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Selberg Trace Formula for a Certain Group Generated by $PSL(2, Z[i])$ and a Reflection

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Introduction

Let H be the three dimensional upper half space consisting of all elements $u=(z, v)$, where z is a complex number and v is a positive real number. Moreover, put $G=PSL(2, C)$ and $\Gamma=PSL(2, Z[i])$. It is well known that Selberg trace formula for the Hilbert space $L^2(\Gamma\backslash H)$ holds, which has already been presented by various methods in [6], [7] or [8] for example. Let now ω: $(z, v)\to(\bar{z}, v)$ be the complex conjugation with respect to the z-part of $u=(z, v)$. Then the space $L^2(\Gamma\backslash H)$ has the direct sum decomposition $L^2(\Gamma\backslash H)=V_e\oplus V_o$ in accordance with the operation of ω, where V_e and V_o are spaces defined respectively by $V_e=\{f\in L^2(\Gamma\backslash H)\mid f(\omega u)=f(u)\}$ and $V_o=\{f\in L^2(\Gamma\backslash H)\mid f(\omega u)=-f(u)\}$.

The purpose of the present paper is to derive trace formulas for V_e and V_o in explicit forms. Let $\tilde{\Gamma}=\langle\Gamma, \omega\rangle$ be the group generated by Γ and ω. Then, the equality $L^2(\tilde{\Gamma}\backslash H)=V_e$ is shown to hold from the definition of them. Hence, it is sufficient to consider the trace formula for the space $L^2(\tilde{\Gamma}\backslash H)$ by means of Selberg's theory to obtain that for V_e. The trace formula for V_o then follows easily from those for $L^2(\Gamma\backslash H)$ and $V_e(=L^2(\tilde{\Gamma}\backslash H))$.

On the other hand, in [10], we investigated the trace formula for the Hilbert space with the group generated by $PSL(2, Z)$ and the reflection with respect to the imaginary axis. Though the material in this article looks analogous to that of [10], the actual analysis involved seems to be rather different. In fact, in the present case, only relatively parabolic elements defined in Section 2 contribute the continuous spectrum, and relatively identical elements in Section 2, which exist clearly, do not contribute the continuous spectrum, while, in [10], the elements, which correspond to our relatively identical elements in the classification in Section 2, contribute the continuous spectrum, and no elements correspond to our relatively parabolic elements.

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In Section 1, trace formulas for V_e and V_o will be stated in rough forms. In Section 2, the concrete decomposition of $\Gamma\omega$ into relative conjugacy classes except for relatively hyperbolic elements will be settled, and, in the last section, we will describe a method for evaluating the traces for V_e and V_o explicitly. Our final results are Theorem 1 and Theorem 2.

§ 1. Weakly symmetric Riemannian space

Let $H=\{u=(z, v)\,|\,z=x+iy \in \boldsymbol{C},\ 0<v \in \boldsymbol{R}\}$ be the three dimensional upper half space having a Riemann structure defined by

$$ds^2=\frac{1}{v^2}(dx^2+dy^2+dv^2). \tag{1.1}$$

As is well known, the group $G=PSL(2, \boldsymbol{C})$ operates onto H transitively by a linear fractional transformation:

$$(z, v)\longrightarrow\left(\frac{(az+b)(\overline{cz+d})+a\bar{c}v^2}{|cz+d|^2+|c|^2v^2},\ \frac{v}{|cz+d|^2+|c|^2v^2}\right)$$

for each element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G. Then, the triple $(G, H, 1)$ becomes a weakly symmetric Riemannian space in the notation of Selberg ([5] or [10]). Let furthermore ω: $(z, v)\rightarrow(\bar{z}, v)$ be the complex conjugation with respect to the z-part of u, and let $\tilde{G}=\langle G, \omega\rangle$ be a group generated by G and ω. From the fact that the metric (1.1) is $\tilde{G}$-invariant and the triple $(G, H, 1)$ is weakly symmetric, we see that the triple $(\tilde{G}, H, 1)$ also turns out to be a weakly symmetric Riemannian space.

The group $\tilde{G}$ has the following properties:

$$\begin{aligned}&\omega^2=\mathrm{id},\\ &\omega\begin{pmatrix} a & b \\ c & d \end{pmatrix}\omega=\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \quad \text{for every} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } G, \text{ i.e.,} \quad \omega G\omega=G,\end{aligned} \tag{1.2}$$

and therefore we have

$$\tilde{G}=G\cup G\omega=G\cup\omega G. \tag{1.3}$$

The Laplace-Beltrami operator and the $\tilde{G}$-invariant measure derived from the metric (1.1) will be denoted by

$$D=v^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial v^2}\right)-v\frac{\partial}{\partial v} \tag{1.4}$$

and

$$d\mu(u)=\frac{dxdydv}{v^3} \tag{1.5}$$

respectively.

Put $\Gamma=PSL(2, \boldsymbol{Z}[i])$. Then Γ operates discontinuously on H, and the fundamental domain $\mathscr{D}=\Gamma\backslash H$ of Γ is given by

$$\mathscr{D}=\{(z, v)\,|\,z=x+iy,\ 0\leqq x\leqq \tfrac{1}{2},\ -\tfrac{1}{2}\leqq y\leqq \tfrac{1}{2},\ x^2+y^2+v^2\geqq 1\}.$$

Let $L^2(\mathscr{D})$ be the Hilbert space of square-integrable functions defined on $\mathscr{D}$ for the measure (1.5), and let $L_0^2(\mathscr{D})$ be its subspace consisting of cusp forms. It is well known that the space $L^2(\mathscr{D})$ has the spectral decomposition with respect to D as in (1.4):

$$L^2(\mathscr{D})=L_0^2(\mathscr{D})\oplus \boldsymbol{C}\oplus L_c^2(\mathscr{D}),$$

where $\boldsymbol{C}$ is the space of constant functions and $L_c^2(\mathscr{D})$ is the continuous part of the spectrum. We can take a set of *Maass wave forms* $\{f_j\}_{j\geqq 1}$ as an orthogonal (but not orthonormal) basis of $L_0^2(\mathscr{D})$, where $Df_j=-(1+r_j^2)f_j$, $r_j>0$, for each f_j.

On the other hand, according to the action of ω, the space $L^2(\mathscr{D})$ has an alternative direct sum decomposition:

$$L^2(\mathscr{D})=V_e\oplus V_0,$$

where $V_e=\{f\in L^2(\mathscr{D})\,|\,f(\omega u)=f(u)\}$ and $V_0=\{f\in L^2(\mathscr{D})\,|\,f(\omega u)=-f(u)\}$. We call the spaces V_e and V_0 even and odd spaces respectively. Corresponding to the last decomposition of $L^2(\mathscr{D})$, the space $L_0^2(\mathscr{D})$ can be decomposed into even and odd spaces as well, which will be denoted by

$$L_{0,e}^2(\mathscr{D})=L_0^2(\mathscr{D})\cap V_e,\quad \{f_{j_1}\}_{j_1\geqq 1}\text{: orthogonal basis of } L_{0,e}^2(\mathscr{D}),$$
$$L_{0,o}^2(\mathscr{D})=L_0^2(\mathscr{D})\cap V_o,\quad \{f_{j_2}\}_{j_2\geqq 1}\text{: orthogonal basis of } L_{0,o}^2(\mathscr{D}),$$

where $\{j\}_{j\geqq 1}=\{j_1\}_{j_1\geqq 1}\cup\{j_2\}_{j_2\geqq 1}$. In view of the fact that $\boldsymbol{C}\oplus L_c^2(\mathscr{D})\subset V_e$, we then find

$$V_e=L_{0,e}^2(\mathscr{D})\oplus \boldsymbol{C}\oplus L_c^2(\mathscr{D}),\qquad V_o=L_{0,o}^2(\mathscr{D}). \tag{1.6}$$

Our aim in this article is to obtain trace formulas for V_e and V_o in explicit forms.

Now consider the group $\tilde{\Gamma}=\langle\Gamma, \omega\rangle$ generated by Γ and ω. It is a discrete subgroup of $\tilde{G}$ and therefore operates discontinuously on H. A fundamental domain $\tilde{\mathscr{D}}=\tilde{\Gamma}\backslash H$ of $\tilde{\Gamma}$ can be written as

$$\tilde{\mathscr{D}}=\{(z, v)\,|\,z=x+iy,\ 0\leqq x\leqq \tfrac{1}{2},\ 0\leqq y\leqq \tfrac{1}{2},\ x^2+y^2+v^2\geqq 1\}.$$

Let furthermore $L^2(\tilde{\mathscr{D}})$ be the Hilbert space of square-integrable functions defined on $\tilde{\mathscr{D}}$. The spectral decomposition with respect to D of the space $L^2(\tilde{\mathscr{D}})$ is expressed, in a fashion similar to $L^2(\mathscr{D})$, as

$$L^2(\tilde{\mathscr{D}})=L_0^2(\tilde{\mathscr{D}})\oplus \boldsymbol{C}\oplus L_c^2(\tilde{\mathscr{D}}),$$

where $L_0^2(\tilde{\mathscr{D}})$ and $\boldsymbol{C}$ are spaces of cusp forms and constant functions respectively, and $L_c^2(\tilde{\mathscr{D}})$ is the continuous part of the spectrum. From the fact that the triple $(\tilde{G}, H, 1)$ is weakly symmetric and $\tilde{\Gamma}$ is a discrete subgroup of $\tilde{G}$, we can apply Selberg's theory to the space $L^2(\tilde{\mathscr{D}})$. However, by definition, the equality $V_e=L^2(\tilde{\mathscr{D}})$ holds clearly. Hence, it is sufficient to consider the trace formula for $L^2(\tilde{\mathscr{D}})$ so as to obtain that of V_e. In addition, it is shown easily that $L_{0,e}^2(\mathscr{D})=L_0^2(\tilde{\mathscr{D}})$ from the above decomposition and (1.6), so that we can take even Maass wave forms $\{f_{j_1}\}_{j_1\geq 1}$ as an orthogonal basis of $L_0^2(\tilde{\mathscr{D}})$.

Set $k(u, u')$ to be a point pair invariant with respect to $\tilde{G}$, i.e., it satisfies the condition $k(\sigma u, \sigma u')=k(u, u')$ for any $\sigma\in\tilde{G}$. Moreover put

$$t(u, u')=\frac{|z-z'|^2+(v-v')^2}{vv'} \quad \text{for } u=(z, v),\ u'=(z', v')\in H. \tag{1.7}$$

Then, since any point pair invariant with respect to G is a function of a positive real variable $t=t(u, u')$ and since $t(\omega u, \omega u')=t(u, u')$, any point pair invariant with respect to $\tilde{G}$ can also be identified with a function of $t=t(u, u')$. This implies that for every point pair invariant $k(u, u')$ with respect to $\tilde{G}$, a function φ may be defined by

$$k(u, u')=\varphi(t(u, u')). \tag{1.8}$$

To simplify the explicit computation of the trace formula, we impose the condition on φ that it is smooth and of compact support. An invariant integral operator with respect to $\tilde{G}$ derived from (1.8) will be denoted by L_φ, i.e.,

$$(L_\varphi f)(u)=\int_H k(u, u')f(u')d\mu(u'),$$

where $k(u, u')=\varphi(t(u, u'))$. From Selberg's theory, it is known that any eigenfunction of D with the eigenvalue $-(1+r^2)$, $r\in\boldsymbol{C}$ becomes an eigenfunction of an arbitrary invariant integral operator simultaneously, whose eigenvalue is usually written as $h(r)$. The eigenvalue $h(r)$, determined only by L_φ and r, is called the Selberg transform. Obviously it is an even function of r, i.e., $h(r)=h(-r)$. The operator L_φ, on $L^2(\tilde{\mathscr{D}})$, turns out to be an integral operator with the kernel function $\tilde{K}(u, u')$, where

$$\tilde{K}(u, u') = \sum_{\sigma \in \tilde{\Gamma}} k(u, \sigma u'). \tag{1.8}$$

As in [2], the Eisenstein series with respect to Γ is defined by

$$E(u, s) = \sum_{\sigma \in \Gamma_0 \backslash \Gamma} v(\sigma u)^s$$

for $u \in H$ and $s \in C$, where $\Gamma_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c = 0 \right\}$ and $v(u)$ is the v-part of u. This series converges absolutely for $\mathrm{Re}\,(s) > 2$, and so becomes a holomorphic function of s in this region. It can be continued meromorphically to the whole complex s-plane, having a simple pole at $s = 2$, and satisfies the functional equation:

$$E(u, s) = \Phi(s) E(u, 2-s),$$

where

$$\Phi(s) = \frac{\pi}{s-1} \cdot \frac{\zeta_k(s-1)}{\zeta_k(s)}$$

and $\zeta_k(s)$ is the Dedekind zeta function of the field $k = Q(i)$. If we put

$$H(u, u') = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(u, 1+ir) E(u', 1-ir) h(r) dr,$$

then, by a similar consideration as in [10: Proposition 2], we have the following

Proposition 1.1. *Let $\tilde{K}^*(u, u')$ be the kernel function defined by*

$$\tilde{K}^*(u, u') = \tilde{K}(u, u') - 2H(u, u'), \tag{1.9}$$

then it is bounded on $\tilde{\mathscr{D}} \times \tilde{\mathscr{D}}$.

Let L_φ^* be an integral operator defined on $L^2(\tilde{\mathscr{D}})$ with the kernel function $\tilde{K}^*(u, u')$. Then, it follows from Selberg's theory that $L_\varphi f_{j_1} = h(r_{j_1}) f_{j_1}$ for each f_{j_1}, $L_\varphi f_0 = h(i) f_0$ for $f_0 \equiv 1$ (since $D f_0 \equiv 0$), and $L_\varphi^* \equiv 0$ on $L_c^2(\tilde{\mathscr{D}})$ (cf. [3: Appendix]). This means that the operator L_φ^* coincides with the discrete part of L_φ and the continuous spectrum of L_φ can be expressed by $2H(u, u')$. Since L_φ^* is completely continuous due to Proposition 1.1, we obtain the following rough trace formula for $L^2(\tilde{\mathscr{D}})$ and therefore for V_e:

$$\sum_{j_1 \geq 1} h(r_{j_1}) + h(i) = \int_{\tilde{\mathscr{D}}} \tilde{K}^*(u, u') d\mu(u). \tag{1.10}$$

Next, if we put $K^*(u, u') = \sum_{\sigma \in \Gamma} k(u, \sigma u') - H(u, u')$ and $K'^*(u, u') =$

$\sum_{\sigma\in\Gamma}k(u, \sigma\omega u')-H(u, u')$ corresponding to a classification $\tilde{\Gamma}=\Gamma\cup\Gamma\omega$ which is readily derived from (1.3), then it follows from (1.8) and (1.9) that the right hand side of (1.10) is equal to

$$(1.11)\qquad \frac{1}{2}\int_{\mathscr{D}} K^*(u, u)d\mu(u)+\frac{1}{2}\int_{\mathscr{D}} K'^*(u, u)d\mu(u).$$

Moreover, the definition of $K^*(u, u')$ means that the integral

$$\int_{\mathscr{D}} K^*(u, u)d\mu(u)$$

in (1.11) is nothing but the trace for $L^2(\mathscr{D})$, namely we have

$$(1.12)\qquad \sum_{j\geqq 1} h(r_j)+h(i)=\int_{\mathscr{D}} K^*(u, u)d\mu(u).$$

Therefore, by using (1.10), (1.11) and (1.12), the trace formula for V_o can be described as

$$(1.13)\qquad \begin{aligned}\sum_{j_2\geqq 1} h(r_{j_2})&=(\sum_{j\geqq 1} h(r_j)+h(i))-(\sum_{j_1\geqq 1} h(r_{j_1})+h(i))\\ &=\frac{1}{2}\int_{\mathscr{D}} K^*(u, u)d\mu(u)-\frac{1}{2}\int_{\mathscr{D}} K'^*(u, u)d\mu(u).\end{aligned}$$

As for the integral $\int_{\mathscr{D}} K^*(u, u)d\mu(u)$, it has already been calculated by [6], [7] or [8] for instance. Hence, in order to obtain trace formulas for V_e and V_o, we have only to compute the integral $\int_{\mathscr{D}} K'^*(u, u)d\mu(u)$ explicitly.

About the relationship between $\varphi(t)$ and $h(r)$, we know the following

Proposition 1.2 ([6], [7] or [8]). *Put*

$$Q(w)=\pi\int_w^\infty \varphi(t)dt, \qquad w\geqq 0,$$

and set $Q(w)=g(u)$ for $w=e^u+e^{-u}-2$. Then we have

$$h(r)=\int_{-\infty}^\infty g(u)e^{iru}du, \qquad r\in \boldsymbol{C}.$$

Conversely,

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{-iru}dr$$

and

$$\varphi(t)=-\frac{1}{\pi}Q'(t).$$

In addition, if φ is smooth and of compact support, then $h(r)$ is a holomorphic function in the whole complex r-plane and especially for $r \in \boldsymbol{R}$: it is of rapid decay as $|r|\to\infty$ (cf. [11]).

§ 2. Decomposition of $\Gamma\omega$ into relative conjugacy classes

For an arbitrary element $\tau\omega$ in $\Gamma\omega$, we define $[\tau\omega]$ and $\Gamma_{\tau\omega}$ by

$$[\tau\omega]=\{\sigma\tau\omega\sigma^{-1}\,|\,\sigma \in \Gamma\} \text{ and } \Gamma_{\tau\omega}=\{\sigma \in \Gamma\,|\,\sigma\tau\omega\sigma^{-1}=\tau\omega\}.$$

Then, we call them the relative conjugacy class and the relative centralizer of $\tau\omega$ respectively. In this section, as a preparation of computing the integral $\int_{\mathscr{D}} K'^*(u, u)d\mu(u)$ explicitly, we will determine relative conjugacy classes in $\Gamma\omega$ and the relative centralizer for each representative of them precisely. Since $\sigma\tau\omega\sigma^{-1}=\sigma\tau\bar{\sigma}^{-1}\omega$ holds due to (1.2), it follows that

$$[\tau\omega]=\{\sigma\tau\bar{\sigma}^{-1}\,|\,\sigma \in \Gamma\}\omega \tag{2.1}$$

for an element τ in Γ. In view of this fact, two elements σ_0, σ_0' in $SL(2, \boldsymbol{C})$ but not in G are said to be relatively conjugate to each other when there exists a certain element T in $SL(2, \boldsymbol{C})$ such that $\sigma_0'=T\sigma_0\bar{T}^{-1}$. Then, it is found that every element σ_0 in $SL(2, \boldsymbol{C})$ is relatively conjugate to one of the following four types of elements:

relatively elliptic element $\Longleftrightarrow$ relatively conjugate to $\pm\begin{pmatrix} & -\bar{\varepsilon} \\ \varepsilon & \end{pmatrix}$ where $|\varepsilon|=1$, $\varepsilon\neq\pm1$, $\varepsilon \in \boldsymbol{C}$
$\Longleftrightarrow$ $\sigma_0\bar{\sigma}_0$ is an elliptic element in $SL(2, \boldsymbol{C})$,

relatively parabolic element $\Longleftrightarrow$ relatively conjugate to $\pm\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ where $x\neq 0$, $x \in \boldsymbol{R}$
$\Longleftrightarrow$ $\sigma_0\bar{\sigma}_0$ is a parabolic element in $SL(2, \boldsymbol{C})$,

relatively identical element $\Longleftrightarrow$ relatively conjugate to $\pm\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $\pm\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$
$\Longleftrightarrow$ $\sigma_0\bar{\sigma}_0$ is equal to $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $-\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ respectively,

relatively hyperbolic element $\Longleftrightarrow$ relatively conjugate to $\pm\begin{pmatrix}\mu & \\ & \mu^{-1}\end{pmatrix}$ where $\mu\neq\pm 1$, $\mu\in \boldsymbol{R}$
$\Longleftrightarrow \sigma_0\bar{\sigma}_0$ is a hyperbolic element in $SL(2, \boldsymbol{C})$.

If we put $\sigma_0=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, then we have

$$\sigma_0\bar{\sigma}_0=\begin{pmatrix}|a|^2+b\bar{c}, & a\bar{b}+b\bar{d}\\ c\bar{a}+\bar{c}d, & |d|^2+\bar{b}c\end{pmatrix}.$$

As is easily seen, the trace of $\sigma_0\bar{\sigma}_0$ is always real, so $\sigma_0\bar{\sigma}_0$ is not loxodromic for any σ_0 in $SL(2, \boldsymbol{C})$.

Now, for an arbitrary element τ_0 in $SL(2, \boldsymbol{Z}[i])$, we put

$$[[\tau_0]]'=\{\sigma_0\tau_0\bar{\sigma}_0^{-1}\,|\,\sigma_0\in SL(2, \boldsymbol{Z}[i])\},$$

which will be called the relative conjugacy class of τ_0 in $SL(2, \boldsymbol{Z}[i])$. Let furthemore π be a mapping from $SL(2, \boldsymbol{Z}[i])$ to Γ defined by $\pi(\tau_0)=\tau_0\,(\mathrm{mod}\ \pm 1)$ for every $\tau_0\in SL(2, \boldsymbol{Z}[i])$. Then, it is obvious that the equality $\tau\omega=\tau'\omega$ holds if and only if $\tau=\tau'$ for $\tau, \tau'\in\Gamma$. Combining this fact with (1.2), we have

$$[\pi(\tau_0)\omega]=[[\tau_0]]'/\{\pm 1\} \qquad \text{for } \tau_0\in SL(2, \boldsymbol{Z}[i]). \tag{2.2}$$

This means that it is enough to classify relative conjugacy classes $[[\tau_0]]'$ in $SL(2, \boldsymbol{Z}[i])$ to obtain relative conjugacy classes in $\Gamma\omega$.

For $\eta_0\in SL(2, \boldsymbol{Z}[i])$, let $[[\eta_0]]$ be an usual conjugacy class, i.e.,

$$[[\eta_0]]=\{\sigma_0\eta_0\sigma_0^{-1}\,|\,\sigma_0\in SL(2, \boldsymbol{Z}[i])\}.$$

Then, it is shown that, for $\tau_0\in SL(2, \boldsymbol{Z}[i])$, the mapping $a_0\to a_0\bar{a}_0$ for $a_0\in[[\tau_0]]'$ induces a surjection from $[[\tau_0]]'$ to $[[\tau_0\bar{\tau}_0]]$ and that $[[\tau_0\bar{\tau}_0]]\neq[[\tau_0'\bar{\tau}_0']]$ implies $[[\tau_0]]'\neq[[\tau_0']]'$ for $\tau_0, \tau_0'\in SL(2, \boldsymbol{Z}[i])$. However, we know that conjugacy classes in $SL(2, \boldsymbol{Z}[i])$ have already been settled for example by [7] in a satisfactory form. Therefore, what we must to carry out are firstly to obtain all τ_0 such that $\tau_0\bar{\tau}_0=\eta_0$ for a representative η_0 of each conjugacy class $[[\eta_0]]$ in $SL(2, \boldsymbol{Z}[i])$, and, secondly to distinguish them with respect to relative conjugation.

According to this plan, we will determine, from now on, relative conjugacy classes and their representatives in $SL(2, \boldsymbol{Z}[i])$ explicitly.

2.1. Relatively elliptic elements

If τ_0 is relatively elliptic, $\tau_0\bar{\tau}_0$ is an elliptic element by definition. We intend to determine the set consisting of all τ_0 such that $\tau_0\bar{\tau}_0=\eta_0$, where η_0 is a certain fixed representative of each conjugacy class $[[\eta_0]]$ of elliptic elements, and its decomposition into relative conjugacy classes. It is well known that every elliptic element in $SL(2, Z[i])$ has an order equal to 4, 3 or 6; we will treat separately the case of order 4 and the case of order 3 or 6.

In case of order 4, by Tanigawa [7: p. 233], we see that there are four conjugacy classes. They are represented by $\eta_{0,\lambda}=\begin{pmatrix} i & \lambda \\ 0 & -i \end{pmatrix}$, where $\lambda=0$, 1, $-i$ or $1-i$, and their centralizers are

$$(2.3) \qquad \{\pm 1, \pm\eta_{0,\lambda}\}$$

respectively. On the other hand, any element τ_0 in $SL(2, Z[i])$ which satisfies a somewhat general condition $\tau_0\bar{\tau}_0=\begin{pmatrix} i & * \\ 0 & -i \end{pmatrix}$ should have the form:

$$(2.4) \qquad \pm\begin{pmatrix} a, & -\dfrac{1+|a|^2}{2}+i\dfrac{1-|a|^2}{2} \\ 1+i, & -i\bar{a} \end{pmatrix} \qquad \text{for } a \in Z[i].$$

Considering these facts, it can be verified that the equality $\tau_0\bar{\tau}_0=\begin{pmatrix} i & \lambda \\ 0 & -i \end{pmatrix}$ for each λ as above is possible only if $\lambda=1-i$, and such τ_0 is given by $\begin{pmatrix} i & -1 \\ 1+i & -1 \end{pmatrix}$ or $-\begin{pmatrix} i & -1 \\ 1+i & -1 \end{pmatrix}$. Moreover these two matrices are not relatively conjugate to each other. Therefore, we obtain the following two relative conjugacy classes:

$$(2.5) \qquad \pm\left[\left[\begin{pmatrix} i & -1 \\ 1+i & -1 \end{pmatrix}\right]\right]'.$$

For the other case, i.e, for elliptic elements of order 3 or 6, again from [7: p. 234], we see that there are four conjugacy classes, whose representatives are given by $\pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\pm\begin{pmatrix} 0 & -i \\ -i & -1 \end{pmatrix}$. The centralizers of $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i \\ -i & -1 \end{pmatrix}$ coincide respectively with

$$(2.6) \qquad \left\langle\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2+i & -2i \\ 2i & 2-i \end{pmatrix}\right\rangle \text{ and } \left\langle\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 2+i & 2 \\ 2 & 2-i \end{pmatrix}\right\rangle.$$

Since $\tau_0\bar{\tau}_0=\bar{\tau}_0\tau_0$ holds in case of $\tau_0\bar{\tau}_0=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, such an element τ_0 has the property $\tau_0^{-1}\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}\tau_0=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$. This implies that τ_0 is on element in the centralizer of $\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$, and therefore is an element of the group $\left\langle\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2+i & -2i\\ 2i & 2-i\end{pmatrix}\right\rangle$ which comes from (2.6). If we put $\sigma_0=\begin{pmatrix}2+i & -2i\\ 2i & 2-i\end{pmatrix}$, then it follows from $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\sigma_0=\sigma_0\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$ that any element of the group $\left\langle\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \sigma_0\right\rangle$ can be expressed as $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^m\sigma_0^n$, $m, n \in \boldsymbol{Z}$, $0\leqq m\leqq 5$. Furthermore $\sigma_0=\bar{\sigma}_0^{-1}$ leads to

$$\left\{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^m\sigma_0^n\right\}\left\{\overline{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^m\sigma_0^n}\right\}=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^{2m}.$$

Thus, we see that the set of all τ_0 such that $\tau_0\bar{\tau}_0=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$ consists of the following elements:

$$\pm\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\sigma_0^n, \qquad n\in \boldsymbol{Z}. \tag{2.7}$$

Next, if we suppose that there exists an element γ_0 in $SL(2, \boldsymbol{Z}[i])$ such that $\tau_0'=\gamma_0\tau_0\bar{\gamma}_0^{-1}$ for two elements τ_0 and τ_0' satisfying $\tau_0\bar{\tau}_0=\tau_0'\bar{\tau}_0'=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$, then obviously γ_0 should be an element of $\left\langle\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \sigma_0\right\rangle$. However, in general, it is verified that

$$\left\{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^{m'}\sigma_0^{n'}\right\}\left\{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^{m}\sigma_0^{n}\right\}\left\{\overline{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^{m'}\sigma_0^{n'}}\right\}^{-1}=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^{m}\sigma_0^{n+2n'}$$

for $m, m', n, n' \in \boldsymbol{Z}$ and $0\leqq m, m'\leqq 5$. Hence, the set consisting of elements in (2.7) can be decomposed into four relative conjugacy classes, which are represented by $\pm\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$ and $\pm\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\sigma_0$. Namely we have the following four relative conjugacy classes:

$$\pm\left[\left[\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right]\right]', \quad \pm\left[\left[\begin{pmatrix}2-i & -2-i\\ 2+i & -2i\end{pmatrix}\right]\right]'. \tag{2.8}$$

For the case of $\tau_0\bar{\tau}_0=\begin{pmatrix}0 & -i\\ -i & -i\end{pmatrix}$, we know firstly

$$\left\{\tau_0\begin{pmatrix}i & \\ & -i\end{pmatrix}\right\}^{-1}\begin{pmatrix}0 & -i\\ -i & -1\end{pmatrix}\left\{\tau_0\begin{pmatrix}i & \\ & -i\end{pmatrix}\right\}=\begin{pmatrix}0 & -i\\ -i & -1\end{pmatrix}.$$

Thus, by (2.6), $\tau_0\begin{pmatrix} i & \\ & -i \end{pmatrix}$ becomes an element of $\left\langle \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 2+i & 2 \\ 2 & 2-i \end{pmatrix} \right\rangle$. Putting $\sigma_0 = \begin{pmatrix} 2+i & 2 \\ 2 & 2-i \end{pmatrix}$ and noting $\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}\sigma_0 = \sigma_0 \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}$, we see secondly that any element of $\left\langle \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \sigma_0 \right\rangle$ can also be expressed as $\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}^m \sigma_0^n$, $m, n \in \boldsymbol{Z}$, $0 \leqq m \leqq 5$. Then, the properties

$$\bar{\sigma}_0 = \begin{pmatrix} i & \\ & -i \end{pmatrix} \sigma_0^{-1} \begin{pmatrix} -i & \\ & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & \\ & -i \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} -i & \\ & i \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$$

yield

$$\left\{ \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}^m \sigma_0^n \begin{pmatrix} -i & \\ & i \end{pmatrix} \right\} \overline{\left\{ \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}^m \sigma_0^n \begin{pmatrix} -i & \\ & i \end{pmatrix} \right\}} = \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}^{2m}.$$

Therefore, the set of all τ_0 such that $\tau_0 \bar{\tau}_0 = \begin{pmatrix} 0 & -i \\ -i & -1 \end{pmatrix}$ consists of

$$(2.9) \qquad \pm \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix} \sigma_0^n \begin{pmatrix} -i & \\ & i \end{pmatrix}, \qquad n \in \boldsymbol{Z}.$$

By a similar consideration as in the preceding case, it is shown finally that the set of (2.9) can be represented by four elements $\pm \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} -i & \\ & i \end{pmatrix}$, $\pm \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}\sigma_0 \begin{pmatrix} -i & \\ & i \end{pmatrix}$ in the sense of relative conjugation. Consequently, we obtain the following four relative conjugacy classes:

$$(2.10) \qquad \pm \left[\left[\begin{pmatrix} -i & 1 \\ -1 & 0 \end{pmatrix} \right]\right]', \quad \pm \left[\left[\begin{pmatrix} -1-2i & 2+i \\ -2-i & 2 \end{pmatrix} \right]\right]'.$$

As for $\tau_0 \bar{\tau}_0 = -\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ or $-\begin{pmatrix} 0 & -i \\ -i & 1 \end{pmatrix}$, non-existence of such τ_0 follows easily.

2.2 Relatively parabolic elements

By [7: p. 233], all of the representatives of conjugacy classes are given by $\pm \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$, where β ranges over the set

$$\{\beta \in \boldsymbol{Z}[i] \mid \operatorname{Im}(\beta) > 0\} \cup \{\beta \in \boldsymbol{Z}[i] \mid \operatorname{Im}(\beta) = 0,\ \operatorname{Re}(\beta) > 0\}.$$

Their centralizers are all equal to

$$\left\{\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}\middle|\gamma\in \boldsymbol{Z}[i]\right\}. \tag{2.11}$$

For any β, there exists no element τ_0 such that $\tau_0\bar{\tau}_0=-\begin{pmatrix}1 & \beta\\ & 1\end{pmatrix}$. Moreover, for the existence of at least one element τ_0 such that $\tau_0\bar{\tau}_0=\begin{pmatrix}1 & \beta\\ & 1\end{pmatrix}$, it is necessary and sufficient that β has the form $2b_1$ or $2ib_1$ for every $b_1\in \boldsymbol{Z}_{\geqq 1}$. So, in case of $\beta=2b_1$, the set of all τ_0 such that $\tau_0\bar{\tau}_0=\begin{pmatrix}1 & 2b_1\\ & 1\end{pmatrix}$ is given by

$$\left\{\pm\begin{pmatrix}1 & b_1+ib_2\\ & 1\end{pmatrix}\middle|b_2\in \boldsymbol{Z}\right\}, \tag{2.12}$$

and for $\beta=2ib$, the set of all τ_0 such that $\tau_0\bar{\tau}_0=\begin{pmatrix}1 & 2ib_1\\ & 1\end{pmatrix}$ is equal to

$$\left\{\pm\begin{pmatrix}i & b_1+ib_2\\ & -i\end{pmatrix}\middle|b_2\in \boldsymbol{Z}\right\}. \tag{2.13}$$

For elements $\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}$ in (2.12), we have

$$\begin{aligned}\left\{\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}\right\}\begin{pmatrix}1 & b_1+ib_2\\ & 1\end{pmatrix}\left\{\overline{\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}}\right\}^{-1}&=\begin{pmatrix}1 & b_1+ib_2+(\gamma-\bar{\gamma})\\ & 1\end{pmatrix},\\ \left\{\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}\right\}\begin{pmatrix}i & b_1+ib_2\\ & -i\end{pmatrix}\left\{\overline{\pm\begin{pmatrix}1 & \gamma\\ & 1\end{pmatrix}}\right\}^{-1}&=\begin{pmatrix}i & b_1+ib_2-i(\gamma+\bar{\gamma})\\ & -i\end{pmatrix}.\end{aligned} \tag{2.14}$$

Therefore, for each fixed $b_1\in \mathbf{Z}_{\geqq 1}$, the elements of (2.12) can be represented by $\pm\begin{pmatrix}1 & b_1\\ & 1\end{pmatrix}$ and $\pm\begin{pmatrix}1 & b_1+i\\ & 1\end{pmatrix}$ in the sense of relative conjugation, and the corresponding result for (2.13) is that we have $\pm\begin{pmatrix}i & b_1\\ & -i\end{pmatrix}$ and $\pm\begin{pmatrix}i & b_1+i\\ & -i\end{pmatrix}$ as representatives. Hence, the full set of representatives of relatively parabolic elements can be given by

$$\left\{\pm\begin{pmatrix}1 & b_1\\ & 1\end{pmatrix},\ \pm\begin{pmatrix}1 & b_1+i\\ & 1\end{pmatrix},\ \pm\begin{pmatrix}i & b_1\\ & -i\end{pmatrix},\ \pm\begin{pmatrix}i & b_1+i\\ & -i\end{pmatrix}\middle|b_1\in \boldsymbol{Z}_{\geqq 1}\right\}. \tag{2.15}$$

2.3. Relatively identical elements

By definition, any relatively identical τ_0 satisfies the condition $\tau_0\bar{\tau}_0=\begin{pmatrix}1 & \\ & 1\end{pmatrix}$ or $-\begin{pmatrix}1 & \\ & 1\end{pmatrix}$.

In case of $\tau_0\bar{\tau}_0=\begin{pmatrix}1 & \\ & 1\end{pmatrix}$, all of such τ_0 must have the forms

(2.16) $$\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix}, \quad |a|^2+b_2c_2=1, \quad a \in \boldsymbol{Z}[i], \quad b_2, c_2 \in \boldsymbol{Z}.$$

In particular, they can be written as $\pm\begin{pmatrix} 1 & ib_2 \\ & 1 \end{pmatrix}$ or $\pm\begin{pmatrix} i & ib_2 \\ & -i \end{pmatrix}$ under the restriction of $c_2=0$. Now, if we consider the elements $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}$, $\begin{pmatrix} i & \\ & -i \end{pmatrix}$ and $\begin{pmatrix} i & i \\ & -i \end{pmatrix}$ in (2.16), a simple calculation shows that they are not relatively conjugate to each other and each of them is relatively conjugate to itself times -1. Actually we have

(2.17) $$\begin{aligned} \begin{pmatrix} i & \\ & -i \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\overline{\begin{pmatrix} i & \\ & -i \end{pmatrix}}^{-1} &= -\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\ \begin{pmatrix} i & 1 \\ & -i \end{pmatrix}\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}\overline{\begin{pmatrix} i & 1 \\ & -i \end{pmatrix}}^{-1} &= -\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}, \\ \begin{pmatrix} i & \\ & -i \end{pmatrix}\begin{pmatrix} i & \\ & -i \end{pmatrix}\overline{\begin{pmatrix} i & \\ & -i \end{pmatrix}}^{-1} &= -\begin{pmatrix} i & \\ & -i \end{pmatrix}, \\ \begin{pmatrix} i & i \\ & -i \end{pmatrix}\begin{pmatrix} i & i \\ & -i \end{pmatrix}\overline{\begin{pmatrix} i & i \\ & -i \end{pmatrix}}^{-1} &= -\begin{pmatrix} i & i \\ & -i \end{pmatrix}. \end{aligned}$$

Then, from the following lemma, it will be found that the number of relative conjugacy classes consisting of relatively identical elements is exactly equal to four, while the above arguments only mean at least four.

Lemma 2.1. *Any element τ_0 as in* (2.16) *is relatively conjugate to one of* $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}$, $\begin{pmatrix} i & \\ & -i \end{pmatrix}$ *and* $\begin{pmatrix} i & i \\ & -i \end{pmatrix}$.

Proof. If c_2 of τ_0 is equal to 0, the assertion readily follows from the form of τ_0 as in (2.16), (2.14) and (2.17).

In case of $c_2 \neq 0$, three important formulas are

(i) $$\begin{pmatrix} 1 & b_0 \\ & 1 \end{pmatrix}\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix}\overline{\begin{pmatrix} 1 & b_0 \\ & 1 \end{pmatrix}}^{-1} = \begin{pmatrix} a+ib_0c_2 & -ab_0-ib_0^2c_2+ib_2+\bar{a}b_0 \\ ic_2 & \bar{a}-ib_0c_2 \end{pmatrix},$$

(ii) $$\begin{pmatrix} 1 & ib_0 \\ & 1 \end{pmatrix}\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix}\overline{\begin{pmatrix} 1 & ib_0 \\ & 1 \end{pmatrix}}^{-1} = \begin{pmatrix} a-b_0c_2 & iab_0-ib_0^2c_2+ib_2+i\bar{a}b_0 \\ ic_2 & \bar{a}-b_0c_2 \end{pmatrix},$$

for an arbitrary $b_0 \in \boldsymbol{Z}$, and

(iii) $$\begin{pmatrix} & i \\ i & \end{pmatrix}\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix}\overline{\begin{pmatrix} & i \\ i & \end{pmatrix}}^{-1} = \begin{pmatrix} -\bar{a} & -ic_2 \\ -ib_2 & -a \end{pmatrix}.$$

If c_2 of τ_0 is equal to 1 or -1, all of such τ_0 can be reduced to $\begin{pmatrix} & i \\ i & \end{pmatrix}$

or $-\begin{pmatrix} & i \\ i & \end{pmatrix}$ respectively by use of (i) and (ii). However, we see

$$\begin{pmatrix}-i & \\ i & i\end{pmatrix}\left\{\pm\begin{pmatrix} & i \\ i & \end{pmatrix}\right\}\overline{\begin{pmatrix}-i & \\ i & i\end{pmatrix}}^{-1}=\pm\begin{pmatrix}i & i \\ & -i\end{pmatrix}.$$

Therefore, by (2.17), they all become relatively conjugate to $\begin{pmatrix}i & i \\ & -i\end{pmatrix}$.

Next, in case of $|c_2|>1$, suppose that, by means of (i) and (ii), τ_0 has been transformed into $\begin{pmatrix}a' & ib_2' \\ ic_2 & \bar{a}'\end{pmatrix}$ with $a'=a_1'+ia_2'$, a_1', $a_2'\in \boldsymbol{Z}$. Then, a suitable choice of b_0 as in (i) and (ii) makes

$$|a_1'|\leqq\frac{|c_2|}{2} \quad\text{and}\quad |a_2'|\leqq\frac{|c_2|}{2},$$

and moreover, because of $|c_2|>1$, it does not happen that both a_1' and a_2' are 0. Considering these facts, we see firstly

$$(*)\qquad a_1'^2+a_2'^2-1=|b_2'||c_2|,$$

which comes from $a_1'^2+a_2'^2-1=-b_2'c_2$. Secondly it is possible to put

$$|a_1'|=\frac{|c_2|}{2}-m,\ |a_2'|=\frac{|c_2|}{2}-n,\ m,\ n\in\frac{1}{2}\boldsymbol{Z}\ \text{and}\ 0\leqq m,\ n\leqq\frac{|c_2|}{2},$$

where at least one of m and n is strictly smaller than $|c_2|/2$ due to $|c_2|>1$. If $|a_1'|\geqq|a_2'|$, then we surely obtain

$$a_2'^2+a_2'^2-1\leqq 2\left(\frac{|c_2|}{2}-m\right)^2-1<\frac{(|c_2|-2m)^2}{2}.$$

Hence, it follows from $(*)$ that

$$(**)\qquad |b_2'|<\frac{(|c_2|-2m)}{2}=\frac{|c_2|}{2}-m,$$

and for $|a_2'|\geqq|a_1'|$, a similar consideration reaches

$$(**)'\qquad |b_2'|<\frac{|c_2|}{2}-n.$$

Now, if we transform $\begin{pmatrix}a' & ib_2' \\ ic_2 & \bar{a}'\end{pmatrix}$ into $\begin{pmatrix}-\bar{a}' & -ic_2 \\ -ib_2' & -a'\end{pmatrix}$ by (iii), then, by virtue of $(**)$ or $(**)'$, (i) and (ii) are again applicable to $\begin{pmatrix}-\bar{a}' & -ic_2 \\ -ib_2' & -a'\end{pmatrix}$ in order to find the new value b_2'' which satisfies $|b_2''|<|b_2'|/2$. Using in this way (i), (ii) and (iii) repeatedly, we can make the absolute value $|b_2|$ gradually smaller, where b_2 comes from (2.16) in connection with (i), (ii) and (iii), and after all, every $\tau_0=\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix}$ such that $|c_2|>1$ can be reduced to one of the following two forms:

$$\begin{pmatrix} a & ib_2 \\ ic_2 & \bar{a} \end{pmatrix} \longrightarrow \begin{pmatrix} a' & i \\ ic_2' & \bar{a}' \end{pmatrix} \text{ or } \begin{pmatrix} a' & -i \\ ic_2' & \bar{a}' \end{pmatrix}, \qquad \longrightarrow \begin{pmatrix} a' & 0 \\ ic_2' & \bar{a}' \end{pmatrix}.$$

In view of (iii), both $\begin{pmatrix} a' & i \\ ic_2' & \bar{a}' \end{pmatrix}$ and $\begin{pmatrix} a' & -i \\ ic_2' & \bar{a}' \end{pmatrix}$ are relatively conjugate to $\begin{pmatrix} i & i \\ & -i \end{pmatrix}$. For the second form, again by (iii), $\begin{pmatrix} a' & 0 \\ ic' & \bar{a}' \end{pmatrix}$ is relatively conjugate to the matrix whose c_2 is equal to 0. If $a'=\pm 1$, $\begin{pmatrix} a' & 0 \\ ic_2' & \bar{a}' \end{pmatrix}$ is relatively conjugate to $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}$ according as c_2' is even or odd, and similarly, if $a'=\pm i$, relatively conjugate to $\begin{pmatrix} i & \\ & -i \end{pmatrix}$ or $\begin{pmatrix} i & i \\ & -i \end{pmatrix}$ according to the value of c_2' being even or odd. This completes the proof.

As a result of the above arguments, the set of all τ_0 such that $\tau_0\bar{\tau}_0=\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ is decomposed into the following four relative conjugacy classes:

$$\text{(2.18)} \qquad \left[\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right]\right]', \quad \left[\left[\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}\right]\right]', \quad \left[\left[\begin{pmatrix} i & \\ & -i \end{pmatrix}\right]\right]', \quad \left[\left[\begin{pmatrix} i & i \\ & -i \end{pmatrix}\right]\right]'.$$

In case of $\tau_0\bar{\tau}_0=-\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, all of such τ_0 must have the forms:

$$\text{(2.19)} \qquad \begin{pmatrix} a & b_1 \\ c_1 & -\bar{a} \end{pmatrix}, \quad -|a|^2-b_1c_1=1, \quad a \in Z[i], \ b_1, \ c_1 \in Z.$$

By an analogous process to the proof of Lemma 2.1, all matrices as in (2.19) can be reduced to $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ or $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, and they are not relatively conjugate to each other. Thus we obtain the following two relative conjugacy classes:

$$\text{(2.20)} \qquad \pm\left[\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right]\right]'.$$

We can not discuss relatively hyperbolic elements in detail because of the difficulty in obtaining the representatives of hyperbolic elements in $SL(2, Z[i])$ explicitly (cf. [4], [7]). But it seems to be of some importance to observe, for example, whether or not the mapping $\tau_0 \to \tau_0\bar{\tau}_0$ from relatively hyperbolic elements τ_0 to conjugacy classes consisting of hyperbolic elements with a certain fixed trace t, which at least one element $\tau_0\bar{\tau}_0$ can take, is surjective.

Now, an element $\tau\omega$ in $\Gamma\omega$ will be called by the same name according as the element $\pi^{-1}(\tau)$ in $SL(2, Z[i])$ is relatively elliptic, parabolic, identical or hyperbolic. As to the relative centralizer, its definition leads to

$$(2.21)\qquad \begin{aligned}\Gamma_{\tau\omega}&=\{\sigma_0 \in SL(2, Z[i]) \mid \sigma_0\tau_0\bar{\sigma}_0^{-1}=\tau_0\}\\ &\cup\{\sigma_0 \in SL(2, Z[i]) \mid \sigma_0\tau_0\bar{\sigma}_0^{-1}=-\tau_0\}/\{\pm 1\}\end{aligned}$$

for $\pi(\tau_0)=\tau$. Then, it is easily seen that the inverse image of the right hand side of (2.21) under the mapping π is included in the centralizer of $\tau_0\bar{\tau}_0$, thus we have

$$(2.22)\qquad \Gamma_{\tau\omega}\subset\{\sigma_0 \in SL(2, Z[i]) \mid \sigma_0\tau_0\bar{\tau}_0\sigma_0^{-1}=\tau_0\bar{\tau}_0\}/\{\pm 1\}$$

for $\pi(\tau_0)=\tau$. By these results and (2.2), the decomposition of $\Gamma\omega$ into relative conjugacy classes and their relative centralizers are now given as follows.

Relatively elliptic elements: there exist five relative conjugacy classes owing to (2.5), (2.8) and (2.10), which are expressed as

$$(2.23)\qquad \begin{aligned}&\left[\begin{pmatrix} i & -1\\ 1+i & -1\end{pmatrix}\omega\right],\ \left[\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}\omega\right],\ \left[\begin{pmatrix} -2+i & 2+i\\ -2-i & 2i\end{pmatrix}\omega\right]\\ &\left[\begin{pmatrix} i & -1\\ 1 & 0\end{pmatrix}\omega\right],\ \left[\begin{pmatrix} -1-2i & 2+i\\ -2-i & 2\end{pmatrix}\omega\right].\end{aligned}$$

Moreover, (2.5), (2.8) and (2.10) mean that their relative centralizers coincide respectively with

$$(2.24)\qquad \Gamma_{\tau\omega}=\left\{id,\ \begin{pmatrix} i & 1-i\\ 0 & -i\end{pmatrix}\right\}\qquad \text{for } \tau=\begin{pmatrix} i & -1\\ 1+i & -1\end{pmatrix},$$

$$(2.25)\qquad \Gamma_{\tau\omega}=\left\langle\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}\right\rangle\qquad \text{for } \tau=\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix} \text{ or } \begin{pmatrix} -2+i & 2+i\\ -2-i & 2i\end{pmatrix}$$

and

$$(2.26)\qquad \Gamma_{\tau\omega}=\left\langle\begin{pmatrix} 1 & -i\\ -i & 1\end{pmatrix}\right\rangle=\begin{pmatrix} \sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}\left\langle\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}\right\rangle\begin{pmatrix} \sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}^{-1}$$

for $\tau=\begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1-2i & 2+i \\ -2-i & 2 \end{pmatrix}$.

Relatively parabolic elements: in view of (2.15), the full set of representatives are written as

$$\left\{\begin{pmatrix} 1 & b_1 \\ & 1 \end{pmatrix}\omega,\ \begin{pmatrix} 1 & b_1+i \\ & 1 \end{pmatrix}\omega,\ \begin{pmatrix} i & b_1 \\ & -1 \end{pmatrix}\omega,\ \begin{pmatrix} i & b+i \\ & -i \end{pmatrix}\omega \,\middle|\, b_1 \in Z_{\geqq 1}\right\}, \tag{2.27}$$

and the relative centralizer of each element in (2.27) is given by

$$\Gamma_{\tau\omega}=\left\{\begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \,\middle|\, \beta \in Z\right\} \quad \text{for } \tau=\begin{pmatrix} 1 & b_1 \\ & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & b_1+i \\ & 1 \end{pmatrix} \tag{2.28}$$

or

$$\Gamma_{\tau\omega}=\left\{\begin{pmatrix} 1 & i\beta \\ & 1 \end{pmatrix} \,\middle|\, \beta \in Z\right\} \quad \text{for } \tau=\begin{pmatrix} i & b_1 \\ & -i \end{pmatrix} \text{ or } \begin{pmatrix} i & b_1+i \\ & -i \end{pmatrix}. \tag{2.29}$$

Here, throughout both cases in the above, all matrices are identified with elements in G.

Relatively identical elements: five relative conjugacy classes are deduced from (2.18) and (2.20), which are expressed as

$$\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega\right],\ \left[\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}\omega\right],\ \left[\begin{pmatrix} i & \\ & -i \end{pmatrix}\omega\right],\ \left[\begin{pmatrix} i & i \\ & -i \end{pmatrix}\omega\right],\ \left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\omega\right], \tag{2.30}$$

and (2.21) shows that their relative centralizers are respectively equal to

$$\Gamma_{\tau\omega}= SL(2, Z) \cup \left\{\begin{pmatrix} ia_2 & ib_2 \\ ic_2 & id_2 \end{pmatrix} \in SL(2, Z[i])\right\} \Big/ \{\pm 1\} \tag{2.31}$$

for $\tau=\pi\left(\pm\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)$,

$$\Gamma_{\tau\omega}=\left\{\begin{bmatrix} a_1+i\dfrac{c_1}{2}, & b_1+i\dfrac{d_1-a_1}{2} \\ c_1 & , d_1-i\dfrac{c_1}{2} \end{bmatrix}, \begin{bmatrix} -\dfrac{c_2}{2}+ia_2, & \dfrac{a_2-d_2}{2}+ib_2 \\ ic_2 & , \dfrac{c_2}{2}+id_2 \end{bmatrix} \in SL(2, Z[i]) \,\middle|\, \begin{matrix} a_1, b_1, c_1, d_1 \\ a_2, b_2, c_2, d_2 \end{matrix} \in Z\right\} \Big/ \{\pm 1\} \tag{2.32}$$

for $\tau=\pi\left(\pm\begin{pmatrix}1 & i\\ & 1\end{pmatrix}\right)$,

$$\text{(2.33)}\qquad \begin{aligned}\Gamma_{\tau_1\omega}&=\begin{pmatrix}\sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}\Gamma_{\tau_2\omega}\begin{pmatrix}\sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}^{-1}\\ \Gamma_{\tau_3\omega}&=\begin{pmatrix}\sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}\Gamma_{\tau_4\omega}\begin{pmatrix}\sqrt{i} & \\ & \sqrt{\bar{i}}\end{pmatrix}^{-1}\end{aligned}$$

for $\tau_1=\begin{pmatrix}i & \\ & -i\end{pmatrix}$, $\tau_2=\begin{pmatrix}1 & \\ & 1\end{pmatrix}$, $\tau_3=\begin{pmatrix}i & i\\ & -i\end{pmatrix}$ and $\tau_4=\begin{pmatrix}1 & i\\ & 1\end{pmatrix}$, and

$$\text{(2.34)}\qquad \Gamma_{\tau\omega}=\left\{id,\ \begin{pmatrix}i & \\ & -i\end{pmatrix},\ \begin{pmatrix} & 1\\ -1 & \end{pmatrix},\ \begin{pmatrix} & i\\ i & \end{pmatrix}\right\}$$

for $\tau=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, where all matrices except for (2.31) and (2.32) are also regarded as elements in G.

It seems to be impossible to determine relative conjugacy classes of relatively hyperbolic elements explicitly by the same reason as in the case of relatively hyperbolic elements in $SL(2, \boldsymbol{Z}[i])$. But, as will be introduced later on, it is possible to define the notion of the '*primitive*' element, which is necessary to compute the trace formula.

§ 3. Trace formulas for V_e and V_o

Trace formulas for V_e and V_o have been stated by (1.10), (1.11) and (1.13) in rough forms. Moreover one knows that it is sufficient to compute the integral

$$\int_{\mathscr{D}} K'^*(u, u)d\mu(u) \text{ with } K'^*(u, u)=\sum_{\sigma\in\Gamma} k(u, \sigma\omega u)-H(u, u)$$

in order to obtain them in explicit forms. We are now in the position to carry out this evaluation. Throughout this section, we suppose that all matrices are identified with elements in G.

Let $[\tau\omega]$ and $\Gamma'_{\tau\omega}$ be the relative conjugacy class and the relative centralizer of $\tau\omega$ respectively. It has been shown that the only set of relatively parabolic elements contributes the continuous spectrum. Therefore, the above integral is naturally decomposed into the sum of components $C(\tau\omega)$ corresponding to relative conjugacy classes of relatively elliptic, identical and hyperbolic elements and a modified component $C(\infty\omega)$ related to relatively parabolic elements. Namely, we have

$$\text{(3.1)}\qquad \int_{\mathscr{D}} K'^*(u, u)d\mu(u)=\sum_{[r.e]} C(\tau\omega)+\sum_{[r.i]} C(\tau\omega)+\sum_{[r.h]} C(\tau\omega)+C(\infty\omega),$$

where

(3.2) $$C(\tau\omega)=\int_{\mathscr{D}_{\tau\omega}} k(u, \tau\omega u)d\mu(u) \qquad \text{for } \mathscr{D}_{\tau\omega}=\Gamma_{\tau\omega}\backslash H,$$

(3.3) $$C(\infty\omega)=\lim_{V\to\infty}\Big[\sum_{[r.p]}\int_{\mathscr{D}^V}\sum_{\sigma} k(\sigma u, \tau\omega\sigma u)d\mu(u)-\int_{\mathscr{D}^V} H(u, u)d\mu(u)\Big]$$

(σ in $\sum_\sigma$ runs over all the representatives of $\Gamma_{\tau\omega}\backslash\Gamma$), $\mathscr{D}^V=\{(z, v)\in\mathscr{D}\,|\,v\leqq V\}$ for $V\gg 0$, and $r.e$, $r.i$, $r.h$ and $r.p$ denote relatively elliptic, identical, hyperbolic and parabolic elements respectively.

From now on, we will bring all of components appearing in (3.1) into their satisfactorily precise expressions.

3.1. Contribution from relatively elliptic elements

Let α_0 be a complex number such that $|\alpha_0|=1$, and put $a_0=-\alpha_0^{-1}\bar{\alpha}_0(=-\bar{\alpha}_0^2)$. As a preliminary step, we consider firstly the integral $\int_H k\left(u, \begin{pmatrix} & -\alpha_0^{-1}\\ \alpha_0 & \end{pmatrix}\omega u\right)d\mu(u)$ under the assumption $\mathrm{Re}\,(a_0)\leqq 0$. By

$$\begin{pmatrix} & -\alpha_0^{-1}\\ \alpha_0 & \end{pmatrix}\omega(z, v)=\left(\frac{a_0 z}{|z|^2+v^2}, \frac{v}{|z|^2+v^2}\right),$$

it follows from (1.8) that

$$\int_H k\left(u, \begin{pmatrix} & -\alpha_0^{-1}\\ \alpha_0 & \end{pmatrix}\omega u\right)d\mu(u)$$
$$=\int_H \varphi\left(\frac{|z|^4-(a_0+\bar{a}_0)|z|^2+1}{v^2}+2(|z|^2-1)+v^2\right)d\mu(u).$$

After a change of variable z to polar coordinates, the right hand side of the above equation coincides with

$$2\pi\int_0^\infty\int_0^\infty \varphi\left(\frac{r^4-(a_0+\bar{a}_0)r^2+1}{v^2}+2(r^2-1)+v^2\right)r\,dr\frac{dv}{v^3}.$$

If we substitute

$$t=\frac{r^4-(a_0+\bar{a}_0)r^2+1}{v^2}+2(r^2-1)+v^2 \quad \text{for } r,$$

analogous calculations to [3: p. 100] and the assumption $\mathrm{Re}\,(a_0)\leqq 0$ show that the last integral is equal to

(3.4) $$\frac{\pi}{2}\int_0^\infty \varphi(t)\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{f(t, v)}}dv\,dt,$$

where

$$f(t, v)=t+2-(a_0+\bar{a}_0)-\frac{4-(a_0+\bar{a}_0)^2}{4}v^2 \quad \text{and} \quad \alpha=(\sqrt{t}+\sqrt{t+4})/2$$

which is the positive root of $t=(\alpha-1/\alpha)^2$ satisfying $\alpha>1$ for any fixed $t>0$. Furthermore, replacing $\varphi(t)$ by $-(1/\pi)Q'(t)$ in Proposition 1.2, we have by partial integration

$$\text{(3.5)} \quad \begin{aligned}&\int_H k\left(u, \begin{pmatrix} & -\alpha_0^{-1} \\ \alpha_0 & \end{pmatrix}\omega u\right)d\mu(u) \\ &= -\frac{1}{2}\left[Q(t)\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{f(t, v)}}dv\Big|_0^{\infty}-\int_0^{\infty}Q(t)d\left(\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{f(t, v)}}dv\right)\right]\end{aligned}$$

where $a_0=-\alpha_0^{-1}\bar{\alpha}_0$, $\mathrm{Re}\,(a_0)\leqq 0$, $\alpha=(\sqrt{t}+\sqrt{t+4})/2$ and $f(t, v)=t+2-(a_0+\bar{a}_0)-\dfrac{4-(a_0+\bar{a}_0)^2}{4}v^2$.

Next, we propose to apply (3.5) to each component $C(\tau\omega)$ represented by an element of (2.23) to express it in terms of $h(r)$. As for $\tau\omega=\begin{pmatrix} i & -1 \\ 1+i & -1\end{pmatrix}\omega$, its relative centralizer is of order 2 owing to (2.24), so it follows that

$$C\left(\begin{pmatrix} i & -1 \\ 1+i & -1\end{pmatrix}\omega\right)=\frac{1}{2}\int_H k\left(u, \begin{pmatrix} i & -1 \\ 1+i & -1\end{pmatrix}\omega u\right)d\mu(u).$$

If we take

$$T=\begin{pmatrix} 2^{1/4} & \\ & 2^{-1/4}\end{pmatrix}\begin{pmatrix} 1 & -(1+i)/2 \\ & 1\end{pmatrix},$$

then

$$T\begin{pmatrix} i & -1 \\ 1+i & -1\end{pmatrix}T^{-1}=\frac{\sqrt{2}}{2}\begin{pmatrix} & -1+i \\ 1+i & \end{pmatrix}$$

holds. Thus, using (3.5) with $\alpha_0=(\sqrt{2}/2)(1+i)$, i.e., $a_0=i$, the expression for $C\left(\begin{pmatrix} i & -1 \\ 1+i & -1\end{pmatrix}\omega\right)$ is transformed into

$$-\frac{1}{4}\left[Q(t)\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{t+2-v^2}}dv\Big|_0^{\infty}-\int_0^{\infty}Q(t)d\left(\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{t+2-v^2}}dv\right)\right].$$

On the other hand, by $\alpha=(\sqrt{t}+\sqrt{t+4})/2$, one has

$$\int_{1/\alpha}^{\alpha} \frac{1}{\sqrt{t+2-v^2}}dv$$

$$= \arcsin \frac{\sqrt{t}+\sqrt{t+4}}{2\sqrt{t+2}} - \arcsin \frac{2}{(\sqrt{t}+\sqrt{t+4})\sqrt{t+2}}.$$

Therefore, substituting $g(u)$ for $Q(t)$ by Proposition 1.2, the component in question turns out to be equal to

$$\frac{1}{2}\int_0^{\infty} g(u)\frac{1}{e^u+e^{-u}}du.$$

Further calculations after this yield

$$(3.6)\qquad \begin{aligned} &C\left(\begin{pmatrix} i & -1 \\ 1+i & -1 \end{pmatrix}\omega\right) \\ &= \frac{1}{16\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+\frac{ir}{4}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{4}\right)\right\}h(r)dr, \end{aligned}$$

where $(\Gamma'/\Gamma)(s)$ is a logarithmic derivative of the gamma function $\Gamma(s)$.

For the remaining elements of (2.23), we see firstly

$$\eta^{-1}\begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}\bar{\eta} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and}$$

$$\eta^{-1}\begin{pmatrix} -1-2i & 2+i \\ -2-i & 2 \end{pmatrix}\bar{\eta} = \begin{pmatrix} -2+i & 2+i \\ -2-i & 2i \end{pmatrix} \quad \text{for } \eta = \begin{pmatrix} \sqrt{i} & \\ & \sqrt{\bar{i}} \end{pmatrix}.$$

Thus, it follows from (2.26) that

$$(3.7)\qquad \begin{aligned} &C\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\omega\right) = C\left(\begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}\omega\right), \\ &C\left(\begin{pmatrix} -2+i & 2+i \\ -2-i & 2i \end{pmatrix}\omega\right) = C\left(\begin{pmatrix} -1-2i & 2+i \\ -2-i & 2 \end{pmatrix}\omega\right). \end{aligned}$$

This implies that it is enough to calculate left hand sides of (3.7) respectively. For the component $C\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\omega\right)$, putting

$$T = \sqrt{\frac{1}{\sqrt{3}}}\sqrt{i}\begin{bmatrix} 1 & -\frac{1}{2}(1+\sqrt{3}i) \\ 1 & -\frac{1}{2}(1-\sqrt{3}i) \end{bmatrix},$$

one has

$$T\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}T^{-1}=\frac{1}{2}\begin{pmatrix} & -\sqrt{3}-i\\ \sqrt{3}-i & \end{pmatrix}.$$

By this property, (2.25) and (3.5) with $a_0=-(1/2)(1+\sqrt{3}i)$, the component $C\left(\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\omega\right)$ is expressed as

$$-\frac{1}{3\sqrt{3}}\left[Q(t)\int_{\sqrt{3}/2\alpha}^{\sqrt{3}\alpha/2}\frac{1}{\sqrt{t+3-v^2}}dv\Big|_0^\infty-\int_0^\infty Q(t)d\left(\int_{\sqrt{3}/2v}^{\sqrt{3}\alpha/2}\frac{1}{\sqrt{t+3-v^2}}dv\right)\right].$$

Similarly to the preceding case, the above formula is reduced to

$$\frac{1}{3}\int_0^\infty g(u)\frac{1}{(e^{u/2}-e^{-u/2})^2+3}du.$$

Here, substituting the Fourier integral of $h(r)$ for $g(u)$ by use of Proposition 1.2 and calculating the residues of the integrand arising from this substitution, we obtain the following formula:

$$C\left(\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\omega\right)=\frac{1}{6\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{-(2\pi/3)r}-e^{-(4\pi/3)r}}{1-e^{-2\pi r}}h(r)dr. \tag{3.8}$$

As for the component $C\left(\begin{pmatrix}-2+i & 2+i\\ -2-i & 2i\end{pmatrix}\omega\right)$, one has

$$T\begin{pmatrix}-2+i & 2+i\\ -2-i & 2i\end{pmatrix}T^{-1}=\frac{1}{2}\begin{pmatrix} & \sqrt{3}+i\\ -\sqrt{3}+i & \end{pmatrix}$$

with

$$T=\sqrt{\frac{1}{\sqrt{3}}}\sqrt{i}\begin{pmatrix}\sqrt{2-\sqrt{3}} & \\ & \sqrt{2-\sqrt{3}}^{-1}\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2}(1+\sqrt{3}i)\\ 1 & -\frac{1}{2}(1+\sqrt{3}i)\end{pmatrix}.$$

Considering this fact, it follows from (2.25) and (3.5) that

$$C\left(\begin{pmatrix}-2+i & 2+i\\ -2-i & 2i\end{pmatrix}\omega\right)=C\left(\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\omega\right). \tag{3.9}$$

Collecting (3.6) through (3.9), the contribution from relatively elliptic elements is now given by

$$\begin{aligned}\sum_{[r.e]}C(\tau\omega)&=\frac{1}{16\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+\frac{ir}{4}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{4}\right)\right\}h(r)dr\\ &\quad+\frac{2}{3\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{-(2\pi/3)r}-e^{-(4\pi/3)r}}{1-e^{-2\pi r}}h(r)dr.\end{aligned} \tag{3.10}$$

3.2. Contribution from relatively identical elements

In the beginning, we restrict our investigation to the components represented by elements of (2.30) except for $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\omega$. It is easily seen that

$$\eta^{-1}\begin{pmatrix} i & \\ & -i \end{pmatrix}\bar{\eta}=\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \text{ and } \eta^{-1}\begin{pmatrix} i & i \\ & -i \end{pmatrix}\bar{\eta}=\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix} \text{ for } \eta=\begin{pmatrix} \sqrt{i} & \\ & \sqrt{i} \end{pmatrix}.$$

Thus, it follows from (2.33) that

$$(3.11)\quad C\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega\right)=C\left(\begin{pmatrix} i & \\ & -i \end{pmatrix}\omega\right),\quad C\left(\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}\omega\right)=C\left(\begin{pmatrix} i & i \\ & -i \end{pmatrix}\omega\right).$$

Namely, it is sufficient to calculate the components $C\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega\right)$ and $C\left(\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}\omega\right)$. In case of $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega$, from (2.31), the fundamental domain $\mathscr{D}_{\tau\omega}=\Gamma_{\tau\omega}\backslash H$ for $\tau\omega=\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega$ can be given, for instance, by

$$\left\{(z, v)\,|\, z=x+iy,\ -\frac{1}{2}\leqq x\leqq\frac{1}{2},\ y\geqq -x,\ x^2+y^2+v^2\geqq 1\right\}.$$

Hence, we have

$$(*)\quad \begin{aligned} C\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\omega\right)&=\int_0^\infty\int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^\infty \varphi\left(\frac{4y^2}{v^2}\right)dy\,dx\frac{dv}{v^3} \\ &\quad+\frac{1}{2}\int_{-1/2}^{1/2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{a_0}^\infty \varphi\left(\frac{4y^2}{v^2}\right)\frac{dv}{v^3}dydx, \end{aligned}$$

where $a_0=\sqrt{1-(x^2+y^2)}$. For the first term on the right hand side of $(*)$, a straight-forward calculation after the substitution $t=4y^2/v^2$ with respect to y shows that

$$\begin{aligned} \int_0^\infty\int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^\infty \varphi\left(\frac{4y^2}{v^2}\right)dy\,dx\frac{dv}{v^3}&=\frac{1}{2}\int_0^\infty\int_0^{1/2}\int_{b_0}^\infty \frac{\varphi(t)}{\sqrt{t}}dtdx\frac{dv}{v^2} \\ &=\frac{1}{2}\int_0^\infty\int_{4/v^2}^\infty \frac{\varphi(t)}{\sqrt{t}}\int_0^{1/2}dxdt\frac{dv}{v^2}+\frac{1}{2}\int_0^\infty\int_{3/v^2}^{4/v^2}\frac{\varphi(t)}{\sqrt{t}}\int_{c_0}^{1/2}dxdt\frac{dv}{v^2}, \end{aligned}$$

where $b_0=4(1-x^2)/v^2$ and $c_0=\sqrt{4-v^2t}/2$. In view of Proposition 1.2, the last formula is reduced to

$$\frac{1}{8\pi}g(0)+\left(\frac{1}{24}g(0)-\frac{1}{8\pi}g(0)\right)=\frac{1}{24}g(0).$$

For the second term on the right hand side of (*), again making the change of variable v to $t=4y^2/v^2$ and using Proposition 1.2, it is found that

$$\frac{1}{2}\int_{-1/2}^{1/2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{a_0}^{\infty}\varphi\left(\frac{4y^2}{v^2}\right)\frac{dv}{v^3}dydx=\frac{1}{4}\int_0^{1/2}\int_0^{\sqrt{1-x^2}}\int_0^{d_0}\varphi(t)dt\frac{dy}{y^2}dx$$

$$=\frac{1}{4}\int_0^{1/2}\int_0^{\infty}\varphi(t)\int_{e_0}^{\sqrt{1-x^2}}\frac{dy}{y^2}dtdx=-\frac{1}{24}g(0)+\frac{\pi}{24}\int_0^{\infty}\varphi(t)\frac{\sqrt{t+4}}{\sqrt{t}}dt,$$

where $a_0=\sqrt{1-(x^2+y^2)}$, $d_0=4y^2/(1-x^2-y^2)$ and $e_0=\sqrt{t(1-x^2)}/\sqrt{t+4}$. Hence, we have

$$C\left(\begin{pmatrix}1 & \\ & 1\end{pmatrix}\omega\right)=\frac{\pi}{24}\int_0^{\infty}\varphi(t)\frac{\sqrt{t+4}}{\sqrt{t}}dt. \tag{3.12}$$

In case of $\begin{pmatrix}1 & i\\ & 1\end{pmatrix}\omega$, by (2.32), the fundamental domain $\mathcal{D}_{\tau\omega}$ for $\tau\omega=\begin{pmatrix}1 & i\\ & 1\end{pmatrix}\omega$ may be chosen, for example, as

$$\left\{(z, v)\,|\,z=x+iy,\ -\frac{1}{2}\leqq x\leqq\frac{1}{2},\ y\geqq\frac{1}{2},\ x^2+\left(y-\frac{1}{2}\right)^2+v^2\geqq\frac{1}{4}\right\}.$$

Thus, a similar calculation as in the preceding case yields

$$C\left(\begin{pmatrix}1 & i\\ & 1\end{pmatrix}\omega\right)=\frac{\pi}{8}\int_0^{\infty}\varphi(t)\frac{\sqrt{t+4}}{\sqrt{t}}dt. \tag{3.13}$$

For the remaining component represented by $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\omega$ of (2.30), (2.34) shows that its relative centralizer is of order 4. Thus we have

$$C\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\omega\right)=\frac{1}{4}\int_H k\left(u, \begin{pmatrix} & 1\\ -1 & \end{pmatrix}\omega u\right)d\mu(u).$$

By means of (3.4) with $a_0=-1$, the above integral is transformed into

$$\frac{\pi}{8}\int_0^{\infty}\varphi(t)\int_{1/\alpha}^{\alpha}\frac{1}{\sqrt{t+4}}dvdt,$$

where $\alpha=(\sqrt{t}+\sqrt{t+4})/2$. Hence, the property $\alpha-1/\alpha=t$ yields

$$C\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\omega\right)=\frac{\pi}{8}\int_0^{\infty}\varphi(t)\frac{\sqrt{t}}{\sqrt{t+4}}dt. \tag{3.14}$$

Since $\varphi(t)$ is of compact support, it is obvious that all the integrals on right hand sides of (3.12), (3.13) and (3.14) are absolutely convergent. They are expressed in terms of $h(r)$ as follows:

$$(3.15)\quad \begin{aligned} &\int_0^\infty \varphi(t)\frac{\sqrt{t+4}}{\sqrt{t}}dt=\frac{1}{\pi}g(0)-\frac{i}{\pi^2}\int_{-\infty}^\infty \left\{\frac{\Gamma'}{\Gamma}\left(1+ir\right)\right\}rh(r)dr,\\ &\int_0^\infty \varphi(t)\frac{\sqrt{t}}{\sqrt{t+4}}dt=\frac{1}{\pi}g(0)-\frac{i}{2\pi^2}\int_{-\infty}^\infty \left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)\right.\\ &\qquad\qquad\qquad\qquad\left.-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right\}rh(r)dr,\end{aligned}$$

where $g(0)=1/2\pi\int_{-\infty}^\infty h(r)dr$ by Proposition 1.2. Thus the expression in terms of $g(0)$ is identical with that of $h(r)$.

Consequently, combining (3.11) through (3.15), the contribution from relatively identical elements is given by

$$(3.16)\quad \begin{aligned}\sum_{[r.i]} C(\tau\omega)=&\frac{11}{24}g(0)-\frac{1}{3\pi}\int_{-\infty}^\infty \left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)\right\}irh(r)dr\\ &-\frac{1}{16\pi}\int_{-\infty}^\infty \left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right\}irh(r)dr.\end{aligned}$$

3.3. Contribution from relatively hyperbolic elements

By an easy consideration, it is shown that the relative centralizer $\Gamma_{\tau\omega}$ of every relatively hyperbolic element $\tau\omega$ turns out to be an infinite cyclic group generated by a hyperbolic element $P_0\in\Gamma$ which is not a power of any element in $\Gamma_{\tau\omega}$ except $P_0^{\pm1}$. Snch an element P_0 will be called a '*primitive*' hyperbolic element '*related to* $\Gamma\omega$'. On the other hand, one knows that a suitable choice of $T\in G$ enables us to have $T\tau\overline{T}^{-1}=\begin{pmatrix}\mu & \\ & \mu^{-1}\end{pmatrix}$ with $|\mu|>1$. If we suppose that $TP_0T^{-1}=\begin{pmatrix}\rho & \\ & \rho^{-1}\end{pmatrix}$ with $|\rho|>1$ for such an element T, then the generator P_0 of $\Gamma_{\tau\omega}$ may be uniquely determined. The number ρ^2 will be called the '*norm*' of P_0 and will be denoted by NP_0. Since $(\tau\omega)^2$ or τ itself is contained in the group $\Gamma_{\tau\omega}$ according to $\tau\omega\neq\omega\tau$ or $\tau\omega=\omega\tau$, there exists a positive integer l which satisfies $\rho^{l/2}=\mu$ or $\rho^l=\mu$, respectively. For convenience of computation, define the positive number l_0 by

$$l_0=\begin{cases}\dfrac{l}{2} & \text{for } \tau\omega\neq\omega\tau,\\ l & \text{for } \tau\omega=\omega\tau.\end{cases}$$

Then, by a straight-forward calculation using

$$\begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix} \omega(z, v) = (NP_0^{l_0} z, NP_0^{l_0} v),$$

we readily obtain the following equation:

$$(3.17) \qquad C(\tau\omega) = \frac{\log NP_0}{NP_0^{l_0} - NP_0^{-l_0}} g(l_0 \log NP_0).$$

Here, it should be noted that, in general, a '*primitive*' hyperbolic element '*related to $\Gamma\omega$*' does not coincide with a '*primitive*' hyperbolic element of Γ in the sense of, for example, [7: p. 234].

3.4. Contribution from relatively parabolic elements

The main purpose in this subsection is to investigate the asymptotic behavior of

$$\sum_{[r.p]} \int_{\mathscr{D}^V} \sum_{\sigma} k(\sigma u, \tau\omega\sigma u) d\mu(u),$$

where σ runs over all representatives of $\Gamma_{\tau\omega} \backslash \Gamma$ in (3.3) as $V \to \infty$. In view of (2.27), the term in question can be written as

$$(3.18) \qquad \sum_{\tau\omega} \sum_{b_1 \geqq 1} \int_{\mathscr{D}^V} \sum_{\sigma} k(\sigma u, \tau\omega\sigma u) d\mu(u),$$

where $\tau\omega$ denotes $\begin{pmatrix} 1 & b_1 \\ & 1 \end{pmatrix}\omega$, $\begin{pmatrix} 1 & b_1+i \\ & 1 \end{pmatrix}\omega$, $\begin{pmatrix} i & b_1 \\ & -i \end{pmatrix}\omega$ or $\begin{pmatrix} i & b_1+i \\ & -i \end{pmatrix}\omega$ depending on $b_1 \geqq 1$ and σ ranges over all representatives of $\Gamma_{\tau\omega} \backslash \Gamma$ for each $b_1 \geqq 1$ and for each $\tau\omega$. Moreover, it follows from (2.28) and (2.29) that the fundamental domain $\mathscr{D}_{\tau\omega} = \Gamma_{\tau\omega} \backslash \Gamma$ can be given, for instance, by

$$\mathscr{D}_{\tau\omega} = \{(z, v) \mid z = x + iy,\ 0 \leqq x \leqq 1,\ -\infty < y < \infty,\ 0 < v < \infty\}$$

for $\tau\omega = \begin{pmatrix} 1 & b_1 \\ & 1 \end{pmatrix}\omega$ or $\begin{pmatrix} 1 & b_1+i \\ & 1 \end{pmatrix}\omega$ and

$$\mathscr{D}_{\tau\omega} = \{(z, v) \mid z = x + iy,\ 0 \leqq y \leqq 1,\ -\infty < x < \infty,\ 0 < v < \infty\}$$

for $\tau\omega = \begin{pmatrix} i & b_1 \\ & -i \end{pmatrix}\omega$ or $\begin{pmatrix} 1 & b_1+i \\ & -i \end{pmatrix}\omega$ respectively. For each $\tau\omega$ of (3.18), we show firstly that

$$(3.19) \qquad \left[\sum_{b_1 \geqq 1} \int_{\mathscr{D}_{\tau\omega}^V} k(u, \tau\omega u) d\mu(u) - \sum_{b_1 \geqq 1} \int_{\mathscr{D}^V} \sum_{\sigma} k(\sigma u, \tau\omega\sigma u)\, d\mu(u) \right] = o(1)$$

as $V \to \infty$, where $\mathscr{D}_{\tau\omega}^V = \{(z, v) \in \mathscr{D}_{\tau\omega} \mid v \leqq V\}$. Secondly we have

$$(3.20)\qquad \sum_{b_1\geqq 1}\int_{\mathscr{D}^V_{\tau\omega}} k(u, \tau\omega u)d\mu(u) = \sum_{b_1\geqq 1}\int_{\mathscr{D}^V_{\tau'\omega}} k(u, \tau'\omega u)d\mu(u)$$

for any $\tau\omega$, $\tau'\omega$ in the summation $\sum_{\tau\omega}$ of (3.18). Hence, it is sufficient to calculate the first term on the left hand side of (3.19) under the condition of $\tau\omega=\begin{pmatrix}1 & b_1\\ & 1\end{pmatrix}\omega$.

To do this, we start from the following equation:

$$(3.21)\qquad \sum_{b_1\geqq 1}\int_{\mathscr{D}^V_{\tau\omega}} k(u, \tau\omega u)d\mu(u)=\sum_{b_1\geqq 1}\int_0^V\int_0^\infty \varphi\left(\frac{y^2+b_1^2}{v^2}\right)\frac{dydv}{v^3}$$

for $\tau\omega=\begin{pmatrix}1 & b_1\\ & 1\end{pmatrix}\omega$. If we put

$$f(x)=\int_0^V\int_0^\infty \varphi\left(\frac{y^2+x^2}{v^2}\right)\frac{dydv}{v^3},$$

then an easy calculation shows that

$$(3.22)\qquad f(x)=\frac{1}{2x}\int_{x^2/V^2}^\infty \varphi(t)\int_{x/V}^{\sqrt{t}}\frac{1}{\sqrt{t-v^2}}dvdt.$$

Then, as is shown in [3: p. 103], a well-known summation formula yields

$$(3.23)\qquad \sum_{b_1\geqq 1} f(b_1)=\frac{1}{2}f(1)+\int_1^\infty f(x)dx+\int_1^\infty f'(x)\{x\}dx,$$

where $\{x\}=x-[x]-1/2$, and $[x]$ denotes the largest integer not greater than x. Since $\sum_{b_1\geqq 1}f(b_1)$ is exactly same as what we want to get, we have only to evaluate each term on the right hand side of (3.23) in terms of $h(r)$.

As for the first term, it follows easily from (3.22) that

$$(3.24)\qquad \frac{1}{2}f(1)=\frac{1}{8}g(0)+o(1)\qquad \text{as } V\to\infty.$$

For the second term on the right hand side of (3.23), it is shown that

$$\int_1^\infty f(x)dx=\frac{g(0)}{4}\log V+\frac{\pi}{4}\int_0^1\int_{1/y^2}^\infty \varphi(t)dt\frac{dy}{y}-\frac{\pi}{4}\int_1^\infty\int_0^{1/y^2}\varphi(t)dt\frac{dy}{y}$$
$$-\frac{1}{2}\int_0^V\int_{1/y^2}^\infty \varphi(t)\int_0^{1/y}\frac{1}{\sqrt{t-v^2}}dvdt\frac{dy}{y}+o(1)$$

as $V\to\infty$. So, by [3: p. 103], one has

$$\frac{\pi}{4}\left\{\int_0^1\int_{1/y^2}^{\infty}\varphi(t)dt\frac{dy}{y}-\int_1^{\infty}\int_0^{1/y^2}\varphi(t)dt\frac{dy}{y}\right\}$$
$$=\frac{1}{4}\left\{\frac{1}{4}h(0)-\gamma g(0)-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)dr\right\},$$

where γ is Euler's constant. Furthermore,

$$\frac{1}{2}\int_0^V\int_{1/y^2}^{\infty}\varphi(t)\int_0^{1/y}\frac{1}{\sqrt{t-v^2}}dv\,dt\frac{dy}{y}$$
$$=\frac{1}{2}\int_0^{\infty}\int_{1/y^2}^{\infty}\varphi(t)\int_0^{1/y}\frac{1}{\sqrt{t-v^2}}dv\,dt\frac{dy}{y}+o(1)$$
$$=\frac{1}{4}\left\{\int_0^{\infty}\varphi(t)\log t\int_0^{\sqrt{t}}\frac{1}{\sqrt{t-v^2}}dvdt\right.$$
$$\left.-\int_0^{\infty}\varphi(t)\int_0^{\sqrt{t}}\frac{\log v}{\sqrt{t-v^2}}dv\,dt\frac{dy}{y}\right\}+o(1)$$
$$=\frac{\log 2}{4}g(0)+o(1)\qquad\text{as } V\to\infty.$$

Thus we have

$$\int_1^{\infty}f(x)dx=\frac{1}{4}\left\{g(0)\log V+\frac{1}{4}h(0)-(\gamma+\log 2)g(0)\right.$$
$$\left.-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)dr\right\}+o(1)\qquad\text{as } V\to\infty. \tag{3.25}$$

For the third term of (3.23), it follows from (3.22) that

$$f'(x)=-\frac{\pi}{4x^2}\int_{x^2/V^2}^{\infty}\varphi(t)dt-\frac{1}{2}\sum_{k=1}^{\infty}\frac{2ka_k}{V^{2k+1}}x^{2k-1}\int_{x^2/V^2}^{\infty}\varphi(t)\frac{1}{t^{k+1/2}}dt,$$

where $a=(2k)!/(2^{2k}(k!)^2(2k+1))$ which is the k-th Taylor coefficient of $\arcsin x$ at $x=0$. By making use of

$$\int_1^{\infty}-\frac{\{x\}}{x^2}dx=\gamma-\frac{1}{2},$$

we obtain finally

$$\int_1^{\infty}f'(x)\{x\}dx=\frac{1}{4}\left(\gamma-\frac{1}{2}\right)g(0)+o(1)\qquad\text{as } V\to\infty. \tag{3.26}$$

The second half of $C(\infty\omega)$ has already been evaluated by [7: p. 241] for example, namely,

(3.27)
$$\int_{\mathscr{D}^V} H(u, u)d\mu(u) = g(0)\log V - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Phi'}{\Phi}(1+ir)h(r) + \frac{1}{4}h(0)\Phi(1) + o(1)$$

as $V \to \infty$.

In view of (3.19), (3.20) ,(3.24) through (3.27) and $\Phi(1) = -1$, the component related to relatively parabolic elements is now expressed as

(3.28)
$$C(\infty\omega) = -\log 2\cdot g(0) + \frac{1}{2}h(0) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Phi'}{\Phi}(1+ir)h(r)dr - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)dr.$$

Summing up (3.10), (3.16), (3.17) and (3.28), we obtain the following

Proposition 3.1. *Let $K'^*(u, u')$ be the kernel function defined by $K'^*(u, u') = \sum_{\sigma\in\Gamma} k(u, \sigma\omega u') - H(u, u')$. Then,*

$$\begin{aligned}
\int_{\mathscr{D}} K'^*(u, u)d\mu(u) &= \frac{11}{24}g(0) - \log 2\cdot g(0) + \frac{1}{2}h(0) + \sum_{[r,h]}\frac{\log NP_0}{NP_0^{l_0} - NP_0^{-l_0}}g(l_0\log NP_0) \\
&+ \frac{2}{3\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{-(2\pi/3)r} - e^{-(4\pi/3)r}}{1-e^{-2\pi r}}h(r)dr + \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Phi'}{\Phi}(1+ir)h(r)dr \\
&+ \frac{1}{16\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+\frac{ir}{4}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{4}\right)\right\}h(r)dr \\
&- \frac{1}{6\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+ir)\right\}(3+2ir)h(r)dr \\
&- \frac{1}{16\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right\}irh(r)dr.
\end{aligned}$$

Set $\gamma_{Q(i)}$ to be the generalized Euler constant of the field $Q(i)$, i.e.,

$$\gamma_{Q(i)} = \lim_{n\to\infty}\left(\sum_x \frac{1}{x} - \pi\log n\right),$$

where x runs over all elements of the set $\{x = m_1^2 + m_2^2 \leq n \mid m_1, m_2 \in \boldsymbol{Z}, (m_1, m_2) \neq (0, 0)\}$ for a fixed $n > 0$. Then, using (1.10) and (1.11), and combining Proposition 3.1 with the trace formula for the space $L^2(\mathscr{D})$ which has been calculated by [6], [7] or [8], we can now write down the explicit trace formula for the even space V_e.

Theorem 1 (Trace formula for V_e). *Let $L^2_{0,e}(\mathscr{D})$ be the space of cusp forms in V_e, and let $\{f_{j_1}\}_{j_1 \geqq 1}$ be the orthogonal basis of $L^2_{0,e}(\mathscr{D})$ consisting of Maass wave forms. If the eigenvalues of each f_{j_1} with respect to D and L^*_φ are denoted by $Df_{j_1} = -(1+r^2_{j_1})f_{j_1}$ and $L^*_\varphi f_{j_1} = h(r_{j_1})f_{j_1}$ respectively, then we have*

$$\begin{aligned}
&\sum_{j_1 \geqq 1} h(r_{j_1}) + h(i) \\
&= \frac{\zeta_{Q(i)}(2)}{4\pi^4}\int_{-\infty}^{\infty} r^2 h(r)\,dr + \frac{1}{2}h(0) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\Phi'}{\Phi}(1+ir)h(r)\,dr \\
&+ \left\{\frac{11}{48} - \frac{11}{32}\log 2 - \frac{\gamma}{4} + \frac{\gamma_{Q(i)}}{8\pi} + \frac{2}{9}\log(2+\sqrt{3})\right\}g(0) \\
&+ \frac{1}{3\sqrt{3}}\int_{-\infty}^{\infty} \frac{e^{-(2\pi/3)r} - e^{-(4\pi/3)r}}{1-e^{-2\pi r}}h(r)\,dr \\
&+ \frac{1}{32\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+\frac{ir}{4}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{4}\right)\right\}h(r)\,dr \\
&+ \frac{1}{2}\sum_{[\tau]} \frac{\log|NP_0|\,g(l_0 \log|NP_0|)}{[\Gamma_\tau : (\Gamma_\tau)']\,|NP_0^{l_0/2} - NP_0^{-l_0/2}|^2} \\
&+ \frac{1}{2}\sum_{[r.h]} \frac{\log NP_0}{NP_0^{l_0} - NP_0^{-l_0}}\,g(l_0 \log NP_0) \\
&- \frac{1}{24\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+ir)\right\}(9+4ir)h(r)\,dr \\
&- \frac{1}{32\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right\}(2+ir)h(r)\,dr,
\end{aligned}$$

where $(\Gamma_\tau)'$ is a free part of Γ_τ, $[\Gamma_\varphi : (\Gamma)']$ means the index between Γ_τ and $(\Gamma_\tau)'$, and $\sum_{[\tau]}$ (resp. $\sum_{[r.h]}$) *ranges over all conjugcay classes of hyperbolic and loxodromic elements* (resp. *all relative conjugacy classes of relatively hyperbolic elements*).

In view of (1.13), again by using the trace formula for $L^2(\mathscr{D})$ and Proposition 3.1, we can also derive the explicit trace formula for the odd space V_o.

Theorem 2 (Trace formula for V_o). *Let $\{f_{j_2}\}_{j_2 \geqq 1}$ be the orthogonal basis of V_o consisting of Maass wave forms. If the eigenvalues of each f_{j_2} with respect to D and L^*_φ are denoted respectively by $Df_{j_2} = -(1+r^2_{j_2})f_{j_2}$ and $L^*_\varphi f_{j_2} = h(r_{j_2})f_{j_2}$, then we have*

$$\sum_{j_2 \geqq 1} h(r_{j_2}) = \frac{\zeta_{Q(i)}(2)}{4\pi^4}\int_{-\infty}^{\infty} r^2 h(r)\,dr$$

$$+\left\{-\frac{11}{48}-\frac{21}{16}\log 2-\frac{\gamma}{4}+\frac{\gamma_{Q(i)}}{8\pi}+\frac{2}{9}\log(2+\sqrt{3})\right\}g(0)$$

$$-\frac{1}{3\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{-(2\pi/3)r}-e^{-(4\pi/3)r}}{1-e^{-2\pi r}}h(r)dr$$

$$-\frac{1}{32\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+\frac{ir}{4}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{4}\right)\right\}h(r)dr$$

$$+\frac{1}{2}\sum_{[\tau]}\frac{\log|NP_0|g(l_0\log|NP_0|)}{[\Gamma_\tau:(\Gamma_\tau)']|NP^{l_0/2}-NP^{-l_0/2}|^2}$$

$$-\frac{1}{2}\sum_{[r.h]}\frac{\log NP_0}{NP_0^{l_0}-NP_0^{-l_0}}g(l_0\log NP_0)$$

$$+\frac{1}{24\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+ir)\right\}(3-4ir)h(r)dr$$

$$-\frac{1}{32\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right\}(2-ir)h(r)dr,$$

where $(\Gamma_\tau)'$, $[\Gamma_\tau:(\Gamma_\tau)']$, $\sum_{[\tau]}$ *and* $\sum_{[r.h]}$ *are as in Theorem* 1.

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