mentioned in the proof of Lemma 6.7, every $k_0 \in K\cap L$ is fixed under the Cayley transform μ. So, by putting $\tilde{\alpha}(x)=(\mu\circ\theta)(x)=k_0\alpha(an_0)$ for $x=k_0an_0 \in L=(K\cap L)A_p(N_m\cap L)$, we can extend $\alpha\,|\,A_p(N_m\cap L)$ to an embedding $\tilde{\alpha}$ from L into K^C. Since $\mathrm{Ad}(k_0)A=A$ for $k_0 \in K\cap L$ (see [II, Lemma 3.5]),

$$L \ni x \longmapsto \langle A^*, \mathrm{Ad}(x)^{-1}A\rangle \in \boldsymbol{R}$$

gives a $(K\cap L)$-biinvariant function on L. Thus (11.26) becomes, by integrating over $K\cap L$ with respect to the Haar measure dk_0 on $K\cap L$ normalized as $\int_{K\cap L} dk_0=1$,

$$(11.27)\qquad \int_{(K\cap L)\times A_p\times(N_m\cap L)} (\exp 2\langle A^*, \mathrm{Ad}(k_0an_0)^{-1}A\rangle) \\ \times\|(\tau_\lambda^*\circ\tilde{\alpha})(k_0an_0)\|^2 a^{2\delta}dk_0dadn_0.$$

Let dx be a Haar measure on L. Then it is expressed, under a suitable normalization, by means of Haar measures dk_0, da and dn_0 as follows (cf. [31, 7.6.4]):

$$(11.28)\qquad dx=a^{2\delta'}dk_0dadn_0 \qquad (x=k_0an_0),$$

where $\delta'(H)=(1/2)\cdot\mathrm{tr}(\mathrm{ad}(H)\,|\,(\mathfrak{n}_m\cap\mathfrak{l}))$ for $H \in \mathfrak{a}_p$. We thus get

$$(11.29)\qquad I[\lambda]=\int_L (\exp 2\langle A^*, \mathrm{Ad}(x)^{-1}A\rangle)\|\tau_\lambda^*(\tilde{\alpha}(x))\|^2 a[x]^{2(\delta-\delta')}dx.$$

Here $a[x]$ denotes the A_p-component of $x \in L$ along the Iwasawa decomposition: $a[x] \in A_p$ such that $x \in (K\cap L)a[x](N_m\cap L)$.

Secondly, let us further rewrite (11.29) to an integral on an open Weyl chamber A_p^+ of A_p with respect to the root system $\Delta(\mathfrak{l})$ of $(\mathfrak{l}, \mathfrak{a}_p)$. For this purpose, we need the following integral formula coming from the Cartan decomopsition of L: $L=(K\cap L)A_p(K\cap L)$.

Lemma 11.9 (cf. [31, 8.16.6]). *There exists a constant $c>0$, depending only on normalization of the Haar measure dx on L, for which*

$$(11.30)\qquad \int_L h(x)dx=c\int_{(K\cap L)\times A_p^+\times(K\cap L)} h(k_0ak_0')D(a)dk_0dadk_0'$$

for any $h \in C_0^\infty(L)$. Here $D(a)$ is given as

(11.31) $$D(a)=\prod_{\psi\in\Lambda(\mathfrak{l})^+}(a^{\psi}-a^{-\psi})^{\dim\mathfrak{g}(\mathfrak{a}_p;\psi)} \qquad (a\in A_p^+),$$

where $\Lambda(\mathfrak{l})^+$ *denotes the positive system of* $\Lambda(\mathfrak{l})$ *associated to the positive Weyl chamber* A_p^+: $\Lambda(\mathfrak{l})^+=\{\psi\in\Lambda(\mathfrak{l});\ a^{\psi}>1$ *for all* $a\in A_p^+\}$.

We now take A_p^+ as

(11.32) $$A_p^+=\{\exp(\textstyle\sum_{1\leq m\leq l}t_mH_m)\in A_p;\ t_1<t_2<\cdots<t_l\}.$$

Then the corresponding positive system $\Lambda(\mathfrak{l})^+$ is given as

(11.33) $$\Lambda(\mathfrak{l})^+=\Lambda(\mathfrak{l})\cap\Lambda^+=\{(\lambda_k-\lambda_m)/2;\ 1\leq m<k\leq l\},$$

whence $\mathfrak{n}_m\cap\mathfrak{l}=\sum_{\psi\in\Lambda(\mathfrak{l})^+}\mathfrak{g}(\mathfrak{a}_p;\psi)$.

Making use of the integral formula (11.30), one gets the following

Proposition 11.10. *The integral* $I[\lambda]$ *in* (11.26) *is expressed as*

$$I[\lambda]=c\int_{A_p^+}(\exp 2\langle A^*,\ \mathrm{Ad}(a)^{-1}A\rangle)\|\tau_\lambda^*(\alpha(a))\|^2a^{2(\delta-\delta')}D(a)da.$$

Proof. We see easily that

$$a[x]^{2(\delta-\delta')}=\det(\mathrm{Ad}(x)\,|\,\mathfrak{n}) \qquad (x\in L),$$

whence $x\mapsto a[x]^{2(\delta-\delta')}$ gives a one-dimensional representation of L with values in $\boldsymbol{R}_+=\{y\in\boldsymbol{R};\ y>0\}$. This implies in particular that

(11.34) $$a[k_0bk_0']^{2(\delta-\delta')}=a[b]^{2(\delta-\delta')}=b^{2(\delta-\delta')}$$

for $k_0,\ k_0'\in K\cap L$ and $b\in A_p$. On the other hand, the function $L\ni x\mapsto(\exp 2\langle A^*,\ \mathrm{Ad}(x)^{-1}A\rangle)\|\tau_\lambda^*(\tilde{\alpha}(x))\|^2$ is $(K\cap L)$-biinvariant. Thus we find that the function in the integrand of (11.29) is $(K\cap L)$-biinvariant. So, the proposition follows from (11.30) and (11.34). Q.E.D.

Now we estimate the integral in Proposition 11.10 using the Weyl character formula, and then prove $I[\lambda]<\infty$. Let Θ_λ be the character of the holomorphic representation τ_λ^* of K^C: $\Theta_\lambda(k)=\mathrm{tr}(\tau_\lambda^*(k))$ $(k\in K^C)$. Since $\tau_\lambda^*(\alpha(a))$ is a positive operator on V_λ^* for every $a\in A_p$, we have

(11.35) $$\|\tau_\lambda^*(\alpha(a))\|^2\leq\Theta_\lambda(\alpha(a)^2),$$

whence

(11.36) $$I[\lambda]\leq c\int_{A_p^+}(\exp 2\langle A^*,\ \mathrm{Ad}(a)^{-1}A\rangle)\Theta_\lambda(\alpha(a)^2)D(a)a^{2(\delta-\delta')}da.$$

In order to show the convergence of this integral, we need some more preparations. Put $\mathfrak{k}'=\mu(\mathfrak{g}(0)_C)\cap\mathfrak{k}$ with $\mathfrak{g}(0)=\mathfrak{l}=\mathrm{Lie}\,L$. Then $\mathfrak{k}'$ is a compact real form of $\mu(\mathfrak{g}(0)_C)$, and so $\mathfrak{k}'_C=\mu(\mathfrak{g}(0)_C)\subseteqq\mathfrak{k}_C$. Further $\mathfrak{k}'_C$ admits the root space decomposition

$$\mathfrak{k}'_C=\mathfrak{t}_C\otimes\sum_{\gamma\in\Sigma_{\mathfrak{t}'}}\mathfrak{g}_C(\mathfrak{t}_C;\gamma), \tag{11.37}$$

where $\Sigma_{\mathfrak{t}'}=\Sigma^+_{\mathfrak{t}'}\cup(-\Sigma^+_{\mathfrak{t}'})$ with

$$\Sigma^+_{\mathfrak{t}'}\equiv C_0\cup(\bigcup_{k>m}C_{km})\subseteqq\Sigma^+_{\mathfrak{t}}. \tag{11.38}$$

If G/K is of tube type, C_m is empty, and so we have $\Sigma^+_{\mathfrak{t}'}=\Sigma^+_{\mathfrak{t}}$, $\mathfrak{k}'_C=\mathfrak{k}_C$. Otherwise, $\mathfrak{k}'_C$ is a Levi subalgebra of the maximal parabolic subalgebra $\mathfrak{k}'_C\oplus\sum_{\gamma\in C_{m},m}\mathfrak{g}_C(\mathfrak{t}_C;\gamma)$ of $\mathfrak{k}_C$. Denote by $(K')^C$ the analytic subgroup of K^C with Lie algebra $\mathfrak{k}'_C$.

Consider the restriction of the irreducible representation $(\tau_\lambda, V_\lambda)$ of K^C onto $(K')^C$, and let

$$(\tau_\lambda\,|\,(K')^C, V_\lambda)\simeq\bigoplus_{1\leqq j\leqq p}(\sigma_{\nu_j}, V'_{\nu_j}) \tag{11.39}$$

be its irreducible decomposition, where $(\sigma_{\nu_j}, V'_{\nu_j})$ denotes an irreducible $(K')^C$-module with $\Sigma^+_{\mathfrak{t}'}$-extreme highest weight $\nu_j\in\mathfrak{t}^*_C$. Clearly each ν_j is of the form

$$\nu_j=\lambda-\sum_{\gamma\in\Sigma^+_{\mathfrak{t}}}n_{j\gamma}\gamma \qquad \text{with integers } n_{j\gamma}\geqq 0. \tag{11.40}$$

Let Θ'_{ν_j} be the character of the representation $\sigma^*_{\nu_j}$ contragredient to σ_{ν_j}. Then $\Theta_\lambda\,|\,(K')^C=\sum_j\Theta'_{\nu_j}$, and so in particular one has

$$\Theta_\lambda(\alpha(a)^2)=\sum_{1\leqq j\leqq p}\Theta'_{\nu_j}(\alpha(a)^2). \tag{11.41}$$

We need the following formula for Θ'_{ν_j}, a consequence of the Weyl character formula.

Lemma 11.11. *Put $\nu=\nu_j$ for any fixed $1\leqq j\leqq p$. Let $W_{\mathfrak{t}'}$ denote the Weyl group of $(\mathfrak{k}'_C, \mathfrak{t}_C)$. For each $s\in W_{\mathfrak{t}'}$, there exists a constant $c_s\in\boldsymbol{C}$ such that*

$$\Theta'_\nu(\alpha(a)^2)\cdot\prod_{\gamma\in\Sigma^+_{\mathfrak{t}'}\setminus C_0}(\mu(a)^\gamma-\mu(a)^{-\gamma})=\sum_{s\in W_{\mathfrak{t}'}}c_s\mu(a)^{2s(\nu+\rho_{\mathfrak{t}'})} \qquad (a\in A_p), \tag{11.42}$$

where $\rho_{\mathfrak{t}'}\equiv(1/2)\cdot\sum_{\gamma\in\Sigma^+_{\mathfrak{t}'}}\gamma$ and $\alpha=\mu\circ\theta$.

Proof. Harish-Chandra gave in [9, VI, Lemma 25] an expression similar to (11.42) for irreducible characters of K^C (not for those of $(K')^C$). But his proof works also in our situation, and so we get the lemma. Q.E.D.

Notice that the subset $\Sigma_{\mathfrak{t}'}^+\setminus C_0 \subseteqq \Sigma_{\mathfrak{t}}^+$ corresponds to $\Delta(\mathfrak{l})^+ \subseteqq \Delta^+$ in (11.33) through the Cayley transform μ: $(\Sigma_{\mathfrak{t}'}^+\setminus C_0)\circ(\mu\,|\,\mathfrak{a}_p)=\Delta(\mathfrak{l})^+$, which implies

$$\prod_{\gamma\in\Sigma_{\mathfrak{t}'}^+\setminus C_0}(\mu(a)^\gamma-\mu(a)^{-\gamma})=D(a) \qquad (a\in A_p). \tag{11.43}$$

On the other hand, we have

$$a^{2(\delta-\delta')}=\mu(a)^{2(\rho-\rho_{\mathfrak{t}'})}=\mu(a)^{2s(\rho-\rho_{\mathfrak{t}'})} \tag{11.44}$$

for all $s\in W_{\mathfrak{t}'}$, since $(\rho-\rho_{\mathfrak{t}'})\circ(\mu\,|\,\mathfrak{a}_p)=\delta-\delta'$ and $s(\Sigma^+\setminus\Sigma_{\mathfrak{t}'}^+)=\Sigma^+\setminus\Sigma_{\mathfrak{t}'}^+$. Thus,

$$\Theta_\nu'(\alpha(a)^2)D(a)a^{2(\delta-\delta')}=\sum_{s\in W_{\mathfrak{t}'}}c_s\mu(a)^{2s(\nu+\rho)}. \tag{11.45}$$

Consequently, by (11.36), (11.41) and (11.45), in order to show $I[\lambda]<\infty$, it is enough to prove that the integral

$$I_{\nu,s}\equiv\int_{A_p}(\exp 2\langle A^*,\ \mathrm{Ad}(a)^{-1}A\rangle)\mu(a)^{2s(\nu+\rho)}da \tag{11.46}$$

converges for every highest weight $\nu=\nu_j$ and every $s\in W_{\mathfrak{t}'}$.

Let us show $I_{\nu,s}<\infty$ in two steps. First, putting $a=\exp(\sum_m t_m H_m)$ with $t_m\in\boldsymbol{R}$ $(1\leqq m\leqq l)$, we can rewrite (11.46) to an integral in $(t_1, t_2, \cdots, t_l)$ as

$$I_{\nu,s}=\prod_{m=1}^{l}\int_{-\infty}^{\infty}\exp(-2be^{-2t_m}+2(\nu+\rho)(s^{-1}H'_{\gamma_m})t_m)dt_m$$

where $b=\|\gamma_m\|^2/2>0$ is independent of m. This implies immediately that $I_{\nu,s}<\infty$ if and only if

$$\langle\nu+\rho,\ s^{-1}\gamma_m\rangle<0 \qquad \text{for } 1\leqq m\leqq l, \tag{11.47}$$

where $\langle\ ,\ \rangle$ denotes the inner product on $(\sqrt{-1}\,\mathfrak{t})^*\subseteqq\mathfrak{t}_C^*$ induced from the Killing form of $\mathfrak{g}_C$.

Second, we show that every $\nu=\nu_j$ and $s\in W_{\mathfrak{t}'}$ satisfy the condition (11.47). For this purpose, we put

$$\mathfrak{p}'_+=\sum_{\gamma\in(\Sigma_{\mathfrak{p}}^+)'}\mathfrak{g}_C(\mathfrak{t}_C;\gamma)$$

with $(\Sigma_{\mathfrak{p}}^+)'\equiv\{\gamma_1,\gamma_2,\cdots,\gamma_l\}\cup(\bigcup_{k>m}P_{km})$. Then $\mathfrak{p}'_+$ is $\mathrm{Ad}((K')^C)$-stable since $\mathrm{Lie}(K')^C=\mu(\mathfrak{g}(0)_C)$ and $\mathfrak{p}'_+=\mu(\mathfrak{g}(2)_C)$, where $\mathfrak{g}=\bigoplus_j\mathfrak{g}(j)$ is the gradation of $\mathfrak{g}$ determined by $\mathrm{ad}(\sum_m H_m)$. This implies in particular that $s^{-1}\gamma_m\in(\Sigma_{\mathfrak{p}}^+)'$ for $s\in W_{\mathfrak{t}'}$ and $1\leqq m\leqq l$. Therefore, to check the condition (11.47), it is enough to prove the following

Lemma 11.12. *Assume that $\lambda \in \Xi_K^+$ satisfies the non-vanishing condition* (7.11). *Then $\langle \nu+\rho, \gamma\rangle<0$ for all $\gamma \in (\Sigma_{\mathfrak{p}}^+)'$, where $\nu=\nu_j$ is any highest weight of $\tau_\lambda|(K')^C$.*

Proof. It follows from the simplicity of $\mathfrak{g}$ that $\mathfrak{k}_C$ acts on $\mathfrak{p}_+$ irreducibly through the adjoint action. We can see from this fact that the subalgebra $\mathfrak{k}'_C \subseteqq \mathfrak{k}_C$ acts on $\mathfrak{p}'_+ \subseteqq \mathfrak{p}_+$ irreducibly, too.

In fact, let $\mathfrak{v}$ be a non-zero ad($\mathfrak{k}'_C$)-stable subspace of $\mathfrak{p}'_+$. We wish to show $\mathfrak{v}=\mathfrak{p}'_+$. For this purpose, we first note that $\mathfrak{k}_C$ admits a triangular decomposition

$$\mathfrak{k}_C=\mathfrak{k}'_-\oplus \mathfrak{k}'_C \oplus \mathfrak{k}'_+, \qquad [\mathfrak{k}'_C, \mathfrak{k}'_\pm]\subseteqq \mathfrak{k}'_\pm,$$

where $\mathfrak{k}'_\pm \equiv \mu(\mathfrak{g}(\pm 1)_C)\cap \mathfrak{k}_C=\sum_{\gamma \in C_m, 1\leqq m\leqq l} \mathfrak{g}_C(\mathfrak{t}_C;\ \pm\gamma)$ are abelian subalgebras of $\mathfrak{k}_C$. Then, $\mathfrak{p}'_+=\mu(\mathfrak{g}(2)_C)$ is transferred under ad($\mathfrak{k}'_\pm$) as

$$[\mathfrak{k}'_+, \mathfrak{p}'_+]\subseteqq \mu([\mathfrak{g}(1)_C, \mathfrak{g}(2)_C])=(0), \tag{11.48}$$

$$\begin{aligned}[\mathfrak{k}'_-, \mathfrak{p}'_+]&\subseteqq \mathfrak{p}_+\cap \mu([\mathfrak{g}(-1)_C, \mathfrak{g}(2)_C])\\ &\subseteqq \mathfrak{p}_+\cap \mu(\mathfrak{g}(1)_C)=\textstyle\sum_{\gamma \in P_m, 1\leqq m\leqq l} \mathfrak{g}_C(\mathfrak{t}_C;\gamma),\end{aligned} \tag{11.49}$$

$$\begin{aligned}[\mathfrak{k}'_-, [\mathfrak{k}'_-, \mathfrak{p}'_+]]&\subseteqq \mathfrak{p}_+\cap \mu([\mathfrak{g}(-1)_C, \mathfrak{g}(1)_C])\\ &\subseteqq \mathfrak{p}_+\cap \mu(\mathfrak{g}(0)_C)=\mathfrak{p}_+\cap \mathfrak{k}'_C=(0).\end{aligned} \tag{11.50}$$

Keeping in mind $U(\mathfrak{k}_C)=U(\mathfrak{k}'_-)U(\mathfrak{k}'_C)U(\mathfrak{k}'_+)$ (by the Poincaré-Birkhoff-Witt theorem), we obtain

$$\begin{aligned}[U(\mathfrak{k}_C), \mathfrak{v}]&=[U(\mathfrak{k}'_-)U(\mathfrak{k}'_C), \mathfrak{v}] \qquad \text{(by (11.48))}\\ &=[U(\mathfrak{k}'_-), \mathfrak{v}] \qquad \text{(since } \mathfrak{v} \text{ is ad}(\mathfrak{k}'_C)\text{-stable)}\\ &\subseteqq \mathfrak{v}\oplus \mathfrak{p}_+\cap \mu(\mathfrak{g}(1)_C) \qquad \text{(by (11.49) and (11.50))}.\end{aligned}$$

The left end must coincide with the whole $\mathfrak{p}_+$ by the irreducibility of ad($\mathfrak{k}_C$)-module $\mathfrak{p}_+$. This implies $\mathfrak{v}=\mathfrak{p}'_+$ as desired.

The highest weight of $\mathfrak{k}'_C$-module $\mathfrak{p}'_+$ is equal to the largest root γ_l. So, any $\gamma \in (\Sigma_{\mathfrak{p}}^+)'$ is expressed as

$$\gamma=\gamma_l-\textstyle\sum_{\beta \in \Sigma_{\mathfrak{k}'}^+} m_{\gamma\beta}\beta \qquad \text{with integers } m_{\gamma\beta}\geqq 0.$$

Since ν is $\Sigma_{\mathfrak{k}'}^+$-dominant, we have $\langle \nu, \beta\rangle \geqq 0$ for $\beta \in \Sigma_{\mathfrak{k}'}^+$. Note that $\langle \rho, \beta\rangle>0$. Then one gets

$$\langle \nu+\rho, \gamma\rangle=\langle \nu+\rho, \gamma_l\rangle-\textstyle\sum_\beta m_{\gamma\beta}\langle \nu+\rho, \beta\rangle \leqq \langle \nu+\rho, \gamma_l\rangle.$$

In view of (11.40), we see

$$\langle \nu+\rho, \gamma_l\rangle = \langle \lambda+\rho, \gamma_l\rangle - \sum_{\eta\in\Sigma_{\mathfrak{k}}^+} n_{j\eta}\langle \eta, \gamma_l\rangle \leqq \langle \lambda+\rho, \gamma_l\rangle$$

because $\langle \eta, \gamma_l\rangle \geqq 0$ (this follows from the maximality of the root γ_l). The right end must be negative by the assumption on λ. We now get the lemma. Q.E.D.

We have thus obtained the "if" part of Theorem 11.5.

Consequently, the proof of Theorem 11.5 and so that of Theorem 11.3 are now completely finished off.

§ 12. Whittaker models for the holomorphic discrete series and irreducible highest weight representations

In this section, we describe, using the results of Sections 10 and 11, embeddings of the holomorphic discrete series, more generally of irreducible admissible highest weight representations, into our generalized Gelfand-Graev representations (=GGGRs) $\Gamma_i = \mathrm{Ind}_N^G(\rho_i)$ in C^∞- or L^2-context. Such an embedding is called a Whittaker model. Theorems 12.6 and 12.10, and 12.13 are our principal results of Part II of this paper.

12.1. Relation between Whittaker models and highest weight vectors. Let λ be a $\mathfrak{k}$-dominant, K-integral linear form on $\mathfrak{t}_C$, i.e., $\lambda \in \Xi_K^+$. Consider the universal admissible $(\mathfrak{g}_C, K)$-module $M[\lambda]'$ with highest weight λ defined by (7.8), and its unique irreducible quotient L_λ (see Proposition 7.1). We wish to describe Whittaker models for L_λ's into GGGRs Γ_i. For this purpose, let us clarify here the relationship between Whittaker models and K-finite highest weight vectors for Γ_i.

We proceed in more general situation. Let Y be any $(\mathfrak{g}_C, K)$-module. Take a surjective $(\mathfrak{g}_C, K)$-module homomorphims $\tilde{\psi}$: $M[\lambda]' \to L_\lambda$. (It is unique up to scalar multiples.) Then $\tilde{\psi}$ induces an embedding of vector space

$$(12.1) \qquad \iota: \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(L_\lambda, Y) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(M[\lambda]', Y)$$

through $\iota(T) \equiv T\circ\tilde{\psi}$ for $T \in \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(L_\lambda, Y)$. Here, for $(\mathfrak{g}_C, K)$-modules A and B, $\mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(A, B)$ denotes the vector space of $(\mathfrak{g}_C, K)$-module homomorphisms from A into B.

Denote by $Y(\lambda)^\dagger$ the space of K-finite, Σ^+-extreme, λ-highest weight vectors in Y(cf. 7.1).

Lemma 12.1. *For every $\lambda \in \Xi_K^+$, one has an isomorphism of vector spaces:*

$$\mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(M[\lambda]', Y) \ni T \longmapsto T(v_0) \in Y(\lambda)^\dagger,$$

where v_0 is a fixed non-zero, λ-highest weight vector in $M[\lambda]'$.

Proof. If $T \in \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(M[\lambda]', Y)$, then obviously $T(v_0) \in Y(\lambda)^\dagger$. Conversely, let $y \in Y(\lambda)^\dagger$. Then the element y generates a λ-highest weight $(\mathfrak{g}_C, K)$-module $U(\mathfrak{g}_C)y \subseteqq Y_K$, where Y_K denotes the space of K-finite vectors in Y. Further, $U(\mathfrak{g}_C)y$ is of K-multiplicity finite, or admissible. In fact, by the decomposition (5.1): $\mathfrak{g}_C = \mathfrak{p}_- \oplus \mathfrak{k}_C \oplus \mathfrak{p}_+$, one gets $U(\mathfrak{g}_C) = U(\mathfrak{p}_-)U(\mathfrak{k}_C)U(\mathfrak{p}_+)$. So we have

$$U(\mathfrak{g}_C)y = U(\mathfrak{p}_-)(U(\mathfrak{k}_C)y), \tag{12.2}$$

because $\mathfrak{p}_+ y = (0)$. Notice that $U(\mathfrak{k}_C)y$ is a (finite-dimensional) irreducible K-module with highest weight λ, and that $U(\mathfrak{p}_-)$ has a structure of admissible K-module through $\mathrm{Ad}(K) \mid U(\mathfrak{p}_-)$. These two facts together with (12.2) imply the admissiblity of $U(\mathfrak{g}_C)y$.

By virtue of the universal property for $M[\lambda]'$ (Proposition 7.2), there exists a unique $T \in \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(M[\lambda]', U(\mathfrak{g}_C)y)$ such that $T(v_0) = y$. We thus obtain the lemma. Q.E.D.

The above lemma combined with (12.1) and Proposition 7.4 yields immediately the following

Proposition 12.2. *The multiplicity of L_λ in Y as submodules is estimated as* $\dim \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(L_\lambda, Y) \leqq \dim Y(\lambda)^\dagger$. *The equality holds if $\lambda \in \Xi_K^+$ has the property $(\mathscr{I})$: $M[\lambda]' \simeq L_\lambda$, in 7.2. Moreover, the equality holds whenever L_λ globalizes to a (limit of) holomorphic discrete series representation.*

If $\lambda \in \Xi_K^+$ does not have the property $(\mathscr{I})$, or $M[\lambda]'$ is not irreducible, then L_λ can not be necessarily embedded into a $(\mathfrak{g}_C, K)$-module Y even when $Y(\lambda)^\dagger \neq (0)$ (see Remark 12.8 below). Nevertheless, taking into account realizations of L_λ as subquotients further, we can deduce the following

Proposition 12.3. *Let Y be any $(\mathfrak{g}_C, K)$-module and $\lambda \in \Xi_K^+$. If $Y(\lambda)^\dagger \neq (0)$, then the irreducible highest weight $(\mathfrak{g}_C, K)$-module L_λ is contained in Y as its subquotient.*

Proof. Take a non-zero highest weight vector $y \in Y(\lambda)^\dagger$. As shown in the proof of Lemma 12.1, the $U(\mathfrak{g}_C)$-module $V_y \equiv U(\mathfrak{g}_C)y \subseteqq Y_K$ generated by y is isomorphic to a certain quotient $M[\lambda]'/M'$. Recall that L_λ is isomorphic to $M[\lambda]'/K'_\lambda$ with the unique largest proper submodule K'_λ

of $M[\lambda]'$ (cf. 7.1). Since $M' \subseteqq K'_\lambda$, we see that L_λ is contained in $V_y \subseteqq Y$ as its quotient, and so in Y as its subquotient. Q.E.D.

Now let $(\pi, \mathscr{H})$ be a unitary representation of G, and $Y \equiv \mathscr{H}^\omega_K \subseteqq \mathscr{H}$ be the $(\mathfrak{g}_C, K)$-module consisting of K-finite analytic vectors for π. In this case, embeddings of irreducible highest weight representations of G into $(\pi, \mathscr{H})$ correspond bijectively to highest weight vectors in Y, as is seen in the following. First, one gets

Lemma 12.4. *Let $\lambda \in \Xi^+_K$. Then, every non-zero λ-highest weight vector $y \in Y(\lambda)^\dagger$, $Y = \mathscr{H}^\omega_K$, generates an irreducible $(\mathfrak{g}_C, K)$-module isomorphic to L_λ: $L_\lambda \simeq U(\mathfrak{g}_C)y \subseteqq Y$.*

Proof. First notice that any λ-highest $\mathfrak{g}_C$-module V has a Jordan-Hölder series of $\mathfrak{g}_C$-submodules with finite length. (Indeed, the Verma module $M[\lambda]$ defined by (7.4) admits such a series, cf. [30, Lemma 2.2.6], and so does its quotient V.) If V is further an admissible $(\mathfrak{g}_C, K)$-module, then its composition series consists of $(\mathfrak{g}_C, K)$-submodules.

On the other hand, as seen in the proof of Lemma 12.1, $U(\mathfrak{g}_C)y$ is an admissible λ-highest weight $(\mathfrak{g}_C, K)$-module for any $y \in Y(\lambda)^\dagger$, Hence, $U(\mathfrak{g}_C)y$ admits a Jordan-Hölder series of $(\mathfrak{g}_C, K)$-submodules. Furthermore, this composition series must split, because the group G acts on the closure of $U(\mathfrak{g}_C)y$ in $\mathscr{H}$ unitarily. Thus we get a direct sum decomposition of $U(\mathfrak{g}_C)y$ into irreducibles: $U(\mathfrak{g}_C)y = \bigoplus_{1 \leqq j \leqq p} Y_j$ with irreducible $(\mathfrak{g}_C, K)$-submodules Y_j.

Along this decomposition, put $y = \sum_j y_j$ with $y_j \in Y_j$. Clearly, all y_j's are λ-highest weight vectors. By the uniqueness of λ-highest weight vectors in $U(\mathfrak{g}_C)y$ (up to scalar multiples), there should be $p=1$, whence $U(\mathfrak{g}_C)y$ is irreducible. Q.E.D.

We see from this lemma that, if $Y(\lambda)^\dagger \neq (0)$, then L_λ is unitarizable and that the closure $\overline{U(\mathfrak{g}_C)y}$ of $U(\mathfrak{g}_C)y$ in $\mathscr{H}$ gives, for every $y \in Y(\lambda)$, $\neq 0$, a globarization of L_λ to an irreducible unitary representation of G. In other words, $\overline{U(\mathfrak{g}_C)y}$ realizes the irreducible unitary highest weight representation π_λ of G in 7.3. Conversely, it is obvious that any embedding of π_λ into $(\pi, \mathscr{H})$ as a unitary representation is obtained in this fashion. Thus we deduce the following

Proposition 12.5. *Let $(\pi, \mathscr{H})$ be a unitary representation of G, and π_λ the irreducible unitary representation of G with highest weight λ. Then, π_λ occurs in π as its subrepresentations exactly* dim $Y(\lambda)^\dagger$ *times, where $Y(\lambda)^\dagger$ is the space of λ-highest weight vectors in $Y = \mathscr{H}^\omega_K$.*

12.2. Whittaker models in C^∞-GGGRs.

Now let us apply Propositions 12.2 and 12.3 to $Y=C^\infty\text{-}\Gamma_i$ $(0\leqq i\leqq l)$, our GGGRs induced in C^∞-context. By virtue of Theorem 10.6, we obtain immediately the following description of Whittaker models for irreducible admissible highest weight modules L_λ, which is our first main result of this Part II.

Theorem 12.6. *Let G be a connected, simple linear Lie group of hermitian type, and l the real rank G. For each integer i, $0\leqq i\leqq l$, denote by $C^\infty\text{-}\Gamma_i$ the C^∞-GGGR associated with the nilpotent* Ad(G)*-orbit ω_i in Theorem* 5.2. *We introduce the following three cases according as the type of hermitian symmetric space G/K (tube or non-tube) and the parameter i of our GGGRs*:

(Case I: $0\leqq i\leqq l$) *G/K is of tube type and i is arbitrary,*
(Case II: $i=0$) *G/K is of non-tube type and $i=0$,*
(Case II: $i\neq 0$) *G/K is of non-tube type and $i\neq 0$.*

(1) *Let λ be a $\mathfrak{k}$-dominant, K-integral linear form on the complexification of the Cartan subalgebra $\mathfrak{t}\subseteq\mathfrak{g}$, i.e., $\lambda\in\Xi_K^+$. Then, the multiplicity of the irreducible admissible λ-highest weight $(\mathfrak{g}_C, K)$-module L_λ in $C^\infty\text{-}\Gamma_i$ as submodules is bounded as*

$$\dim \operatorname{Hom}_{\mathfrak{g}_C\text{-}K}(L_\lambda, C^\infty\text{-}\Gamma_i)\begin{cases}\leqq \dim \tau_\lambda & \textit{for } (\text{Case I}: 0\leqq i\leqq l) \\ & \quad \textit{and } (\text{Case II}: i=0), \\ =0 & \textit{for } (\text{Case II}: i\neq 0).\end{cases} \tag{12.3}$$

Here τ_λ denotes the irreducible finite-dimensional K-module with highest weight λ.

(2) *Assume that $\lambda\in\Xi_K^+$ has the property $(\mathscr{S})$: $M[\lambda]'\simeq L_\lambda$, in* 7.2. *(This assumption is fulfilled whenever λ corresponds to a member of (limit of) the holomorphic discrete series.) Then, the equality holds in* (12.3).

(3) *Consider* (Case I: $0\leqq i\leqq l$) *or* (Case II: $i=0$). *Then L_λ is contained in the GGGR $C^\infty\text{-}\Gamma_i$ as its subquotient for every $\lambda\in\Xi_K^+$.*

Although we treated in this theorem their multiplicities only, we can further describe explicitly embeddings of highest weight modules L_λ into GGGRs $C^\infty\text{-}\Gamma_i$. In fact, (i) we now know an exact expression of highest weight vectors for GGGRs (Theorem 10.6), and (ii) any Whittaker model for L_λ is, if exist, characterized by such a highest weight vector (see 12.1). Therefore, we can say that Theorem 12.6(2) gives, together with these informations (i) and (ii), a complete description of Whittaker models for L_λ's into our C^∞-GGGRs, for λ's with the property $(\mathscr{S})$.

However, for $\lambda\in\Xi_K^+$ such that $M[\lambda]'\not\simeq L_\lambda$, our result is not so perfect, and there rests a problem still open. More precisely, to any embedding of L_λ into C^∞-Γ_i, one can attach canonically a K-finite λ-highest weight vector $F \in C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$, $\neq 0$, in C^∞-Γ_i. Is the converse true? Or does any non-zero element $F \in C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ produce a Whittaker model $L_\lambda \hookrightarrow C^\infty$-$\Gamma_i$? If $M[\lambda]' \simeq L_\lambda$, the answer is "yes", which yields Theorem 12.6(2). However, if $M[\lambda]' \not\simeq L_\lambda$, it is not necessarily "yes" in general (see Rremark 12.8). So, there arises the following

Problem 12.7. Determine which K-finite highest weight vectors $F \in C^\infty(G; \rho_i \| \Sigma^+; \lambda)^\dagger$ give actually Whittaker models for the irreducible highest weight modules L_λ.

Solving this problem, one may reach to a final goal for describing C^∞-Whittaker models: $L_\lambda \hookrightarrow C^\infty$-$\Gamma_i$.

Remark 12.8. Toward this problem, Matumoto's result [21, Cor. B], on the existence of Whittaker models in connection with Gelfand-Kirillov dimensions of Harish-Chandra modules, tells us the following:

If L_λ can be embedded into C^∞-Γ_i for some $0 \leqq i \leqq l$, then necessarily

$$\mathrm{Dim}(L_\lambda) \geqq \dim \mathfrak{p}_+ \qquad \textit{with } \mathfrak{p}_+ = \textstyle\sum_{\gamma \in \Sigma_\mathfrak{p}^+} \mathfrak{g}_C(\mathfrak{t}_C; \gamma), \tag{12.4}$$

where $\mathrm{Dim}(L_\lambda)$ *denotes the Gelfand-Kirillov dimension of* L_λ.

In particular, assume that $\lambda \in \Xi_K^+$ is dominant with respect to the whole $\Sigma^+ \supseteqq \Sigma_\mathfrak{t}^+$. Then L_λ is finite-dimensional, and so $\mathrm{Dim}(L_\lambda) = 0$. Hence (12.4) implies that such an L_λ can never occur in our GGGRs as submodules, although it is contained in C^∞-Γ_i, $i=0$, as its subquotient (see Theorem 12.6(3)). It should be remarked that this non-existence property of Whittaker models can be obtained more directly by making use of the following fact: On a finite-dimensional $\mathfrak{g}_C$-module, any nilpotent element of $\mathfrak{g}_C$ is represented by a nilpotent operator.

On the other hand, if $\lambda \in \Xi_K^+$ has the property $(\mathscr{I})$, then

$$\mathrm{Dim}(L_\lambda) = \dim \mathfrak{p}_+. \tag{12.5}$$

From this equality together with our theorem, we find that the estimate (12.4) is best possible.

Remark 12.9. We have modified in Theorem 12.6 Hashizume's result [11, Theorem 4.4 and Cor. 4.5] on Whittaker models, in our setting of GGGRs C^∞-Γ_i. (See 8.1, Remarks 10.2 and 10.9)

12.3. Whittaker models in L^2-GGGRs.

We now present our second main result of Part II, which gives a complete description of Whittaker models for highest weight representations in the unitarily induced GGGRs.

Theorem 12.10. *Let $\lambda \in \Xi_K^+$ and $0 \leqq i \leqq l$. Consider the irreducible admissible λ-highest weight representation π_λ of G, and the unitary GGGR $L^2\text{-}\Gamma_i = L^2\text{-}\mathrm{Ind}_N^G(\rho_i)$. Then, π_λ occurs in $L^2\text{-}\Gamma_i$ as its subrepresentation if and only if $i=0$ and π_λ is a member of the holomorphic discrete series: $\pi_\lambda = D_\lambda$ (see 7.3). In such a case, the multiplicity of D_λ in $L^2\text{-}\Gamma_i$, $i=0$, coincides with* dim τ_λ, *the dimension of the irreducible K-module τ_λ with highest weight λ.*

Proof. The statement follows immediately from Theorem 11.3, Propositions 11.1 and 12.5. Q.E.D.

Remark 12.11. If G/K is of tube type, the discrete series representation D_λ occurs dim τ_λ times in C^∞-GGGR $C^\infty\text{-}\Gamma_i$ for every $0 \leqq i \leqq l$. Nevertheless, it does not appear in the unitary GGGRs $L^2\text{-}\Gamma_i$ unless $i=0$.

Remark 12.12. We can describe Whittaker models for lowest weight representations, too. Such a description is obtained by replacing the terminology: highest weight modules, the holomorphic discrete series and GGGRs Γ_i respectively by lowest weight modules, the anti-holomorphic discrete series and GGGRs Γ_{l-i}, in Theorems 12.6 and 12.10.

So in particular, we thus obtain a perfect answer to Problem EDS (Embeddings of Discrete Series) in the beginning of Part II, for the holomorphic and anti-holomorphic discrete series.

12.4. Whittaker models in reduced GGGRs (=RGGGRs).

Now consider the most interesting case $i=0$. Then, our GGGR $\Gamma_0 = \mathrm{Ind}_N^G(\rho_0)$ is decomposed into a direct sum of the corresponding RGGGRs $\Gamma_0(c) = \mathrm{Ind}_{(K\cap L)N}^G(\tilde{c} \otimes \tilde{\rho}_0)$ (see (5.18)), where c ranges over irreducible finite-dimensional representations of the maximal compact subgroup $K \cap L$ of L.

By the method of highest weight vectors, we can also describe Whittaker models for highest weight representations in these RGGGRs.

Theorem 12.13. (1) *Let λ be a $\mathfrak{k}$-dominant, K-integral linear form on $\mathfrak{t}_C$ with the property $(\mathscr{I})$: $M[\lambda]' \simeq L_\lambda$, in 7.2. Then the multiplicity of L_λ in the C^∞-induced RGGGR $C^\infty\text{-}\Gamma_0(c)$ is given as*

$$(12.6) \qquad \dim \mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(L_\lambda, C^\infty\text{-}\Gamma_0(c)) = \dim \mathrm{Hom}_{K\cap L}(c, \tau_\lambda|(K\cap L))$$

for every $c \in (K\cap L)^\wedge$, where $\tau_\lambda|(K\cap L)$ denotes the restriction of the *irreducible representation τ_λ of K onto $K \cap L$.*

(2) *The irreducible λ-highest weight representation π_λ of G occurs in a unitary RGGGR L^2-$\Gamma_0(c)$ for some $c \in (K \cap L)^\wedge$ if and only if $\pi_\lambda = D_\lambda$, a member of the holomorphic discrete series. Moreover, for each $c \in (K \cap L)^\wedge$, the multiplicity of D_λ in L^2-$\Gamma_0(c)$ coincides with that of c in $\tau_\lambda | (K \cap L)$.*

This theorem is obtained analogously to the case of non-reduced GGGRs. One can carry out the proof by repreating the argument in Sections 10–12 for RGGGRs $\Gamma_0(c)$ instead of GGGRs Γ_i. So we omit it here.

12.5. Remarks on our method.

Our method of highest weight vectors is the most direct and the most elementary way to describe Whittaker models for highest weight representations including the holomorphic discrete seires. But, our result on L^2-Whittaker model (Theorem 12.10) has a certain connection with Rossi-Vergne's result that describes the restriction of holomorphic discrete series of G to an Iwasawa subgroup $S \equiv A_p N_m$ of G (see Theorem 12.14 below). We clarify here the relationship between these two results, and comment on our method.

For this purpose, one needs detailed informations on representations of the exponential solvable Lie group S (cf. [II, 4.1]). Let $\mathfrak{s} = \mathfrak{a}_p \oplus \mathfrak{n}_m$ denote the Lie algebra of S, and $\mathfrak{s}^*$ the dual space of $\mathfrak{s}$ on which the group S acts through the coadjoint action Ad^*. The Kirillov-Bernat correspondence (cf. [4]) sets up a bijection between the coadjoint orbit space $\mathfrak{s}^*/\mathrm{Ad}^*(S)$ and the unitary dual $\hat{S}$ of S. Put $l = \dim A_p$. Then, there exist exactly 2^l-number of open coadjoint orbits in $\mathfrak{s}^*$ described as follows. Using the root vectors $E_k \in \mathfrak{g}(\mathfrak{a}_p; \lambda_k) \subseteqq \mathfrak{n}_m$ $(1 \leqq k \leqq l)$ defined by (5.11), we set $A_\varepsilon = \sum_{1 \leqq k \leqq l} \varepsilon_k E_k \in \mathfrak{n}_m$ for every $\varepsilon = (\varepsilon_k)_{1 \leqq k \leqq l} \in \{1, -1\}^l$, $\varepsilon_k = \pm 1$. Through the Killing form B of $\mathfrak{g}$ and the Cartan involution θ, A_ε gives rise to an element $A_\varepsilon^* \in \mathfrak{s}^*$ via $\langle A_\varepsilon^*, Z \rangle = B(Z, \theta A_\varepsilon)$ $(Z \in \mathfrak{s})$. According to Nomura [23, Prop. 1.4], $\varepsilon \mapsto \mathrm{Ad}^*(S) A_\varepsilon^*$ gives a one-to-one correspondence from $\{1, -1\}^l$ onto the set of open $\mathrm{Ad}^*(S)$-orbits in $\mathfrak{s}^*$.

Let ζ_ε denote an irreducible unitary representation of S associated with the orbit $\mathrm{Ad}^*(S) A_\varepsilon^*$ through the Kirillov-Bernat correspondence. Applying Dulfo-Raïs's result on the Plancherel formula for a solvable Lie group [7, p. 132], one finds out that (i) ζ_ε belongs to the discrete series for the left regular representation λ_S of S on $L^2(S)$, and that (ii) λ_S is decomposed into irreducibles as

$$\lambda_S \simeq \oplus_{\varepsilon \in \{1,-1\}^l} [\infty] \cdot \zeta_\varepsilon. \tag{12.7}$$

Now consider the unitarily induced representation L^2-$\mathrm{Ind}_S^G(\zeta_\varepsilon)$ for

$\varepsilon=(\varepsilon_k)\in\{1,-1\}^l$. As shown in [II, 4.1], this representation is unitarily equivalent to our GGGR:

$$(12.8)\qquad L^2\text{-}\mathrm{Ind}_S^G(\zeta_\varepsilon)\simeq L^2\text{-}\Gamma_{i(\varepsilon)}=L^2\text{-}\mathrm{Ind}_N^G(\rho_{i(\varepsilon)}),$$

where $i(\varepsilon)$ denotes the number of the integers k, $1\leqq k\leqq l$, for which $\varepsilon_k=-1$. (This equivalence together with (12.7) produces Theorem 5.3.)

With (12.7) and (12.8) in mind, let us apply Anh reciprocity [1, Cor. 1.10] to the pair (G, S), $G\supseteqq S$. Then we find out in particular that, for any discrete series representation D of G, its restriction to the subgroup S is described as

$$(12.9)\qquad D\,|\,S\simeq\bigoplus\nolimits_{\varepsilon\in\{1,-1\}^l}[M(D, i(\varepsilon))]\cdot\zeta_\varepsilon,$$

where,

$$M(D, i(\varepsilon))\equiv\mathrm{mtp}(D;\,L^2\text{-}\mathrm{Ind}_S^G(\zeta_\varepsilon))=\mathrm{mtp}(D;\,L^2\text{-}\Gamma_{i(\varepsilon)})$$

denotes the multiplicity of D in the unitary GGGR $L^2\text{-}\Gamma_{i(\varepsilon)}$. We have thus clarified the relationship between L^2-Whittaker model for the discrete series D and its restriction $D\,|\,S$ to the Iwasawa subgroup S.

Using (12.9), we deduce immediately from Theorem 12.10 (L^2-Whittaker model) the following result of Rossi and Vergne.

Theorem 12.14 [28, Cor. 5.23]. *Let D_λ be the holomorphic discrete series representation of G with highest weight $\lambda\in\mathfrak{t}_C^*$. Then, its restriction to S is given as*

$$(12.10)\qquad D_\lambda\,|\,S\simeq[\dim\tau_\lambda]\cdot\zeta_{\mathbf{1}}\qquad \textit{with } \mathbf{1}\equiv(1, 1, \cdots, 1)\in\{1,-1\}^l.$$

Conversely, our description of L^2-Whittaker model can be obtained from this theorem through (12.9). Therefore, Theorem 12.10 is equivalent to the above result (12.10) of Rossi and Vergne.

However, to derive our theorem from (12.10), the reciprocity (12.9) is needed, and further, in order to prove (12.9), one need to sum up many results on representations of S by different authors: for instance,

(1) Kirillov-Bernat correspondence (cf. [4]),

(2) description of open coadjoint orbits in $\mathfrak{s}^*$ (Nomura),

(3) Plancherel theorem for exponential solvable Lie groups (Duflo and Raïs),

(4) Forbenius reciprocity theorem due to Anh [1] (see also [27]), and so on. Furthermore, Rossi-Vergne's result (12.10) is a consequence of a detailed study on the holomorphic discrete series for S in connection such discrete series for G (see [28]).

For this reason, we wanted to take a short cut and give a more elementary proof of Theorem 12.10. Our method of highest weight vectors have realized this hope satisfactorily. Further our proof is independent of the result (12.10) or Rossi and Vergne. We utilized rather the classical technique of Harish-Chandra in [9, VI], where the non-vanishing condition was studied for the holomorphic discrete series of G.

Our method of highest weight vectors is also applicable to describe embeddings of irreducible highest weight $(\mathfrak{g}_C, K)$-modules into the principal series (see Theorem 8.1), which will be discussed elsewhere.

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