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A Tripling Symbol for Central Extensions of Algebraic Number Fields

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Let K/k be a finite abelian extension of a finite algebraic number field and M be a Galois extension of k which contains K. Denote by $\hat{K}_{M/k}$ and $K^*_{M/k}$ the maximal central extension of K/k in M and the genus field of K/k in M. Since K/k is abelian, $K^*_{M/k}$ coincides with the maximal abelian extension of k in M. In general, the Galois group $G(\hat{K}_{M/k}/K^*_{M/k})$ is isomorphic to a quotient group of the dual $M(G)=H^{-3}(G, Z)$ of the Schur multiplier $H^2(G, Q/Z)$ of G. If M is enough large, $G(\hat{K}_{M/k}/K^*_{M/k})$ is isomorphic to $M(G)$. In such a case, we call M abundant for K/k.

Furuta [2] gives a prime decomposition symbol $[d_1, d_2, p]$ which indicates the decomposition in $\hat{K}_{M/k}/K^*_{M/k}$ of a prime p which is degree 1 in $K^*_{M/k}$, where $k=Q$, $K=Q(\sqrt{d_1}, \sqrt{d_2})$ and M is a ray class field of K which is abundant for K/k. Also it proves the inversion formula $[p_1, p_2, p_3]=[p_1, p_3, p_2]$ except only a case.

Akagawa [1] extended this symbol to $(x, y, z)_n$ for any kummerian bicyclic extension $K=k(\sqrt[n]{x}, \sqrt[n]{y})$ over any base field k with serveral conditions which make $(x, y, z)_n$ and $(x, z, y)_n$ defined and the inversion formula $(x, y, z)_n(x, z, y)_n=1$ be true. This contains the proof of the excepted case of Furuta [2].

In this paper, we extend the symbol [, ,] as a character of the number knot modulo $\mathfrak{m}$ of K/k with $\mathfrak{m}$ being a Scholz conductor of K/k which is defined in Heider [4]. The character is defined by using the inverse map $H^{-1}(G, C_K)\cong H^{-3}(G, Z)$ (of Tate's isomorphism), which is obtained by translating the norm residue map of Furuta [3], which is written in ideal theoretic, into idele theoretic. In our definition, the extension K/k may be any bicyclic extension $K=k_{\chi_1}\cdot k_{\chi_2}$ with χ_1, χ_2 being global characters. But the symbol is of type (χ_1, χ_2, c), where c is contained in the number knot. So we can consider the inversion formula only in the case when χ_1 and χ_2 are Kummer characters $\chi_a^{(n)}$ and $\chi_b^{(n)}$. When that is the case, we put $(a, b, c)_n=(\chi_a^{(n)}, \chi_b^{(n)}, c)$ and calculate $(a, b, c)_n+(a, c, b)_n$ (which are written additively in this paper). We approach this result to a necessary and sufficient condition of the inversion formula $(a, b, c)_n+(a, c, b)_n=0$, by

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representing explicitly the components of $(a, b, c)_n+(a, c, b)_n$ at the primes $\mathfrak{p}$ of k where $k_{\mathfrak{p}}(\sqrt[n]{a}, \sqrt[n]{b})$ and $k_{\mathfrak{p}}(\sqrt[n]{a}, \sqrt[n]{c})$ are of degree $\leqq n$ or $\mathfrak{p}$ not dividing n (Theorem 1 and Corollary 1). This gives the explicit value of $(a, b, c)_n+(a, c, b)_n$ when $k_{\mathfrak{p}}(\sqrt[n]{a})$, $k_{\mathfrak{p}}(\sqrt[n]{b})$ and $k_{\mathfrak{p}}(\sqrt[n]{c})$ are tamely ramified. For the components dividing n, it is difficult to write down them explicitly in general. So we calculate it only in the case $k=\boldsymbol{Q}$ and $n=2$. (Theorem 2)

In the final section, we compare this symbol with the one [, ,] defined in Furuta [2]. But the comparison with the one in Akagawa [1] becomes too cumbersome, and it is so delicate that we omit it with saying here that they are essentially the same.

§ 1. Homomorphisms $\varphi_{K/k}$ and $\psi_{K/k}$

For an algebraic number field F, we denote by $F^{\times}$, J_F and C_F the multiplicative group of F, the group of ideles and idele classes of F. For an integral divisor $\mathfrak{m}$ of F, we denote the ray modulo $\mathfrak{m}$ of J_F and $F^{\times}$ by $J_F(\mathfrak{m})$ and $F^{\times}(\mathfrak{m})$.

For a finite group G, let I_G be the augmentation ideal of the group ring $\boldsymbol{Z}[G]$. For a finite extension K/k, let $N_{K/k}$ be the norm map.

Let K be a finite abelian extension of a finite algebraic number field k with group G. When G is abelian, the Pontrjagin dual $M(G)=H^{-3}(G, \boldsymbol{Z})$ of the Schur multiplier of G is isomorphic to the exterior product $\Lambda(G)= G\wedge G\ (=G\otimes G/\langle g\otimes g;\ g\in G\rangle)$. Let $\xi(\sigma, \tau)$ be the canonical 2-cocycle of K/k and take a transversal $\{u_\sigma;\ \sigma\in G\}$ of G in the Weil group $G_{K,k}$ of K/k. We define an isomorphism $\varphi_{K/k}$ from $\Lambda(G)$ to $N_{K/k}^{-1}(1)/I_G C_K=H^{-1}(G, C_K)$ by

$$\begin{aligned}\varphi_{K/k}(\sigma\wedge\tau) &\equiv u_\sigma^{-1}u_\tau^{-1}u_\sigma u_\tau \\ &\equiv \xi(\sigma, \tau)\xi(\tau, \sigma)^{-1} \bmod I_G C_K.\end{aligned}$$

Let α be an epimorphism and M be the Galois extension corresponding to $\operatorname{Ker}\alpha$. Then α determines an epimorphism $\Lambda(\alpha)\colon \Lambda(G)\to\Lambda(H)$ naturally, and it gives a commutative diagram

$$\begin{array}{ccc} \Lambda(G) & \overset{\varphi_{K/k}}{\cong} & N_{K/k}^{-1}(1)/I_G C_K \\ \Lambda(\alpha)\wr\| & & \wr\| \text{ induced by } N_{K/M} \\ \Lambda(H) & \cong & N_{M/k}^{-1}(1)/I_H C_M. \end{array}$$

Since G is abelian, it can be decomposed into cyclic groups G_i as $G=G_1\times\cdots\times G_r$ such that $|G_j|$ divides $|G_i|$ for $i<j$. Let K_i be the Galois extension of k corresponding to $G_1\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_r$ with group

G_i, and put $K_{ij}=K_i\cdot K_j$ and $G_{ij}=G_i\times G_j$. Then the above diagram implies

$$\begin{array}{ccc} \Lambda(G) & \overset{\varphi_{K/k}}{\cong} & N_{K/k}^{-1}(1)/I_G C_K \\ \wr\Vert & & \wr\Vert \\ \prod_{i<j}\Lambda(G_{ij}) & \overset{\varphi_{K_{ij}/k}}{\cong} & \prod_{i<j} N_{K_{ij}/k}^{-1}(1)/I_{G_{ij}}C_{K_{ij}} \end{array}$$

Proposition 1. *Let F/k be a finite cyclic extension with group $G(F/k)=\langle\sigma\rangle$, and L/k be a finite abelian extension containing F with group H. Then*

$$N_{L/k}^{-1}(1)/N_{L/F}^{-1}(1)I_H C_L\cong C_F/C_k\cdot N_{L/F}C_L\cong G(F'/F),$$

where F' is the abelian extension of F corresponding to $C_k\cdot N_{L/F}C_L$ and contained in L. For each $A\in N_{L/k}^{-1}(1)$, taking $b\in C_F$ so that $b^{\sigma-1}=N_{L/F}A$, the above isomorphism $N_{L/k}^{-1}(1)/N_{L/F}^{-1}(1)\cong G(F'/F)$ is given by A mod $N_{L/F}^{-1}(1)I_H C_L\mapsto(b, F'/F)$, where $(\ , F'/F)$ is the global norm residue symbol for F'/F.

Proof. $N_{L/F}(N_{L/k}^{-1}(1))=N_{F/k}^{-1}(1)=C_F^{\sigma-1}$ and $N_{L/F}(I_H C_L)=N_{L/F}C_L^{\sigma-1}$ are immediate. Since the kernel of $\sigma-1\colon C_F\to C_F^{\sigma-1}$ is C_K, naturally $N_{L/k}^{-1}(1)/N_{L/F}^{-1}(1)I_H C_L\cong C_F^{\sigma-1}/N_{L/F}C_L^{\sigma-1}\cong C_F/C_k\cdot N_{L/F}C_L$. So the proposition implied.

Put now $L=K_{ij}$ and $F=K_i$, then $N_{L/F}^{-1}(1)I_H C_L=I_{G_{ij}}\cdot C_{K_{ij}}$. If we compare the degrees, $F'=K_{ij}$ is clear. So the above proposition gives an isomorphism

$$\psi_{K_{ij}/k}\colon N_{K_{ij}/k}^{-1}(1)/I_{G_{ij}}C_{K_j}\cong G(K_{ij}/K_i)\cong G(K_j/k)\cong\Lambda(G_{ij})$$

by using a fixed generator σ_i. For $A\in N_{K_{ij}/k}^{-1}(1)$, take $b\in C_{K_i}$ such that $N_{K_{ij}/K_i}A=b^{\sigma-1}$, then

$$\psi_{K_{ij}/k}(A \bmod I_{G_{ij}}C_{K_{ij}})=\sigma_i\wedge(N_{K_i/k}b,\ K_j/k).$$

Now we define $\psi_{K/k}\colon N_{K/k}^{-1}(1)/I_G C_K\cong\Lambda(G)$ by

$$\psi_{K/k}(A \bmod I_G C_K)=\sum_{i<j}\psi_{K_{ij}/k}(N_{K/K_{ij}}A \bmod I_{G_{ij}}C_{K_{ij}})$$

for $A\in N_{K/k}^{-1}(1)$. Then the following proposition shows

$$\psi_{K/k}=\varphi_{K/k}^{-1}.$$

Proposition 2. $\psi_{K_{ij}/k}(\varphi_{K_{ij}/k}(\sigma_i\wedge\sigma_j))=\sigma_i\wedge\sigma_j.$

Proof. Put $G'_{K_{ij},k}=G(K_{ij}/k)$ and let $\varphi_{K_{ij},k}\colon G_{K_{ij},k}\to G'_{K_{ij},k}$ be the natural epimorphism of Weil groups. Denote by $V_{K_i,k}\colon G_{K_i,k}\to C_k$ and $V_{K_{ij},K_i}\colon G_{K_{ij},K_i}\to C_{K_i}$ the group transfers from $G_{K_i,k}$ to C_{K_i} and from G_{K_{ij},K_i} to $C_{K_{ij}}$ respectively. Put $H=\varphi^{-1}_{K_{ij},k}(G'_{K_{ij},K_i})$ and let $\lambda\colon G_{K_{ij},k}\to G_{K_{ij},k}/H^c$ be the canonical epimorphism modulo the topological commutator H^c of H. Moreover let $\eta\colon G_{K_{ij},k}/H^c\cong G_{K_i,k}$ be the natural isomorphism of Weil groups.

Take a transversal u_σ; $\sigma\in G_{ij}$ of G_{ij} in $G_{K_{ij},k}$. Then

$$\begin{aligned}\varphi_{K_{ij}/k}(\sigma_i\wedge\sigma_j)&\equiv u_{\sigma_i}^{-1}u_{\sigma_j}^{-1}u_{\sigma_i}u_{\sigma_j}\\&\equiv u_{\sigma_i}u_{\sigma_j}u_{\sigma_i}^{-1}u_{\sigma_j}^{-1}\bmod I_{G_{ij}}C_{K_{ij}}.\end{aligned}$$

$$\begin{aligned}N_{K_{ij}/K_i}(u_{\sigma_i}u_{\sigma_j}u_{\sigma_i}^{-1}u_{\sigma_j}^{-1})&=V_{K,K_i}(u_{\sigma_i}u_{\sigma_j}u_{\sigma_i}^{-1}u_{\sigma_j}^{-1})\\&=\eta\circ\lambda(u_{\sigma_i}u_{\sigma_j}u_{\sigma_i}^{-1}u_{\sigma_j}^{-1})\\&=\eta\circ\lambda(u_{\sigma_i})\eta\circ\lambda(u_{\sigma_j})\eta\circ\lambda(u_{\sigma_i})^{-1}\eta\circ\lambda(u_{\sigma_j})^{-1}\\&=\eta\circ\lambda(u_{\sigma_j})^{\sigma_i-1}\end{aligned}$$

because $\eta\circ\lambda(u_{\sigma_j})\in C_{K_i}$ and $\eta\circ\lambda(u_{\sigma_i})$ is a representative of σ_i in $G_{K_i,k}$.

So we can take the element $b\in C_{K_i}$ in the definition of $\psi_{K_{ij}/k}(\varphi_{K_{ij}/k}(\sigma_i\wedge\sigma_j))$ so that $b=\eta\circ\lambda(u_{\sigma_j})$.

Now we have the following commutative diagram, denoting by res the restriction maps and $\bar\sigma_j=\varphi_{K,k}(u_{\sigma_j})$ a prolongation of σ_j to $G'_{K_{ij},k}$:

$$\begin{array}{ccccccccc}
u_{\sigma_j} & \mapsto & b & \mapsto & N_{K_i/k}b & \mapsto & (N_{K_i/k}b,\ K_i/k) & \mapsto & \psi_{K_{ij}/k}(\varphi_{K_{ij}/k}(\sigma_i\wedge\sigma_j))\\
G_{K_{ij},k} & \longrightarrow & G_{K_i,k} & \longrightarrow & C_k & \longrightarrow & G(K_j/k) & \longrightarrow & \Lambda(G)\\
\downarrow{\scriptstyle\varphi_{K_{ij},k}} & & \downarrow{\scriptstyle\varphi_{K_i,k}} & & \downarrow{\scriptstyle\varphi_{k,k}} & & \| & & \|\\
G'_{K_{ij},k} & \xrightarrow{\text{res}} & G'_{K_i,k} & \xrightarrow{\text{res}} & G'_{k,k} & \xrightarrow{\text{res}} & G_j & \longrightarrow & \Lambda(G)\\
\bar\sigma_j & \mapsto & \bar\sigma_j|k^{\text{ab}} & \mapsto & \bar\sigma_j|k^{\text{ab}} & \mapsto & \sigma_j|_{K_j}=\sigma_j & \mapsto & \sigma_i\wedge\sigma_j
\end{array}\qquad(\circlearrowleft)$$

Since the above diagram is commutative, we have

$$\psi_{K_{ij}/k}(\varphi_{K_{ij}/k}(\sigma_i\wedge\sigma_j))=\sigma_i\wedge\sigma_j.$$

§ 2. Tripling (a, b, c)

For a G-invariant integral divisor $\mathfrak{m}$ of K, we call it a Scholz-conductor of K/k, when the mapping $H^2(G, J_K(\mathfrak{m}))\to H^2(G, C_K)$ induced by $J_K(\mathfrak{m})\to J_K\twoheadrightarrow C_K$ is zero mapping (See Heider [4]). Since $J_K=J_K(\mathfrak{m})\cdot K^\times$, $N^{-1}_{K/k}(1)/I_GC_K=N^{-1}_{K/k}(k^\times)/k^\times I_GJ_K\cong J_K(\mathfrak{m})\cap N^{-1}_{K/k}(k^\times)/K^\times(\mathfrak{m})I_GJ_K(\mathfrak{m})$. And the condition that $\mathfrak{m}$ is a Scholz-conductor is equivalent to $J_K(\mathfrak{m})\cap N^{-1}_{K/k}(k^\times)/K^\times(\mathfrak{m})I_GJ_K(\mathfrak{m})\cong N_{K/k}J_K(\mathfrak{m})\cap k^\times/N_{K/k}K^\times(\mathfrak{m})$ by means of the norm map $N_{K/k}$.

Now let χ_1 and χ_2 be global characters of J_k (i.e. Ker $\chi_i \supset k^\times$) such that ord χ_2 divides ord χ_1, and $K_i = k_{\chi_i}$ be the cyclic extensions of k corresponding to Ker χ_i. For $\chi_i: J_k \twoheadrightarrow G(K_i/k) \cong (1/\mathrm{ord}\,\chi_i)\mathbf{Z}/\mathbf{Z}$, we take $\sigma_i \in G(K_i/k)$ whose image is $(1/\mathrm{ord}\,\chi_i) \bmod \mathbf{Z}$ $(i=1, 2)$. Put $K = K_1 \cdot K_2$.

When $K_1 \cap K_2 = k$, we connect the mapping ψ of Section 1 with the above isomorphism. Namely, for $c \in N_{K/k}J_K(\mathfrak{m}) \cap k^\times$, taking $\mathfrak{C} \in J_K(\mathfrak{m})$ and $C \in K_1^\times$ with $N_{K/k}\mathfrak{C} = c$ and $N_{K_1/k}C = c$, put $(\chi_1, \chi_2, c) = \chi_2(N_{K_1/k}\mathfrak{c})$, where $\mathfrak{c} \in J_{K_1}$ with $\mathfrak{c}^{\sigma_1 - 1} = C^{-1} \cdot N_{K/K_1}\mathfrak{C}$. It gives

$$N_{K/k}J_K(\mathfrak{m}) \cap k^\times / N_{K/k}K^\times(\mathfrak{m}) \cong \frac{1}{|M(G)|}\mathbf{Z}/\mathbf{Z} \subseteqq \mathbf{Q}/\mathbf{Z},$$

and the image c_0 of $\varphi_{K/k}(\sigma_1 \wedge \sigma_2)$ by $N_{K/k}^{-1}(1)/I_G C_K \cong N_{K/k}J_K(\mathfrak{m}) \cap k^\times / N_{K/k}K^\times(\mathfrak{m})$ corresponds to $(1/|M(G)|) \bmod \mathbf{Z}$.

Definition. When $K_1 \cap K_2 = k$, we put

$$(\chi_1, \chi_2, c) = \chi_2(N_{K_1/k}\mathfrak{c}) \qquad \text{for} \quad c \in N_{K/k}J_K(\mathfrak{m}) \cap k^\times.$$

Remark. As far as the symbol (χ_1, χ_2, c) is defined, its value is independent on $\mathfrak{m}$. Scholz-conductor has the smallest element, so we use it throughout in the following. Instead of $J_K(\mathfrak{m})$, we can use any G-invariant closed subgroup $\tilde{J}$ of J_K such that $H^{-1}(G, \tilde{J}) \to H^{-1}(G, C_K)$ is zero mapping. But if we used $\tilde{J}$, the value (χ_1, χ_2, c) should depend on the choice of $\tilde{J}$. So we don't use this $\tilde{J}$ for the simplicity.

The following proposition implies immediately from the definition.

Proposition 3. i) *Let* χ_2' *be another global character such that* ord χ_2' *divides* ord χ_1. *When* (χ_1, χ_2, c), (χ_1, χ_2, c) *and* $(\chi_1, \chi_2 + \chi_2', c)$ *are all defined, it holds*

$$(\chi_1, \chi_2 + \chi_2', c) = (\chi_1, \chi_2, c) + (\chi_1, \chi_2', c).$$

ii) *If* ord $\chi_1 =$ ord χ_2 *and* (χ_1, χ_2, c) *is defined, then* (χ_2, χ_1, c) *is also defined and*

$$(\chi_1, \chi_2, c) + (\chi_2, \chi_1, c) = 0.$$

When k contains a primitive root ζ_n of 1 (fix it throughout this section), each a of k defines the Kummer character χ_a of degree n.

Definition. Assume $k \ni \zeta_n$. When $a, b \in k^\times$ satisfy ord $\chi_a |$ ord χ_b and $k_{\chi_a} \cap k_{\chi_b} = k$, we put

$$(a, b, c)_n = (\chi_a, \chi_b, c)$$

for $c \in N_{k_{\chi_a}k_{\chi_b}/k}J_{k_{\chi_a}k_{\chi_b}}(\mathfrak{m}) \cap k^{\times}$ where $\mathfrak{m}$ is the Scholz-conductor of $k_{\chi_a}k_{\chi_b}/k$.

Let $\left(\frac{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}}{\mathfrak{p}}\right)$ be the Hilbert symbol and $\omega_n\colon \langle\zeta_n\rangle \to (1/n)\boldsymbol{Z}/\boldsymbol{Z} \subseteqq \boldsymbol{Q}/\boldsymbol{Z}$ be the homomorphism given by $\zeta_n \mapsto (1/n) \bmod \boldsymbol{Z}$. Then $(\alpha, \beta)_{k,n} = \sum_{\mathfrak{p}} \omega_n\left(\frac{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}}{\mathfrak{p}}\right)$ gives the Kummer pairing for $\alpha, \beta \in J_k$, where $\mathfrak{p}$ runs over all the prime divisors of k and $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}$ are the $\mathfrak{p}$-components of α, β respectively.

For each $c \in N_{k_{\chi_a}k_{\chi_b}/k}J_{k_{\chi_a}k_{\chi_b}}(\mathfrak{m}) \cap k^{\times}$, taking $\mathfrak{C} \in J_{k_{\chi_a}k_{\chi_b}}(m)$, $C \in k_{\chi_a}$ and $\mathfrak{c} \in J_{k_{\chi_a}}$ such that $N_{k_{\chi_a}k_{\chi_b}/k}\mathfrak{C} = N_{k_{\chi_a}/k}C = c$ and $\mathfrak{c}^{\sigma_a - 1} = C^{-1} \cdot N_{k_{\chi_a}k_{\chi_b}/k_{\chi_a}}\mathfrak{C}$, we get $(a, b, c)_n = \chi_b(N_{k_{\chi_a}/k}\mathfrak{c}) = (N_{k_{\chi_a}/k}\mathfrak{c}, b)_{k,n}$, where σ_a is the element of $G(k_{\chi_a}/k)$ whose image by χ_a is $(1/\mathrm{ord}\,\chi_a) \bmod \boldsymbol{Z}$.

In the following, we consider only the case $\mathrm{ord}\,\chi_a = n$ for the simplicity and we write σ instead of σ_a. Put $K_1 = k_{\chi_a}$, $K_2 = k_{\chi_b}$, $K_3 = k_{\chi_c}$, $K = K_1K_2$ and $K' = K_1K_3$.

Proposition 4. *Assume $(a, b, c)_n$ and $(a, c, b)_n$ are defined with the Scholz-conductors $\mathfrak{m}$ and $\mathfrak{m}'$ respectively. Take $\mathfrak{C} \in J_K(\mathfrak{m})$ and $\mathfrak{B} \in J_{K'}(\mathfrak{m}')$ such that $N_{K/k}\mathfrak{C} = c$ and $N_{K'/k}\mathfrak{B} = b$, and put $\delta = (n-1)\sigma + (n-2)\sigma^2 + \cdots + \sigma^{n-1} \in \boldsymbol{Z}/n\boldsymbol{Z}[G(K_1/k)]$. Then*

$$(a, b, c)_n + (a, c, b)_n = (N_{K'/K_1}\mathfrak{B}^{\delta}, N_{K/K_1}\mathfrak{C})_{K_1,n}.$$

Proof. Take the element $C, B \in K_1^{\times}$ and $\mathfrak{c}, \mathfrak{b} \in J_{K_1}$ so that $N_{K_1/k}C = c$, $N_{K_1/k}B = b$, $\mathfrak{c}^{\sigma-1} = C^{-1}N_{K/K_1}\mathfrak{C}$ and $\mathfrak{b}^{\sigma-1} = B^{-1} \cdot N_{K/K_1}\mathfrak{B}$. Then

$$(a, b, c)_n = (N_{K_1/k}\mathfrak{c}, b)_{k,n} = (\mathfrak{c}, b)_{K_1,n}.$$

Since $\delta(1-\sigma^{-1}) = 1 + \sigma + \cdots + \sigma^{n-1}$ in $\boldsymbol{Z}/n\boldsymbol{Z}[(G(K_1/k)]$,

$$(\mathfrak{c}, b)_{K_1,n} = (\mathfrak{c}, (B^{-\delta})^{\sigma^{-1}-1})_{K_1,n}.$$

Moreover

$$(\mathfrak{c}, (B^{-\delta})^{\sigma^{-1}-1})_{K_1,n} = (\mathfrak{c}^{\sigma-1}, B^{-\delta})_{K_1,n}$$

owing to $(\alpha^{\sigma}, \beta^{\sigma})_{K_1,n} = (\alpha, \beta)_{K_1,n} (\alpha, \beta \in J_{K_1})$. As $B^{-\delta}$ and $C^{-1} \in K_1^{\times}$, we have

$$(\mathfrak{c}^{\sigma-1}, B^{-\delta})_{K_1,n} = (N_{K/K_1}\mathfrak{C}, B^{-\delta})_{K_1,n}.$$

Now it follows from $B = N_{K'/K_1}\mathfrak{B} \cdot \mathfrak{b}^{1-\sigma}$ that

$$\begin{aligned}(N_{K/K_1}\mathfrak{C}, B^{-\delta})_{K_1,n} &= (N_{K/K_1}\mathfrak{C}, \mathfrak{b}^{-\delta\cdot(1-\sigma)})_{K_1,n} + (N_{K/K_1}\mathfrak{C}, N_{K'/K_1}\mathfrak{B}^{-\delta})_{K_1,n} \\ &= (N_{K/K_1}\mathfrak{C}, N_{K_1/k}\mathfrak{b})_{K_1,n} + (N_{K'/K_1}\mathfrak{B}^{\delta}, N_{H/K_1}\mathfrak{C})_{K_1,n}\end{aligned}$$

$$=(c,\ N_{K_1/k}b)_{k,n}+(N_{K'/K_1}\mathfrak{B}^{\delta}+N_{K/K_1}\mathfrak{C})_{K_1,n}$$
$$=-(a,\ c,\ b)_n+(N_{K'/K_1}\mathfrak{B}^{\delta},\ N_{K/K_1}\mathfrak{C})_{K_1,n},$$

and the proposition is proved.

It is a problem when the inversion formula $(a, b, c)_n+(a, c, b)_n=0$ holds. We shall treat it in the following section.

§ 3. Inversion formula

When $n=\prod_i p_i^{m_i}$ where p_i are prime numbers,

$$(a,\ b,\ c)_{p_i^{m_i}}=\frac{n}{p_i^{m_i}}(a,\ b,\ c)_n \quad \text{with} \quad \zeta_{p_i^{m_i}}=\zeta_n^{n/p_i^{m_i}}.$$

So it is enough to consider only when n is a prime power p^m.

We assume $(a, b, c)_n$ and $(a, c, b)_n$ are defined. Then $\left(\frac{a,\ b}{\mathfrak{p}}\right)_n=\left(\frac{a,\ c}{\mathfrak{p}}\right)_n=\left(\frac{b,\ c}{\mathfrak{p}}\right)_n=1$ are every prime $\mathfrak{p}$ of k. When this is the case, we call that a, b, c are orthogonal (See Akagawa [1]). As in the previous section, let ord $\chi_a=n$, $K_1=k_{\chi_a}$, $K_2=k_{\chi_b}$, $K_3=k_{\chi_c}$, $K=K_1K_2$ and $K'=K_1K_3$. For each prime divisor $\mathfrak{p}$ of k, take prime divisors $\mathfrak{P}_1$, $\mathfrak{P}$ and $\mathfrak{P}'$ of $\mathfrak{p}$ in K_1, K and K' satisfying $\mathfrak{P}|\mathfrak{P}_1$ and $\mathfrak{P}'|\mathfrak{P}_1$. When $\mathfrak{P}^e\|\mathfrak{m}$, put $K_{\mathfrak{P}}^{(\mathfrak{m})}=U_{\mathfrak{P}}^{(e)}$ for $e\geqq 1$ and $K_{\mathfrak{P}}^{(\mathfrak{m})}=K_{\mathfrak{P}}$ for $e=0$.

Take $\mathfrak{C}_{\mathfrak{P}}\in K_{\mathfrak{P}}^{(\mathfrak{m})}$ and $\mathfrak{B}_{\mathfrak{P}'}\in K_{\mathfrak{P}'}^{\prime(\mathfrak{m}')}$ so that $N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}\mathfrak{C}_{\mathfrak{P}}=c$ and $N_{K'_{\mathfrak{p}'}/k_{\mathfrak{p}}}\mathfrak{B}_{\mathfrak{p}'}=b$. Then we can take $\mathfrak{C}\in J_K(\mathfrak{m})$ and $\mathfrak{B}\in J_{K'}(\mathfrak{m}')$ in Proposition 4 with components 1 except $\mathfrak{C}_{\mathfrak{P}}$ and $\mathfrak{B}_{\mathfrak{P}'}$ at $\mathfrak{P}$ and $\mathfrak{P}'$ for each $\mathfrak{p}$, respectively.

Put $n_{\mathfrak{p}}=[K_{1_{\mathfrak{P}_1}}:k_{\mathfrak{p}}]$ and let $\delta(n_{\mathfrak{p}})=\sum_{i=1}^{n_{\mathfrak{p}}-1}(n/n_{\mathfrak{p}})(n_{\mathfrak{p}}-i)\sigma^{(n/n_{\mathfrak{p}})i}$ (of course $\delta(1)=0$). Now we consider the components of $(N_{K'/K_1}B^{\delta},\ N_{K/K_1}\mathfrak{C})_{K_1,n}$ in order to estimate the value $(a, b, c)_n+(a, c, b)_n$ by Proposition 4. The components at the prime divisors of $\mathfrak{p}$ are 1 except at $\mathfrak{P}_1$, and the component at $\mathfrak{P}_1$ is

$$\omega_n\left(\frac{N_{K'_{\mathfrak{P}'}/K_{1_{\mathfrak{P}_1}}}\mathfrak{B}_{\mathfrak{P}'}^{\delta(n_{\mathfrak{p}})},\ N_{K_{\mathfrak{P}}/K_{1_{\mathfrak{P}_1}}}\mathfrak{C}_{\mathfrak{P}}}{\mathfrak{P}_1}\right)=\omega_n\left(\frac{N_{K_{\mathfrak{P}}/K_{1_{\mathfrak{P}_1}}}\mathfrak{C}^{\delta(n_{\mathfrak{p}})},\ N_{K_{\mathfrak{P}}/K_{1_{\mathfrak{P}_1}}}\mathfrak{P}_{\mathfrak{P}'}}{\mathfrak{P}_1}\right).$$

The equality is immediate from $(\alpha^{\sigma n/n_{\mathfrak{p}}},\ \beta^{\sigma n/n_{\mathfrak{p}}}/P_1)=(\alpha,\ \beta/P_1)$ for $\alpha, \beta\in K_{1_{\mathfrak{P}_1}}$, we denote this component by $\gamma_{\mathfrak{p}}$.

For infinite, $\mathfrak{p}$, $n_{\mathfrak{p}}=1$ or 2. When $n_{\mathfrak{p}}=2$, $\mathfrak{P}_1$ is complex and the Hilbert symbol is trivial. In case of $n_{\mathfrak{p}}=1$, the above term is 0 since $\delta(n_{\mathfrak{p}})=0$.

We consider the component $\gamma_{\mathfrak{p}}$ at finite $\mathfrak{p}$ under the condition $[K_{\mathfrak{P}}:k_{\mathfrak{p}}]\leqq n$ and $[K'_{\mathfrak{p}'}:k_{\mathfrak{p}}]\leqq n$. Then the homomorphism $\Lambda(G(K_{\mathfrak{P}}/k_{\mathfrak{p}}))\to\Lambda(G)$ induced by the inclusion $G(K_{\mathfrak{P}}/k_{\mathfrak{p}}) \to G$ is zero mapping and $\mathfrak{P} \nmid \mathfrak{m}$ for every $\mathfrak{P}|\mathfrak{p}$. So $K_{\mathfrak{P}}^{(\mathfrak{m})} = K_{\mathfrak{P}}$ and $K_{\mathfrak{P}'}^{\prime(\mathfrak{m}')} = K'_{\mathfrak{P}'}$. If $\sqrt[n_{\mathfrak{p}}]{b} \notin K_{1\mathfrak{P}_1}$ then $[K_{\mathfrak{P}} : k_{\mathfrak{p}}] > n$, which contradicts. Hence $K_{1_{\mathfrak{P}_1}}$ contains $\sqrt[n_{\mathfrak{p}}]{b}$ and $\sqrt[n_{\mathfrak{p}}]{c}$.

At first we assume $p \neq 2$. Then, since $\left(\frac{\sqrt[n_{\mathfrak{p}}]{c}, b}{\mathfrak{P}_1}\right) = \left(\frac{c, b}{\mathfrak{p}}\right) = 1$, $\sqrt[n_{\mathfrak{p}}]{c} \in N_{K_{\mathfrak{P}}/K_{1\mathfrak{P}_1}} K_{\mathfrak{P}}$ and we can set $N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}} \mathfrak{C}_{\mathfrak{P}} = \sqrt[n_{\mathfrak{p}}]{c}$. When $\sqrt[n_{\mathfrak{p}}]{c}^{\,\sigma(n/n_{\mathfrak{p}})} = \zeta_n^u$ $(u \in Z/nZ)$, we have $\sqrt[n_{\mathfrak{p}}]{c}^{\,\delta(n_{\mathfrak{p}})} = \zeta_n^{-(1/6)un(n_{\mathfrak{p}}-1)} \cdot c^{(n/n_{\mathfrak{p}})(n_{\mathfrak{p}}-1)/2} \in k_{\mathfrak{p}}$ and

$$\gamma_{\mathfrak{p}} = \omega_n\left(\frac{\sqrt[n_{\mathfrak{p}}]{c}^{\,\delta(n_{\mathfrak{p}})}, b}{\mathfrak{p}}\right) = \omega_n\left(\frac{\zeta_n^{-(1/6)un(n_{\mathfrak{p}}-1)(2n_{\mathfrak{p}}-1)}, b}{\mathfrak{P}}\right).$$

The last term is equal to zero when $p \neq 3$ or $n_{\mathfrak{p}} < n$.

Let $p=3$, $n_{\mathfrak{p}} = n$ and $\zeta_3 = \zeta_n^{n/3}$. Then there exist $b_{\mathfrak{p}}$ and $c_{\mathfrak{p}}$ in Z/nZ such that $b \equiv a^{b_{\mathfrak{p}}} \bmod k_{\mathfrak{p}}^n$ and $c \equiv a^{c_{\mathfrak{p}}} \bmod k_{\mathfrak{p}}^n$. Of course $u = c_{\mathfrak{p}}$ and

$$\gamma_{\mathfrak{p}} = \omega_n\left(\frac{\zeta_3^{c_{\mathfrak{p}}}, a^{b_{\mathfrak{p}}}}{\mathfrak{p}}\right) = c_{\mathfrak{p}} \cdot b_{\mathfrak{p}} \omega_n\left(\frac{\zeta_3, a}{\mathfrak{p}}\right).$$

If $p \nmid 3$, then evidently $\gamma_{\mathfrak{p}} = w_{\mathfrak{p}}(a) w_{\mathfrak{p}}(b) w_{\mathfrak{p}}(c) \omega_n(\zeta_3/\mathfrak{p})$, where $w_{\mathfrak{p}}$ is the normalized additive valuation of $k_{\mathfrak{p}}$.

Now we assume $p=2$, $[K_{\mathfrak{P}} : k_{\mathfrak{p}}] \leqq n$ and $[K'_{\mathfrak{P}'} : k_{\mathfrak{p}}] \leqq n$. If $c \in k_{\mathfrak{p}}^2$, we can set $N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}} \mathfrak{C}_{\mathfrak{P}} = \sqrt[n_{\mathfrak{p}}]{c}$ and $\sqrt[n_{\mathfrak{p}}]{c}^{\,\sigma(n_{\mathfrak{p}})} = \sqrt{c}^{\,(n_{\mathfrak{p}}-1)n/n_{\mathfrak{p}}} \in k_{\mathfrak{p}}$. So

$$\gamma_{\mathfrak{p}} = \frac{n}{n_{\mathfrak{p}}} \omega_n\left(\frac{\sqrt{c}, b}{\mathfrak{p}}\right).$$

If $b \in k_{\mathfrak{p}}^2$, then similarly

$$\gamma_{\mathfrak{p}} = \frac{n}{n_{\mathfrak{p}}} \omega_n\left(\frac{\sqrt{b}, c}{\mathfrak{p}}\right).$$

If $n_{\mathfrak{p}} < n$ and $c \notin k_{\mathfrak{p}}^2$, we can set $N_{K_{\mathfrak{p}}/k_{\mathfrak{P}}} \mathfrak{C}_{\mathfrak{p}} = \zeta_n^{n/2n_{\mathfrak{p}}} \sqrt[n_{\mathfrak{p}}]{c}$ and $(\zeta_n^{n/2n_{\mathfrak{p}}} \sqrt[n_{\mathfrak{p}}]{c})^{\delta(n_{\mathfrak{p}})} = (-1)^{n/2n_{\mathfrak{p}}} c^{(n_{\mathfrak{p}}-1)n/2n_{\mathfrak{p}}} \in k_{\mathfrak{p}}$. Then

$$\gamma_{\mathfrak{p}} = \frac{n}{2n_{\mathfrak{p}}}(n-1)\omega_n\left(\frac{-1, b}{\mathfrak{p}}\right).$$

If $b \notin k_{\mathfrak{p}}^2$, $c \notin k_{\mathfrak{p}}^2$ and $n_{\mathfrak{p}} = n$, then there exist $b_{\mathfrak{p}}, c_{\mathfrak{p}} \in Z/nZ$; $u, v \in k_{\mathfrak{p}}$ such that $b = a^{b_{\mathfrak{p}}} u^n$, $c = a^{c_{\mathfrak{p}}} v^n$. Take an element $\alpha \in K_{1_{\mathfrak{P}_1}}$ such that $N_{K_{1\mathfrak{P}_1}/k_{\mathfrak{p}}} \alpha = -1$. Then

$$\gamma_{\mathfrak{p}} = \omega_n\left(\frac{(\alpha \cdot \sqrt[n_{\mathfrak{p}}]{a})^{b_{\mathfrak{p}}\delta} u^{(n-1)n/2}, (\alpha \cdot \sqrt[n_{\mathfrak{p}}]{a})^{c_{\mathfrak{p}}} v}{\mathfrak{P}_1}\right).$$

Since

$$\left(\frac{\alpha^{b_{\mathfrak{p}}\delta},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)=\left(\frac{\alpha^{b_{\mathfrak{p}}},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,-c_{\mathfrak{p}}\delta}}{\mathfrak{P}_1}\right)$$

$$=\left(\frac{\alpha^{c_{\mathfrak{p}}},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,-b_{\mathfrak{p}}\delta}}{\mathfrak{P}_1}\right)=\left(\frac{{}^{n_{\mathfrak{p}}}\sqrt{a}^{\,b_{\mathfrak{p}}\delta},\ \alpha^{c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)$$

and $\alpha^2=\beta^{\sigma-1}$ for some $\beta\in K_{1\mathfrak{p}_1}$, we have

$$\left(\frac{\alpha^{b_{\mathfrak{p}}\delta},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)\cdot\left(\frac{{}^{n_{\mathfrak{p}}}\sqrt{a}^{\,b_{\mathfrak{p}}\delta},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)=\left(\frac{(\alpha^2)^{b_{\mathfrak{p}}\delta},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)$$

$$=\left(\frac{\beta^{(\sigma-1)\delta\cdot b_{\mathfrak{p}}},\ {}^{n_{\mathfrak{p}}}\sqrt{a}^{\,c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)=\left(\frac{N_{K_{1\mathfrak{P}_1}/k_{\mathfrak{p}}}\beta^{b_{\mathfrak{p}}},\ (-a)^{c_{\mathfrak{p}}}}{\mathfrak{P}_1}\right)$$

$$=\left(\frac{N_{K_{1\mathfrak{P}_1}/k_{\mathfrak{p}}}\beta,\ -1}{\mathfrak{p}}\right)^{b_{\mathfrak{p}}c_{\mathfrak{p}}}\left(\frac{\alpha^2,\ \alpha}{\mathfrak{P}_1}\right)^{b_{\mathfrak{p}}c_{\mathfrak{p}}}$$

$$=\left(\frac{-1,\ \alpha}{\mathfrak{P}_1}\right)^{2b_{\mathfrak{p}}c_{\mathfrak{p}}}=1.$$

Next we calculate $b_{\mathfrak{p}}\cdot c_{\mathfrak{p}}\omega_n\left(\frac{\alpha,{}^{\delta}\,\alpha}{\mathfrak{P}_1}\right)$. Take $\alpha\in k_{\mathfrak{p}}(\sqrt{a})$ such that $N_{k_{\mathfrak{p}}(\sqrt{a})/k_{\mathfrak{p}}}\cdot\alpha=\zeta_n$. Then $N_{K_{1\mathfrak{P}_1}/k_{\mathfrak{p}}}\alpha=\zeta_n^{n/2}=-1$. So we can use this α. Now $\alpha^{\delta}=\alpha^{((n-1)+\cdots+1)\delta}\alpha^{(n-2)+\cdots+0}=\alpha^{(n/2)^2\sigma}\alpha^{(n/2)((n/2)-1)}=(\alpha^{\sigma+1})^{(n/2)^2}\alpha^{-n/2}=\zeta_n^{(n/2)^2}\alpha^{-n/2}=(-1)^{n/2}\alpha^{-n/2}=(-\alpha)^{-n/2}$. So

$$b_{\mathfrak{p}}\cdot c_{\mathfrak{p}}\omega_n\left(\frac{\alpha^{\delta},\ \alpha}{\mathfrak{P}_1}\right)=b_{\mathfrak{p}}c_{\mathfrak{p}}\omega_n\left(\frac{(-\alpha)^{-n/2},\ \alpha}{\mathfrak{P}_1}\right)=0\quad\text{and}\quad\gamma_{\mathfrak{p}}=\omega_n\left(\frac{u^{c_{\mathfrak{p}}}v^{b_{\mathfrak{p}}},\ a^{n/2}}{\mathfrak{p}}\right).$$

Theorem 1. *We assume* $[K_{\mathfrak{P}}: k_{\mathfrak{p}}]\leqq n$ *and* $[K'_{\mathfrak{P}'}: k_{\mathfrak{p}}]\leqq n$. *Then* $\gamma_{\mathfrak{p}}$ *has following values*:

i) $\gamma_{\mathfrak{p}}=0$ *for infinite* $\mathfrak{p}$.

ii) $\gamma_{\mathfrak{p}}=0$ *if* $p\neq 2, 3$.

iii) *When* $n_{\mathfrak{p}}\ (=[K_{1\mathfrak{P}_1}: k_{\mathfrak{p}}])<n$, $\gamma_{\mathfrak{p}}=0$ *unless* $p=2$ *and* $1<n_{\mathfrak{p}}=n/2$. *If* $p=2$ *and* $1<n_{\mathfrak{p}}=n/2$, *then* $\gamma_{\mathfrak{p}}=\omega_n\left(\frac{-1,\ b}{\mathfrak{p}}\right)=\omega_n\left(\frac{-1,\ c}{\mathfrak{p}}\right)$.

iv) *When* $n_{\mathfrak{p}}=n$, *there exist* $b_{\mathfrak{p}}$ *and* $c_{\mathfrak{p}}$ *in* $\boldsymbol{Z}/n\boldsymbol{Z}$ *such that* $b\equiv a^{b_{\mathfrak{p}}} \bmod k_{\mathfrak{p}}^{n}$ *and* $c\equiv a^{c_{\mathfrak{p}}} \bmod k_{\mathfrak{p}}^{n}$. *If* $p=3$, $\gamma_{\mathfrak{p}}=c_{\mathfrak{p}}\cdot b_{\mathfrak{p}}\omega_n\left(\frac{\zeta_3,\ a}{\mathfrak{p}}\right)$. *If* $p=2$ *and* $b_{\mathfrak{p}}$, $c_{\mathfrak{p}}\equiv 0$ mod 2, *then* $\gamma_{\mathfrak{p}}=0$. *If* $p=2$ *and* $b_{\mathfrak{p}}$ *or* $c_{\mathfrak{p}}\not\equiv 0$ mod 2, *then*

$$\gamma_{\mathfrak{p}}=\omega_n\left(\frac{\sqrt{b^{c_{\mathfrak{p}}}\cdot c^{b_{\mathfrak{p}}}},\ a}{\mathfrak{p}}\right).$$

If $p\neq 2$ and $\mathfrak{p}\nmid p$ then orthogonality implies $[K_{\mathfrak{P}}: k_{\mathfrak{p}}]\leqq n$ and $[K'_{\mathfrak{P}'}: k_{\mathfrak{p}}]\leqq n$. When $p=2$ and $\mathfrak{p}\nmid p$, $[K_{\mathfrak{P}}: k_{\mathfrak{p}}]\leqq n$ unless $w_{\mathfrak{p}}(a)\equiv w_{\mathfrak{p}}(b)\equiv 1 \bmod 2$ and $k_{\mathfrak{p}}\not\ni\zeta_{2n}$. If $p=2$, $w_p(a)\equiv w_{\mathfrak{p}}(b)\equiv 1 \bmod 2$ and $k_{\mathfrak{p}}\not\ni\zeta_n$, then $[K_{\mathfrak{P}}: k_{\mathfrak{p}}]=2n$ and $K_{\mathfrak{P}}^{(m)}=U_{\mathfrak{P}}^{(1)}=(U_{\mathfrak{P}}^{(1)})^n$, so $\gamma_{\mathfrak{p}}=0$.

Corollary 1. *Assume* $\mathfrak{p}\nmid p$. *Then* $\gamma_{\mathfrak{p}}=0$ *expect the following four cases.*

i) *If* $p=3$, $k_{\mathfrak{p}}\not\ni\zeta_{3n}$ *and* $w_{\mathfrak{p}}(a)w_{\mathfrak{p}}(b)w_{\mathfrak{p}}(c)\not\equiv 0 \bmod 3$, *then*

$$\gamma_{\mathfrak{p}}=w_{\mathfrak{p}}(a)w_{\mathfrak{p}}(b)w_{\mathfrak{p}}(c)\omega_n\left(\frac{\zeta_3}{\mathfrak{p}}\right).$$

ii) *If* $p=2$, $1<n_{\mathfrak{p}}=n/2$, $k_{\mathfrak{p}}\not\ni\zeta_{2n}$ *and* $w_{\mathfrak{p}}(b)w_{\mathfrak{p}}(c)\equiv 1 \bmod 2$, *then*

$$\gamma_{\mathfrak{p}}=\tfrac{1}{2} \bmod \boldsymbol{Z}.$$

iii) *If* $p=2$ *and* $n_{\mathfrak{p}}=[K_{\mathfrak{P}}: k_{\mathfrak{p}}]=[K'_{\mathfrak{P}'}: k_{\mathfrak{p}}]=n$ *and* $k_{\mathfrak{p}}\ni\zeta_4$, *then*

$$\gamma_{\mathfrak{p}}=\frac{[k_{\mathfrak{p}}(\sqrt{a}): k_{\mathfrak{p}}(\sqrt{c})]}{[k_{\mathfrak{p}}(\sqrt[2n]{b}, \sqrt[2n]{a}): k_{\mathfrak{p}}(\sqrt[2n]{a})]}+\frac{[k_{\mathfrak{p}}(\sqrt{a}): k_{\mathfrak{p}}(\sqrt{b})]}{[k_{\mathfrak{p}}(\sqrt[2n]{c}, \sqrt[2n]{a}): k_{\mathfrak{p}}(\sqrt[2n]{a})]} \bmod \boldsymbol{Z}.$$

iv) *If* $p=n=n_{\mathfrak{p}}=[K_{\mathfrak{P}}: k_{\mathfrak{p}}]=[K'_{\mathfrak{P}'}: k_{\mathfrak{p}}]=2$ *and* $k_{\mathfrak{p}}\not\ni\zeta_4$, *then*

$$\gamma_{\mathfrak{p}}=\frac{[k_{\mathfrak{p}}(\sqrt{a}): k_{\mathfrak{p}}(\sqrt{c})]w_{\mathfrak{p}}(b)+[k_{\mathfrak{p}}(\sqrt{a}): k_{\mathfrak{p}}(\sqrt{b})]w_{\mathfrak{p}}(c)}{4} \bmod \boldsymbol{Z}.$$

Proof. $\gamma_{\mathfrak{p}}=0$ except i), ii), iii) and iv) is already proved. i) and ii) is evident from Theorem 1. At first we consider the case iii). If $k_{\mathfrak{p}}\not\ni\zeta_{2n}$ and $w_{\mathfrak{p}}(a)\equiv 1 \bmod 2$, then $[K_{\mathfrak{P}}: k_{\mathfrak{p}}]=[K'_{\mathfrak{P}'}: k_{\mathfrak{p}}]=n$ shows b and c are contained in $k_{\mathfrak{p}}^2$ and $\gamma_{\mathfrak{p}}=0$. Hence above equation holds. So we may assume $k_{\mathfrak{p}}\ni\zeta_{2n}$ or $w_{\mathfrak{p}}(a)\equiv 0 \bmod 2$. Then $k_{\mathfrak{p}}(\sqrt{a})\ni\zeta_{2n}$, and $k_{\mathfrak{p}}(\sqrt[2n]{b}, \sqrt[2n]{a})$, $k_{\mathfrak{p}}(\sqrt[2n]{c}, \sqrt[2n]{a})$ are uniquely determined. Since $k_{\mathfrak{p}}\ni\zeta_4$, $k_{\mathfrak{p}}(\sqrt[2n]{a})$ is a cyclic extension of cegree $2n$. Put $b=a^{b_{\mathfrak{p}}}u^n$ and $c=a^{c_{\mathfrak{p}}}v^n$. Then

$$\gamma_{\mathfrak{p}}=\omega_n\left(\frac{u^{c_{\mathfrak{p}}}v^{b_{\mathfrak{p}}},\ a^{n/2}}{\mathfrak{p}}\right).$$

So it is sufficient to show

$$\omega_n\left(\frac{u,\ a^{n/2}}{p}\right)=\frac{1}{[k_{\mathfrak{p}}(\sqrt[2n]{b}, \sqrt[2n]{a}): k_{\mathfrak{p}}(\sqrt[2n]{a})]} \bmod \boldsymbol{Z}$$

and

$$\omega_n\left(\frac{v,\ a^{n/2}}{\mathfrak{p}}\right)=\frac{1}{[k_{\mathfrak{p}}(\sqrt[2n]{c}, \sqrt[2n]{a}): k_{\mathfrak{p}}(\sqrt[2n]{a})]} \bmod \boldsymbol{Z}.$$

These terms take values in 0 and $\frac{1}{2}$ mod $\boldsymbol{Z}$, and

$$\omega_n\left(\frac{u,\ a^{n/2}}{\mathfrak{p}}\right)=1 \Longleftrightarrow k_{\mathfrak{p}}(\sqrt{u})\subset k_{\mathfrak{p}}(\sqrt{a})$$
$$\Longleftrightarrow k_{\mathfrak{p}}(\sqrt{u})\subset k_{\mathfrak{p}}(\sqrt[2n]{a})$$
$$\Longleftrightarrow k_{\mathfrak{p}}(\sqrt[2n]{b})\subset k_{\mathfrak{p}}(\sqrt[2n]{a})$$

show the first equality. The second equality is all the same.

Next we consider the case iv). If $w_{\mathfrak{p}}(a)\equiv 1$ mod 2 then b and c are in $k_{\mathfrak{p}}^2$ and $\gamma_{\mathfrak{p}}=0$. We assume $w_{\mathfrak{p}}(a)\equiv 0$ mod 2. Then $a\equiv -1$ mod $k_{\mathfrak{p}}^2$. Put $=ab^{b_{\mathfrak{p}}}u^2$ and $c=a^{c_{\mathfrak{p}}}v^2$.

$$\gamma_{\mathfrak{p}}=\omega_n\left(\frac{u^{c_{\mathfrak{p}}}v^{b_{\mathfrak{p}}},\ -1}{\mathfrak{p}}\right)=\frac{1}{2}w_{\mathfrak{p}}(u^{c_{\mathfrak{p}}}v^{b_{\mathfrak{p}}}) \bmod Z.$$

$$\begin{aligned}2w_{\mathfrak{p}}(u^{c_{\mathfrak{p}}}v^{b_{\mathfrak{p}}})&=w_{\mathfrak{p}}(u^{2c_{\mathfrak{p}}}v^{2b_{\mathfrak{p}}})\\ &\equiv w_{\mathfrak{p}}((a^{b_{\mathfrak{p}}}u^2)^{c_{\mathfrak{p}}}(a^{c_{\mathfrak{p}}}v^2)^{b_{\mathfrak{p}}})\\ &\equiv w_{\mathfrak{p}}(b^{c_{\mathfrak{p}}}c^{b_{\mathfrak{p}}})\\ &\equiv c_{\mathfrak{p}}w_{\mathfrak{p}}(b)+b_{\mathfrak{p}}w_{\mathfrak{p}}(c)\\ &\equiv [k_{\mathfrak{p}}(\sqrt{a}):k_{\mathfrak{p}}(\sqrt{c})]w_{\mathfrak{p}}(b)+[k_{\mathfrak{p}}(\sqrt{a}):k_{\mathfrak{p}}(\sqrt{c})]w_{\mathfrak{p}}(c) \bmod 4\end{aligned}$$

shows iv).

Especially if K_1, K_2 and K_3 are tame, then all the components are calculated.

Corollary 2. *Assume K_1, K_2 and K_3 are all tame. When $p=2$, let*

$$P_1=\{\mathfrak{p}:\text{ finite prime of } k\,|\,k_{\mathfrak{p}}\ni\zeta_4,\ k_{\mathfrak{p}}^2\not\ni a\equiv c \bmod k_{\mathfrak{p}}^2,\ k_{\mathfrak{p}}(\sqrt[2n]{b})\not\subset k_{\mathfrak{p}}(\sqrt[2n]{a}),\ \mathfrak{p}\nmid 2\},$$
$$P_2=\{\mathfrak{p}:\text{ finite prime of } k\,|\,k_{\mathfrak{p}}\ni\zeta_4,\ k_{\mathfrak{p}}^2\not\ni a\equiv b \bmod k_{\mathfrak{p}}^2,\ k_{\mathfrak{p}}(\sqrt[2n]{c})\not\subset k_{\mathfrak{p}}(\sqrt[2n]{a}),\ \mathfrak{p}\nmid 2\},$$

and if $n>2$ then put

$P_3=\{\mathfrak{p}:$ finite prime of $k\,|\,k_{\mathfrak{p}}^2\ni a\notin k_{\mathfrak{p}}^4,\ w_{\mathfrak{p}}(b)\cdot w_{\mathfrak{p}}(c)=1 \bmod 2,\ \mathfrak{p}\nmid 2\}$.

Then

$$(a,b,c)_n+(a,c,b)_n=0 \qquad \text{if } p\neq 2,3.$$

If $p=3$

$$(a,b,c)_n+(a,c,b)_n=\sum_{\mathfrak{p};\ \text{finite}\nmid p} w_{\mathfrak{p}}(a)w_{\mathfrak{p}}(b)w_{\mathfrak{p}}(c)\omega_n\left(\frac{\zeta_3}{\mathfrak{p}}\right).$$

If $p=2$ *and* $n>2$,

$$(a, b, c)_n+(a, c, b)_n$$
$$=\sum_{\mathfrak{p}\mid p}\frac{[k_\mathfrak{p}(\sqrt{a}): k_\mathfrak{p}(\sqrt{b})]w_\mathfrak{p}(c)+[k_\mathfrak{p}(\sqrt{a}): k_\mathfrak{p}(\sqrt{c})]w_\mathfrak{p}(b)}{2n}$$
$$+\tfrac{1}{2}({}^\#P_1+{}^\#P_2+{}^\#P_3) \bmod \boldsymbol{Z}.$$

If $p=2$

$$(a, b, c)_n+(a, b, c)_n$$
$$=\sum_{\mathfrak{p}}\frac{[k_\mathfrak{p}(\sqrt{a}): k_\mathfrak{p}(\sqrt{b})]w_\mathfrak{p}(c)+[k_\mathfrak{p}(\sqrt{a}): k_\mathfrak{p}(\sqrt{c})]w_\mathfrak{p}(b)}{4}$$
$$+\tfrac{1}{2}({}^\#P_1+{}^\#P_2) \bmod \boldsymbol{Z}$$

where $\mathfrak{p}$ *runs over all the finite prime of* k *which divides* p *or satisfies* $k_\mathfrak{p} \not\ni \zeta_4$, $a \notin k_\mathfrak{p}^2$ *and* $w_\mathfrak{p}(a)\equiv 0 \bmod 2$.

At $\mathfrak{p}$ dividing p, if $[K_\mathfrak{P}: k_\mathfrak{p}]$ or $[K'_{\mathfrak{P}'}: k_\mathfrak{p}]>n$, it is difficult to determine $\gamma_\mathfrak{p}$ explicitly, because we must determine the minimal Scholz-conductor or take the elements $N_{K'_{\mathfrak{P}'}/K_{1\mathfrak{P}_1}}\mathfrak{B}_{\mathfrak{P}'}$ and $N_{K_\mathfrak{p}/K_{1\mathfrak{P}_1}}\mathfrak{C}_\mathfrak{P}$ of $K_{1\mathfrak{P}_1}$ which have norms b and c to $k_\mathfrak{p}$ and are contained in the smallest i_b-th and i_c-th unit group $U_{\mathfrak{P}_1}^{(i_b)}$ and $U_{\mathfrak{P}_1}^{(i_c)}$ respectively.

So we calculate it in Section 5 only when $k=\boldsymbol{Q}$ (of course $n=2$).

§ 4. Inversion formula over $\boldsymbol{Q}$

We calculate γ_2 in the case of $k=\boldsymbol{Q}$, when $K_\mathfrak{P}$ or $K'_{\mathfrak{P}'}$ is bicyclic over $\boldsymbol{Q}_2$. Then there are 22 cases by separating a, b and c modulo $\boldsymbol{Q}_2^2$.

I_a. $a\equiv -1$, $b\equiv 5$, $c\equiv 5$

I_b. $a\equiv -1$ $\begin{cases} b\equiv 5, & c\equiv 1 \\ b\equiv 1, & c\equiv 5 \end{cases}$

I_c. $a\equiv -5$, $b\equiv 5$, $c\equiv 5$

I_d. $a\equiv -5$ $\begin{cases} b\equiv 5, & c\equiv 1 \\ b\equiv 1, & c\equiv 5 \end{cases}$

I_e. $a\equiv 5$ $\begin{cases} b\equiv -1, & c\equiv 5 \\ b\equiv 5, & c\equiv -1 \end{cases}$

I_f. $a\equiv 5$ $\begin{cases} b\equiv -1, & c\equiv 1 \\ b\equiv 1, & c\equiv -1 \end{cases}$

$$
\begin{array}{lll}
\mathrm{I}_g. & a\equiv 5 & \begin{cases} b\equiv -5, & c\equiv 5 \\ b\equiv 5, & c\equiv -5 \end{cases} \\
\mathrm{I}_h. & a\equiv 5 & \begin{cases} b\equiv -5, & c\equiv 1 \\ b\equiv 1, & c\equiv -5 \end{cases} \\
\mathrm{II}_a. & a\equiv -1 & \begin{cases} b\equiv 2, & c\equiv 1 \\ b\equiv 1, & c\equiv 2 \end{cases} \\
\mathrm{II}_b. & a\equiv 2 & \begin{cases} b\equiv -1, & c\equiv 1 \\ b\equiv 1, & c\equiv -1 \end{cases} \\
\mathrm{III}_a. & a\equiv -1 & \begin{cases} b\equiv 2\cdot 5, & c\equiv 1 \\ b\equiv 1, & c\equiv 2\cdot 5 \end{cases} \\
\mathrm{III}_b. & a\equiv 2\cdot 5 & \begin{cases} b\equiv -1, & c\equiv 1 \\ b\equiv 1, & c\equiv -1 \end{cases} \\
\mathrm{IV}_a. & a\equiv -5 & \begin{cases} b\equiv -2, & c\equiv 1 \\ b\equiv 1, & c\equiv -2 \end{cases} \\
\mathrm{IV}_b. & a\equiv -2 & \begin{cases} b\equiv -5, & c\equiv 1 \\ b\equiv 1, & c\equiv -5 \end{cases} \\
\mathrm{V}_a. & a\equiv -5 & \begin{cases} b\equiv -2\cdot 5 & c\equiv 1 \\ b\equiv 1, & c\equiv -2\cdot 5 \end{cases} \\
\mathrm{V}_b. & a\equiv -2\cdot 5 & \begin{cases} b\equiv -5, & c\equiv 1 \\ b\equiv 1, & c\equiv -5 \end{cases} \\
\mathrm{VI}_a. & a\equiv 2 & \begin{cases} b\equiv -2, & c\equiv 1 \\ b\equiv 1, & c\equiv -2 \end{cases} \\
\mathrm{VI}_b. & a\equiv -2 & \begin{cases} b\equiv 2, & c\equiv 1 \\ b\equiv 1, & c\equiv 2 \end{cases} \\
\mathrm{VII}_a. & a\equiv -2 & \begin{cases} b\equiv -2\cdot 5, & c\equiv 1 \\ b\equiv 1, & c\equiv -2\cdot 5 \end{cases} \\
\mathrm{VII}_b. & a\equiv -2\cdot 5 & \begin{cases} b\equiv -2, & c\equiv 1 \\ b\equiv 1, & c\equiv -2 \end{cases} \\
\mathrm{VIII}_a. & a\equiv 2\cdot 5 & \begin{cases} b\equiv -2\cdot 5, & c\equiv 1 \\ b\equiv 1, & c\equiv -2\cdot 5 \end{cases} \\
\mathrm{VIII}_b. & a\equiv -2\cdot 5 & \begin{cases} b\equiv 2\cdot 5, & c\equiv 1 \\ b\equiv 1, & c\equiv 2\cdot 5 \end{cases}
\end{array}
$$

Here, when two conditions contained in a case like I_b, we consider only the upper one, because of exchanging b and c.

In case of $\mathrm{VI}_a \cdots \mathrm{VIII}_b$,

$$\gamma_2=\frac{w_2(c-1)}{16} \bmod \boldsymbol{Z}.$$

In case of I_e, I_g,

$$\gamma_2=\frac{w_2(b)}{4} \bmod \boldsymbol{Z}.$$

Otherwise $\gamma_2=0$.

Theorem 2. *Assume $k=\boldsymbol{Q}$ and both $(a, b, c)_2$ and $(a, c, b)_2$ are defined. Put*

$$P_1=\{p\colon \text{prime number}\equiv 1 \bmod 4 \mid \boldsymbol{Q}_p^2 \not\ni a\equiv c \bmod \boldsymbol{Q}_p^2,\ \boldsymbol{Q}_p(\sqrt[4]{b})\not\subset \boldsymbol{Q}_p(\sqrt[4]{a})\},$$
$$P_2=\{p\colon \text{prime number}\equiv 1 \bmod 4 \mid \boldsymbol{Q}_p^2 \not\ni a\equiv c \bmod \boldsymbol{Q}_p^2,\ \boldsymbol{Q}_p(\sqrt[4]{c})\not\subset \boldsymbol{Q}_p(\sqrt[4]{a})\},$$
$$P=\{p\colon \text{prime number}\equiv 3 \bmod 4 \mid a\equiv -1 \bmod \boldsymbol{Q}_p^2\}.$$

Then

$$(a, b, c)_2+(a, c, b)_2 = \sum_{p\in P}\frac{[k_p(\sqrt{a})\colon k_p(\sqrt{b})]w_p(c)+[k_p(\sqrt{a})\colon k_p(\sqrt{c})]w_p(b)}{4}+\tfrac{1}{2}({}^{\#}P_1+{}^{\#}P_2)+\gamma_2 \bmod \boldsymbol{Z},$$

where γ_2 takes value as follows:

$\gamma_2=\frac{1}{2}$ *when in the upper cases of* VI_a, VI_b, VII_a, VII_b, VIII_a, VIII_b *with* $b\equiv 9 \bmod 8$, *and when in the lower cases of* $\mathrm{VI}_a, \cdots, \mathrm{VIII}_b$ *with* $c\equiv 9 \bmod 8$, *and when in the upper cases of* I_e, I_g *with* $w_p(b)\equiv 2 \bmod 4$, *and when in the lower cases of* I_e, I_g *with* $w_p(c)\equiv 2 \bmod 4$,

$$\gamma_2=\omega_2\left(\frac{\sqrt{c}, b}{2}\right) \quad \text{if } \boldsymbol{Q}_2^2 \not\ni a\equiv b \bmod \boldsymbol{Q}_2^2 \quad \text{and} \quad c \in \boldsymbol{Q}_2^2,$$

$$\gamma_2=\omega_2\left(\frac{c, \sqrt{b}}{2}\right) \quad \text{if } \boldsymbol{Q}_2^2 \not\ni a\equiv c \bmod \boldsymbol{Q}_2^2 \quad \text{and} \quad b \in \boldsymbol{Q}_2^2,$$

$$\gamma_2=\omega_2\left(\frac{a, \sqrt{bc}}{2}\right) \quad \text{if } \boldsymbol{Q}_2^2 \not\ni a\equiv b\equiv c \bmod \boldsymbol{Q}_2^2,$$

$$\gamma_2=0 \qquad \text{otherwise.}$$

§ 5. About the prime decomposition symbol in Furuta [2]

We assume that $\mathfrak{m}$ is abundant for K/k, i.e. the ray class field $H_K(\mathfrak{m})$ modulo $\mathfrak{m}$ over K is abundant for K/k. Put $\hat{K}(\mathfrak{m})=\hat{K}_{H_K(\mathfrak{m})/k}$ and $K^*(\mathfrak{m}) =K^*_{H_K(\mathfrak{m})/k}$. The other notations are the same as the beginning of Section 2.

Moreover let $I^{\mathfrak{m}}_{K'(\mathfrak{m})}$ be the group of ideals of $K^*(\mathfrak{m})$ which are prime to $\mathfrak{m}$. Taking representatives S_{σ_1} and S_{σ_2} of σ_1 and σ_2 in $G(\hat{K}(\mathfrak{m})/k)$, define an isomorphism

$$\chi\colon G(\hat{K}(\mathfrak{m})/K^*(\mathfrak{m}))\cong\frac{1}{\operatorname{ord}\chi_2}\boldsymbol{Z}/\boldsymbol{Z}$$

by $S_{\sigma_1}^{-1}S_{\sigma_2}^{-1}S_{\sigma_1}S_{\sigma_2}\mapsto(1/\operatorname{ord}\chi_2) \bmod \boldsymbol{Z}$, where of course $S_{\sigma_1}^{-1}S_{\sigma_2}^{-1}S_{\sigma_1}S_{\sigma_2}= (\varphi_{K/k}(\sigma_1\wedge\sigma_2), \hat{K}(\mathfrak{m})/k)$.

Definition. For each $\mathfrak{q}\in N_{K^*(\mathfrak{m})/k}I^{\mathfrak{m}}_{K^*(\mathfrak{m})}$, take an element $\mathfrak{Q}\in I^{\mathfrak{m}}_{K^*(\mathfrak{m})}$ such that $N_{K^*(\mathfrak{m})/k}\mathfrak{Q}=\mathfrak{q}$. Then we define

$$[\chi_1, \chi_2, \mathfrak{q}]=\chi\left(\frac{\hat{K}(\mathfrak{m})/K^*(\mathfrak{m})}{\mathfrak{Q}}\right),$$

where $\left(\frac{\hat{K}(\mathfrak{m})/K^*(\mathfrak{m})}{}\right)$ is the Artin symbol.

Remark. If $[\chi_1, \chi_2, \mathfrak{q}]$ is defined modulo $\mathfrak{m}$, then there exists an element $q\in k^\times$ such that $\mathfrak{q}=(q)$ and (χ_1, χ_2, q) is defined modulo $\mathfrak{m}$. For any such q, $[\chi_1, \chi_2, \mathfrak{q}]=(\chi_1, \chi_2, q)$. However even though $\mathfrak{q}=(q)$ and (χ_1, χ_2, q) is defined, if (χ_1, χ_2, q) is not defined modulo $\mathfrak{m}$, the values (χ_1, χ_2, q) and $[\chi_1, \chi_2, q]$ may not be equal.

Especially in the case that $k=\boldsymbol{Q}$ and $n=2$, $\mathfrak{m}\cdot\mathfrak{p}_\infty$ is abundant for K/k, where $\mathfrak{m}$ is the maximal Scholz-conductor of K/k, and $\mathfrak{p}_\infty$ is the product of all the real primes of K if $\mathfrak{m}$ contains no infinite primes, otherwise $\mathfrak{p}_\infty=1$. Since any finite prime divisor of $\mathfrak{m}$ is ramified in $\hat{K}(\mathfrak{m})/K^*(\mathfrak{m})$, if the prime decomposition symbol $[d_1, d_2, a]$ of Furuta [2] is defined, then $(d_1, d_2, |a|)_2$ is defined and

$$[d_1, d_2, a]=(-1)^{(d_1, d_2, |a|)_2}.$$

So Theorem 2 contains the following

Corollary. *Assume that* d_1, d_2 *and* $d_3\in\boldsymbol{Z}$ *are relatively prime and* $d_2, d_3>0$. *If the symbol* $[d_1, d_2, d_3]$ *and* $[d_1, d_3, d_2]$ *of Furuta* [2] *are defined, then*

$$[d_1, d_2, d_3]=[d_1, d_3, d_2].$$

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