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On an Application of Zagier's Method in the Theory of Selberg's Trace Formula

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Introduction

Let H be the complex upper half plane, and put $G=PSL(2, \boldsymbol{R})$, $\Gamma=PSL(2, \boldsymbol{Z})$. Then, the well-known Selberg trace formula holds for the Hilbert space $L^2(\Gamma\backslash H)$. Let furthermore $\omega: z\to -\bar{z}$ be the reflection with respect to the imaginary axis, and let $\tilde{G}=\langle G, \omega\rangle$ be the group generated by G and ω. Then, the triple $(\tilde{G}, H, 1)$ turns out to be a weakly symmetric Riemannian space in the notation of Selberg (§ 1). Therefore, it is possible to investigate the trace formula for the Hilbert space $L^2(\tilde{\Gamma}\backslash H)$ with $\tilde{\Gamma}=\langle\Gamma, \omega\rangle$.

The space $L^2(\Gamma\backslash H)$ has the direct sum decomposition $L^2(\Gamma\backslash H)=V_e \oplus V_o$, where V_e and V_o are defined by $V_e=\{f\in L^2(\Gamma\backslash H)|f(\omega z)=f(z)\}$, $V_o=\{f\in L^2(\Gamma\backslash H)|f(\omega z)=-f(z)\}$ respectively, in accordance with the operation of ω. Since it is clear that $V_e=L^2(\tilde{\Gamma}\backslash H)$, the trace formulas for $L^2(\tilde{\Gamma}\backslash H)$ and for V_e are the same.

In fact, Venkov [8: Chap. 6] presented trace formulas for V_e and V_o in more general cases where the discontinuous group has an ω-invariant fundamental domain.

On the other hand, Zagier [10] gave a new method to derive the trace formulas in the case of $\Gamma=PSL(2, \boldsymbol{Z})$, considering an integral of the form $I(s)=\int_{\Gamma\backslash H} K_0(z, z)E(z, s)dz$ (§ 2).

In the present paper, we shall prove the trace formula for V_e, i.e., for $L^2(\tilde{\Gamma}\backslash H)$ by means of Zagier's method in the case of $\Gamma=PSL(2, \boldsymbol{Z})$ (§ 3 and Theorem 2), and add an explicit form of the trace formula for V_o as a direct consequence of the trace formulas for $L^2(\Gamma\backslash H)$ and V_e (Theorem 3).

§ 1. Weakly symmetric Riemannian space

Let S be a Riemannian manifold with a positive definite metric $ds^2=\sum g_{ij}dx^i dx^j$. The mapping of S onto itself is called an isometry if it holds

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the metric invariant. Let Ω be a group consisting of isometries which operate on S transitively. If we have an isometry μ of S which satisfies

(i) $\mu^{-1}\Omega\mu=\Omega$, $\mu^2 \in \Omega$,

(ii) there exists an element m in Ω such that $(\mu x, \mu y)=(my, mx)$ for any $x, y \in S$,

we call the triple (Ω, S, μ) a weakly symmetric Riemannian space.

Let H be the complex upper half plane $\{z=x+iy \in \boldsymbol{C} \mid \operatorname{Im} z=y>0\}$, to which we give a Riemann structure defined by

$$ds^2=\frac{1}{y^2}(dx^2+dy^2). \tag{1.1}$$

Let furthermore ω: $z\rightarrow -\bar{z}$ be the reflection with respect to the imaginary axis, and putting $G=PSL(2, \boldsymbol{R})$, $\tilde{G}=\langle G, \omega\rangle$ be the group generated by G and ω. From the fact that the metric (1.1) is invariant under the actions of G and ω, and the triple $(G, H, 1)$ is weakly symmetric, the triple $(\tilde{G}, H, 1)$ also turns out to be a weakly symmetric Riemannian space. The metric (1.1) gives rise naturally to a $\tilde{G}$-invariant measure on H whose explicit form is

$$dz=\frac{dxdy}{y^2}. \tag{1.2}$$

It can be easily seen that

(i) $\omega^2=id$

(ii) $\omega\begin{pmatrix} a & b \\ c & d \end{pmatrix}\omega=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Hence G is a normal subgroup of $\tilde{G}$ with index 2. Namely we have

$$\tilde{G}=G\cup\omega G=G\cup G\omega. \tag{1.3}$$

Let $f(z)$ be a complex valued function on H. For $\sigma \in \tilde{G}$, the mapping $f(z)\rightarrow f(\sigma z)$ defines a linear operator. This will be denoted by T_σ. A linear operator T is called an invariant operator with respect to $\tilde{G}$, if it commutes with all T_σ $(\sigma \in \tilde{G})$, i.e., if we have $T(f(\sigma z))=(Tf)(\sigma z)$.

The Laplace-Beltrami operator induced from (1.1) on H is

$$D=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \tag{1.4}$$

and D is a generator of the commutative ring of invariant differential operators with respect to $\tilde{G}$.

§ 2. Selberg transform, Selberg kernel function

Let L be an integral operator defined by

$$(2.1)\qquad (Lf)(z)=\int_H k(z,z')f(z')dz'$$

with a kernel function $k(z, z')$. In order that an integral operator L defined by $k(z, z')$ is invariant with respect to $\tilde{G}$, it is necessary and sufficient that $k(z, z')$ satisfies the condition

$$(2.2)\qquad k(\sigma z, \sigma z')=k(z, z') \qquad \text{for every } \sigma \in \tilde{G},$$

and such a function $k(z, z')$ is called a point pair invariant with respect to $\tilde{G}$.

Now we put

$$(2.3)\qquad t(z,z')=\frac{|z-z'|^2}{yy'},\quad z=x+iy,\quad z'=x'+iy' \in H.$$

Since any point pair invariant with respect to G is a function of a positive real variable $t=t(z, z')$, and since $t(\omega z, \omega z')=t(z, z')$, any point pair invariant with respect to $\tilde{G}$ can also be identified with a function of $t=t(z, z')$. Therefore, for a point pair invariant $k(z, z')$ with respect to $\tilde{G}$, we set

$$(2.4)\qquad \varphi(t(z,z'))=k(z,z'),$$

and furthermore we impose the following condition on φ:

(2.5) *$\varphi(t)$ is a smooth function with compact support of a positive real variable t*

An invariant operator with respect to $\tilde{G}$ derived from such a function φ will be denoted by L_φ.

Theorem 1 (c.f., [6: p. 55] or [3: Theorem 1.3.2]). *Suppose that the function f on H is an eigenfunction of D with the eigenvalue $-(\frac{1}{4}+r^2)$, $r \in C$. Then, f is an eigenfunction of an arbitrary invariant integral operator L_φ with respect to $\tilde{G}$. More precisely, we have $L_\varphi f=h(r)f$, $r \in C$.*

The eigenvalue $h(r)$, determined only by L_φ and r, is called the Selberg transform. Obviously it is an even function of r, i.e., $h(r)=h(-r)$.

Proposition 1. *Let φ be such a function as in* (2.5). *Then the Selberg transform can be computed as follows. Set*

$$(2.6)\qquad Q(w)=\int_{-\infty}^{\infty}\varphi(w+v^2)dv=\int_w^\infty \frac{\varphi(t)}{\sqrt{t-w}}dt \qquad (w\geqq 0)$$

and define $g(u)$ *by*

$$(2.7)\qquad Q(w)=g(u)\quad \text{with}\quad w=e^{u}+e^{-u}-2.$$

Then we have

$$(2.8)\qquad h(r)=\int_{-\infty}^{\infty} g(u)e^{iru}du,\qquad r\in \boldsymbol{C}.$$

Conversely it holds that

$$(2.9)\qquad g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}dr,$$

$$(2.10)\qquad Q(w)=g\left(\log\frac{w+2+\sqrt{w^2+4w}}{2}\right),$$

and

$$(2.11)\qquad \varphi(t)=-\frac{1}{\pi}\int_{t}^{\infty}\frac{dQ(w)}{\sqrt{w-t}}.$$

Combining (2.8), (2.9), (2.10) *and* (2.11), *we obtain*

$$(2.12)\qquad \varphi(t)=\frac{1}{4\pi}\int_{-\infty}^{\infty} P_{-(1/2)+ir}\left(1+\frac{t}{2}\right)r\tanh \pi r h(r)dr,$$

where $P_\nu(z)$ ($\nu\in \boldsymbol{C}$, $z\in \boldsymbol{C}-(0, 1]$) *denotes a Legendre function of the first kind. Moreover* $h(r)$ *is a holomorphic function in the whole complex r-plane, and for* $r\in \boldsymbol{R}$ *it is of rapid decay as* $|r|\to\infty$.

For the proof, we refer to [3: Theorem 5.3.1] and [10: p. 319].

Put $\Gamma=PSL(2, \boldsymbol{Z})$, and let $\tilde{\Gamma}=\langle\Gamma, \omega\rangle$ be the group generated by Γ and ω. From (1.3), we have

$$(2.13)\qquad \tilde{\Gamma}=\Gamma\cup\omega\Gamma=\Gamma\cup\Gamma\omega.$$

Γ and $\tilde{\Gamma}$ are discrete subgroups of G and $\tilde{G}$ respectively, which operate on H discontinuously. The fundamental domain $\mathscr{D}$ and $\tilde{\mathscr{D}}$ of Γ and $\tilde{\Gamma}$ are given, in a standard form, by

$$\mathscr{D}=\{z\in H\,|\,|z|\geqq 1,\ -\tfrac{1}{2}\leqq \mathrm{Re}\,(z)\leqq\tfrac{1}{2}\},$$
$$\tilde{\mathscr{D}}=\{z\in H\,|\,|z|\geqq 1,\ 0\leqq \mathrm{Re}\,(z)\leqq\tfrac{1}{2}\},$$

respectively.

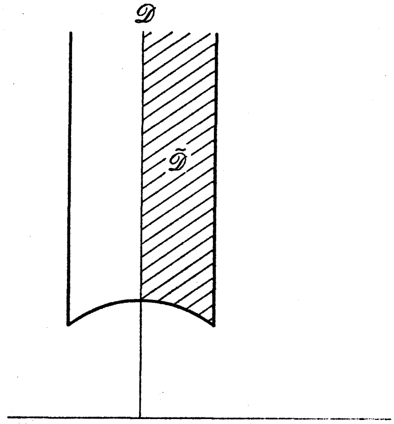

Fig. 1

Let $L^2(\tilde{\mathscr{D}})$ be the Hilbert space of measurable functions such that

(i) $f(\sigma z)=f(z)$ for all $\sigma \in \tilde{\Gamma}$,

(ii) $\int_{\tilde{\mathscr{D}}} |f(z)|^2 dz < \infty$.

Let $L_0^2(\tilde{\mathscr{D}})$ be the subspace of $L^2(\tilde{\mathscr{D}})$ satisfying the additional condition

(iii) $\int_0^{1/2} f(z)dx = \frac{1}{2}\int_0^1 f(z)dx = 0$.

The space $L^2(\tilde{\mathscr{D}})$ has the spectral decomposition with respect to D

$$L^2(\tilde{\mathscr{D}}) = L_0^2(\tilde{\mathscr{D}}) \oplus \boldsymbol{C} \oplus L^2_{conti}(\tilde{\mathscr{D}}), \tag{2.14}$$

where $\boldsymbol{C}$ is the space of constant functions, and $L^2_{conti}(\tilde{\mathscr{D}})$ is the continuous part of the spectrum.

The operator L_φ is, on $L^2(\tilde{\mathscr{D}})$, an integral operator with the kernel function $\tilde{K}(z, z')$, where

$$\tilde{K}(z, z') = \sum_{\sigma \in \tilde{\Gamma}} k(z, \sigma z'). \tag{2.15}$$

If we put $K(z, z') = \sum_{\sigma \in \Gamma} k(z, \sigma z')$ and $K'(z, z') = \sum_{\sigma \in \Gamma} k(z, \sigma\omega z')$, then from (2.13), we have

$$\tilde{K}(z, z') = K(z, z') + K'(z, z'). \tag{2.16}$$

For $z \in H$, $s \in \boldsymbol{C}$, the Eisenstein series with respect to Γ is defined by

$$E(z, s) = \sum_{\sigma \in \Gamma_0 \backslash \Gamma} \operatorname{Im}(\sigma z)^s, \tag{2.17}$$

where $\Gamma_0 = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \middle| n \in \boldsymbol{Z} \right\}$. This series converges absolutely and uniformly for $\operatorname{Re}(s)>1$ and therefore defines a holomorphic function in s which is real-analytic and Γ-invariant in z. The function (2.17) can be continued meromorphically to the whole complex s-plane, which has a simple pole at $s=1$, and satisfies a functional equation

$$E^*(z, s)=E^*(z, 1-s), \tag{2.18}$$

where

$$E^*(z, s)=\pi^{-s}\Gamma(s)\zeta(2s)E(z, s)=\zeta^*(2s)E(z, s), \tag{2.19}$$

and $\zeta(s)$ is the Riemann zeta-function. The residue at $s=1$ is

$$\operatorname*{res}_{s=1} E(z, s)=\frac{6}{\pi}\operatorname*{res}_{s=1} E^*(z, s)=\frac{3}{\pi}. \tag{2.20}$$

Now we put

$$H(z, z')=\frac{1}{4\pi}\int_{-\infty}^{\infty} E(z, \tfrac{1}{2}+ir)E(z', \tfrac{1}{2}-ir)h(r)dr. \tag{2.21}$$

Then we see that the continuous spectrum of L_φ on $L^2(\tilde{\mathscr{D}})$ can be expressed by $2H(z, z')$. Actually we have the following

Proposition 2. *Let $\tilde{K}^*(z, z')$ be the kernel function defined by*

$$\tilde{K}^*(z, z')=\tilde{K}(z, z')-2H(z, z'). \tag{2.22}$$

Then, it is bounded on $\tilde{\mathscr{D}}\times\tilde{\mathscr{D}}$.

Proof. From the definition of $\tilde{K}^*(z, z')$ and (2.16), we have

$$\tilde{K}^*(z, z')=(K(z, z')-H(z, z'))+(K'(z, z')-H(z, z')).$$

It follows from [3: Theorem 5.3.3] that $K(z, z')-H(z, z')$ is bounded on $\tilde{\mathscr{D}}\times\tilde{\mathscr{D}}$. Moreover, we can obtain the boundedness of $K'(z, z')-H(z, z')$ by a similar consideration as in the proof of [3: Theorem 5.3.3]. Namely it is sufficient to observe the following two cases:

(a) z is in a compact subset of $\tilde{\mathscr{D}}$ and z' tends to ∞,

(b) both z and z' tend to ∞.

Separating the terms with $\sigma\in\Gamma_0$ and $\sigma\notin\Gamma_0$, we have

$$K'(z, z')=\sum_{n\in Z} k(z, \omega(z'+n))+\sum_{\substack{\sigma\in\Gamma\\ \sigma\notin\Gamma_0}} k(z, \sigma\omega z').$$

Since $k(z, z')$ has a compact support by (2.5), $\sum_{\substack{\sigma\in\Gamma\\ \sigma\notin\Gamma_0}} k(z, \sigma\omega z')$ is bounded in both cases (a) and (b). Furthermore, as in the proof of [3: Theorem 5.3.3], $H(z, z')-\sqrt{yy'}\, g(\log y-\log y')$ is also bounded in both cases (a) and (b). Hence it is enough to show that

$$(*) \qquad \sum_{n\in Z} k(z, \omega(z'+n))-\sqrt{yy'}\, g(\log y-\log y')$$

is bounded.

From [3: Theorem 5.3.2], we have $\int_{-\infty}^{\infty} k(z, z'+b)db=\sqrt{yy'}\, g(\log y-\log y')$, and we easily find that $\int_{-\infty}^{\infty} k(z, z'+b)db=\int_{-\infty}^{\infty} k(z, \omega z'+b)db$. Therefore, $(*)$ is equal to

$$(**) \qquad \sum_{b\in Z} k\left(\frac{z+x'}{y'},\ i+\frac{b}{y'}\right)-y'\int_{-\infty}^{\infty} k\left(\frac{z+x'}{y'},\ i+t\right)dt.$$

However, in general, if $f(t)$ is any C^∞ function of a real variable with compact support of euclidean measure M, then f satisfies

$$\left|\frac{1}{y}\sum_{b\in Z} f\left(\frac{b}{y}\right)-\int_{-\infty}^{\infty} f(t)dt\right| \leqq \frac{M}{y}\max\left|\frac{d}{dt}f(t)\right|.$$

Applying this fact to $k((z+x')/y',\ i+b/y')$, we have

$$\frac{1}{y'}\sum_{b\in Z} k\left(\frac{z+x'}{y'},\ i+\frac{b}{y'}\right)-\int_{-\infty}^{\infty} k\left(\frac{z+x'}{y'},\ i+t\right)dt=O\left(\frac{1}{y'}\right)$$

uniformly for z as $y'\to\infty$. This implies that $(**)$ is bounded in both cases (a) and (b).

Let $L^2(\mathscr{D})$ be the Hilbert space consisting of square-integrable functions on $\mathscr{D}$. (For a detailed definition, which is essentially identical with that of $L^2(\tilde{\mathscr{D}})$, see for example [3: Chap. 5]). Let $L_0^2(\mathscr{D})$ be the space of cusp forms in $L^2(\mathscr{D})$. Then, the space $L^2(\mathscr{D})$ also has the spectral decomposition with respect to D,

$$(2.23) \qquad L^2(\mathscr{D})=L_0^2(\mathscr{D})\oplus \boldsymbol{C}\oplus L^2_{conti}(\mathscr{D}),$$

where $\boldsymbol{C}$ is the space of constant functions, and $L^2_{conti}(\mathscr{D})$ is the continuous part of the spectrum given by integrals of Eisenstein series. As is well known, we can take *Mass wave forms* $\{f_j\}_{j\geqq 1}$ as an orthogonal (but not orthonormal) basis of $L_0^2(\mathscr{D})$ ([3: Theorem 5.2.2]), i.e.,

$$f_j(z)=\sum_{n\neq 0} y^{1/2}a_j(n)K_{ir_j}(2\pi|n|y)e^{2\pi i n x},\ Df_j=-(\tfrac{1}{4}+r_j^2)f_j,\ r_j>0,$$

where $K_\nu(z)$ is the K-Bessel function defined by

$$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh \nu t dt, \qquad (\nu,\ z\in C,\ \mathrm{Re}(z)>0).$$

On the other hand, the space $L^2(\mathscr{D})$ has the direct sum decomposition in accordance with the operation of ω

$$L^2(\mathscr{D})=V_e\oplus V_o,$$

where $V_e=\{f\in L^2(\mathscr{D})|f(\omega z)=f(z)\}$ and $V_o=\{f\in L^2(\mathscr{D})|f(\omega z)=-f(z)\}$. We call the spaces V_e and V_o even and odd spaces respectively. Now if we put

$$L^2_{0,e}(\mathscr{D})=L^2_0(\mathscr{D})\cap V_e,\quad \{f_{j_1}\}_{j_1\geqq 1};\quad \text{orthogonal basis of } L^2_{0,e}(\mathscr{D}),$$
$$L^2_{0,o}(\mathscr{D})=L^2_0(\mathscr{D})\cap V_o,\quad \{f_{j_2}\}_{j_2\geqq 1};\quad \text{orthogonal basis of } L^2_{0,o}(\mathscr{D}),$$

where $\{j\}_{j\geqq 1}=\{j_1\}_{j_1\geqq 1}\cup\{j_2\}_{j_2\geqq 1}$, then, on account of $C\oplus L^2_{conti}(\mathscr{D})\subset V_e$, we have

$$V_e=L^2_{0,e}(\mathscr{D})\oplus C\oplus L^2_{conti}(\mathscr{D}),\qquad V_0=L^2_{0,o}(\mathscr{D}). \tag{2.24}$$

Moreover, since $L^2(\tilde{\mathscr{D}})=V_e$ is clear from the definition of $L^2(\tilde{\mathscr{D}})$, we obtain

$$L^2_0(\tilde{\mathscr{D}})=L^2_{0,e}(\mathscr{D}),\qquad L^2_{conti}(\tilde{\mathscr{D}})=L^2_{conti}(\mathscr{D}). \tag{2.25}$$

Therefore, we can take $\{f_{j_1}\}_{j_1\geqq 1}$ as an orthogonal basis of $L^2_0(\tilde{\mathscr{D}})$.

Let L^*_φ be an integral operator on $L^2(\tilde{\mathscr{D}})$ with a kernel function $\tilde{K}^*(z,z')$. From the fact that L^*_φ is completely continuous on $L^2(\tilde{\mathscr{D}})$, which comes from Proposition 2, and from the fact $L^*_\varphi f_{j_1}=h(r_{j_1})f_{j_1}$, we have

$$\tilde{K}^*(z,z')=\sum_{j_1=0}^\infty \frac{h(r_{j_1})}{(f_{j_1},f_{j_1})_{\tilde{\mathscr{D}}}} f_{j_1}(z)\overline{f_{j_1}(z')}, \tag{2.26}$$

where $f_0\equiv 1$ (constant), $r_0=i/2$ (since $Df_0\equiv 0$), and $(f_{j_1},f_{j_1})_{\tilde{\mathscr{D}}}=\int_{\tilde{\mathscr{D}}}|f_{j_1}(z)|^2dz$. Consequently, the above results imply the following trace formula

$$\sum_{j_1\geqq 0} h(r_{j_1})=\int_{\tilde{\mathscr{D}}}\tilde{K}^*(z,z)\,dz. \tag{2.27}$$

Venkov [8: §6.4, §6.4] presented the calculation of an integral in (2.27) by Selberg's original method in more general discontinuous groups including Γ. Here, according to Zagier [10], we will consider the integral (2.27) by the Rankin-Selberg method.

Let $\tilde{K}_0(z,z')$ be a kernel function on $L^2(\tilde{\mathscr{D}})$ such that

$$\tilde{K}_0(z,z')=\tilde{K}^*(z,z')-\frac{h(i/2)}{(f_0,f_0)_{\tilde{\mathscr{D}}}}=\sum_{j_1\geqq 1}\frac{h(r_{j_1})}{(f_{j_1},f_{j_1})_{\tilde{\mathscr{D}}}}f_{j_1}(z)\overline{f_{j_1}(z')}, \tag{2.28}$$

and put

$$(2.29)\qquad \tilde{I}(s)=\int_{\tilde{\mathscr{D}}}\tilde{K}_0(z,z)E(z,s)dz,$$

and

$$(2.30)\qquad \tilde{I}^*(s)=\int_{\tilde{\mathscr{D}}}\tilde{K}_0(z,z)E^*(z,s)dz.$$

By(2.28), we see that $\tilde{K}_0(z,z')$ is of rapid decay, hence both $\tilde{I}(s)$ and $\tilde{I}^*(s)$ can be continued to the whole complex s-plane, and have a simple pole at $s=1$. Then, by making use of $(f_0,f_0)_{\tilde{\mathscr{D}}}=\frac{1}{2}(f_0,f_0)_{\mathscr{D}}$ and (2.20), the residue of $\tilde{I}^*(s)$ at $s=1$ can be given by

$$\begin{aligned}\operatorname*{res}_{s=1}\tilde{I}^*(s)&=\frac{1}{2}\int_{\tilde{\mathscr{D}}}\tilde{K}_0(z,z)dz\\&=\frac{1}{2}\left\{\int_{\tilde{\mathscr{D}}}\tilde{K}^*(z,z)dz-h\left(\frac{i}{2}\right)\right\}.\end{aligned}$$

Namely, we have

$$(2.31)\qquad \int_{\tilde{\mathscr{D}}}\tilde{K}^*(z,z)dz=2\operatorname*{res}_{s=1}\tilde{I}^*(s)+h\left(\frac{i}{2}\right).$$

If we put

$$(2.32)\qquad \begin{aligned}K_0(z,z')&=K(z,z')-\frac{3}{\pi}h\left(\frac{i}{2}\right)-H(z,z'),\\K'_0(z,z')&=K'(z,z')-\frac{3}{\pi}h\left(\frac{i}{2}\right)-H(z,z'),\end{aligned}$$

then from (2.16), (2.22), (2.28) and $(f_0,f_0)_{\mathscr{D}}=\pi/3$ we have

$$\begin{aligned}\tilde{I}(s)&=\int_{\tilde{\mathscr{D}}}K_0(z,z)E(z,s)dz+\int_{\tilde{\mathscr{D}}}K'_0(z,z)E(z,s)dz\\&=\frac{1}{2}\left\{\int_{\mathscr{D}}K_0(z,z)E(z,s)dz+\int_{\mathscr{D}}K'_0(z,z)E(z,s)dz\right\}.\end{aligned}$$

Furthermore set

$$(2.33)\qquad I(s)=\int_{\mathscr{D}}K_0(z,z)E(z,s)dz,\qquad I'(s)=\int_{\mathscr{D}}K'_0(z,z)E(z,s)dz,$$

then, we easily obtain

(2.34) $$\operatorname*{res}_{s=1} \tilde{I}(s)=\tfrac{1}{2}(\operatorname*{res}_{s=1} I(s)+\operatorname*{res}_{s=1} I'(s)).$$

Similarly, if we put

(2.35) $$I^*(s)=\int_{\mathcal{D}} K_0(z, z)E^*(z, s)dz, \qquad I'^*(s)=\int_{\mathcal{D}} K_0'(z, z)E^*(z, s)dz,$$

then we have

(2.36) $$\operatorname*{res}_{s=1} \tilde{I}^*(s)=\tfrac{1}{2}(\operatorname*{res}_{s=1} I^*(s)+\operatorname*{res}_{s=1} I'^*(s)).$$

§ 3. Computation of $\tilde{I}(s)$ and its residue at $s=1$

3.1. Computation of $I'(s)$

From the definition of $I'(s)$, we have

(3.1) $$I'(s)=\int_0^\infty \mathcal{K}'(y)y^{s-2}dy \qquad \text{for } \operatorname{Re}(s)>1,$$

where

(3.2) $$\mathcal{K}'(y)=\int_0^1 K_0'(z, z)dx.$$

According to Zagier [10: p. 323 or p. 352], we decompose $\mathcal{K}'(y)$ into four parts, i.e.,

$$\mathcal{K}'(y)=\sum_{i=1}^{4} \mathcal{K}_i'(y)$$

with

$$\mathcal{K}_1'(y)=\int_0^1 \sum_{\substack{\sigma\in\Gamma \\ \sigma\notin\Gamma_0}} k(z, \sigma\omega z)dx,$$

$$\mathcal{K}_2'(y)=\int_0^1 \sum_{\sigma\in\Gamma_0} k(z, \sigma\omega z)dx-\frac{y}{2\pi}\int_{-\infty}^{\infty} h(r)dr,$$

$$\mathcal{K}_3'(y)=-\frac{y}{2\pi}\int_{-\infty}^{\infty} y^{2ir}\frac{\zeta^*(1+2ir)}{\zeta^*(1-2ir)}h(r)dr-\frac{3}{\pi}h\left(\frac{i}{2}\right),$$

$$\mathcal{K}_4'(y)=-\frac{2y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\zeta^*(1+2ir)\zeta^*(1-2ir)}\left(\sum_{n=1}^{\infty}\tau_{ir}^2(n)\,K_{ir}^2(2\pi ny)\right)h(r)dr,$$

where

$$\tau_\nu(n)=|n|^\nu \sum_{\substack{d|n\\ d>0}} d^{-2\nu}, \qquad (n \in \boldsymbol{Z}-\{0\},\ \nu \in \boldsymbol{C}).$$

If we write $I'_i(s)=\int_0^1 \mathscr{K}'_i(y)y^{s-2}dy$ $(i=1,\cdots,4)$, then $I'(s)=\sum_{i=1}^4 I'_i(s)$ follows easily, and if we furthermore set $I'^*_i(s)=\zeta^*(2s)I'_i(s)$ $(i=1,\cdots,4)$, then we get $I'^*(s)=\sum_{i=1}^4 I'^*_i(s)$.

Now, we will calculate $I'_i(s)$ $(i=1,\cdots,4)$ separately.

(i) $I'_2(s)$.

Since $\sum_{\sigma\in\Gamma_0} k(z,\sigma\omega z)=\sum_{n=-\infty}^{\infty}\varphi(|2x-n|^2/y^2)$, we have

$$\int_0^1 \sum_{\sigma\in\Gamma_0} k(z,\sigma\omega z)dx=\int_{-\infty}^{\infty}\varphi\left(\frac{x^2}{y^2}\right)dx.$$

However, in view of (2.9), (2.7) and (2.6), we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)dr=\frac{1}{y}\int_{-\infty}^{\infty}\varphi\left(\frac{x^2}{y^2}\right)dx.$$

This implies that $\mathscr{K}'_2(y)\equiv 0$, namely

$$I'_2(s)\equiv 0. \tag{3.3}$$

(ii) $I'_3(s)$ and $I'_4(s)$.

By definition, $I'_3(s)$ and $I'_4(s)$ are equal to $I_3(s)$ and $I_4(s)$ in [10: Theorem 2], respectively. Hence, we have by [10: (3.4)]

$$I'_4(s)=-\frac{1}{4\pi}\frac{\zeta^{*2}(s)}{\zeta^*(2s)}\int_{-\infty}^{\infty}\frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}h(r)dr$$

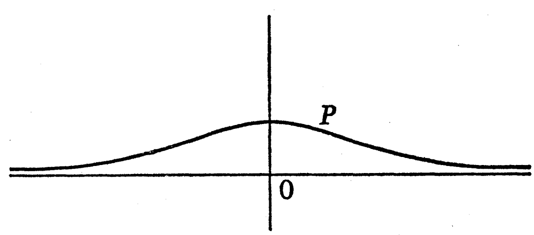

Fig. 2

for Re $(s)>1$. Next, let P be a smooth curve which is sufficiently close to the real axis such that all zeroes of the Riemann zeta-function on the left of $1+2iP$ and $\zeta(1+2ir)^{-1}=O(|r|^\varepsilon)$ for $r\in P$, $\varepsilon>0$, and put

$$J_P(s)=\int_P \frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}h(r)dr.$$

Then, from [10: (4.8)], $I_4'^*(s)$ can be continued holomorphically to a sufficiently small neighbourhood U of the point $s=1$ by the following identity:

$$I_4'^*(s) = -\frac{1}{4\pi}\zeta^{*2}(s)J_P(s) - \frac{1}{4}\frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)}h\left(i\frac{s-1}{2}\right) \qquad \text{in } s \in U.$$

Thus, considering an expansion

(3.4) $\zeta^*(s) = (s-1)^{-1} + \frac{1}{2}(\gamma - \log 4\pi) + O(s-1)$ (γ: Euler constant),

we obtain the Laurent expansion of $I_4'^*(s)$ at $s=1$:

$$(3.5) \qquad I_4'^*(s) = -\kappa(s-1)^{-2} + \left\{-\kappa(\gamma - \log 4\pi) + \frac{h(0)}{8} - \frac{1}{4\pi}\int_{-\infty}^{\infty} z(r)h(r)dr\right\}(s-1)^{-1} + O(1),$$

where

$$\kappa = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)dr = \frac{1}{2}g(0)$$

and

$$z(r) = \frac{\zeta^{*\prime}}{\zeta^*}(1+2ir) + \frac{\zeta^{*\prime}}{\zeta^*}(1-2ir),$$

([10: p. 340]).

As for $I_3'(s)$, we see by [10: (3.5)] that

$$I_3'(s) = -\frac{1}{2}\frac{\zeta^*(s)}{\zeta^*(s+1)}h\left(\frac{is}{2}\right) \qquad \text{for } \mathrm{Re}(s) > 1.$$

Since $h(r)$ is a holomorphic function in the whole complex r-plane, $I_3'(s)$ can be continued meromorphically to the whole complex s-plane by the right hand side of the above equality. Therefore, we obtain

$$(3.6) \qquad \operatorname*{res}_{s=1} I_3'^*(s) = \operatorname*{res}_{s=1}(\zeta^*(2s)I_3'(s)) = -\frac{1}{2}h\left(\frac{i}{2}\right), \qquad \text{([10: p. 340]).}$$

(iii) $I_1'(s)$.

It can be seen that $I_1'(s)$ coincides with the case of $m=-1$ in [10: (5.6)], hence we obtain

$$(3.7) \qquad I_1'(s) = \sum_{t=-\infty}^{\infty} \frac{\zeta(s, t^2+4)}{\zeta(2s)} V_{-}(s, t) \qquad \text{for } \mathrm{Re}(s) > 1,$$

where

$$V_-(s,t)=\int_H \varphi\left(\frac{(|z|^2-(\Delta/4))^2}{y^2}+t^2\right)y^s dz, \qquad (\Delta=t^2+4),$$

and $\zeta(s, t^2+4)$ is a zeta-function defined by Zagier [10: (1.12)] or [9: (6)]. To explain $\zeta(s, t^2+4)$ more precisely, consider a binary quadratic form

$$Q(u,v)=au^2+buv+cv^2, \qquad (a, b, c \in \boldsymbol{Z}),$$

on which the group $SL(2, \boldsymbol{Z})$ operates by

$$(\gamma \circ Q)(u,v)=Q(au+cv, bu+dv), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z}),$$

and let $|Q|=b^2-4ac=D$ be the discriminant of Q. Then, the zeta-function $\zeta(s, D)$ is defined by

$$(3.8) \qquad \zeta(s, D)=\sum_{\substack{[Q] \\ |Q|=D}} \sum_{\substack{(m,n)\in \boldsymbol{Z}^2/\mathrm{Aut}\,Q \\ Q(m,n)>0}} \frac{1}{Q(m,n)^s} \qquad \text{for } \mathrm{Re}\,(s)>1,$$

where the first sum ranges over $SL(2, \boldsymbol{Z})$-equivalence classes of quadratic forms Q with discriminant D, and

$$(3.9) \qquad \mathrm{Aut}\, Q=\{\gamma \in SL(2, \boldsymbol{Z}) | \gamma \circ Q=Q\}.$$

Transforming z into $(\sqrt{\Delta}/4)(z+1)/(-z+1)$, we find

$$V_-(s,t)=\Delta^{s/2}\int_H \varphi\left(\frac{\Delta x^2+t^2y^2}{y^2}\right)\frac{y^s}{|1-x-iy|^{2s}}\frac{dxdy}{y^2}$$

$$=\Delta^{s/2}\int_{-\infty}^{\infty}\frac{\varphi(\Delta u^2+t^2)}{(1+u^2)^{s/2}} \cdot \int_0^{\infty}\frac{v^{s-1}}{\left(1-\frac{2u}{\sqrt{u^2+1}}v+v^2\right)^s}dvdu,$$

putting $\quad u=\dfrac{y}{x}, \quad v=\sqrt{x^2+y^2}$.

By using [1:2.12(10), 2.1.5 (28)], we get

$$(3.10) \quad V_-(s,t)=\frac{1}{2}\frac{\Gamma(s/2)^2}{\Gamma(s)}\Delta^{s/2}\int_{-\infty}^{\infty}\frac{\varphi(\Delta u^2+t^2)}{(1+u^2)^{s/2}}F\left(\frac{s}{2}, \frac{s}{2}; \frac{1}{2}; \frac{u^2}{u^2+1}\right)du,$$

where $F={}_2F_1$ is a hypergeometric function. The integral in (3.10) converges absolutely for all $s \in \boldsymbol{C}$, and by a similar consideration as in [10: p. 335], $I_1'(s)$ can be continued meromorphically to the whole complex s-plane, which has at most a 2-order pole at $s=1$.

Further computation of $V_{-}(s, t)$.
In view of (2.12) and (3.10), we can write

$$(3.11)\quad V_{-}(s,t) = \frac{\Delta^{s/2}}{8\pi}\frac{\Gamma(s/2)^2}{\Gamma(s)}\int_{-\infty}^{\infty} r\tanh \pi r h(r)\int_0^1 P_{-(1/2)+ir}\left(-1+\frac{\Delta}{2(1-\xi)}\right) \times(1-\xi)^{(s-3)/2}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}dr.$$

From [1:3.2 (18)] or [10: p. 353], we have

$$P_{-(1/2)+ir}\left(-1+\frac{\Delta}{2(1-\xi)}\right) = \mathscr{S}_r\left[\frac{\Gamma(2ir)}{\Gamma(\frac{1}{2}+ir)^2}\left(\frac{4}{\Delta}\right)^{(1/2)-ir}F\left(\frac{1}{2}-ir,\frac{1}{2}-ir;1-2ir;\frac{4(1-\xi)}{\Delta}\right)\right],$$

where $\mathscr{S}_r[f(r)]=f(r)+f(-r)$ for any function f. Thus, using the hypergeometric series, we find that

$$\int_0^1 P_{-(1/2)+ir}\left(-1+\frac{\Delta}{2(1-\xi)}\right)(1-\xi)^{(s-3)/2}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}$$
$$=\mathscr{S}_r\left[\frac{\Gamma(2ir)}{\Gamma(\frac{1}{2}+ir)^2}\left(\frac{4}{\Delta}\right)^{(1/2)-ir}\int_0^1(1-\xi)^{(s/2)-1-ir} \times F\left(\frac{1}{2}-ir,\frac{1}{2}-ir;1-2ir;\frac{4(1-\xi)}{\Delta}\right)F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}\right]$$
$$=\mathscr{S}_r\left[\frac{\Gamma(2ir)\Gamma(1-2ir)}{\Gamma(\frac{1}{2}+ir)^2\Gamma(\frac{1}{2}-ir)^2}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(n+\frac{1}{2}-ir)^2}{\Gamma(n+1-2ir)} \times\left(\frac{\Delta}{4}\right)^{-n-(1/2)+ir}\int_0^1(1-\xi)^{(s/2)+n-1-ir}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}\right].$$

Then, by [1: 2.4 (2)]

$$\int_0^1(1-\xi)^{(s/2)+n-1-ir}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}} = \frac{\Gamma(\frac{1}{2})\Gamma((s/2)+n-ir)}{\Gamma((1+s)/2+n-ir)}F\left(\frac{s}{2},\frac{s}{2};\frac{1+s}{2}+n-ir;1\right)$$

for $\mathrm{Re}\,(s)>0$, also by [1: 2.8 (46)]

$$F\left(\frac{s}{2}, \frac{s}{2}; \frac{1+s}{2}+n-ir; 1\right)=\frac{\Gamma((1+s)/2-ir+n)\Gamma((1-s)/2-ir+n)}{\Gamma(\frac{1}{2}-ir+n)^2}$$

for $\mathrm{Re}(s)<1$.

Therefore, we show that

$$\int_0^1 P_{-(1/2)+ir}\left(-1+\frac{\Delta}{2(1-\xi)}\right)(1-\xi)^{(s-3)/2}F\left(\frac{s}{2}, \frac{s}{2}; \frac{1}{2}; \xi\right)\frac{d\xi}{\sqrt{\xi}}$$

$$(*) \qquad =\mathscr{S}_r\left[\frac{\coth \pi r}{2i\sqrt{\pi}}\frac{\Gamma(s/2-ir)\Gamma((1-s)/2-ir)}{\Gamma(1-2ir)}\left(\frac{\Delta}{4}\right)^{ir-1/2}\right.$$
$$\left.\times F\left(\frac{s}{2}-ir, \frac{1-s}{2}-ir; 1-2ir; \frac{4}{\Delta}\right)dr\right]$$

for $0<\mathrm{Re}(s)<1$ (c.f., [10: p. 353]). Substituting $(*)$ into (3.11), we obtain finally

$$(3.12) \qquad V_-(s, t)=\frac{\Delta^{s/2}}{8\pi i\sqrt{\pi}}\frac{\Gamma(s/2)^2}{\Gamma(s)}\int_{-\infty}^{\infty} rh(r)\frac{\Gamma(s/2-ir)\Gamma((1-s)/2-ir)}{\Gamma(1-2ir)}$$
$$\times\left(\frac{\Delta}{4}\right)^{ir-(1/2)}F\left(\frac{s}{2}-ir, \frac{1-s}{2}-ir; 1-2ir; \frac{4}{\Delta}\right)dr$$

for $0<\mathrm{Re}(s)<1$.

In view of (3.7), we have

$$I_1'^*(s)=\sum_{t=-\infty}^{\infty}\pi^{-s}\Gamma(s)\zeta(s, t^2+4)V_-(s, t).$$

From now on, we will investigate the residue or the Laurent expansion of the above series at $s=1$, separating the terms with $t\neq 0$ and $t=0$.

1) In the case of $t\neq 0$, it follows from [9: Proposition 3] that $\zeta(s, t^2+4)$ has a simple pole at $s=1$, thus

$$(3.13) \quad \operatorname*{res}_{s=1}\left(\sum_{t\neq 0}\pi^{-s}\Gamma(s)\zeta(s, t^2+4)V_-(s, t)\right)=\frac{1}{\pi}\sum_{t\neq 0}V_-(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2+4).$$

Then, by (3.10)

$$V_-(1, t)=\frac{\pi}{2}\Delta^{1/2}\int_{-\infty}^{\infty}\frac{\varphi(\Delta u^2+t^2)}{(1+u^2)^{1/2}}F\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{u^2}{u^2+1}\right)du.$$

By making use of [1: 2.1.4 (22)], i.e., $(1+u^2)^{-1/2}F(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; u^2/(u^2+1))=F(\frac{1}{2}, 0; \frac{1}{2}; -u^2)$, $F(a, 0; c; x)=1$ and (2.6), we obtain

(3.14) $$V_{-}(1, t)=\frac{\pi}{2}\int_{t^2}^{\infty}\frac{\varphi(x)}{\sqrt{x-t^2}}dx.$$

2) In the case of $t=0$, clearly $\Delta=4$, thus we have by (3.12)

$$V_{-}(s, 0)=\frac{4^{s/2}}{8\pi i\sqrt{\pi}}\frac{\Gamma(s/2)^2}{\Gamma(s)}\int_{-\infty}^{\infty}rh(r)\frac{\Gamma(s/2-ir)\Gamma((1-s)/2-ir)}{\Gamma(1-2ir)}$$
$$\times F\left(\frac{s}{2}-ir, \frac{1-s}{2}-ir; 1-2ir; 1\right)dr.$$

Utilizing [1: 2.8 (46)], we see that

$$F\left(\frac{s}{2}-ir, \frac{1-s}{2}-ir; 1-2ir; 1\right)=\frac{\Gamma(\frac{1}{2})\Gamma(1-2ir)}{\Gamma(-s/2-ir+1)\Gamma(s/2-ir+\frac{1}{2})}.$$

Hence, we obtain

(3.15) $$V_{-}(s, 0)$$
$$=\frac{4^{s/2}}{8\pi i}\frac{\Gamma(s/2)^2}{\Gamma(s)}\int_{-\infty}^{\infty}\frac{\Gamma(s/2-ir)\Gamma((1-s)/2-ir)}{\Gamma(-s/2-ir+1)\Gamma(s/2-ir+\frac{1}{2})}rh(r)dr$$

for $0<\mathrm{Re}(s)<1$.

To derive a Laurent expansion of $\pi^{-s}\Gamma(s)\zeta(s, 4)V_{-}(s, 0)$ at $s=1$, we must settle the analytic continuation of $V_{-}(s, 0)$ to a neighbourhood U of the point $s=1$, and to do this, we use a similar method as in the case of $I_4'(s)$. Put

$$F(s, r)=\frac{\Gamma(s/2-ir)\Gamma((1-s)/2-ir)}{\Gamma(-s/2-ir+1)\Gamma(-s/2-ir+\frac{1}{2})}rh(r).$$

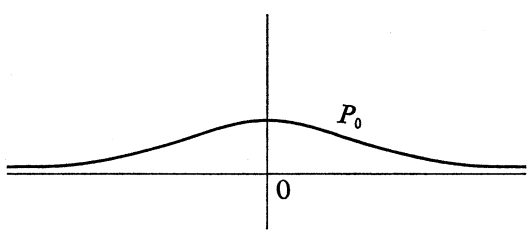

Fig. 3

Let P_0 be a smooth curve which is sufficiently close to the real axis and let

$$J(s)=\int_{-\infty}^{\infty}F(s, r)dt, \qquad J_{P_0}(s)=\int_{P_0}F(s, r)dr.$$

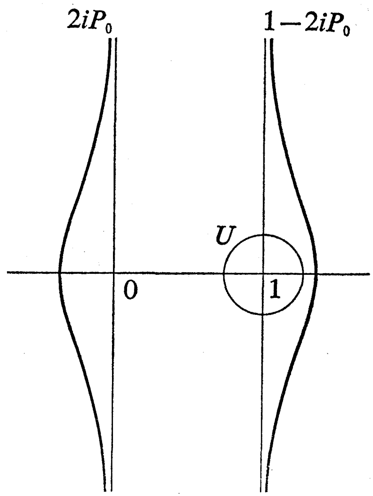

Fig. 4

Then, as is easily seen, $F(s, r)$ has a pole with respect to r in the region enclosed by P_0 and the real axis if and only if s lies in the domain between $2iP_0$ and the imaginary axis or in the domain between $1-2iP_0$ and the line $\sigma=\mathrm{Re}(s)=1$ as in Fig. 4. Thus,

$$J(s)=J_{P_0}(s) \qquad \text{in } 0<\mathrm{Re}(s)<1.$$

On the other hand, $J_{P_0}(s)$ is holomorphic in the region $2iP_0<\mathrm{Re}(s)<1-2iP_0$, therefore putting

$$J(s)=J_{P_0}(s) \qquad \text{in } U,$$

we can give the analytic continuation of $J(s)$ to a neighbourhood U of the point $s=1$. Furthermore it follows from [9: Proposition 3] that $\zeta(s, 4)=\zeta^2(s)(1+2^{1-2s}-2^{-s})$. Hence, we have

$$\text{(3.16)} \quad \pi^{-s}\Gamma(s)\zeta(s,4)V_-(s,0)=\frac{1}{8\pi i}\zeta^{*2}(s)(-1+2^s+2^{1-s})J_{P_0}(s) \qquad \text{in } U.$$

Laurent expansion of $\pi^{-s}\Gamma(s)\zeta(s,4)V_-(s,0)$ *at* $s=1$.

By (3.4), we have

$$\zeta^{*2}(s)=(s-1)^{-2}+(\gamma-\log 4\pi)(s-1)^{-1}+O(1),$$

and also

$$(-1+2^s+2^{1-s})=2+\log 2\cdot(s-1)+O((s-1)^2).$$

Moreover, for $J_{P_0}(s)$, we find

$$J_{P_0}(1)=\int_{P_0}\frac{\Gamma(-ir)}{\Gamma(1-ir)}rh(r)dr=-\frac{1}{i}\int_{P_0}h(r)dr$$
$$=-\frac{1}{i}\int_{-\infty}^{\infty}h(r)dr=2\pi ig(0),$$

and

$$J'_{P_0}(1)=\frac{1}{i}\int_{P_0}\left\{\frac{\Gamma'}{\Gamma}(1-ir)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-ir\right)\right\}h(r)dr-\frac{1}{2}\int_{P_0}\frac{h(r)}{r}dr.$$

Using $\int_{P_0}(h(r)/r)dr=-\pi ih(0)$ and $h(r)=h(-r)$, we see that the last expression is equal to

$$\frac{1}{i}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+ir)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\right\}h(r)dr+\frac{\pi i}{2}h(0).$$

It follows from these facts that the Laurent expansion of $\pi^{-s}\Gamma(s)\zeta(s, 4)\times V_{-}(s, 0)$ at $s=1$ can be written as

$$\begin{aligned}(3.17)\qquad &\pi^{-s}\Gamma(s)\zeta(s, 4)V_{-}(s, 0)\\ &=\frac{1}{8\pi i}\{(s-1)^{-2}+(\gamma-\log 4\pi)(s-1)^{-1}\}\times\{2+\log 2\cdot(s-1)\}\\ &\quad\times\{2\pi ig(0)+J'_{P_0}(1)(s-1)\}+O(1)\\ &=\frac{1}{2}g(0)(s-1)^{-2}+\frac{g(0)}{4}\{\log 2+2(\gamma-\log 4\pi)\}(s-1)^{-1}\\ &\quad+\frac{1}{8}h(0)(s-1)^{-1}-\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+ir)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\right\}\\ &\quad\times h(r)dr\cdot(s-1)^{-1}+O(1).\end{aligned}$$

Now, since $\operatorname{res}_{s=1}I'^*(s)=\sum_{i=1}^{4}\operatorname{res}_{s=1}I_i'^*(s)$, adding up (3.3), (3.5), (3.6), (3.13) and (3.17), we obtain the following

Proposition 3. *Let $K_0'(z, z')$ be the kernel function defined by* (2.32), *and put $I'^*(s)=\int_{\mathcal{D}}K_0'(z, z)E^*(z, s)dz$. Then, $\operatorname{res}_{s=1}I'^*(s)$ can be expressed as*

$$\operatorname*{res}_{s=1}I'^*(s)=\frac{\log 2}{4}g(0)+\frac{1}{4}h(0)-\frac{1}{2}h\left(\frac{i}{2}\right)$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{z(r)+\frac{\Gamma'}{\Gamma}(1+ir)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\right\}h(r)dr$$

$$+\frac{1}{\pi}\sum_{t\neq 0}V_{-}(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2+4),$$

where $z(r)=(\zeta^{*\prime}/\zeta^*)(1+2ir)+(\zeta^{*\prime}/\zeta^*)(1-2ir)$ *and* $V_{-}(1, t)$ *is as in* (3.14).

3.2. Computation of $I(s)$

According to its definition, $I(s)$ coincides with that of Zagier [10] completely, thus we have the following

Proposition 4 ([10: p. 342]). *Let* $K_0(z, z')$ *be the kernel function defined by* (2.32), *and put* $I^*(s)=\int_{\mathscr{D}} K_0(z, z)E^*(z, s)dz$. *Then,* $\operatorname{res}_{s=1} I^*(s)$ *can be expressed as*

$$\operatorname*{res}_{s=1} I^*(s)=-\frac{\log 2}{2}g(0)+\frac{1}{4}h(0)-\frac{1}{2}h\left(\frac{i}{2}\right)$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{z(r)+\frac{\Gamma'}{\Gamma}(1+ir)\right\}h(r)dr$$

$$+\frac{1}{24}\int_{-\infty}^{\infty}\tanh \pi r r h(r)dr$$

$$+\frac{1}{\pi}\sum_{t^2\neq 4}V(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2-4),$$

where

$$V(1, t)=\begin{cases}\dfrac{\pi}{2}\displaystyle\int_0^{\infty}\frac{\varphi(x)}{\sqrt{x+4-t^2}}dx & |t|<2,\\[2ex] \dfrac{\pi}{2}\displaystyle\int_{t^2-4}^{\infty}\frac{\varphi(x)}{\sqrt{x+4-t^2}}dx & |t|>2.\end{cases} \tag{3.18}$$

§ 4. Trace formula

We have by [10: (4.13)]

$$\operatorname*{res}_{s=1}\zeta(s, D)=\begin{cases}\dfrac{2\pi}{\sqrt{|D|}}\displaystyle\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=D}}\frac{1}{|\mathrm{Aut}\, Q|} & D<0,\\[2ex] \dfrac{1}{\sqrt{|D|}}\displaystyle\sum_{\substack{Q \bmod SL(2,Z\\ |Q|=D}}\log \varepsilon_Q & D>0,\end{cases} \tag{4.1}$$

where Aut Q is the group defined by (3.9) and ε_Q is the largest eigenvalue of the matrix M which is a generator of Aut Q up to $\{\pm 1\}$ with positive trace, i.e, Aut $Q=\{\pm M^n | n \in Z\}$.

To make the correspondence between Selberg's method and Zagier's method clear in the computation of the integral in (2.27), we will calculate $V_-(1, t)\operatorname{res}_{s=1}\zeta(s, t^2+4)$ and $V(1, t)\operatorname{res}_{s=1}\zeta(s, t^2-4)$ in a more explicit form. Since t^2+4 is always positive, using (3.14) and (4.1), we can write

$$V_-(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2+4)=\frac{\pi}{4}\frac{1}{\sqrt{t^2+4}}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2+4}}\log\varepsilon_Q^2\cdot\int_{t^2}^{\infty}\frac{\varphi(x)}{\sqrt{x-t^2}}dx.$$

Then, from the fact that M is a generator of Aut Q up to $\{\pm 1\}$, it follows that there exists a positive number $l\,(=l_Q \in \frac{1}{2}Z)$such that $\varepsilon_Q^l-\varepsilon_Q^{-l}=t$ corresponding to each Q. Thus, using (2.6) and (2.7), we have

$$V_-(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2+4)=\frac{\pi}{4}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2+4\\ \varepsilon_Q^l-\varepsilon_Q^{-l}=t}}\frac{\log\varepsilon_Q^2}{\varepsilon_Q^l+\varepsilon_Q^{-l}}g(l\log\varepsilon_Q^2). \tag{4.2}$$

In the case of $t^2-4>0$, a similar consideration as for t^2+4 is possible, namely there exists a positive integer $l\,(=l_Q \in Z\geqq 1)$ such that $\varepsilon_Q^l+\varepsilon_Q^{-l}=t$ corresponding to each Q, therefore it follows from (3.18) and (4.1) that

$$V(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2-4)=\frac{\pi}{4}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2+4\\ \varepsilon_Q^l+\varepsilon_Q^{-l}=t}}\frac{\log\varepsilon_Q^2}{\varepsilon_Q^l-\varepsilon_Q^{-l}}g(l\log\varepsilon_Q^2). \tag{4.3}$$

As for $t^2-4<0$, further calculations after (3.18) yield

$$\begin{aligned} &V(1, t)\operatorname*{res}_{s=1}\zeta(s, t^2-4)\\ &=\frac{\pi}{\sqrt{4-t^2}}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2-4}}\frac{1}{|\mathrm{Aut}\, Q|}\cdot\int_{-\infty}^{\infty}\frac{e^{-2\alpha r}}{1+e^{-2\pi r}}h(r)dr, \end{aligned} \tag{4.4}$$

where $|t|=2\cos\alpha$, $0<\alpha\leqq\pi/2$.

Combining (2.27), (2.31) and (2.36), we find that

$$\sum_{j_1\geqq 0}h(r_{j_1})=\int_{\tilde{\mathscr{D}}}\tilde{K}^*(z, z)dz=\operatorname*{res}_{s=1}I^*(s)+\operatorname*{res}_{s=1}I'^*(s)+h\left(\frac{i}{2}\right). \tag{4.5}$$

Hence, by using Propositions 3, 4, we obtain a trace formula on the even space V_e.

Theorem 2 (Trace formula on V$_e$). *Let $L_{0,e}(\mathscr{D})$ be the space of cusp forms in V_e, and let $\{f_{j_1}\}_{j_1\geq 1}$ be the orthogonal basis of $L_{0,e}(\mathscr{D})$ consisting of*

Maass wave forms. If the eigenvalue of each f_{j_1} *with respect to D is given by* $Df_{j_1}=-(\frac{1}{4}+r_{j_1}^2)f_{j_1}$, *then we obtain*

$$\sum_{j_1\geqq 0} h(r_{j_1})$$
$$=-\frac{\log 2}{4}g(0)+\frac{1}{2}h(0)+\frac{1}{24}\int_{-\infty}^{\infty}\tanh \pi r r h(r)dr$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{z(r)+\frac{\Gamma'}{\Gamma}(1+ir)\right\}h(r)dr+\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)h(r)dr$$
$$+\sum_{|t|<2}\frac{1}{\sqrt{4-t^2}}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2-4}}\frac{1}{|\mathrm{Aut}\,Q|}\cdot\int_{-\infty}^{\infty}\frac{e^{-2\cos^{-1}(|t|/2)r}}{1+e^{-2\pi r}}h(r)dr$$
$$+\frac{1}{4}\sum_{|t|>2}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2-4\\ \varepsilon_Q^l+\varepsilon_Q^{-l}=t}}\frac{\log \varepsilon_Q^2}{\varepsilon_Q^l-\varepsilon_Q^{-l}}g(l\log \varepsilon_Q^2)$$
$$+\frac{1}{4}\sum_{t\neq 0}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2+4\\ \varepsilon_Q^l-\varepsilon_Q^{-l}=t}}\frac{\log \varepsilon_Q^2}{\varepsilon_Q^l+\varepsilon_Q^{-l}}g(l\log \varepsilon_Q^2),$$

where $z(r)=(\zeta^{*\prime}/\zeta^*)(1+2ir)+(\zeta^{*\prime}/\zeta^*)(1-2ir)$ *and* Aut Q *is as in* (3.9).

As is proved in Zagier [10], the trace formula on $L^2(\mathscr{D})$ is

$$\sum_{j\geqq 0} h(r_j)=2\operatorname*{res}_{s=1} I^*(s)+h\left(\frac{i}{2}\right).$$

Thus, we have by (4.5)

$$\sum_{j_2\geqq 1} h(r_{j_2})=\sum_{j\geqq 0} h(r_j)-\sum_{j_1\geqq 0} h(r_{j_1})=\operatorname*{res}_{s=1} I^*(s)-\operatorname*{res}_{s=1} I'^*(s).$$

Again, by using Propositions 3, 4, we have a trace formula on the odd space V_o.

Theorem 3 (Trace formula on V_o). *Let* $\{f_{j_2}\}_{j_2\geqq 1}$ *be the orthogonal basis of the space* V_o *consisting of Maass wave forms. If the eigenvalue of each* f_{j_2} *with respect to D is given by* $Df_{j_2}=-(\frac{1}{4}+r_{j_2}^2)f_{j_2}$, *then we obtain*

$$\sum_{j_2\geqq 1} h(r_{j_2})$$
$$=-\frac{3}{4}\log 2\cdot g(0)+\frac{1}{24}\int_{-\infty}^{\infty}\tanh \pi r r h(r)dr-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)h(r)dr$$
$$+\sum_{|t|<}\frac{1}{\sqrt{4-t^2}}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2-4}}\frac{1}{|\mathrm{Aut}\,Q|}\cdot\int_{-\infty}^{\infty}\frac{e^{-2\cos^{-1}(|t|/2)r}}{1+e^{-2\pi r}}h(r)dr$$

$$+\frac{1}{4}\sum_{|t|>2}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2-4\\ \varepsilon_Q^l+\varepsilon_Q^{-l}=t}}\frac{\log \varepsilon_Q^2}{\varepsilon_Q^l-\varepsilon_Q^{-l}}g(l\log \varepsilon_Q^2)$$

$$-\frac{1}{4}\sum_{t\neq 0}\sum_{\substack{Q \bmod SL(2,Z)\\ |Q|=t^2+4\\ \varepsilon_Q^l-\varepsilon_Q^{-l}=t}}\frac{\log \varepsilon_Q^2}{\varepsilon_Q^l+\varepsilon_Q^{-l}}g(l\log \varepsilon_Q^2),$$

where Aut Q *is as in* (3.9).

The formula of Theorem 3 is just the same as the formula of Venkov [8: Theorem 6.5.4] in the case of $\Gamma=PSL(2, Z)$.

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