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On a Classical Theta-Function, II

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The present paper, containing a partial and expository reconstruction of the results which were known since [3], is written for the purpose of stating some basic facts on a classical theta-function in a form which is possibly convenient in investigations related positively to metaplectic groups.

Since this paper is a continuation of [2], the ordinals of all sections, theorems, propositions and formulas follow those of [2], while references and footnotes are numbered anew, and the only theorem in [2] is quoted as Theorem 1.

§ 3. Eisenstein series $E(z, s)$

Having finished the investigation of the automorphic factors of the theta function (1), we are naturally led to the following Eisenstein series:

$$E(z, s)=\sum_{\Gamma_0\backslash\Gamma} \chi(\sigma, 1)e^{-(1/2)i \arg (cz+d)}\frac{y^{s/2}}{|cz+d|^s}. \tag{14}$$

Here, z is a point in the upper half plane H, s is a complex number, Γ_0 is the group consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c=0$, and $\chi(\sigma, 1)$ is as in Theorem 1. Moreover, $\arg(cz+d)$ is always normalized by

$$-\pi \leqq \arg(cz+d) < \pi \tag{15}$$

in accordance with (2). The series (14) is absolutely convergent for $\operatorname{Re} s>2$, and satisfies the transformation formula

$$E(z, s)=\chi(\sigma, 1)e^{-(1/2)i \arg (cz+d)}E(\sigma z, s), \qquad (\sigma \in \Gamma). \tag{16}$$

Therefore one can expect that $E(z, s)$ may coincide with $\vartheta(z)$ at $s=\frac{1}{2}$. That this is actually the case will be shown in Section 7.

In this section, we shall observe the effect on $E(z, s)$ of the invariant differential operator

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(17) $$D=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

with respect to $SL(2, \boldsymbol{R})$.

Because of the Iwasawa decomposition

(18) $$\omega=\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \frac{1}{\sqrt{y}}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},$$

every element $\omega \in G$ is in one to one correspondence with a pair of $z=x+iy \in H$ and $\theta \bmod 2\pi$. We normalize this θ by $-\pi \leqq \theta < \pi$, denote it by $\theta(\omega)$, and we set

$$f(\tilde{\omega})=e^{(1/2)i(\theta(\omega)+(1-\varepsilon)\pi)}=\varepsilon e^{(1/2)i\theta(\omega)}$$

for an element $\tilde{\omega}=(\omega, \varepsilon)$ of the covering group $\tilde{G}$, introduced in Section 1, of G. $f(\tilde{\omega})$ is a continuous function on $\tilde{G}$. On the other hand, let $\tilde{\sigma}=(\sigma, \varepsilon') \in \tilde{\Gamma}$, $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma$. Then,

$$\theta(\sigma\omega)=\theta(\omega)+\arg(cz+d)-A(\theta(\omega), \arg(cz+d))\cdot 2\pi$$

holds with

$$A(\theta, \psi)=\begin{cases}0, & -\pi \leqq \theta+\psi < \pi,\\ 1, & \text{otherwise,}\end{cases}$$

while

$$2A(\theta(\omega), \arg(cz+d))=1-a(\sigma, \omega)$$

follows from (11) and (18). Therefore,

$$\begin{aligned} f(\tilde{\sigma}\tilde{\omega}) &= e^{(1/2)i(\theta(\omega)+\arg(cz+d)-(1-a(\sigma,\omega))\pi+(1-\varepsilon\varepsilon' a(\sigma,\omega))\pi)}\\ &= e^{(1/2)i(\theta(\omega)+\arg(cz+d)+a(\sigma,\omega)(1-\varepsilon\varepsilon')\pi)}\\ &= \varepsilon\varepsilon' e^{(1/2)i(\theta(\omega)+\arg(cz+d))}=\varepsilon' e^{(1/2)i\arg(cz+d)}f(\tilde{\omega}). \end{aligned}$$

Hence, setting

$$g(\tilde{\omega})=\varepsilon y^{s/2}e^{-(1/2)i\theta},$$

we have

$$g(\tilde{\sigma}\tilde{\omega})=\varepsilon\varepsilon'\frac{y^{s/2}}{|cz+d|^s}e^{-(1/2)i\arg(cz+d)}e^{-(1/2)i\theta},$$

which is a term of our series (14).

Now, the invariant differential operators on G with respect to the operation of G are polynomials of

$$D'=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+y\frac{\partial^2}{\partial x\partial\theta}+\frac{5}{4}\frac{\partial^2}{\partial\theta^2}$$

and $\partial/\partial\theta$ with constant coefficients[1]. $\tilde{G}$ has the same invariant differential operators as G, and $g(\tilde{\omega})$ is an eigenfunction of D', i.e.,

$$D'g=\left(\lambda-\frac{5}{16}\right)g, \tag{19}$$

$$\lambda=\frac{s}{2}\left(\frac{s}{2}-1\right). \tag{20}$$

From these facts follows that $g(\tilde{\sigma}\tilde{\omega})$ also satisfies the differential equation (19).

Let g temporarily stand for an arbitrary solution of the differential equation (19), and put $g_1=ge^{(1/2)i\theta}$. Then

$$D'g=D'g_1e^{-(1/2)i\theta}=(Dg_1)e^{-(1/2)i\theta}-\frac{i}{2}y\left(\frac{\partial}{\partial x}g_1\right)e^{-(1/2)i\theta}$$
$$-\frac{5}{16}g_1e^{-(1/2)i\theta}.$$

This implies

$$Dg_1-\frac{i}{2}y\frac{\partial}{\partial x}g_1=\lambda g_1.$$

Thus we obtain

Proposition 3. *The Eisenstein series* (14) *satisfies* (*termwise*) *the differential equation*

$$\left(D-\frac{i}{2}y\frac{\partial}{\partial x}\right)E(z,s)=\lambda E(z,s),$$

where D resp. λ is defined by (17) *resp.* (20).

§ 4. Fourier expansion of the Eisenstein series

Our Eisenstein series (14) has of course a Fourier expansion of the form

[1] D′ is the Laplacian of the metric of [4], p. 81.

$$E(z, s)=\sum_{m=-\infty}^{\infty} a_m(y, s)e^{\pi i m x}, \qquad z=x+iy. \tag{21}$$

We now propose to observe $a_m(y, s)$. As for the constant term $a_0(y, s)$, we have the following

Proposition 4. *An explicit form of the constant term of the Fourier series* (21) *is given by*

$$\begin{aligned} a_0(y, s) &= \frac{1}{2}\int_0^2 E(z, s)dx \\ &= 2y^{s/2}+\sqrt{2\pi}\, y^{1-s/2}\left(1+\frac{1}{1+2^{s-1/2}}\right)\frac{\zeta(2s-2)}{\zeta(2s-1)}\,\frac{\Gamma(s-1)}{\Gamma(s-1/2)}, \end{aligned}$$

where ζ is Riemann's zeta function.

Proof. Theorem 1 shows that $\chi(\sigma, 1)$ depends only on c and d, when $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, we write $\chi(c, d)$ instead of $\chi(\sigma, 1)$, so that (14) gives rise to

$$E(z, s)=\sum_{\substack{(c,d)=1 \\ cd\equiv 0 \ (\mathrm{mod}\ 2)}} \chi(c, d)e^{-(1/2)i \arg (cz+d)}\frac{y^{s/2}}{|cz+d|^s}. \tag{22}$$

If $c=0$, then $d=\pm 1$. So, the partial sum of (22) for $c=0$ is

$$y^{s/2}+(-i)e^{-(1/2)i(-\pi)}y^{s/2}=2y^{s/2}. \tag{23}$$

On the other hand, denote by $\sum_1$ the partial sum of (22) for $c\neq 0$. Then,

$$\begin{aligned} \int_0^2 \textstyle\sum_1 dx &= y^{s/2}\int_0^2 \textstyle\sum_1 \dfrac{e^{-(1/2)i \arg c}}{|c|^s}\chi(c, d)\dfrac{e^{-(1/2)i \arg (z+d/c)}}{|z+d/c|^s}dx \\ &= y^{s/2}\sum_{c\neq 0}\frac{e^{-(1/2)i \arg c}}{|c|^s}\left(\textstyle\sum_2 \chi(c, d)\right)\displaystyle\int_{-\infty}^{\infty}\frac{e^{-(1/2)i \arg z}}{|z|^s}dx, \end{aligned} \tag{24}$$

where $\sum_2$ is the sum over d mod $2c$ with $(d, c)=1$, $dc\equiv 0$ (mod 2). Theorem 1 yields

$$\textstyle\sum_2 \chi(c, d)=\begin{cases} \eta^{\mathrm{sgn}\, c}\varphi(c) & c\not\equiv 0 \ (\mathrm{mod}\ 2),\ |c| \text{ square}, \\ 0 & \quad ''\quad ,\ |c| \text{ not square}, \\ & \text{for} \\ \sqrt{2}\,\eta^{\mathrm{sgn}\, c}\varphi(c) & c\equiv 0 \ (\mathrm{mod}\ 2),\ |2c| \text{ square}, \\ 0 & \quad ''\quad ,\ |2c| \text{ not square}, \end{cases}$$

with $\eta = e^{\pi i/4}$, and consequently

$$\sum_{c\neq 0} \frac{e^{-(1/2)i \arg c}}{|c|^s} (\textstyle\sum_2 \chi(c, d))$$
$$= \sum_{\substack{(c,2)=1\\ c>0}} \frac{2\eta}{c^{2s}} \varphi(c^2) + \sum_{c>0} \frac{2\sqrt{2}\,\eta}{2^s c^{2s}} \varphi(2c^2).$$

Hence, by virtue of

$$\sum_{\substack{(c,2)=1\\ c>0}} \frac{\varphi(c^2)}{c^{2s}} = \sum \frac{\varphi(c)}{c^{2s-1}} = \frac{2^{2s-1}-2}{2^{2s-1}-1} \frac{\zeta(2s-2)}{\zeta(2s-1)}$$

and

$$\sum_{c>0} \frac{\varphi(2c^2)}{2^s c^{2s}} = \sum_{\substack{(c,2)=1\\ c>0}} \frac{\varphi(c^2)}{2^s c^{2s}} + \sum_{\substack{c\equiv 0 (\mathrm{mod}\ 2)\\ c>0}} \frac{2\varphi(c^2)}{2^s c^{2s}}$$
$$= \left\{2^{-s}\frac{2^{2s-1}-2}{2^{2s-1}-1} + 2^{1-s}\left(1 - \frac{2^{2s-1}-2}{2^{2s-1}-1}\right)\right\} \frac{\zeta(2s-2)}{\zeta(2s-1)}$$
$$= \frac{2^{s-1}}{2^{2s-1}-1} \frac{\zeta(2s-2)}{\zeta(2s-1)},$$

we see

$$\sum_{c\neq 0} \frac{e^{-(1/2)i \arg c}}{|c|^s} (\textstyle\sum_2 \chi(c, d)) = 2\eta\left(1 + \frac{1}{1+2^{s-1/2}}\right) \frac{\zeta(2s-2)}{\zeta(2s-1)}. \tag{25}$$

Next, an ordinary calculation shows

$$\int_{-\infty}^{\infty} \frac{e^{-(1/2)i \arg z}}{|z|^s} dx = y^{1-s} \int_{-\infty}^{\infty} \frac{e^{-(1/2)i \arg (t+i)}}{(t^2+1)^{s/2}} dt$$
$$= y^{1-s} \int_{-\infty}^{\infty} \frac{e^{-\pi i/4 + (1/2)i \arctan t}}{(t^2+1)^{s/2}} dt, \quad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right),$$

and

$$\int_{-\infty}^{\infty} \frac{e^{i/2 \arctan t}}{(t^2+1)^{s/2}} dt = \int_{-\infty}^{\infty} \frac{\cos (1/2 \arctan t)}{(t^2+1)^{s/2}} dt$$
$$= 2\int_0^1 \sqrt{\frac{1+u}{2}}\, u^s \frac{du}{u^2\sqrt{1-u^2}} = \sqrt{2} \int_0^1 u^{s-2}(1-u)^{-1/2} du$$
$$= \sqrt{2\pi}\, \frac{\Gamma(s-1)}{\Gamma(s-1/2)}, \qquad (u = \cos (\arctan t)).$$

Hence,

(26) $$\int_{-\infty}^{\infty} \frac{e^{-(1/2)i \arg z}}{|z|^s} dx = y^{1-s}\eta^{-1}\sqrt{2\pi}\,\frac{\Gamma(s-1)}{\Gamma(s-1/2)}.$$

The proposition follows now at once from (23), (24), (25), (26).

The nature of Fourier coefficients $a_m(y, s)$ other than the constant term is considerably complicated, and will be treated in Section 5. But, we shall perform here some preliminaries.

The meanings of $\sum_1$ and $\sum_2$ being as above, and m being not 0, it follows from a direct computation that

$$\begin{aligned}\int_0^2 E(z, s)e^{-\pi i m x}dx &= \int_0^2 \textstyle\sum_1 \chi(c, d)\, e^{-(1/2)i \arg (cz+d)} \dfrac{y^{s/2}}{|cz+d|^s} e^{-\pi i m x}dx \\ &= y^{s/2} \textstyle\sum_1 \dfrac{e^{-(1/2)i \arg c}}{|c|^s} \chi(c, d) \dfrac{e^{-(1/2)i \arg (z+d/c)}}{|z+d/c|^s} e^{-\pi i m x}dx \\ &= y^{s/2} \displaystyle\sum_{c\neq 0} \dfrac{e^{-(1/2)i \arg c}}{|c|^s} (\textstyle\sum_2 \chi(c, d) e^{\pi i m d/c}) \displaystyle\int_{-\infty}^{\infty} \dfrac{e^{-(1/2)i \arg z}}{|z|^s} e^{-\pi i m x}dx \\ &= y^{1-s/2} \displaystyle\sum_{c\neq 0} \dfrac{e^{-(1/2)i \arg c}}{|c|^s} (\textstyle\sum_2 \chi(c, d)\, e^{\pi i m d/c}) \\ &\quad \times \displaystyle\int_{-\infty}^{\infty} \dfrac{e^{-(1/2)i \arg (t+i)}}{(t^2+1)^{s/2}} e^{-\pi i (my) t}dt.\end{aligned}$$

So, if we put

(27) $$\tau_m(c) = e^{-(1/2)i \arg c}(\textstyle\sum_2 \chi(c, d)e^{\pi i m d/c}),$$

and

(28) $$w(u, s) = \int_{-\infty}^{\infty} \frac{e^{(1/2)i \arctan t}}{(t^2+1)^{s/2}} e^{-\pi i u t}dt, \qquad -\frac{\pi}{2} < \arctan t < \frac{\pi}{2},$$

u being a positive or negative real number, then

(29) $$a_m(y, s) = y^{1-s/2}\frac{1}{2\eta}\left(\sum_{c\neq 0} \frac{\tau_m(c)}{|c|^s}\right) w(my, s).$$

Because of Proposition 3, $a_m(y, s)$ satisfies the differential equation

(30) $$\frac{d^2 a_m}{dy^2} - \left(\pi^2 m^2 - \frac{\pi m}{2y} + \frac{\lambda}{y^2}\right)a_m = 0,$$

where λ is as in (20). Accordingly, our Fourier coefficients are all expressed by the Whittaker function.

The integral in (28) is not absolutely convergent unless $\operatorname{Re} s>1$. But, by means of the recursive formula

$$\int_{-\infty}^{\infty} \frac{t^k e^{(1/2)i\arctan t}}{(t^2+1)^{s/2+k}} e^{-\pi i u t} dt = \frac{1}{\pi i u}\int_{-\infty}^{\infty}\left[\left\{\frac{k}{t^2}-(s+2k)\right\}e^{(1/2)i\arctan t}+\frac{i}{2t}\right]\frac{t^{k+1}e^{-\pi i u t}}{(t^2+1)^{s/2+k+1}}dt \tag{31}$$

for $k=0, 1, 2, \cdots$, one gets an analytic continuation of $w(u, s)$ which is an entire function on the whole s-plane. Furthermore, for an arbitrary s, it follows from (31) and from the properties of Fourier integrals that $w(u, s)$ tends to 0 as $|u|\to\infty$. Recalling the differential equation (30), we obtain a more precise result that $|w(u, s)|$ decreases with the order of $e^{-\pi|u|}$ as $|u|\to\infty$. Although $w(u, s)$ is analytic with respect to s, it is not analytic with respect to u at $u=0$.

§ 5. Computation of Dirichlet series

We have already discussed in part the Fourier coefficients in (21) of the Eisenstein series (14); one remaining part of number-theoretical importance is the Dirichlet series on the right hand side of (29). In this section, we shall show that the Dirichlet series can be expressed by a combination of ordinary zeta and L-functions in spite of the appearance of Gauss sums $\tau_m(c)$.

To do this, we must first determine $\tau_m(c)$ completely. For the sake of simplicity, most of our arguments will be done under the assumption $c>0$. The general case can be treated quite incidentally.

Set $c=2^r c'$, $(c', 2)=1$, and $\varepsilon=(-1)^{(c'-1)/2}$. Then, (27) implies

$$\tau_m(c)=\eta^{\varepsilon}\sum_{\substack{d \bmod c\\ (d,c)=1}}\left(\frac{d}{c}\right)e^{2\pi i m d/c}, \qquad (r=0), \tag{32}$$

$$\begin{aligned}\tau_m(c)=&\textstyle\sum_{3,1}(c', d)\left(\frac{2^{r+1}c'}{d}\right)e^{2\pi i m d/2^{r+1}c'}\\ &+i\textstyle\sum_{3,-1}(c', d)\left(\frac{2^{r+1}c'}{d}\right)e^{2\pi i m d/2^{r+1}c'}, \qquad (r>0),\end{aligned} \tag{33}$$

where $\sum_{3,\alpha}$ is the sum over $d \bmod 2^{r+1}c'$ with $(d, c')=1$, $d\equiv\alpha \pmod 4$. If $d=2^{r+1}d_1+c'd_2$, then

$$\begin{aligned}&\textstyle\sum_{3,\alpha}(c', d)\left(\frac{2^{r+1}c'}{d}\right)e^{2\pi i m/2^{r+1}c'}\\ &=\textstyle\sum_{3,\alpha}(c'\varepsilon, d)\left(\frac{c'\varepsilon}{d}\right)\cdot(\varepsilon, d)\left(\frac{2^{r+1}\varepsilon}{d}\right)e^{2\pi i(md_1/c'+md_2/2^{r+1})}\end{aligned}$$

$$=\sum_{\substack{d_1 \bmod c' \\ (d_1,c')=1}} \left(\frac{2^{r+1}d}{c'}\right) e^{2\pi i m d_1/c'} \cdot \sum_{\substack{d_2 \bmod 2^{r+1} \\ d_2\equiv \varepsilon\alpha (\bmod 4)}} (\varepsilon, c'd_2)\left(\frac{2^{r+1}\varepsilon}{c'd_2}\right) e^{2\pi i m d_2/2^{r+1}}$$

$$=\varepsilon \sum_{\substack{d_1 \bmod c' \\ (d_1,c')=1}} \left(\frac{d}{c'}\right) e^{2\pi i m d_1/c'} \cdot \sum_{\substack{d_2 \bmod 2^{r+1} \\ d_2\equiv \varepsilon\alpha (\bmod 4)}} (\varepsilon, d_2)\left(\frac{2^{r+1}\varepsilon}{d_2}\right) e^{2\pi i m d_2/2^{r+1}}.$$

Hence, if we put

$$\tau(c, m)=\sum_{\substack{d \bmod c \\ (d,c)=1}} \left(\frac{d}{c}\right) e^{2\pi i m d/c}, \qquad (c, 2)=1, \tag{34}$$

$$\tau_\alpha(2^{r+1}\varepsilon, m)=\sum_{\substack{d \bmod 2^{r+1} \\ d\equiv \alpha (\bmod 4)}} (\varepsilon, d)\left(\frac{2^{r+1}\varepsilon}{d}\right) e^{2\pi i m d/2^{r+1}}, \tag{35}$$

then (32) yields

$$\tau_m(c)=\eta^\varepsilon \tau(c, m), \qquad (c, 2)=1, \tag{36}$$

and (33) yields

$$\tau_m(c)=\varepsilon\tau(c', m)(\tau_\varepsilon(2^{r+1}\varepsilon, m)+i\tau_{-\varepsilon}(2^{r+1}\varepsilon, m)), \quad 2\,|\,c. \tag{37}$$

Thus the determination of $\tau_m(c)$ is reduced to the determination of two kinds of sums (34), (35).

Proposition 5. *Let $c>0$ be an odd natural number, and m be a non-zero rational integer. Denote, in general, by l an odd prime number dividing m, and by p a prime number not dividing m. Furthermore, let p^r resp. l^r be the p- resp. l-component of c, let l^e be the l-component of m, and let m_0 be the non-square kernel of m. Finally, define the following notations*:

for p

$$\tau(c, m)_p=\begin{cases} 1, & r=0, \\ \left(\frac{m_0}{p}\right)\sqrt{p}, & r=1, \\ 0, & r>1. \end{cases}$$

for l with odd e

$$\tau(c, m)_l=\begin{cases} 1, & r=0, \\ (l-1)l^{r-1}, & 0<r\leqq e-1,\ r \text{ even}, \\ -l^e, & r=e+1, \\ 0, & r>e+1, \text{ or } r \text{ odd}. \end{cases}$$

for l with even e

$$\tau(c, m)_l = \begin{cases} 1, & r=0, \\ (l-1)l^{r-1}, & 0<r\leqq e,\ r \text{ even}, \\ \left(\frac{m_0}{l}\right) l^e \sqrt{l}, & r=e+1, \\ 0, & r>e+1, \text{ or } r \text{ odd} (\neq e+1). \end{cases}$$

Then, we obtain a decomposition

$$\tau(c, m) = \eta^{1-\varepsilon} \prod_l \tau(c, m)_l \cdot \prod_p \tau(c, m)_p$$

of $\tau(c, m)$ in (34) *with $\varepsilon=(-1)^{(c-1)/2}$, $\eta=e^{\pi i/4}$.*

Proof. This proposition is contained in the results of [1]. So, we state here only an outline of the proof. Let $c=c_1c_2$ be a decomposition of c into two mutually prime natural numbers, and put $d=c_2d_1+c_1d_2$. Then the quadratic reciprocity law shows that it is enough to prove the proposition for c_1 and c_2 instead of c. Therefore, the proof is reduced to the case where c is a power of a prime number p. In this case, the assertion of the proposition follows from the fact that the sum of the values of a non-trivial character over a finite abelian group is 0, and from the well-known classical result[2] on the value of the Gauss sum $\tau(p, 1)$.

Proposition 6. *Let $m\neq 0$ be a rational integer, and put $m=2^e m'$, $((m', 2)=1)$, $\varepsilon=\pm 1$, $\varepsilon_m=(-1)^{(m'-1)/2}$, and $\eta=e^{\pi i/4}$. Then, the value of the sum $\tau_\alpha(2^{r+1}\varepsilon, m)$ in* (35) *is as in the following table:*

conditions on r		*value of* $\tau_\alpha(2^{r+1}\varepsilon, m)$
$0<r\leqq e-1$,	r *even*	0
〃	, r *odd*	$2^{r-1}(\varepsilon, \alpha)$
$r=e$	, r *even*	0
〃	, r *odd*	$-2^{e-1}(\varepsilon, \alpha)$
$r=e+1$	, r *even*	0
〃	, r *odd*	$2^e(\varepsilon, \alpha) i^{\varepsilon_m \alpha}$
$r=e+2$	, r *even*	$2^{e+1}(\varepsilon, \alpha)\left(\frac{2}{m'}\right)\eta^{\varepsilon_m \alpha}$
〃	, r *odd*	0

2) Contained in the proof of Proposition 1.

Proof. As in the preceding proposition, use the fact that the sum of the values of a non-trivial character over a finite abelian group is 0. Then, it is easy to see that non-trivial assertions in the proposition are only those concerning $r=e$, $r=e+1$, and $r=e+2$. But, the assertions for these cases also reduce to observations of sums over a prime residue system mod 8, and are simply treated by direct computations.

This proposition immediately entails

Corollary. *Notations being as in Proposition* 6, *put*

$$\tau(2^{r+1}\varepsilon, m)=\varepsilon(\tau_{\varepsilon}(2^{r+1}\varepsilon, m)+i\tau_{-\varepsilon}(2^{r+1}\varepsilon, m)).$$

Then, we have the following result:

	conditions on r		value of $\tau(2^{r+1}\varepsilon, m)$
for odd e			
	$0<r\leqq e-1,$	r odd	$\eta^{\varepsilon}2^{r-1}\sqrt{2}$
	″ ,	r even	0
	$r=e$		$-\eta^{\varepsilon}2^{e-1}\sqrt{2}$
	$r>e$		0
for even e			
	$0<r\leqq e,$	r odd	$\eta^{\varepsilon}2^{r-1}\sqrt{2}$
	″ ,	r even	0
	$r=e+1$		$\eta^{\varepsilon}2^{e}\sqrt{2}\,\varepsilon_m$
	$r=e+2$		$\eta^{\varepsilon}2^{e+1}(1+\varepsilon_m)\left(\dfrac{2}{m'}\right)$
	$r>e+2$		0

Now, denote anew by $c>0$ an arbitrary natural number, and by $m\neq 0$ a rational integer, set $c=2^r c'$, $((c', 2)=1)$, $\varepsilon=(-1)^{(c'-1)/2}$, $m=2^e m'$, $((m', 2)=1)$, and $\varepsilon_m=(-1)^{(m'-1)/2}$. Furthermore, let $\tau(c, m)_p$ and $\tau(c, m)_l$ be as in Proposition 5 for odd primes p and l, and, using the notations in Corollary to Proposition 6, put

$$\tau(c, m)_2=\begin{cases}\eta^{-\varepsilon}\tau(2^{r+1}\varepsilon, m), & r>0,\\ 1, & r=0.\end{cases} \tag{38}$$

Then, from (36), (37), Proposition 5 and Corollary to Proposition 6, it follows that the component decomposition

$$\tau_m(c)=\eta\prod_q \tau(c, m)_q \tag{39}$$

holds, the product being extended over all prime numbers q.

From now on, we drop the condition $c>0$; every notation which we have ever defined has a definite meaning even if c is a negative rational integer. Suppose that c is negative, and $(c, 2)=1$. Then, by definition,

$$\begin{aligned}\tau_m(c)&=i\eta^{\varepsilon}\tau(c, m)=i\eta^{\varepsilon}\left(\frac{-1}{c}\right)\tau(|c|, m)\\&=i\eta^{2\varepsilon}\left(\frac{-1}{c}\right)\tau_m(|c|)=\tau_m(|c|).\end{aligned} \tag{40}$$

Therefore, $\tau_m(c)$ depends only on $|c|$, when $(c, 2)=1$. If $2|c$ and $c<0$, (39) and (40) imply

$$\begin{aligned}\tau_m(c)&=i\eta^{\varepsilon}\tau(c', m)\cdot\tau(c, m)_2\\&=\tau_m(|c'|)\cdot\tau(c, m)_2,\end{aligned}$$

and it is clear by Corollary to Proposition 6 that $\tau(c, m)_2=\tau(|c|, m)_2$. So, $\tau_m(c)=\tau_m(|c|)$ also in this case. On the other hand, Proposition 5 shows that $\tau(c, m)_q$ for $q\neq 2$ depends only on $|c|$, too. Hence, in considering $\tau_m(c)$ and its component decomposition, we may always replace c by $|c|$.

Because of the component decomposition (39), the Dirichlet series on the right hand side of (29) posesses an Euler product;

$$\sum_{c\neq 0}\frac{\tau_m(c)}{|c|^s}=2\eta\prod A_{m,q}(s) \tag{41}$$

with

$$A_{m,q}(s)=\sum_{r=1}^{\infty}\frac{\tau(c, q^r)_q}{q^{rs}}.$$

We now propose to determine each q-component of this Euler product by means of Proposition 5 and Corollary to Proposition 6.

If $p\neq 2$, $(p, m)=1$, then

$$A_{m,p}(s)=1+\left(\frac{m_0}{p}\right)\frac{1}{p^{s-1/2}}, \tag{42}$$

where m_0 is the non-square kernel of m.

If $l\neq 2$, $m=l^e m'$, and $(m', l)=1$, then

$$A_{m,l}(s)=1+\frac{(l-1)l}{l^{2s}}+\frac{(l-1)l^3}{l^{4s}}+\cdots+\frac{(l-1)l^{e-2}}{l^{(e-1)s}}-\frac{l^e}{l^{(e+1)s}}$$
$$(43)\qquad =1+\frac{(l-1)(l^{(e-1)(1-s)}-1)}{l-l^{2s-1}}-\frac{l^e}{l^{(e+1)s}}$$

for odd e, and

$$A_{m,l}(s)=1+\frac{(l-1)l}{l^{2s}}+\frac{(l-1)l^3}{l^{4s}}+\cdots+\frac{(l-1)l^{e-1}}{l^{es}}$$
$$(44)\qquad +\left(\frac{m_0}{l}\right)\frac{l^{e+1/2}}{l^{(e+1)s}}$$
$$=1+\frac{(l-1)(l^{e(1-s)}-1)}{l-l^{2s-1}}+\left(\frac{m_0}{l}\right)\frac{l^{e+1/2}}{l^{(e+1)s}}$$

for even e. As for 2, we must recall (38). If in this case $m=2^e m'$, $(m', 2)=1$, then

$$A_{m,2}(s)=1+\frac{\sqrt{2}}{2^s}+\frac{2^2\sqrt{2}}{2^{3s}}+\cdots+\frac{2^{e-3}\sqrt{2}}{2^{(e-2)s}}-\frac{2^{e-1}\sqrt{2}}{2^{es}}$$
$$(45)\qquad =1+2^{s-1/2}\frac{2^{(e-1)(1-s)}-1}{2-2^{2s-1}}-\frac{1}{2^{es-e+1/2}}$$

for odd e, and

$$A_{m,2}(s)=1+\frac{\sqrt{2}}{2^s}+\frac{2^2\sqrt{2}}{2^{3s}}+\cdots+\frac{2^{e-2}\sqrt{2}}{2^{(e-1)s}}$$
$$(46)\qquad +\frac{2^e\sqrt{2}}{2^{(e+1)s}}+\frac{2^{e+1}(1+\varepsilon_m)}{2^{(e+2)s}}\left(\frac{2}{m'}\right)$$
$$=1+2^{s-1/2}\frac{2^{e(1-s)}-1}{2-2^{2s-1}}+\frac{\varepsilon_m}{2^{(e+1)s-e-1/2}}+\frac{1+\varepsilon_m}{2^{(e+2)s-e-1}}\left(\frac{2}{m'}\right),$$
$$\varepsilon_m=(-1)^{(m'-1)/2}$$

for even e. Let now χ_m be the class-field-theoretical character with respect to $F=\boldsymbol{Q}(\sqrt{m})$, that is, for a prime number q, let

$$(47)\qquad \chi_m(q)=\begin{cases}1, & q \text{ is a product of primes of degree 1 in } F,\\ -1, \quad \text{if} & q \text{ is a prime of degree 2 in } F,\\ 0, & q \text{ is ramified in } F.\end{cases}$$

Furthermore, set

$$
(48) \qquad A_{m,q}(s)=\begin{cases} A'_{m,q}(s)\left(1+\dfrac{\chi_m(q)}{q^{s-1/2}}\right), & \chi_m(q)\neq 0, \\ & \text{for} \\ A'_{m,q}(s)\left(1-\dfrac{1}{q^{2s-1}}\right), & \chi_m(q)=0. \end{cases}
$$

Then, a series of calculations using (43), (44), (45), (46), and (48) yields the following expressions of $A'_{m,q}(s)$:

<u>for $p\neq 2$, $(p, m)=1$</u>

$$
(49) \qquad A'_{m,p}(s)=1.
$$

<u>*for* $l\neq 2$, $m=l^e m'$, $(m', l)=1$, and e odd</u>

$$
(50) \qquad \begin{aligned} A'_{m,l}(s) &= \frac{1-l^{-(e+1)(s-1)}}{1-l^{-2(s-1)}} \\ &= l^{-((e-1)/2)(s-1)} \sinh\left\{\frac{e+1}{2}(s-1)\log l\right\} \\ &\quad /\sinh\{(s-1)\log l\}. \end{aligned}
$$

<u>for $l\neq 2$, $\mathrm{m}=l^e m'$, $(m', l)=1$, and e even</u>

$$
(51) \qquad \begin{aligned} A'_{m,l}(s) &= \left(1-\left(\frac{m_0}{l}\right)l^{-s+1/2}\right)\frac{1-l^{-e(s-1)}}{1-l^{-2(s-1)}}+l^{-e(s-1)} \\ &= l^{-(l/2)(s-1)}\left\{\sinh\left(\left(\frac{e}{2}+1\right)(s-1)\log l\right)\right. \\ &\quad \left. -\left(\frac{m_0}{l}\right)\frac{1}{\sqrt{l}}\sinh\left(\frac{e}{2}(s-1)\log l\right)\right\} \\ &\quad /\sinh((s-1)\log l). \end{aligned}
$$

<u>for $m=2^e m'$, $(m', 2)=1$, and e odd</u>

$$
(52) \qquad \begin{aligned} A'_{m,2}(s) &= 2^{s-1/2}\left(-\frac{1}{1+2^{-(s-1/2)}}+\frac{1-2^{-(e+1)(s-1)}}{1-2^{-2(s-1)}}\right) \\ &= \frac{2^{-((e-3)/2)(s-1)}}{1+2^{s-1/2}}\left\{2\sinh\left(\frac{e-1}{2}(s-1)\log 2\right)\right. \\ &\quad \left. +\sqrt{2}\sinh\left(\frac{e+1}{2}(s-1)\log 2\right)\right\} \\ &\quad /\sinh((s-1)\log 2). \end{aligned}
$$

for $m=2^e m'$, $(m', 2)=1$, e even and $\varepsilon_m=(-1)^{(m'-1)/}=-1$[3]

$$A'_{m,2}(s)=2^{s-1/2}\left(-\frac{1}{1+2^{-(s-1/2)}}+\frac{1-2^{-(e+2)(s-1)}}{1-2^{-2(s-1)}}\right)$$

(53)
$$=\frac{2^{(-(e-2)/2)(s-1)}}{1+2^{s-1/2}}\left\{2\sinh\left(\frac{e}{2}(s-1)\log 2\right)\right.$$
$$\left.+\sqrt{2}\sinh\left(\left(\frac{e}{2}+1\right)(s-1)\log 2\right)\right\}$$
$$/\sinh((s-1)\log 2).$$

for $m=2^e m'$, $(m', 2)=1$, e even, and $\varepsilon_m=1$

$$A'_{m,2}(s)=2^{s-1/2}\left(1-\left(\frac{2}{m'}\right)2^{-(s-1/2)}\right)$$
$$\times\left(-\frac{1}{1+2^{-(s-1/2)}}+\frac{1-2^{-(e+2)(s-1)}}{1-2^{-2(s-1)}}\right)+2^{(e+1)(1-s)+1/2}$$

(54)
$$=\frac{2^{-(e/2)(s-1)}}{1+2^{s-1/2}}\left[2\sinh\left(\left(\frac{e}{2}+1\right)(s-1)\log 2\right)\right.$$
$$+\sqrt{2}\sinh\left(\left(\frac{e}{2}+2\right)(s-1)\log 2\right)$$
$$-\left(\frac{2}{m'}\right)\left\{\sqrt{2}\sinh\left(\frac{e}{2}(s-1)\log 2\right)\right.$$
$$\left.\left.+\sinh\left(\left(\frac{e}{2}+1\right)(s-1)\log 2\right)\right\}\right]$$
$$/\sinh((s-1)\log 2).$$

Let $L(s, \chi_m)$ be Dirichlet's L-function containing the character χ_m defined by (47). Then, formulas (41) and (48) immediately imply

$$\sum_{c\neq 0}\frac{\tau_m(c)}{|c|^s}=2\eta\frac{L(s-1/2, \chi_m)}{\zeta(2s-1)}\prod_q A'_{m,q}(s).$$

Using this and (29), we have the following

Theorem 2. *Except the constant term $a_0(y, s)$ given by Proposition* 4, *the Fourier coefficients in the Fourier expansion* (21) *of the Eisenstein series* (14) *are*

[3] In this and in the next case, it is convenient for our calculation to utilize the resemblance between (46) and (45). Note also that $\varepsilon_m=-1$ or 1 according to $\chi_m(2)=0$ or $\chi_m(2)=(2/m')\neq 0$.

$$a_m(y,s)=y^{1-s/2}w(my,s)\frac{L(s-1/2,\chi_m)}{\zeta(2s-1)}\prod_q A'_{m,q}(s),$$

where w *is defined by the integral in* (28), $A'_{m,q}(s)$ *is as in* (49), (50), (51), (52), (53), *and* (54), *and* q *runs over all prime numbers.*

§ 6. Functional equation of the Eisenstein series

From the concrete expression in Theorem 2 of the Fourier coefficients of our Eisenstein series, we see that, whenever s is in a compact region, the product $\prod_q A'_{m,q}(s)$ is at most of the order of a power of m as $m\to\infty$. On the other hand, $w(my, s)$ decreases exponentially as $m\to\infty$. Consequently, $E(z, s)$ is a meromorphic function in the whole s-plane, which is regular in a domain that does not contain any pole of $L(s,\chi_m)$, $\zeta(2s-1)^{-1}$, or the constant term of $E(z, s)$ given by Proposition 4. Of course, $E(z, s)$ is single valued.

To prove that $E(z, s)$ satisfies a functional equation as in the theory of Selberg, we put here

$$\varphi(s)=\sqrt{\frac{\pi}{2}}\left(1+\frac{1}{1+2^{s-1/2}}\right)\frac{\zeta(2s-2)}{\zeta(2s-1)}\,\frac{\Gamma(s-1)}{\Gamma(s-1/2)}, \tag{55}$$

so that Proposition 4 turns out

$$a_0(y,s)=2(y^{s/2}+\varphi(s)y^{1-s/2}). \tag{56}$$

Put $s=1+it$, ($t\in \boldsymbol{R}$). Then, the functional equation of $\zeta(s)$ yields

$$\begin{aligned}|\zeta(2it)\Gamma(it)|&=|\zeta(-2it)\Gamma(-it)|\\&=\left|\zeta(1+2it)\Gamma\left(\frac{1+2it}{2}\right)\pi^{-it}\pi^{-(i+2it)/2}\right|.\end{aligned}$$

So, by (55), we have

$$\begin{aligned}\varphi(1+it)&=\sqrt{\frac{\pi}{2}}\left|1+\frac{1}{1+2^{1/2+it}}\right|\left|\frac{\zeta(2it)\Gamma(it)}{\zeta(1+2it)\Gamma(1/2+it)}\right|\\&=\left|\frac{2^{1/2}+2^{it}}{1+2^{1/2+it}}\right|=\left|\frac{1+2^{1/2-it}}{1+2^{1/2+it}}\right|=1.\end{aligned}$$

This means that $\varphi(s)$ satisfies the functional equation

$$\varphi(s)\varphi(2-s)=1. \tag{57}$$

Consequently, the constant term of the Fourier expansion of

$$b(z, s)=E(z, s)-\varphi(s)E(z, 2-s) \tag{58}$$

is 0. Since, however, our discontinuous group Γ has two cusps ∞ and 1, what we have shown is merely that $b(z, s)$ vanishes at the cusp ∞. Therefore we must examine the other cusp 1. Let $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ, and set

$$j_1(\sigma, z)=e^{-(i/2)\arg(cz+d)}, \qquad (z \in H).$$

Then, it follows from (12) and (13) that

$$j_1(\sigma, \tau z)j_1(\tau, z)=a(\sigma, \tau)j_1(\sigma\tau, z)$$

holds for $\sigma, \tau \in \Gamma$; $a(\sigma, \tau)$ being the factor set defined by (11). Set now $\rho=\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$, $\tau=\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$, then $\rho\tau\rho^{-1}=\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ and from Theorem 1 follows

$$E(\rho\tau\rho^{-1}z, s)=-ij_1(\rho\tau\rho^{-1}, z)^{-1}E(z, s), \tag{59}$$

while $a(\rho, \tau\rho^{-1})$, $a(\tau, \rho^{-1})$, and $a(\rho^{-1}, \rho)$ are all 1 because of $\rho^{-1}=\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$, $\tau\rho^{-1}=\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$. Hence,

$$\begin{aligned} j_1(\rho\tau\rho^{-1}, z) &= j_1(\rho, \tau\rho^{-1}z)j_1(\tau\rho^{-1}, z) \\ &= j_1(\rho, \tau\rho^{-1}z)j_1(\tau, \rho^{-1}z)j_1(\rho^{-1}, z), \end{aligned}$$

and

$$j_1(\rho\tau\rho^{-1}, z)=j_1(\rho, \tau z)j_1(\rho, z)^{-1}.$$

This, together with (59), shows

$$j_1(\rho, \tau z)E(\rho\tau z, s)=-ij_1(\rho, z)E(\rho z, s). \tag{60}$$

The behavior of $E(z, s)$ in the neighborhood of $z=1$ is indicated by the function $j_1(\rho, z)E(\rho z, s)$. Since (60) shows that the function is a periodic function of period 8 with the multiplicator $-i$ with respect to the transformation $z \to z+2$, the constant term of the Fourier expansion, similar to (21), of $j_1(\rho, z)E(\rho z, s)$ must be 0. Thus the function $b(z, s)$ in (58), vanishing also at the cusp 1, is square integrable on a fundamental domain of Γ, i.e., $b(z, s)$ is a so-called cusp form. In this situation, it is no longer difficult to prove that $b(z, s)$ is identically 0, if we adopt some arguments from Selberg's work. Notations being as in Section 3, $b'=b(z, s)f(\tilde{\omega})^{-1}$ belongs to $L^2(\tilde{\Gamma}\backslash\tilde{G})$, and is an eigenfunction of all invariant differential operators of G. Since, however, the eigenvalue of b' with respect to D' depends continuously and analytically on s, the self-adjointness of D' entails that b' is orthogonal to all of those functions in $L^2(\tilde{\Gamma}\backslash\tilde{G})$ which are eigenfunctions of all invariant differential operators.

Thus we attained our aim in this section to obtain the following

Theorem 3. *The Eisenstein series $E(z, s)$ satisfies the functional equation*

$$E(z, s)=\varphi(s)E(z, 2-s)$$

containing the function $\varphi(s)$ of (55).

§ 7. $E(z, s)$ at $s=1/2$

We consider the function $\theta(z)=(1/2)y^{-1/4}E(z, 1/2)$. First of all, we intend to prove that $\theta(z)$ is an analytic function of z. To do this, we observe the function

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)y^{-s/2}E(z, s) \tag{61}$$

which is given by a series for Re $s>2$, and show that the analytic continuation, with respect to s, of this function vanishes at $s=1/2$. Since by definition

$$y^{-s/2}E(z, s)=\sum_{\Gamma_0\backslash\Gamma}\chi(\sigma, 1)\frac{1}{\sqrt{cz+d}}\frac{1}{|cz+d|^{s-1/2}},$$

we have

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)y^{-s/2}E(z, s)=\left(s-\frac{1}{2}\right)E_1(z, s) \tag{62}$$

with

$$E_1(z, s)=\sum_{\Gamma_0\backslash\Gamma}\chi(\sigma, 1)e^{-(3/2)i\arg(cz+d)}\frac{c}{|cz+d|^{s+1}}. \tag{63}$$

$E_1(z, s)$ has period 2 with respect to z, and has consequently a Fourier expansion with respect to the orthogonal basis $\{e^{\pi imx}\}$. Let us now investigate the Fourier coefficients. While a calculation similar to the proof of Proposition 4 yields

$$\int_0^2 E_1(z, s)dx=\sum_{c\neq 0}\left(\frac{c}{|c|^{s+1}}e^{-(3/2)i\arg c}\sum\nolimits_2\chi(c, d)\right)\int_{-\infty}^{\infty}\frac{e^{-(3/2)i\arg z}}{|z|^{s+1}}dx$$

$$=-\eta\sum_{c\neq 0}\frac{e^{-(1/2)i\arg c}}{|c|^s}\left(\sum\nolimits_2\chi(c, d)\right)\int_{-\infty}^{\infty}\frac{e^{(3/2)i\arctan t}}{(t^2+1)^{(s+1)/2}}dt,$$

the sum over $c\neq 0$ here is nothing else than (25) appearing at the corresponding place of Proposition 4. On the other hand,

$$\int_{-\infty}^{\infty}\frac{e^{(3/2)i\arctan t}}{(t^2+1)^{(s+1)/2}}dt=2\int_0^{\infty}\frac{\cos((3/2)\arctan t)}{(t^2+1)^{(s+1)/2}}dt$$
$$=2\int_0^1\left(u\sqrt{\frac{1+u}{2}}-\sqrt{1-u^2}\sqrt{\frac{1-u}{2}}\right)u^{s+1}\frac{du}{u^2\sqrt{1-u^2}}$$
$$=\sqrt{2}\int_0^1\{u^s(1-u)^{-1/2}-u^{s-1}(1-u)^{1/2}\}du$$
$$=\sqrt{2}\,\frac{\Gamma(s+1)\Gamma(1/2)-\Gamma(s)\Gamma(3/2)}{\Gamma(s+3/2)}=\sqrt{2\pi}\left(s-\frac{1}{2}\right)\frac{\Gamma(s)}{\Gamma(s+3/2)}.$$

Therefore,

$$\int_0^2 E_1(z,s)dx=-2\sqrt{2\pi}\,iy^{-s}\left(1+\frac{1}{1+2^{s-1/2}}\right)\left(s-\frac{1}{2}\right)\frac{\zeta(2s-2)}{\zeta(2s-1)}\frac{\Gamma(s)}{\Gamma(s+3/2)}.$$

Hence, the constant term in the Fourier expansion of $E_1(z,s)$ is 0 at $s=1/2$. Next, again some calculations similar to those in Section 4 show

$$\int_0^2 E_1(z,s)e^{-\pi imx}dx=\int_0^2\sum\nolimits_1\chi(c,d)e^{-(3/2)i\arg(cz+d)}\frac{c}{|cz+d|^{s+1}}e^{-\pi imx}dx$$
$$=-\eta y^{-s}\sum_{c\neq 0}\frac{e^{-(1/2)i\arg c}}{|c|^s}\left(\sum\nolimits_2\chi(c,d)e^{\pi imd/c}\right)$$
$$\times\int_{-\infty}^{\infty}\frac{e^{(3/2)i\arctan t}}{(t^2+1)^{s/2}}e^{-\pi i(my)t}dt,$$

and here, too, the Dirichlet series defined by the sum over $c\neq 0$ completely coincides with (41) in Section 5. Hence, it follows from Theorem 2 that

$$\int_0^2 E_1(z,s)e^{-\pi imx}dx=-2iy^{-s}\frac{L(s-1/2,\chi_m)}{\zeta(2s-1)}\prod A'_{m,q}(s)\times\int_{-\infty}^{\infty}\frac{e^{(3/2)i\arctan t}}{(t^2+1)^{s/2}}e^{-\pi i(my)t}dt \tag{64}$$

for $m\neq 0$. Moreover, although

$$w_1(u,s)=\int_{-\infty}^{\infty}\frac{e^{(3/2)i\arctan t}}{(t^2+1)^{s/2}}e^{-\pi iut}dt$$

is somewhat different from $w(u,s)$ in (28), no essentially new circumstance arises in verifying that $w_1(u,s)$ has almost the same properties as $w(u,s)$, in particular that, through a recursive formula similar to (31), $w_1(u, s)$ has an analytic continuation in the whole s-plane which is an entire function of s. Hence, it follows from (64) that all Fourier coefficients of $E_1(z, s)$ different from the constant term are regular at $s=1/2$. Thus we have proved that $E_1(z, s)$ is regular at $s=1/2$. Consequently, (61) becomes 0 at $s=1/2$ because of (62). This proves that our function $\theta(z)$ is an analytic function of z.

Now, Proposition 4 assures that the constant term of the Fourier expansion of $\theta(z)$ is 1. On the other hand, the same is true for the function $\vartheta(z)$ of (1). Furthermore, notations being as in Section 6, we have already shown in the proof of Theorem 3 that the constant term in the Fourier expansion of the function $j_1(\rho, z)E(\rho z, s)$ is 0. Since $y^{1/4}\vartheta(z)$ and $E(z, 1/2)$ have one and the same transformation formula with respect to the elements of Γ, the constant term of the Fourier expansion of $j_1(\rho, z)(y^{1/4}\vartheta(z))_{z\to\rho z}$ is also 0. From these facts, we can conclude that $\theta(z)-\vartheta(z)$ vanishes at two cusps 1, ∞ of Γ. Consider now the function $(\theta(z)-\vartheta(z))^4$. This is, by Theorem 1, an ordinary, analytic modular form of weight 2 for the congruence subgroup Γ_2 mod 2 of $SL(2, \boldsymbol{Z})$, and is besides a cusp form. Since, however, the genus of the fundamental domain of Γ_2 is 0, $(\theta(z)-\vartheta(z))^4$ must be 0. Thus, we have

Theorem 4. *The Eisenstein series $E(z, s)$ at $s=1/2$ is combined with the theta function* (1) *by the relation*

$$\frac{1}{2}y^{-1/4}E\left(z, \frac{1}{2}\right)=\vartheta(z).$$

The Fourier coefficients of $\vartheta(z)$ with respect to the orthogonal basis $\{e^{\pi imx}\}$ is, by definition, 0 unless m is a square. This corresponds through Theorem 2 to the fact that the value $L(0, \chi_m)$ of Dirichlet's L-function is 0 for $m>0$ unless χ_m is trivial.

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