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Swan Conductors with Differential Values

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In this paper, we give a refinement of the classical theory of Swan conductors for discrete valuation rings, and refer to geometric applications.

Classically, the Swan conductor of a character of the Galois group takes values in $\boldsymbol{Z}$. Our Swan conductor takes values in some extension S of $\boldsymbol{Z}$. Let K be a complete discrete valuation field. We consider the following two cases; totally ramified Galois extensions of K (called Case I in this paper), and Galois extensions of ramification index one whose residue extension is purely inseparable and generated by one element (called Case II). In Case I, our group S is $K^\times/U_K^{(1)}$ ($U_K^{(1)}$ is the group of units which are $\equiv 1 \bmod m_K$, the maximal ideal of K). In Case II, our S is a certain group isomorphic to $K^\times/U_K^{(1)} \oplus \boldsymbol{Z}$, and S has elements written as $[\omega]$ for some non-zero differentials ω of the residue field (this is the reason of the title of this paper). The principle is that for a Galois extension L/K and for $\sigma \in \mathrm{Gal}(L/K)$, $\sigma \neq 1$, it is fruitful to consider not only the ideal I_σ of O_L generated by $\{a-\sigma(a);\ a \in O_L\}$ as in the definition of the classical Swan character, but also the homomorphism

$$\varphi_\sigma\colon \Omega^1_{O_L/O_K} \longrightarrow I_\sigma/I_\sigma^2;\ adb \longmapsto a(b-\sigma(b)).$$

In this paper, we define our Swan character as the pair $(I_\sigma, \varphi_\sigma \bmod m_L)$. (Perhaps, it will give a better theory to consider modulo higher powers of m_L).

Most classical results (relations with subgroups and quotient groups, the integrality of Hasse-Arf$\cdots$) are generalized to our Swan conductors. As in the classical case, our conductor is related to the local class field theory. If the residue field is finite, our Swan conductor of a wildly ramified character χ: $\mathrm{Gal}(L/K) \to \boldsymbol{C}^\times$ of degree one describes not only the maximal integer i such that $\chi(U_K^{(i)}) \neq \{1\}$, but also the homomorphism $U_K^{(i)}/U_K^{(i+1)} \to \boldsymbol{C}^\times$ induced by χ. This relation is generalized to higher local fields (Theorem (3.7)) and essentially to all K (Theorem (3.6)).

As the Swan conductor, the different also has a refinement with value in S (§2). This "refined different" already appeared in the work of

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Ihara [9] in a certain case of Case II (not from the standpoint of "theory of different", but in connection with liftings of frobeniuses and with congruence relations): The differential of degree $q-1$ of the residue field associated to a lifting φ of the q-th power frobenius ([9] §2) gives the refined different for the extension of Case II defined by φ (cf. (2.8)).

In Section 4 and Section 5, we introduce without proofs some results on vanishing cycles in relative dimension one, in which our theory is applied to describe the ramification of the special fiber.

Classically, the Swan *character* is related to Weil's trace formula for the cohomology of curves, and the Swan *conductor* is related to the dimension formula of Grothendieck-Ogg-Shafarevich for etale sheaves on a curve. In Section 4, we see that our Swan character in Case II is related to the trace formula of Takeshi Saito [23] on vanishing cycles, and our Swan conductor in Case II is related to the dimension formula in [14] on vanishing cycles of sheaves.

In the last section, we refer to a result (5.6) of a joint work with Takeshi Saito. This result is an attempt to generalize the theory of Laumon [20] (3.4) concerning the relation between local constants and the Galois action on some space of vanishing cycles. Here the differential in the group S in Case II appears as the differential in local constants. The proof and the details will be given in the joint paper [17].

I express my sincere gratitude to Osamu Hyodo and Masato Kurihara from whom I learned that there is a good ramification theory in Case II. I remark that this paper was written long after I learned the ramification theory in the paper Hyodo [6], and I was inspired largely by results and methods in [6]. The definition of our refined different in Section 2 is a modification of the definition of his generalized different (called depth in [6]). I also express my sincere gratitude to Takeshi Saito for valuable advice on Section 4 and Section 5.

Conventions

In this paper, K denotes a complete discrete valuation field with residue field F, and F is always assumed to be of characteristic $p>0$. The normalized additive valuation of K is denoted by v_K. We denote by O_K, m_K, m_K^i $(i \in \boldsymbol{Z})$, U_K, and $U_K^{(i)}$ $(i \geqq 1)$, the set of all elements x of K such that $v_K(x) \geqq 0$, $v_K(x) \geqq 1$, $v_K(x) \geqq i$, $v_K(x)=0$, and $v_K(x-1) \geqq i$, respectively. For $x \in O_K$, $\bar{x} \in F$ denotes the residue class of x.

If L is a finite extension of K, we denote the residue field of L by E.

For a ring B over a ring A, $\Omega^1_{B/A}$ denotes the differential module. The absolute differential module $\Omega^1_{B/\boldsymbol{Z}}$ is denoted by Ω^1_B.

For a set X, $\sharp(X)$ denotes the cardinal number of X.

§ 1. Swan characters

(1.1) For a field k and for one dimensional k-vector spaces $V_1,\cdots, V_r$, we denote by $k\langle V_1,\cdots, V_r\rangle$ the k-algebra

$$\bigoplus_{(i_1,\dots,i_r)\in \boldsymbol{Z}^r} V_1^{\otimes i_1}\otimes\cdots\otimes V_r^{\otimes i_r}.$$

There is a non-canonical isomorphism of k-algebras

$$k\langle V_1,\dots, V_r\rangle\cong k[T_1,\dots, T_r, T_1^{-1},\dots, T_r^{-1}]$$

and hence

$$(k\langle V_1,\dots, V_r\rangle)^\times \cong k^\times\oplus \boldsymbol{Z}^r.$$

For a non-zero element x of V_i $(1\leqq i\leqq r)$, we denote the corresponding element of $(k\langle V_1,\cdots, V_r\rangle)^\times$ by $[x]$.

(1.2) Since m_K/m_K^2 is a one dimensional F-vector space, the F-algebra $F\langle m_K/m_K^2\rangle$ is defined as in (1.1). We denote the group $(F\langle m_K/m_K^2\rangle)^\times$ by R_K, and for some reasons, we denote the group law of R_K additively.

The following lemma is clear.

Lemma (1.3). *There is a canonical isomorphism*

$$K^\times/U_K^{(1)}\cong R_K$$

having the following characterization. For $u\in U_K$, for a prime element π of K, and for $n\in \boldsymbol{Z}$, the image of $u\pi^n$ mod $U_K^{(1)}$ *in R_K is equal to*

$$u\pi^n \bmod m_K^{n+1}\in m_K^n/m_K^{n+1}=(m_K/m_K^2)^{\otimes n}\subset F\langle m_K/m_K^2\rangle .$$

For $a\in K^\times$, we denote the image of a mod $U_K^{(1)}$ in R_K by $[a]$. By our convention, we have $[ab]=[a]+[b]$.

(1.4) Let L be a finite separable extension of K with residue field E. Then we identify R_K with a subgroup of R_L via the following injection. Let e be the ramification index of L/K. Then the canonical isomorphism

$$E\otimes_F(m_K/m_K^2)\cong m_L^e/m_L^{e+1}=(m_L/m_L^2)^{\otimes e} \tag{1.4.1}$$

induces an injective homomorphism of F-algebras

$$F\langle m_K/m_K^2\rangle \longrightarrow E\langle m_L/m_L^2\rangle$$

and hence an injection $R_K\xrightarrow{\subset} R_L$ (Another definition is that this injection corresponds to the canonical map $K^\times/U_K^{(1)}\to L^\times/U_L^{(1)}$ via (1.3)).

(1.5) Let L be a finite separable extension of K with residue field E.

We say that we are in Case I if L/K is totally ramified (i.e. $E=F$). We say that we are in Case II if the ramification index of L/K is one and E/F is a purely inseparable non trivial extension generated by one element. Of course, Case I and Case II are particular cases and never cover all possibilities.

(1.6) Assume that we are in Case II. Then we define groups $S_{K,L}$ and $S_{L/K}$ as follows.

Let $V=\mathrm{Ker}(\Omega_F^1 \to \Omega_E^1)$. Then, V is a one dimensional F-vector space and $\Omega_{E/F}^1$ is a one dimensional E-vector space. Let

$$S_{K,L}=(F\langle m_K/m_K^2,\ V\rangle)^\times, \quad S_{L/K}=(E\langle m_L/m_L^2,\ \Omega_{E/F}^1\rangle)^\times.$$

We denote the group laws of $S_{K,L}$ and $S_{L/K}$ additively. We regard $S_{K,L}$ as a subgroup of $S_{L/K}$ as follows. Let $n=[E:F]$ and let $f\colon E\to F$ be the homomorphism $x \longmapsto x^n$. First we remark that there exists a canonical isomorphism

$$E \otimes_F V \cong (\Omega_{E/F}^1)^{\otimes n} \tag{1.6.1}$$

(V is as above). Indeed, the canonical map induces an isomorphism

$$F \otimes_{f(E)} \Omega^1_{f(E)/f(F)} \xrightarrow{\cong} V \subset \Omega_F^1,$$

and hence

$$E \otimes_F V \cong E \otimes_{f(E)} \Omega^1_{f(E)/f(F)} \cong (\Omega_{E/F}^1)^{\otimes n}$$

where the last isomorphism is

$$x \otimes f(y)df(z) \longmapsto xy^n(dz)^{\otimes n}.$$

Now the isomorphisms (1.6.1) and (1.4.1) (with $e=1$) induce a homomorphism of F-algebras

$$F\langle m_K/m_K^2,\ V\rangle \longrightarrow E\langle m_L/m_L^2,\ \Omega_{E/F}^1\rangle$$

and hence induces an injection $S_{K,L}\to S_{L/K}$ which we shall regard as "inclusion".

Consequently the inclusion maps induce a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_K & \longrightarrow & S_{K,L} & \xrightarrow{\alpha} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle n} & & \\
0 & \longrightarrow & R_L & \longrightarrow & S_{L/K} & \xrightarrow{\beta} & Z & \longrightarrow & 0
\end{array}$$

where $\alpha(x)=1$ for $x\in V-\{0\}$ and $\beta(x)=1$ for $x\in\Omega_{E/F}^1-\{0\}$.

Let M be a subfield of L containing K. Then, when $M \neq L$ (resp. $M \neq K$), we regard $S_{M,L}$ (resp. $S_{M/K}$) as a subgroup of $S_{L/K}$ containing $S_{K,L}$, as follows. If $M=K$ (resp. $M=L$) we do this by the identity $S_{M,L}=S_{K,L}$ (resp. $S_{M/K}=S_{L/K}$). If $K \subsetneqq M \subsetneqq L$, the canonical maps

$$\operatorname{Ker}(\Omega^1_{\bar F} \longrightarrow \Omega^1_{\bar M}) \longrightarrow \operatorname{Ker}(\Omega^1_{\bar F} \longrightarrow \Omega^1_{\bar E}),$$

$$\Omega^1_{\bar E/\bar F} \longrightarrow \Omega^1_{\bar E/\bar M}, \text{ and } \quad \operatorname{Ker}(\Omega^1_{\bar M} \longrightarrow \Omega^1_{\bar E}) \longrightarrow \Omega^1_{\bar M/\bar F}$$

are isomorphisms ($\overline{M}$ denotes the residue field of M), and give the following identifications:

$$S_{K,L}=S_{K,M} \subset S_{M/K}=S_{M,L} \subset S_{L/M}=S_{L/K}.$$

Lemma (1.7). *In Case* I (*resp.* II), *the quotient group* R_L/R_K (*resp.* $S_{L/K}/S_{K,L}$) *is annihilated by* $[L\colon K]$.

(1.8) Assume L is a finite Galois extension of K with Galois group G, and that we are in Case I (resp. II). For $\sigma \in G$, we define

$$s_G(\sigma) \in R_L \quad (\text{resp. } S_{L/K})$$

as follows.

Assume first $\sigma \neq 1$. In Case I, take a prime element h of L, and let

$$s_G(\sigma) = -[1-\sigma(h)h^{-1}] \in R_L.$$

In Case II, take an element h of O_L such that $E=F(\bar h)$, and let

$$s_G(\sigma)=[d\bar h]-[h-\sigma(h)]$$

($[d\bar h]$ is the element of $S_{L/K}$ corresponding to $d\bar h \in \Omega^1_{\bar E/\bar F}-\{0\}$, see (1.1)).

An intrinsic definition of $s_G(\sigma)$ $(\sigma \neq 1)$, from which it is seen that $s_G(\sigma)$ is independent of the choice of h, is as follows. Let I_σ be the ideal of O_L generated by $\{x-\sigma(x);\, x \in O_L\}$. Then we have a surjective homomorphism

$$\varphi_\sigma\colon \Omega^1_{O_L/O_K} \longrightarrow I_\sigma/I_\sigma^2\colon xdy \longmapsto x(y-\sigma(y)).$$

In Case I (resp. II), by composing $\varphi_\sigma \otimes_{O_L} \bar E$ with the canonical isomorphism

$$m_L/m_L^2 \xrightarrow{\cong} \Omega^1_{O_L/O_K}/m_L\Omega^1_{O_L/O_K}; \quad x \longmapsto dx$$
$$(\text{resp. } \Omega^1_{O_L/O_K}/m_L\Omega^1_{O_L/O_K} \xrightarrow{\cong} \Omega^1_{\bar E/\bar F}),$$

we have an isomorphism

$$m_L/m_L^2 \xrightarrow{\cong} I_\sigma/m_L I_\sigma \quad (\text{resp. } \Omega^1_{\bar E/\bar F} \xrightarrow{\cong} I_\sigma/m_L I_\sigma).$$

If $I_\sigma = m_L^i$, tensoring this isomorphism with $(m_L/m_L^2)^{\otimes(-1)}$ (resp. $(\Omega^1_{F/F})^{\otimes(-1)}$) defines a basis of the E-vector space m_L^{i-1}/m_L^i (resp. $m_L^i/m_L^{i+1} \otimes (\Omega^1_{E/F})^{\otimes(-1)}$) which is, when regarded as an element of R_L (resp. $S_{L/K}$), equal to $-s_G(\sigma)$.

In both Case I and Case II, we define

$$s_G(1) = -\sum_{\substack{\sigma \in G \\ \sigma \neq 1}} s_G(\sigma)$$

Proposition (1.9). *Let L be a finite Galois extension of K with Galois group G, and assume that we are in Case* I (*resp.* II). *Let H be a normal subgroup of G. Then for any element τ of $G/H-\{1\}$, we have*

$$\sum_{\substack{\sigma \in G \\ \sigma \mapsto \tau}} s_G(\sigma) = s_{G/H}(\tau) \quad \text{in} \quad R_L \quad (\text{resp. } S_{L/K}).$$

Proof (cf. the proof of [25] Ch. IV §1 Proposition 3). The proofs for Case I and for Case II are similar to each other, and hence we give here the proof for Case II. Let h be an element of O_L such that $E = F(h)$. Let M be the subfield of L corresponding to H and let

$$P(T) = T^r + c_1 T^{r-1} + \cdots + c_r, \quad c_i \in M, \quad r = [L: M]$$

be the characteristic polynomial of h over M. Then $c_1, \ldots, c_r \in m_M$ and $h^r \equiv -c_r \bmod m_L$. Since $P(T) = \prod_{\rho \in H} (T - \rho(h))$, we have $\tau(P)(T) = \prod_{\substack{\sigma \in G \\ \sigma \mapsto \tau}} (T - \sigma(h))$. Thus

$$[\tau(P)(h)] = \sum_{\sigma \mapsto \tau} [h - \sigma(h)] = r[d\bar{h}] - \sum_{\sigma \mapsto \tau} s_G(\sigma).$$

On the other hand

$$\tau(P)(h) = \tau(P)(h) - P(h) = \sum_{i=1}^{r} (\tau(c_i) - c_i) h^{r-i}.$$

Since c_r generates O_M over O_K, we have

$$\tau(c_i) - c_i \in (\tau(c_r) - c_r) m_M$$

and hence

$$\left[\sum_{i=1}^{r} (\tau(c_i) - c_i) h^{r-i}\right] = [\tau(c_r) - c_r].$$

From this we have

$$\begin{aligned}[\tau(P)(h)] = [\tau(c_r) - c_r] &= [-d\bar{c}_r]_M - s_{G/H}(\tau) \\ &= r[d\bar{h}] - s_{G/H}(\tau) \quad \text{in} \quad S_{L/K}.\end{aligned}$$

Here $[\ ]_M$ emphasizes that the differential $-dc_r$ is regarded as a non-zero element of Ω_M^1 but not as an element of Ω_E^1.

§ 2. Differents

(2.1) Let L be a finite separable extension of K. We denote by $R_{L/K}$ the set of all generators of the invertible $E\langle m_L/m_L^2\rangle$-module $E\langle m_L/m_L^2\rangle \otimes_E \mathrm{Hom}_F(E, F)$ (Hom_F here means F-linear homomorphisms). We define the different $\mathfrak{D}(L/K) \in R_{L/K}$ as follows. Let i be the maximal integer such that $\mathrm{Tr}_{L/K}(m_L^i) = O_K$ where $\mathrm{Tr}_{L/K}$ means the trace. The surjection

$$\mathrm{Tr}_{L/K}\colon m_L^i/m_L^{i+1} \longrightarrow O_K/m_K = F$$

induces an E-isomorphism $m_L^i/m_L^{i+1} \cong \mathrm{Hom}_F(E, F)$ and hence defines a basis of $(m_L/m_L^2)^{\otimes(-i)} \otimes_E \mathrm{Hom}_F(E, F)$. We define $\mathfrak{D}(L/K)$ to be this element of $R_{L/K}$. The following result is proved easily.

Proposition (2.2). *If $L \supset M \supset K$, we have*

$$\mathfrak{D}(L/K) = \mathfrak{D}(L/M) + \mathfrak{D}(M/K).$$

Here we denote additively the map $R_{L/M} \times R_{M/K} \to R_{L/K}$ induced by the canonical homomorphisms

$$\mathrm{Hom}_{\bar{M}}(E, \bar{M}) \otimes_{\bar{M}} \mathrm{Hom}_F(\bar{M}, F) \longrightarrow \mathrm{Hom}_F(E, F),$$

$$\bar{M}\langle m_M/m_M^2\rangle \longrightarrow E\langle m_L/m_L^2\rangle.$$

(2.3) In Case I, $\mathrm{Hom}_F(E, F)$ is identified with E and hence $R_{L/K}$ is identified with R_L. In Case II, we identify $R_{L/K}$ with a subset of $S_{L/K}$ as follows. Let $n = [L\colon K]$ and let $\mathrm{Tr}_{E/F}\colon \Omega_E^1 \to \Omega_F^1$ be the trace map (The trace map of differential modules for an inseparable finite extension is not popular. However the norm map $TCK_2(E) \to TCK_2(F)$ (Bloch [1] Ch. II §7) induces on gr^1 of TCK_2 ([1] Ch. II §7, [11] §2.2 Proposition 2) the desired trace map $\Omega_E^1 \to \Omega_F^1$. It is an F-linear map characterized by the property

$$\mathrm{Tr}_{E/F}\left(\frac{dx}{x}\right) = \frac{dx^n}{x^n}, \quad \mathrm{Tr}_{E/F}\left(x^i \frac{dx}{x}\right) = 0 \quad \text{for} \quad 1 \leqq i \leqq n-1$$

for $x \in E^\times$). Then the image of $\mathrm{Tr}_{E/F}$ is $V = \mathrm{Ker}(\Omega_F^1 \to \Omega_E^1)$. Hence we have

(2.3.1) $$\Omega_{E/F}^1 \cong \mathrm{Hom}_F(E, V);\ \omega \mapsto (a \longmapsto \mathrm{Tr}_{E/F}(a\omega)).$$

Thus

$$\mathrm{Hom}_F(E, F) \xrightarrow[(2.3.1)]{\cong} \Omega^1_{E/F} \otimes_F V^{\otimes(-1)} \xrightarrow[(1.6.1)]{\cong} \Omega^1_{E/F} \otimes_E (\Omega^1_{E/F})^{\otimes(-n)}$$
$$=(\Omega^1_{E/F})^{\otimes(1-n)}.$$

Via this isomorphism, $E\langle m_L/m_L^2\rangle \otimes_E \mathrm{Hom}_F(E, F)$ is regarded as a sub-$E\langle m_L/m_L^2\rangle$-module of $E\langle m_L/m_L^2, \Omega^1_{E/F}\rangle$, and $R_{L/K}$ is thus regarded as a subset of $S_{L/K}$

Proposition (2.4). *Let L be a finite Galois extension of K with Galois group G. In Case* I (*resp. Case* II), *we have*

$$s_G(1)=\mathfrak{D}(L/K) \quad \text{in} \quad R_L \quad (\text{resp. } S_{L/K})$$

Proof. In Case I (resp. II), let h be a prime element of L (resp. an element of O_L such that $E=F(\bar{h})$). Then $O_L=O_K[h]$. Let $n=[L\colon K]$ and let $P(T)$ be the characteristic polynomial of h over K. Then in both cases, we have

$$\mathrm{Tr}\,(h^i P'(h)^{-1})=\begin{cases} 0 & \text{if} \quad 0\leq i<n-1 \\ 1 & \text{if} \quad i=n-1 \end{cases} \tag{2.4.1}$$

(P' is the derivative of P, see [25] Ch. III §6 Lemma 2). In Case I, this shows

$$\mathfrak{D}(L/K)=-[h^{n-1}P'(h)^{-1}]$$

and the right hand side is equal to

$$-[h^{n-1}\prod_{\substack{\sigma\in G\\ \sigma\neq 1}}(h-\sigma(h))^{-1}]=-\sum_{\substack{\sigma\in G\\ \sigma\neq 1}} s_G(\sigma)=s_G(1).$$

In Case II, if $P'(h)O_L=m_L^i$, we have by (2.4.1) that $\mathfrak{D}(L/K)$ is equal to the element

$$(P'(h) \bmod m_L^{i+1})\otimes\theta \in E\langle m_L/m_L^2\rangle \otimes_E \mathrm{Hom}_F(E, F)$$

of $R_{L/K}$, where $\theta\colon E\to F$ is the map

$$\sum_{0\leq i<n} x_i\bar{h}^i \longmapsto x_{n-1} \quad (x_i\in F).$$

Since

$$\mathrm{Tr}_{E/F}(xd\dot{\bar{h}})=\theta(x)d(\bar{h}^n) \quad \text{in} \quad \Omega^1_F \qquad \text{for any} \quad x\in E,$$

θ is identified with the element $(1-n)[d\bar{h}]$ of $S_{L/K}$. Thus

$$\mathfrak{D}(L/K)=[P'(h)]+(1-n)[d\hat{h}]$$
$$=\sum_{\substack{\sigma\in G\\ \sigma\neq 1}}([h-\sigma(h)]-[d\bar{h}])=-\sum_{\substack{\sigma\in G\\ \sigma\neq 1}} s_G(\sigma)=s_G(1).$$

Proposition (2.5). *In Case I, if L/K is tame, we have*

$$\mathfrak{D}(L/K)=[n] \quad \textit{where} \quad n=[L:K]$$

(n is regarded as an element of $F^{\times}$).

Proof. Straightforwards.

Remark (2.6). Let $\mathfrak{D}\subset O_L$ be the classical different ideal of L/K. Then the maximal integer i such that $\mathrm{Tr}_{L/K}(m_L^i)=O_K$ is equal to $-\mathrm{ord}_L(\mathfrak{D})+e-1$, where e is the ramification index of L/K. In Hyodo [6], the integer $\mathrm{ord}_L(\mathfrak{D})-e+1$ and certain generalization of it are studied for wild ramifications of (not necessarily discrete) valuations, and our definition of $\mathfrak{D}(L/K)$ was strongly inspired by [6].

Remark (2.7). It seems strange that Swan characters work only in Case I or in Case II whereas differents work well for any finite extensions.

(2.8) The different in this section is related to the differential associated to a lifting of the frobenius in Ihara [9] as follows.

Let k be a subfield of K and assume the following (i) and (ii).

(i) k is a complete discrete valuation field with respect to the valuation of K, and a prime element of k is a prime in K.

(ii) $[F: F^p]=p$ and the residue field $\bar{k}$ of k is perfect.

Then, to a lifting φ: $K\to K$ of the q-th power homomorphism $F\to F$; $x\longmapsto x^q$ (q is a power of p), an element ω of $(\Omega_F^1)^{\otimes(q-1)}$ (well defined modulo $\bar{k}^{\times}$) is associated in [9], and is used for the study of congruence relations. The definition of ω is $\omega=\bar{\eta}^{\otimes q}f(\bar{v})^{\otimes -1}$ under the notation of (2.9) below, where we take as L the latter K in φ: $K\to K$ regarded as an extension of degree q of the former K via φ. As in (2.9), ω gives the essential part of $\mathfrak{D}(L/K)$. For the properties and applications of ω, see also Koike [18]. In this connection, [13] (5.7) gives a ramification theoretic interpretation to the formula [9] Theorem 3.

Proposition (2.9). *Assume we are given a subfield k of K, and assume that the conditions* (i) (ii) *in* (2.8) *are satisfied. Let L be a finite extension of K of degree n in Case II. Let η be a generator of the invertible O_K-module $\hat{\Omega}^1_{O_K/O_k}=\varprojlim \Omega^1_{O_K/O_k}/m_k^i\Omega^1_{O_K/O_k}$ and write $\eta=cv$ in $\hat{\Omega}^1_{O_L/O_k}$ where $c\in O_K$, $c\neq 0$ and where v is a generator of the invertible O_L-module $\hat{\Omega}^1_{O_L/O_k}$. Then*

$$\mathfrak{D}(L/K)=[c]+[\bar{v}]-n[f^{-1}(\bar{\eta})] \quad \textit{in} \quad S_{L/K}$$

where f is the isomorphism $\Omega^1_{E/F}=\Omega^1_E\xrightarrow{\cong}\Omega^1_F$ induced by $E\xrightarrow{\cong}F$; $x\longmapsto x^n$.

Proof. The surjectivity of $\mathrm{Tr}_{E/F}\colon \Omega_E^1 \to \Omega_F^1$ shows the surjectivity of $\mathrm{Tr}_{L/K}\colon \hat{\Omega}^1_{O_L/O_k} \to \hat{\Omega}^1_{O_K/O_k}$, and hence we have

$$\hat{\Omega}^1_{O_L/O_k} \xrightarrow{\cong} \mathrm{Hom}_{O_K}(O_L, \hat{\Omega}^1_{O_K/O_k});\ \theta \longmapsto (a \longmapsto \mathrm{Tr}_{L/K}(a\theta)).$$

From this, we see that the $i \in \boldsymbol{Z}$ such that $m_L^i = c^{-1}O_L$ is the unique integer satisfying $\mathrm{Tr}_{L/K}(m_L^i) = O_K$. Hence we have

$$\mathrm{Tr}_{L/K}(x)\cdot\bar{\eta} = \mathrm{Tr}_{E/F}(\overline{cx}\cdot\bar{\nu}) \quad \text{in} \quad \Omega_F^1 \quad \text{for all} \quad x \in m_L^i,$$

which proves (2.9).

§ 3. Swan conductors and class field theory

In this section, L denotes a finite Galois extension of K with Galois group G. We assume that we are either in Case I or Case II.

Fix an algebraically closed field Λ of characteristic zero and let ζ be a primitive p-th root of 1. The aim of this section is to define a Swan conductor $\mathrm{sw}_\zeta(\chi) \in R_K$ in Case I (resp. $\mathrm{sw}_\zeta(\chi) \in S_{K,L}$ in Case II) for a virtual character χ of G over Λ and give "class field theoretic interpretations" of $\mathrm{sw}_\zeta(\chi)$.

(3.1) We denote by $R(G)$ the Grothendieck group of finitely generated $\Lambda[G]$-modules. Let $\tilde{\boldsymbol{Z}}$ be the integral closure of $\boldsymbol{Z}$ in Λ. We identify an element χ of $R(G)$ with the corresponding virtual character $G \to \tilde{\boldsymbol{Z}}$.

For $\chi \in R(G)$, in Case I (resp. II), we define an element $s_G(\chi)$ of $R_L \otimes \tilde{\boldsymbol{Z}}$ (resp. $S_{L/K} \otimes \tilde{\boldsymbol{Z}}$) by

$$s_G(\chi) = \sum_{\sigma\in G} s_G(\sigma)\otimes\chi(\sigma).$$

We modify this "primitive Swan conductor" $s_G(\chi)$ as follows to obtain an elaborate Swan conductor $\mathrm{sw}_\zeta(\chi)$ (ζ is a primitive p-th root of 1 in Λ). Let

$$\varepsilon(\zeta) = \sum_{r\in \boldsymbol{F}_p^\times} [r]\otimes\zeta^r \in R_K \otimes \tilde{\boldsymbol{Z}}.$$

We have

$$\varepsilon(\zeta^r) = [r] + \varepsilon(\zeta) \qquad \text{for} \quad r \in \boldsymbol{F}_p^\times.$$

Let P be the unique p-Sylow subgroup of G. For $\chi \in R(G)$, in Case I (resp. II), we define the element $\mathrm{sw}_\zeta(\chi)$ of $R_L \otimes \tilde{\boldsymbol{Z}}$ (resp. $S_{L/K} \otimes \tilde{\boldsymbol{Z}}$) by

$$\mathrm{sw}_\zeta(\chi) = s_P(\chi\,|\,_P) + (\chi(1) - \chi^P(1))\cdot\varepsilon(\zeta^{\#(G/P)})$$

where $\chi\,|\,_P$ denotes the restriction of χ to P (s_P is defined with respect to the Galois extension L/L^P where L^P is the fixed subfield of L by P) and χ^P is the virtual character of G/P defined by $\chi^P(\tau)=\sharp(P)^{-1}\sum_{\substack{\sigma\in G\\ \sigma\mapsto\tau}}\chi(\sigma)$ for $\tau\in G/P$. (That is, if χ corresponds to a finitely generated $\Lambda[G]$-module M, χ^P corresponds to the $\Lambda[G/P]$-module $M^P=\{x\in M;\ \sigma(x)=x\ {}^\forall\sigma\in P\}$.

Lemma (3.2). *Let P be the unique p-Sylow subgroup of G and let $m=\sharp(G/P)$.*

(1) $\mathrm{sw}_{\zeta^r}(\chi)=\mathrm{sw}_\zeta(\chi)+(\chi(1)-\chi^P(1))[r]$ *for any* $r\in \boldsymbol{F}_p^\times$.

(2) $s_G(\chi)=s_P(\chi\mid_P)+s_{G/P}(\chi^P)+(\chi(1)-\chi^P(1))[m]$.

Here, in Case II, $G/P=\{1\}$ *and* $s_{G/P}$ *is the zero function* (*In Case* II, *the trivial extension K/K corresponding to G/P is in Case* I, *and hence $s_{G/P}$ is defined to be the zero function $G/P\to R_K$*).

(3) $\mathrm{sw}_\zeta(\chi)=\mathrm{sw}_\zeta(\chi\mid_P)+(\chi(1)-\chi^P(1))[m]$.

(4) $\mathrm{sw}_\zeta(\chi)=0$ *if χ is tame* (*i.e. if χ belongs to the image of the canonical map $R(G/P)\to R(G)$*).

The proofs of these formulas are easy and we omit them.

The classical formulas of the Swan conductors concerning subgroups and quotient groups are generalized as follows.

Proposition (3.3). *Let H be a subgroup of G.*

(1) *Assume H is normal in G, let $\chi\in R(G/H)$ and let $\chi'\in R(G)$ be the image of χ under the canonical map $R(G/H)\to R(G)$* (*That is, $\chi'(\sigma)=\chi(\sigma \bmod H)$ for all $\sigma\in G$*). *Then we have*

$$s_G(\chi')=s_{G/H}(\chi)\quad \textit{and}\ \mathrm{sw}_\zeta(\chi')=\mathrm{sw}_\zeta(\chi).$$

(2) *Let $\chi\in R(H)$ and let $\tilde\chi\in R(G)$ be the induced virtual representation. Then*

$$s_G(\tilde\chi)=\sharp(G/H)\,(s_H(\chi)+\chi(1)\mathfrak{D}(L^H/K))$$

$$\mathrm{sw}_\zeta(\tilde\chi)=\sharp(G/H)\,(\mathrm{sw}_\zeta(\chi)+\chi(1)\mathfrak{D}(L^H/K)-\chi^{H\cap P}(1)\mathfrak{D}(L^{HP}/K))$$

where P is the p-Sylow subgroup of G and L^H (*resp. L^{HP}*) *denotes the fixed subfield in L by H* (*resp. HP*) (*Note $H\cap P$ is the p-Sylow subgroup of H*).

Proof. For s_G, these are formal consequences of (1.9) (2.2) (2.4). The formulas for sw_ζ are deduced from those for s_G and from (2.5) (3.2). We omit the details of the proof since they are straightforwards.

The following result is a generalization of the classical theorem of Artin that Artin character is indeed a character (or of the equivalent result, the theorem of Hasse-Arf).

Theorem (3.4). *In case* I (*resp.* II), *we have*

$$\mathrm{sw}_\zeta(\chi) \in R_K \quad \text{in} \quad R_L \otimes \tilde{\boldsymbol{Z}}$$
$$(\text{resp. } \mathrm{sw}_\zeta(\chi) \in S_{K,L} \text{ in } S_{L/K} \otimes \tilde{\boldsymbol{Z}})$$

for any $\chi \in R(G)$.

The corresponding fact for $s_G(\chi)$ does not hold. The necessity of the modification of $s_G(\chi)$ into $\mathrm{sw}_\zeta(\chi)$ was pointed out by O. Hyodo to the author.
The proof of (3.4) is given later.

(3.5) We shall give class field theoretic interpretations (3.6) (3.7) of the Swan conductor $\mathrm{sw}_\zeta(\chi)$, generalizing the fact that the classical Swan conductor of a character χ of degree one is the minimal integer $i \geqq 0$ such that $\chi(U_K^{(i+1)}) = \{1\}$. Here we give some necessary reviews on Galois cohomology and on higher dimensional local fields (cf. [13]).

For a field k and for an integer n which is invertible in k, let $H_n^q(k)$ be the Galois cohomology group $H^q(k, \boldsymbol{Z}/n\boldsymbol{Z}(q-1))$ where $(q-1)$ means the Tate-twist. If $\mathrm{char}(k) = p > 0$ and $n = n'p^m$ with $p \nmid n'$ and $m \geqq 0$, let $H_n^q(k) = H_{n'}^q(k) + H_{p^m}^q(k)$ where $H_{n'}^q(k)$ is as above and

$$H_{p^m}^q(k) = \mathrm{Coker}\,(W_m\Omega_k^{q-1} \xrightarrow{F-1} W_m\Omega_k^{q-1}/dW_m\Omega_k^{q-2}).$$

Here $W_m\Omega_k^{\cdot}$ is the de Rham-Witt complex.

We define

$$H^q(k) = \varinjlim_n H_n^q(k).$$

Then we have canonical isomorphisms

(3.5.1) $$H^2(k) \cong \mathrm{Br}\,(k), \text{ the Brauer group of } k,$$

(3.5.2) $$H^1(k) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}\,(k^{\mathrm{ab}}/k), \boldsymbol{Q}/\boldsymbol{Z})$$

where k^{ab} denotes the maximum abelian extension of k.

Let $K_*^M(k)$ be Milnor's K-group of k ([19]). Then we have a pairing

(3.5.3) $$\{\ ,\ \}: H_n^q(k) \otimes K_r^M(k) \longrightarrow H_n^{q+r}(k)$$

and its limit

(3.5.4) $$\{\ ,\ \}: H^q(k) \otimes K_r^M(k) \longrightarrow H^{q+r}(k)$$

as follows. If n is invertible in k, (3.5.3) is defined by the cohomological symbol map

$$K_r^M(k) \longrightarrow H^r(k, \boldsymbol{Z}/n\boldsymbol{Z}(r))$$

and the cup product. If $\mathrm{ch}(k)=p>0$ and n is a power p^m of p, (3.5.3) is defined by

$$K_r^M(k) \longrightarrow W_m\Omega_k^r;\ \{a_1,\cdots,a_r\} \longmapsto d\log(a_1)\cdots d\log(a_r)$$

and the product structure of $W_m\Omega_k^{\cdot}$.

In the case of a complete discrete valuation field k with residue field $\bar{k}$, we have a canonical homomorphism

$$i_q\colon H_n^q(\bar{k}) \longrightarrow H_n^q(k),$$

for which

$$\begin{aligned} H_n^q(\bar{k})\oplus H_n^{q-1}(\bar{k}) &\longrightarrow H_n^q(k) \\ (x, y) &\longmapsto i_q(x)+\{i_{q-1}(y), \pi\} \end{aligned} \tag{3.5.5}$$

is injective for any prime element π of k. This map (3.5.5) is bijective if n is invertible in $\bar{k}$, or if $\mathrm{char}(\bar{k})=p>0$ and $[\bar{k}\colon \bar{k}^p]\leqq p^{q-2}$. If $\mathrm{char}(\bar{k})=p>0$, the composite map

$$H_n^q(\bar{k})\oplus H_n^{q-1}(\bar{k}) \longrightarrow H_n^q(k) \longrightarrow H^q(k)$$

is still injective if n is a power of p.

In the case $\mathrm{char}(\bar{k})=p>0$, we denote the composite

$$\Omega_{\bar{k}}^{q-1} \longrightarrow H_p^q(\bar{k}) \xrightarrow{i_q} H_p^q(k) \longrightarrow H^q(k),$$

where the first map comes from the definition of $H_p^q(\bar{k})$, also by i_q. We then have

$$i_q\left(x\frac{d\bar{u}_1}{\bar{u}_1}\wedge\cdots\wedge\frac{d\bar{u}_{q-1}}{\bar{u}_{q-1}}\right)=\{i_1(x), u_1,\cdots,u_{q-1}\}$$

for $u_1,\cdots,u_{q-1}\in U_k$.

We call a field k an N-dimensional local field if a sequence of fields $k_0,\cdots,k_N$ is given satisfying the following conditions.

(i) k_0 is a finite field.

(ii) For $1\leqq i\leqq N$, k_i is a complete discrete valuation field with residue field k_{i-1}.

(iii) $k=k_N$.

For an N-dimensional local field k, we obtain a canonical isomorphism

$$H_n^{N+1}(k)\cong \boldsymbol{Z}/n\boldsymbol{Z} \tag{3.5.6}$$

by induction on N and by (3.5.5). This gives a canonical pairing

$$H^1(k)\otimes K_N^M(k) \longrightarrow H^{N+1}(k)\cong \boldsymbol{Q}/\boldsymbol{Z}$$

and hence by (3.5.2), the reciprocity map

$$(3.5.7) \qquad K_N^M(k) \longrightarrow \mathrm{Gal}(k^{\mathrm{ab}}/k).$$

of the class field theory of k.
If further $\mathrm{char}(k)=p>0$, the composite

$$\Omega_k^N \longrightarrow H_p^{N+1}(k) \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z}$$

is denoted by Res (called the residue map) and has an explicit description as in [11] §2.

Theorem (3.6). *Let L/K and G be as before and let $\chi\colon G\to\Lambda^\times$ be a character of degree one which is not tamely ramified. Fix an isomorphism $\boldsymbol{Q}/\boldsymbol{Z}\cong(\Lambda^\times)_{\mathrm{tor}}$. Regard χ as an element of $H^1(K)$ via this isomorphism, and let ζ be the primitive p-th root of 1 which corresponds to $\frac{1}{p}$ mod $\boldsymbol{Z}$ via this isomorphism.*

(1) *Assume that we are in Case I, let $\mathrm{sw}_\zeta(\chi)=[c]$, $c\in K^\times$, and let $m=v_K(c)$. Then $m\geqq 1$, and*

$$\{\chi,\ \}\colon K^\times=K_1^M(K)\longrightarrow H^2(K) \qquad (3.5.4)$$

annihilates $U_K^{(m+1)}$. We have

$$\{\chi, 1-cz\}=\{i_1(\bar z), \pi\} \quad \text{in} \quad H^2(K)$$

for any $z\in O_K$ and for any prime element π of K such that $\pi\in N_{L/K}(L^\times)$.

(2) *Assume we are in Case II. Then $\mathrm{sw}_\zeta(\chi)=[c]-[\omega]$ for some $c\in K^\times$ and $\omega\in\Omega_F^1-\{0\}$. Let $m=v_K(c)$. Then, $m\geqq 1$ and $\{\chi,\ \}\colon K^\times\to H^2(K)$ annihilates $U_K^{(m+1)}$. We have*

$$\{\chi, 1-cz\}=i_2(\bar z\omega) \quad \text{in} \quad H^2(K) \qquad \text{for} \quad z\in O_K.$$

In the following, for a discrete valuation field k, we denote by $U^mK_q^M(k)$ $(m, q\geqq 1)$ the subgroup of $K_q^M(k)$ generated by elements of the form $\{x, y_1,\dots, y_{q-1}\}$ such that $x\in k$, $v_k(x-1)\geqq m$, and $y_1,\dots, y_{q-1}\in k^\times$.

Theorem (3.7). *Let K be an N-dimensional local field with $N\geqq 1$ with residue field F $(=k_{N-1})$ of characteristic $p>0$. Let L/K and G be as before, and let $\chi\colon G\to\Lambda^\times$ be a character of degree one which is not tamely ramified. Denote also by χ the homomorphism $K_N^M(K)\to\Lambda^\times$ which is induced by χ and (3.5.7).*

(1) *In Case* I, *let* $\mathrm{sw}_\zeta(\chi)=[c]$ $(c\in K^\times)$ *and* $m=v_K(c)$. *Then χ annihilates $U^{m+1}K_N^M(K)$ and the composite*

$$\Omega_F^{N-1} \xrightarrow{\varphi_c} gr^m K_N^M(K) \xrightarrow{\chi} \Lambda^\times,$$

where $\varphi_c\left(\bar{y}\dfrac{d\bar{z}_1}{\bar{z}_1}\wedge\cdots\wedge\dfrac{d\bar{z}_{N-1}}{\bar{z}_{N-1}}\right)=\{1-yc,\ z_1,\cdots,\ z_{N-1}\}$ $(y\in O_K,\ z_1,\cdots,$ $z_{N-1}\in U_K)$, *coincides with*

$$\theta \longmapsto \zeta^{(-1)^{N-1}\mathrm{Res}(\theta)}.$$

(2) *In Case* II, *let* $\mathrm{sw}_\zeta(\chi)=[c]-[\omega]$ $(c\in K^\times,\ \omega\in\Omega_F^1-\{0\})$, *and let* $m=v_K(c)$. *Then,* χ *annihilates* $U^{m+1}K_N^M(K)$, *and for any prime element* π *of* K, *the composite*

$$\Omega_F^{N-2} \xrightarrow{\varphi_c} gr^m K_N^M(K) \xrightarrow{\chi} \Lambda^\times,$$

where $\varphi_c\left(\bar{y}\dfrac{d\bar{z}_1}{\bar{z}_1}\wedge\cdots\wedge\dfrac{d\bar{z}_{N-2}}{\bar{z}_{N-2}}\right)=\{1-cy,\ z_1,\cdots,\ z_{N-2},\ \pi\}$ $(y\in O_K,\ z_1,\cdots,$ $z_{N-2}\in U_K)$, *coincides with*

$$\theta \longmapsto \zeta^{(-1)^{N}\mathrm{Res}(\theta\wedge\omega)}.$$

This (3.7) is easily deduced from (3.6) by using the facts in (3.5).

Remark (3.8). For K and χ as in Theorem (3.7), $\mathrm{sw}_\zeta(\chi)$ is characterized by its property stated in (3.7). On the other hand, for more general K and χ considered in (3.6), the statement of (3.6) (1) (resp. (2)) becomes very weak if $H_p^1(F)=0$ (resp. $H_p^2(F)=0$) (for example if F is separably closed). However (3.6) gives a characterization of $\mathrm{sw}_\zeta(\chi)$ in the following sense. For an extension F' of F preserving p-basis (this means that a p-basis ([7] Ch. 0 §21) of F is still a p-basis in F', or equivalently, that the map

$$F'\otimes_{F\,\nearrow f} F \longrightarrow F';\ x\otimes y \longmapsto x^p y$$

is bijective where $f(x)=x^p$), there is a complete discrete valuation field K' over K such that the restriction of $v_{K'}$ to K coincides with v_K and such that the residue field of K' is isomorphic to F' over F (See [12] Lemma 1, such K' is unique in a very strong sense). Then $L'=L\otimes_K K'$ is a Galois extension of K' whose Galois group G' is canonically isomorphic to G. If L/K is in Case I (resp. II), so is L'/K'. Furthermore we have $s_{G'}(\sigma')=s_G(\sigma)$ for any $\sigma\in G$ and for the corresponding $\sigma'\in G'$, and hence

$$s_{G'}(\chi')=s_G(\chi),\quad \mathrm{sw}_\zeta(\chi')=\mathrm{sw}_\zeta(\chi)$$

for any $\chi\in R(G)$ and for the corresponding $\chi'\in R(G')$. By the following Lemma (3.9), when F' ranges over all extensions of F preserving p-basis, (3.6) gives a characterization of $\mathrm{sw}_\zeta(\chi)$.

Lemma (3.9). *Let k be a field of characteristic $p>0$, let $r\geqq 0$, and let $\omega\in\Omega_k^r$, $\omega\neq 0$. Then there exist an extension k'/k preserving p-basis and $a\in k'$ such that the class of $a\omega$ in $H_p^{r+1}(k')$ is not zero.*

Proof. By a direct computation, one sees easily the following fact. If $\mathrm{char}(k)=p>0$ and $[k\colon k^p]=p^r<\infty$, and if $(b_i)_{1\leqq i\leqq r}$ is a p-basis of k, the class of $T^{-1}b_1^{-1}db_1\wedge\cdots\wedge b_r d^{-1}b_r$ in $H_p^{r+1}(k((T)))$ is not zero. Now let k and ω be as in (3.9), and let $k'=\bigcup_{i\geqq 0} k(T^{p^{-i}})$ (T is a variable). Then the extension k'/k preserves p-basis. Let $(b_\lambda)_{\lambda\in I}$ be a p-basis of k where I is a totally ordered set, and write

$$\omega=\sum_s x_s b_{s(1)}^{-1}db_{s(1)}\wedge\cdots\wedge b_{s(r)}^{-1}db_{s(r)}$$

where s ranges over all strictly increasing functions $\{1,\cdots,r\}\to I$. Fix s such that $x_s\neq 0$ and let $a=(Tx_s)^{-1}$. We show that the class of $a\omega$ in $H_p^{r+1}(k')$ is not zero. To see this, it is sufficient to show that the class of $a\omega$ in $H_p^{r+1}(k(T^{p^{-i}}))$ is not zero for any i. Let

$$k''=k(b_\lambda^{p^{-i}};\ \lambda\in I-\{s(1),\cdots,s(r)\},\ i\geqq 0).$$

Then $[k''\colon (k'')^p]=p^r$. In $H_p^{r+1}(k''((T^{p^{-i}})))$, the class of $a\omega$ is equal to the class of $T^{p^{-i}}b_{s(1)}^{-1}db_{s(1)}\wedge\cdots\wedge b_{s(r)}^{-1}db_{s(r)}$ and this class is not zero by the remark at the beginning of the proof.

Now we prove Theorem (3.4) and Theorem (3.6). As in the classical case, the proof of Theorem (3.4) is reduced to the case where χ is a character of degree one. Indeed, by the theory of Brauer ([26] §10), $R(G)$ is generated by characters which are induced by characters of degree one of subgroups of G. But if H is a subgroup of G and if (3.4) holds for $\chi\in R(H)$, then (3.4) holds for $\tilde{\chi}\in R(G)$ by (3.3) (2) and (1.6) (1.7).

Our task is now to prove (3.4) and (3.6) for a character of degree one. The reduction to the case $G=P$ is easy, and so we assume that the order of $\chi\colon G\to\Lambda^\times$ is a power of p.

We consider first the case where $\chi\colon G\to\Lambda^\times$ is of order p. In this case, (3.4) and (3.6) follow from the following (3.10) and (3.11).

Lemma (3.10). *Assume that L/K is of degree p, $\chi\colon G\to\Lambda^\times$ a non-trivial homomorphism, σ a generator of G, and let $\zeta=\chi(\sigma)$. In case* I *(resp.* II*), let h be a prime element of L (resp. an element of O_L such that $E=F(\bar{h})$). Let*

$$a=1-\sigma(h)h^{-1},\quad b=N_{L/K}(a).$$

Then, in Case I (*resp.* II), *we have*

$$\mathrm{sw}_\zeta(\chi)=[-b]$$

(*resp.* $\mathrm{sw}_\zeta(\chi)=[-b]-[x^{-1}dx]_K$ *where* $x=\bar{h}^p$ *and where* $[\ ]_K$ *emphasizes that the differential is considered as an element of* Ω^1_F (*not of* $\Omega^1_{E/F}$).

Proof. The proofs for Case I and for Case II are similar, and so we give here the proof for Case II. By definition,

$$s_G(\sigma^i)=-[ia]+[\bar{h}^{-1}d\bar{h}]_L \qquad \text{for} \quad 1\leqq i<p.$$

So,

$$\begin{aligned} s_G(\chi)&=\sum_{i=0}^{p-1} s_G(\sigma^i)\otimes\zeta^i=\sum_{i=1}^{p-1} s_G(\sigma^i)\otimes(\zeta^i-1)\\ &=-\varepsilon(\zeta)+[-1]+p[a]-p[\bar{h}^{-1}d\bar{h}]_L\\ &=-\varepsilon(\zeta)+[-b]-[x^{-1}dx]_K \quad (x=\bar{h}^p), \end{aligned}$$

and hence $\mathrm{sw}_\zeta(\chi)=[-b]-[x^{-1}dx]_K$.

Lemma (3.11). *Let* L/K, χ, σ *be as in* (3.10), *and let* h *be any element of* $L^\times$. *Let* $a=1-\sigma(h)h^{-1}$ *and* $b=N_{L/K}(a)$. *Then*

$$\{\chi,\ 1+bz\}=\{i_1(\bar{z}),\ N_{L/K}(h)\} \quad \text{in} \quad H^2(K)$$

for any $z\in O_K$.

For the proof, see [11] §3.3 Lemma 15.

Next we consider homomorphisms $\chi\colon G\to\Lambda^\times$ of order p^n ($n\geqq 1$) by induction on n. By (3.3) (1), we may assume χ is injective.

Lemma (3.12). *Let* L/K *be a cyclic extension of* K *of degree a power of* p, H *a subgroup of* G *such that* $H\neq\{1\}$, *and let* M *be the subfield of* L *corresponding to* H. *Then for any injective homomorphism* $\chi\colon G\to\Lambda^\times$, *we have*

$$\mathrm{sw}_\zeta(\chi)=\mathrm{sw}_\zeta(\chi|_H)+\mathfrak{D}(M/K).$$

Proof. Since

$$\mathrm{sw}_\zeta(\chi)-\mathrm{sw}_\zeta(\chi|_H)=s_G(1)-s_H(1)+\sum_{\sigma\in G-H} s_G(\sigma)\otimes\chi(\sigma)$$

and $s_G(1)-s_H(1)=\mathfrak{D}(M/K)$ ((2.2) (2.4)), it is sufficient to prove $\sum_{\sigma\in G-H} s_G(\sigma)\otimes\chi(\sigma)=0$. For this, it suffices to prove for $\sigma\in G-H$ of order p^r ($r\geqq 2$), $\sum_{i\in(\mathbf{Z}/p^r)^\times} s_G(\sigma^i)\otimes\chi(\sigma)^i=0$. Since $s_G(\sigma^i)=[i]+s_G(\sigma)$, we are reduced to the easy facts

$$\sum_{i\in(\boldsymbol{Z}/p^r)^\times}\eta^i=0,\qquad \sum_{\substack{i\in(\boldsymbol{Z}/p^r)^\times\\ i\mapsto j}}\eta^i=0\qquad\text{for}\quad j\in\boldsymbol{F}_p^\times$$

for a primitive p^r-th root η of 1 with $r\geqq 2$.

Assume now that L/K is a cyclic extension of degree p^n ($n\geqq 2$). We prove (3.4) (3.6) for an injective homomorphism $\chi\colon G\to\Lambda^\times$ (This is sufficient to finish the proofs of (3.4) (3.6)). Let M/K be the subextension of L/K of degree p. We assume that (3.4) and (3.6) are already proved for L/M and M/K. In Case I (resp. II), let $t=v_M(1-\sigma(h)h^{-1})$ where h is a prime element of M (resp. an element of O_M such that $\bar{M}=F(\bar{h})$). Then t is independent of the choice of h. The study of the norm map $N_{M/K}$: $M^\times\to K^\times$ in [25] Ch. V §3 (for Case I) and in [10] §1 Remark 1 (for Case II) shows the followings. In Case I,

$$N_{M/K}(U_M^{(t+pi)})=U_K^{(t+i)},\quad N_{M/K}(U_M^{(t+pi+1)})=U_K^{(t+i+1)}$$

$$N_{M/K}(1+x)\equiv 1+\mathrm{Tr}_{M/K}(x)\mod m_K^{t+i+1}$$

for any $i\geqq 1$ and for any $x\in m_M^{t+pi}$. In Case II,

$$N_{M/K}(U_M^{(t+i)})=U_K^{(pt+i)}$$

$$N_{M/K}(1+x)\equiv 1+\mathrm{Tr}_{M/K}(x)\mod m_K^{(pt+i+1)}$$

for any $i\geqq 1$ and for any $x\in m_M^{t+i}$ (In fact, in [10] §1, we considered only the case $[F\colon F^p]=p$. But the above results on $N_{M/K}$ is reduced to this case by adding to K the p^j-th roots ($j\geqq 0$) of a lifting of a p-basis of F over $\bar{M}^p$).

We consider Case I. Let $\mathrm{sw}_\zeta(\chi\,|_H)=[b]$, $b\in M^\times$, and let $m'=v_K(b)$. By using (1.9), we can show $m'>t$ (we omit the details). We can prove further $m'\equiv t \bmod p$ as follows. By (3.9), we may assume $H_p^1(F)\neq 0$. If $t+pi+1\leqq m'<t+p(i+1)$ for some $i\geqq 1$, we have $N_{M/K}(U_M^{(m')})=N_{M/K}(U_M^{(m'+1)})$ by the above remark on $N_{M/K}$. But this contradicts $H_p^1(F)\neq 0$ since in $H^2(K)$,

$$\{\chi,\ N_{M/K}(U_M^{(m'+1)})\}=\mathrm{Cor}_{M/K}(\{\chi\,|_H,\ U_M^{(m'+1)}\})=0,$$

$$\begin{aligned}\{\chi,\ N_{M/K}(U_M^{(m')})\}&=\mathrm{Cor}_{M/K}(\{\chi\,|_H,\ U_M^{(m')}\})\\ &=\mathrm{Cor}_{M/K}(\{H_p^1(F),\ N_{L/M}(\pi_L)\})=\{H_p^1(F),\ N_{L/K}(\pi_L)\}\neq 0\end{aligned}$$

where $\mathrm{Cor}_{M/K}$ is the corestriction map and π_L is a prime element of L. Now by $m'>t$ and $m'\equiv t \bmod p$, and by the above remark on $N_{M/K}$, if we put $c=\mathrm{Tr}_{M/K}(b)$ and $m=v_K(c)$, we have $m'-t=p(m-t)$, $N_{M/K}(1-bz)\equiv 1-cz \bmod m_K^{m+1}$ for $z\in O_K$, and

$$\mathfrak{D}(M/K)=[b^{-1}c].$$

This last equation and (3.12) prove $\mathrm{sw}_\zeta(\chi)=[c]$. Furthermore,

$$\{\chi, U_K^{(m+1)}\}=\{\chi, N_{M/K}(U_M^{(m'+1)})\}=\mathrm{Cor}_{M/K}(\{\chi\mid_H, U_M^{(m'+1)}\})=0,$$

$$\begin{aligned}\{\chi,\ 1-cz\}&=\mathrm{Cor}_{M/K}(\{\chi|_H,\ 1-bz\})=\mathrm{Cor}_{M/K}(\{i_1(\bar z),\ N_{L/M}(\pi_L)\})\\ &=\{i_1(\bar z),\ N_{L/K}(\pi_L)\}\end{aligned}$$

for $z\in O_K$ and for a prime element π_L of L.

We next consider Case II. Let $\mathrm{sw}_\zeta(\chi\mid_H)=[b]-[\omega]_M$, $b\in M^\times$, $\omega\in\mathrm{Ker}(\Omega^1_{\bar M}\to\Omega^1_{\bar E})$, $\omega\neq 0$. Let $m'=v_K(b)$ and let $m=pt+(m'-t)$. By using (1.9), we have $m'>t$. Hence by the above description of $N_{M/K}$, we see that

$$\mathfrak{D}(M/K)=[c]-[b]+[\omega']_M-[\eta]_K$$

for some $c\in K^\times$ such that $v_K(c)=m$, and for some non-zero elements $\omega'\in\Omega^1_{\bar M/F}$ and $\eta\in\mathrm{Ker}(\Omega^1_F\to\Omega^1_{\bar M})$, and that

$$N_{M/K}(1-by)\equiv 1-cz \quad \mathrm{mod}\ m_K^{m+1}$$

for $y\in O_M$ and for $z\in O_K$ satisfying

$$\mathrm{Tr}_{M/F}(\bar y\omega')=\bar z\eta.$$

We have for such y and z,

$$(*)\qquad \begin{aligned}\{\chi,\ 1-cz\}&=\mathrm{Cor}_{M/K}(\{\chi\mid_H,\ 1-by\})\\ &=\mathrm{Cor}_{M/K}(i_2(\bar y\omega))=i_2(\mathrm{Tr}_{\bar M/F}(\bar y\omega))\end{aligned}$$

by the commutativity of the diagram (easy to prove)

$$\begin{array}{ccc}\Omega^1_{\bar M} & \xrightarrow{\ i_2\ } & H^2(M)\\ {\scriptstyle \mathrm{Tr}_{\bar M/F}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cor}_{M/K}}\\ \Omega^1_F & \xrightarrow{\ i_2\ } & H^2(K).\end{array}$$

We show that $\omega'\in F\omega$ in $\Omega^1_{\bar M/F}$ (This corresponds to the "Hasse-Arf property" $m'\equiv t \bmod p$ in Case I). To prove this, by (3.9), we may assume that the image of $V=\mathrm{Ker}(\Omega^1_F\to\Omega^1_{\bar E})$ in $H^2_p(F)$ is not zero. Since $\mathrm{Tr}_{\bar M/F}$ factors as

$$\Omega^1_{\bar M}\longrightarrow\Omega^1_{\bar M/F}/d\bar M\xrightarrow{\ \cong\ }V$$

(*) shows

$$\bar{y}\omega' \in d\overline{M} \quad \text{in} \quad \Omega^1_{\overline{M}/F} \Longrightarrow \bar{y}\,\omega \in d\overline{M} \quad \text{in} \quad \Omega^1_{\overline{M}/F},$$

and hence shows $\omega' \in F\omega$ in $\Omega^1_{\overline{M}/F}$. Now by writing $\omega' = \bar{a}\omega$ in $\Omega^1_{\overline{M}/F}$ with $a \in U_K$, we have

$$\mathfrak{D}(M/K) = [c] - [b] + [\omega]_M - [\bar{a}^{-1}\eta]_K.$$

By (3.12), we have

$$\mathrm{sw}_\zeta(\chi) = [c] - [\bar{a}^{-1}\eta]_K$$

proving (3.4), and now (*) shows

$$\{\chi, 1 - cz\} = i_2(z\bar{a}^{-1}\eta) \qquad \text{for} \quad z \in O_K$$

proving (3.6).

Remark (3.13). If F is perfect, $\mathrm{sw}_\zeta(\chi)$ for a character χ of degree one is explained by the local class field theory of Serre [24] and of Hazewinkel [5]. Assume that F is perfect and we are in Case I. Let $\underline{U}_K$ be the pro-algebraic group over F associated to U_K. Then a homomorphism χ: $\mathrm{Gal}(L/K) \to \Lambda^\times$ induces

$$\chi\colon \pi_1(\underline{U}_K) \longrightarrow \Lambda^\times$$

(see [24] [5]). If χ is not tame, there exists $m \geqq 1$ such that

$$\chi(\pi_1(\underline{U}_K^{(m)})) \neq \{1\}, \quad \chi(\pi_1(\underline{U}_K^{(m+1)})) = \{1\}.$$

Fix a primitive p-th root ζ of 1 in Λ and a prime element π of K. Then χ and the homomorphism

$$\mathbf{G}_a \longrightarrow \underline{U}_K^{(m)}/\underline{U}_K^{(m+1)};\ \bar{a} \longmapsto 1 - a\pi^m$$

induce a non-zero continuous homomorphism

$$\mathrm{Hom}(F, \mathbf{Z}/p\mathbf{Z}) \cong \pi_1(\mathbf{G}_a) \cong \pi_1(\underline{U}_K^{(m)}/\underline{U}_K^{(m+1)}) \longrightarrow \Lambda^\times,$$

and this homomorphism must have the form

$$\mathrm{Hom}(F, \mathbf{Z}/p\mathbf{Z}) \longrightarrow \Lambda^\times;\ h \longmapsto \zeta^{h(u)}$$

for a unique $u \in F^\times$. Then our $\mathrm{sw}_\zeta(\chi)$ is equal to $[\pi^m] - [u]$.

By using a generalization [15] of the local class field theory [24] [5], we can generalize the above fact as follows: If $[F: F^p] = p^r < \infty$ and χ is an element of $H^{r+1}(K)$ whose image in $H^{r+1}(K_{nr})$ has order a multiple of p (i.e. χ is not "tame"; here K_{nr} denotes the maximal unramified extension of K), then there exists a unique element $\mathrm{sw}(\chi)$ of $(F\langle m_K/m_K^2, \Omega_F^r\rangle)^\times$ having the following property: $\mathrm{sw}_\zeta(\chi)$ has the form $[c]-[\omega]$ where $c\in K^\times$, $v_K(c)\geqq 1$, $\omega\in\Omega_F^r-\{0\}$, and for any extension F' of F preserving p-basis,

$$\{\chi,\ 1-cz\}=\{i_{r+1}(\bar{z}\omega),\ \pi\}\quad\text{in}\quad H^{r+2}(K')$$

for any $z\in O_K$, and for any prime element π of K where K' is as in (3.8).

Remark (3.14). There is an analogue of the argument in Section 2 for division algebras. Let D be a central division algebra over K of dimension n^2, and assume that the residue skew field E of D is commutative and is a purely inseparable non-trivial extension of F generated by one element. Then, $[E\colon F]=n$ and the ramification index of D/K is n. Let $V=\operatorname{Ker}(\Omega_F^1\to\Omega_E^1)$ and let

$$S_{K,D}=(F\langle m_K/m_K^2,\ V\rangle)^\times,\quad S_{D/K}=(E\langle m_D/m_D^2,\ \Omega_{E/F}^1\rangle)^\times.$$

Then the canonical isomorphisms $E\otimes_F m_K/m_K^2\cong m_D^n/m_D^{n+1}$ and $E\otimes_F V\cong(\Omega_{E/F}^1)^{\otimes n}$ (1.6.1) induce an inclusion map $S_{K,D}\xrightarrow{\subset}S_{D/K}$. We can define the different $\mathfrak{D}(D/K)\in S_{D/K}$ by using the reduced trace map $D\to K$ in the same way as in Section 2.

In the case $[F\colon F^p]=p$, this different is related to the Swan conductor for $H^2(K)$ in (3.13) as follows (cf. [16]). If χ denotes the element of $H^2(K)=Br(K)$ corresponding to D, then χ is of order n and

$$\mathfrak{D}(D/K)=(p-1)\sum_i p^i\,\mathrm{sw}\,((np^{-1-i})\chi)$$

where i ranges over all integers $\geqq 0$ such that $p^{i+1}\mid n$. This is a non-commutative analogue of [6] (3.4).

Remark (3.15). The assumption of this section that L/K is in Case I or II is slightly weakened as follows. Let L/K be a finite Galois extension with Galois group G and with the maximum unramified subextension K'/K, and assume that L/K' is in Case I or II. For $\chi\in R(G)$ and a primitive p-th root ζ of 1 in Λ, define

$$\mathrm{sw}_\zeta(\chi)=\mathrm{sw}_\zeta(\chi\mid_{\mathrm{Gal}(L/K')}).$$

Then, we see easily that $\mathrm{sw}_\zeta(\chi)$ is fixed by $\mathrm{Gal}(K'/K)$ and thus obtain

(3.15.1) if L/K' is in Case I (resp. II), $\mathrm{sw}_\zeta(\chi)$ belongs to R_K (resp. to $S_{K,L}=(F\langle m_K/m_K^2,\ V\rangle)^\times$ where $V=\operatorname{Ker}(\Omega_F^1\to\Omega_E^1)$).

Furthermore, by using a standard corestriction argument, we see that Theorem (3.6) and Theorem (3.7) are generalized to L/K of this type.

Remark (3.16). Finally we remark that the assumption of this section that Λ is of characteristic zero is generalized to $\mathrm{char}(\Lambda)\neq p$. Let

Λ be an algebraically closed field such that $\mathrm{char}(\Lambda) \neq 0, p$, and let Λ' be the algebraic closure of the field of fractions of the Witt ring $W(\Lambda)$. Then as in [26] §15, we have a canonical surjection $R_{\Lambda'}(G) \to R_\Lambda(G)$ where $R_\Lambda(G)$ (resp. $R_{\Lambda'}(G)$) denotes the Grothendieck group of finitely generated modules over $\Lambda[G]$ (resp. $\Lambda'[G]$). For $\chi \in R_\Lambda(G)$ and a primitive p-th root ζ of 1 in Λ, let $\hat{\chi}$ be any element of $R_{\Lambda'}(G)$ with image χ in $R_\Lambda(G)$ and let $\hat{\zeta}$ be the unique primitive p-th root of 1 in Λ' with reduction ζ, and define

$$\mathrm{sw}_\zeta(\chi) = \mathrm{sw}_{\hat{\zeta}}(\hat{\chi}).$$

Then, $\mathrm{sw}_\zeta(\chi)$ is independent of the choice of $\hat{\chi}$ as is seen from the definition in (3.1) and from [26] §18.3. It is easy to see that the theorems (3.4) (3.6) (3.7) (in the situation of (3.15) also) are generalized to this $\mathrm{sw}_\zeta(\chi)$ (In (3.6), $\boldsymbol{Q}/\boldsymbol{Z}$ should be replaced with the prime to $\mathrm{char}(\Lambda)$-part of $\boldsymbol{Q}/\boldsymbol{Z}$).

§4. Formulas for vanishing cycles

(4.1) In this section, k denotes a complete discrete valuation field with algebraically closed residue field. Let A be a two dimensional normal henselian local ring over O_k obtained as the henselization of $\mathcal{O}_{X,x}$ where X is flat of finite type over O_k and x is a closed point of X lying over m_k. We assume further

(4.1.1) $\quad A \otimes_{O_k} O_k/m_k$ is reduced.

(4.1.2) $\quad A \otimes_{O_k} k$ is essentially smooth over k.

We denote by P the finite set of all prime ideals of height one of A lying over m_k. We apply the results in Section 1–Section 3 to the study of wild ramification of the discrete valuation rings $A_\mathfrak{p}$ for $\mathfrak{p} \in P$.

For the formalism of vanishing cycles, see [3]. In particular, for a sheaf $\mathcal{F}$ on $\mathrm{Spec}(A \otimes_{O_k} k)$ for the etale topology, we use the notations

$$R^i\psi(\mathcal{F}) = H^i(\mathrm{Spec}(A \otimes_{O_k} k^{\mathrm{sep}})_{\mathrm{et}}, \mathcal{F}) \quad (i \in \boldsymbol{Z})$$

$$R\psi(\mathcal{F}) = R\Gamma(\mathrm{Spec}(A \otimes_{O_k} k^{\mathrm{sep}})_{\mathrm{et}}, \mathcal{F})$$

where k^{sep} denotes the separable closure of k. We have $R^i\psi(\mathcal{F}) = 0$ for $i \neq 0, 1$.

We denote by l a prime number which is different from the characteristic of the residue field of k.

(4.2) Let $\sigma: A \to A$ be a non-trivial O_k-automorphism of finite order. We introduce a result (4.3) of T. Saito which describes the trace

$$\mathrm{Tr}(\sigma: R\psi(\boldsymbol{Q}_l)) = \sum_i (-1)^i \,\mathrm{Tr}(\sigma: R^i\psi(\boldsymbol{Q}_l)) = 1 - \mathrm{Tr}(\sigma: R^1\psi(\boldsymbol{Q}_l))$$

(note $R^0\psi(\boldsymbol{Q}_l)=\boldsymbol{Q}_l$).

Let P' (resp. P'') be the set of all prime ideals $\mathfrak{p}\in P$ such that $\sigma(\mathfrak{p})=\mathfrak{p}$ and such that the induced map on the residue field $\sigma\colon \kappa(\mathfrak{p})\to\kappa(\mathfrak{p})$ is the identity (resp. not the identity). For $\mathfrak{p}\in P'\cup P''$, we define the integer $s_{\mathfrak{p}}$ as follows.

Let $\mathfrak{p}\in P'$, let L be the field of fractions of the completion of the discrete valuation ring $A_{\mathfrak{p}}$, and let $K=\{a\in L;\ \sigma(a)=a\}$. Then, L/K is an extension in Case II. So we have an element $s_G(\sigma)\in S_{L/K}$ where $G=\mathrm{Gal}(L/K)$, and we can write

$$(*)\qquad s_G(\sigma)=[\omega]-[c],\quad \omega\in\Omega^1_{\kappa(\mathfrak{p})}-\{0\},\quad c\in k^\times.$$

We define the integer $s_{\mathfrak{p}}$ to be the order of the differential ω with respect to the discrete valuation ring $\widetilde{A/\mathfrak{p}}\subset\kappa(\mathfrak{p})$ where $\widetilde{A/\mathfrak{p}}$ is the normalization of $A/\mathfrak{p}$. As is easily seen, $s_{\mathfrak{p}}$ is independent of the expression (*).

On the other hand, for $\mathfrak{p}\in P''$, let $s_{\mathfrak{p}}$ be the classical Swan character of the induced map $\sigma\colon\kappa(\mathfrak{p})\to\kappa(\mathfrak{p})$ with respect to the discrete valuation ring $\widetilde{A/\mathfrak{p}}$.

Finally, let i_η be the dimension of the finite dimensional k-vector space $(A\otimes_{O_k}k)/I_\sigma$ where I_σ denotes the ideal of $A\otimes_{O_k}k$ generated by $\{a-\sigma(a);\ a\in A\otimes_{O_k}k\}$.

Theorem (4.3) (Saito [23]). *Let the notations be as above. Then we have*

$$\mathrm{Tr}\,(\sigma\colon R\psi(\boldsymbol{Q}_l))=i_\eta+\sharp(P')+\sum_{\mathfrak{p}\in P'\cup P''}s_{\mathfrak{p}}\,.$$

(4.4) We turn to the dimension formula. Let Λ be an algebraically closed field of characteristic l and let $\mathscr{F}$ be a locally constant etale sheaf of Λ-modules of finite rank on a non-empty open subset U of $\mathrm{Spec}\,(A\otimes_{O_k}k)$. We assume that the following (4.4.1) holds for any $\mathfrak{p}\in P$.

(4.4.1) Let $\mathfrak{p}\in P$, and let K be the field of fractions of the completion of $A_{\mathfrak{p}}$. Then the representation of $\mathrm{Gal}\,(K^{\mathrm{sep}}/K)$ over Λ defined by $\mathscr{F}$ factors through a quotient $\mathrm{Gal}\,(L/K)$ such that L is a finite Galois extension of K of ramification index one.

By Epp [4], this assumption (4.4.1) becomes satisfied after a finite extension k'/k of the base field (replacing A by $A\otimes_{O_k}O_{k'}$) without changing the space $R^i\psi(\mathscr{F})$.

Let $u\colon U\to\mathrm{Spec}\,(A\otimes_{O_k}k)$ be the inclusion map. We describe a formula (4.5) in [14] §6 for

$$\begin{aligned}\dim R\psi u_!(\mathscr{F})&=\sum_i(-1)^i\dim R^i\psi u_!(\mathscr{F})\\&=\dim R^0\psi u_!(\mathscr{F})-\dim R^1\psi u_!(\mathscr{F}),\end{aligned}$$

a generalization of a formula of Deligne in [19] (5.1.1) which considers the case where $\mathscr{F}$ is unramified at P.

For $\mathfrak{p} \in P$, we define the integers $s_{\mathfrak{p}}$ and $\bar{s}_{\mathfrak{p}}$ as follows. Take K and L as in (4.4.1), and let $G = \mathrm{Gal}(L/K)$. Then, for a primitive p-th root ζ of 1 in Λ, the element of $R_{\Lambda}(G)$ corresponding to $\mathscr{F}$ defines an element $\mathrm{sw}_{\zeta}(\mathscr{F}) \in S_{K,L}$ (cf. (3.15) (3.16)) and we can write

$$(*) \qquad \mathrm{sw}_{\zeta}(\mathscr{F}) = [c] - \sum_{i=1}^{r} [\omega_i],$$

where $c \in k^{\times}$, $\omega_i \in \Omega^1_{\kappa(\mathfrak{p})} - \{0\}$, $r \geq 0$. Define

$$s_{\mathfrak{p}} = -\sum_{i=1}^{r} \mathrm{ord}(\omega_i)$$

where ord is the order of the differential with respect to the discrete valuation ring $\tilde{A}_{\mathfrak{p}}$. As is easily seen, $s_{\mathfrak{p}}$ is independent of the expression (*) and of the choice of ζ. Next, let $\bar{\mathscr{F}}$ be the restriction to $\mathrm{Spec}(\kappa(\mathfrak{p}))$ of the direct image of $\mathscr{F}$ under $\mathrm{Spec}(K) \to \mathrm{Spec}(O_K)$. We define $\bar{s}_{\mathfrak{p}}$ to be the sum of $\mathrm{rk}(\bar{\mathscr{F}})$ and the classical Swan conductor of $\bar{\mathscr{F}}$ with respect to $\tilde{A}_{\mathfrak{p}}$.

On the other hand, let s_{η} be the sum of the classical Swan conductors of $\mathscr{F}$ at all the maximal ideals of the Dedekind domain $A \otimes_{O_k} k^{\mathrm{alg}}$, where k^{alg} denotes the algebraic closure of k. Finally, writing $A/m_k A$ by R and the normalization of R by $\tilde{R}$, let δ be the length of the R-module $\tilde{R}/R$.

Theorem (4.5). *Let the assumptions and the notations be as above. Then*

$$\dim R\psi u_!(\mathscr{F}) = -s_{\eta} + \Big(\sum_{\mathfrak{p} \in P} (s_{\mathfrak{p}} + \bar{s}_{\mathfrak{p}})\Big) - \mathrm{rk}(\mathscr{F})(2\delta + N)$$

where $N = \sharp(\mathrm{Spec}(A \otimes_{O_k} k^{\mathrm{alg}}) - U \otimes_k k^{\mathrm{alg}})$.

Remark (4.6). Classically, Weil's trace formula for the cohomology of a curve is essentially equivalent to the formula of Grothendieck-Ogg-Shafarevich on the dimension of the cohomology of an etale sheaf on a curve. Similarly, the theorems (4.3) and (4.5) are essentially equivalent. However, the proof of (4.3) in Saito [23] is different from the proof of (4.5) in [14] (the latter uses the formula of Deligne in [19] (5.1.1) whereas the former uses the stable reduction theorem) and provides a new proof of the formula of Deligne (cf. also [22] §4).

§ 5. Local constants

The result of this section was obtained in the collaboration with Takeshi Saito, and the proof will be given in the joint paper [17].

In [20] Theorem (3.4), Laumon obtained an important formula which represents the local constant of a local field of positive characteristic as a determinant of the Galois action on certain vanishing cycles. In this section, by using our Swan conductor in Case II, we give a generalization of his formula (but only for local constants of characters of degree one). Here we can treat vanishing cycles in mixed characteristics (but the local constants considered are for local fields of positive characteristic as in [20]).

(5.1) Let k be a complete discrete valuation field with finite residue field $\boldsymbol{F}_q$, and let $A=O_k\{T\}$ be the henselization of $O_k[T]$ at the maximal ideal generated by T and m_k. We denote the field of fractions of A by M.

Let l be a prime number which is different from $p=\mathrm{char}(\boldsymbol{F}_q)$, and let Λ be the algebraic closure of $\boldsymbol{Q}_l$. For smooth Λ-sheaves $\mathscr{F}$ and $\mathscr{G}$ on a non-empty open subset U of $\mathrm{Spec}(A\otimes_{O_k}k)$, we define the homomorphism

$$\psi(\mathscr{F},\mathscr{G})\colon \mathrm{Gal}(k^{\mathrm{ab}}/k)\longrightarrow \Lambda^{\times}$$

to be the product

$$\det(R\psi u_!(\mathscr{F}\otimes\mathscr{G}))\cdot\det(R\psi u_!(\Lambda))^{\mathrm{rk}(\mathscr{F})\mathrm{rk}(\mathscr{G})}$$
$$\cdot\det(R\psi u_!(\mathscr{F}))^{-\mathrm{rk}(\mathscr{G})}\cdot\det(R\psi u_!(\mathscr{G}))^{-\mathrm{rk}(\mathscr{F})}.$$

Here $u\colon U\overset{\subset}{\longrightarrow}\mathrm{Spec}(A\otimes_{O_k}k)$ is the inclusion map, and det means the determinant of the Galois action. If U' is a non-empty open subscheme of U, then $\psi(\mathscr{F},\mathscr{G})=\psi(\mathscr{F}\,|_{U'},\mathscr{G}\,|_{U'})$. So, $\psi(\mathscr{F},\mathscr{G})$ depends only on the representations of $\mathrm{Gal}(M^{\mathrm{sep}}/M)$ over Λ defined by $\mathscr{F}$ and $\mathscr{G}$, respectively. We seek a formula for $\psi(\mathscr{F},\mathscr{G})$.

We assume the following (5.1.1) and (5.1.2). Let $\mathfrak{p}$ be the prime ideal m_kA, and let K be the field of fractions of the completion of the discrete valuation ring $A_{\mathfrak{p}}$.

(5.1.1) $\mathscr{F}$ is extended to a smooth Λ-sheaf on an open subscheme of $\mathrm{Spec}(A)$ containing $\mathfrak{p}$.

(5.1.2) The representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ over Λ defined by $\mathscr{G}$ factors through $\mathrm{Gal}(L/K)$ for some finite Galois extension L of K of ramification index one, and this representation has no non-zero fixed vector by the inertia group of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

The following conjecture relates $\psi(\mathscr{F},\mathscr{G})$ to a certain local constant $\bar{\varepsilon}(\mathscr{F}\,|_{\mathfrak{p}},\mathrm{sw}_{s,\pi}(\mathscr{G}))$ whose definition will be given later.

Conjecture (5.2). Assume (5.1.1) and (5.1.2), and assume that at any closed point of $\mathrm{Spec}(A\otimes_{O_k}k)$, at least one of $\mathscr{F}$ and $\mathscr{G}$ is smooth. Then

$$\bar{\varepsilon}(\mathscr{F}\,|_{\mathfrak{p}},\mathrm{sw}_{s,\pi}(\mathscr{G}))=\psi(\mathscr{F},\mathscr{G})(\pi)\cdot\det(\mathscr{F},\mathrm{sw}_{\eta}(\mathscr{G}))(\pi)\cdot\det(\mathscr{G},\mathrm{sw}_{\eta}(\mathscr{F}))(\pi)$$

for any prime element π of k.

The notations here are explained in (5.3) below.

(5.3) First we introduce a notation for local constants. Fix a primitive p-th root ζ of 1 in Λ. Let F be the residue field $\kappa(\mathfrak{p})$ of K. Then F is the field of fractions of $\boldsymbol{F}_q\{T\}$. For a Λ-sheaf $\mathscr{H}$ on $\mathrm{Spec}(F)$ and for $\omega\in\Omega_F^1-\{0\}$, let

$$\bar{\varepsilon}(\mathscr{H},\omega)=(-1)^{\mathrm{sw}(\mathscr{H})}\varepsilon_0(\mathscr{H},dx,\underline{\omega})\varepsilon_0(\Lambda,dx,\underline{\omega})^{-\mathrm{rk}(\mathscr{H})}\in\Lambda^\times,$$

where $\mathrm{sw}(\mathscr{H})\in\boldsymbol{Z}$ is the classical Swan conductor of $\mathscr{H}$ with respect to the discrete valuation field F, dx is a Haar measure on the completion $\hat{F}$ of F, $\underline{\omega}$ is the additive character

$$\hat{F}\longrightarrow\Lambda^\times;\ a\longmapsto\zeta^{\mathrm{Tr}_{\boldsymbol{F}_q/\boldsymbol{F}_p}(\mathrm{res}(a\omega))}$$

associated to ω (res denotes the residue), and ε_0 is the local constant in Deligne [2] §3.4. As is easily seen, $\bar{\varepsilon}(\mathscr{H},\omega)$ is independent of the Haar measure dx.

Now we return to (5.2). By the assumption (5.1.2), we can apply the results of Section 1–Section 3 to the discrete valuation field K to obtain the element $\mathrm{sw}_\zeta(\mathscr{G})$ of $(F\langle m_K/m_K^2,\Omega_F^1\rangle)^\times$. For a prime element π of k, by writing

$$(*)\qquad \mathrm{sw}_\zeta(\mathscr{G})=[(-\pi)^m]-\sum_{i=1}^{r}[\omega_i]$$

($m\in\boldsymbol{Z}$, $\omega_i\in\Omega_F^1-\{0\}$, $r=rk(\mathscr{G})$), we define

$$\bar{\varepsilon}(\mathscr{F},\mathrm{sw}_{s,\pi}(\mathscr{G}))=\prod_{i=1}^{r}\bar{\varepsilon}(\mathscr{F}\mid_{\mathfrak{p}},\omega_i)\in\Lambda^\times.$$

As is easily seen, this element is independent of the expression (*). ($\mathscr{F}\mid_{\mathfrak{p}}$ denotes the restriction to $\mathrm{Spec}(F)$ of a smooth extension of $\mathscr{F}$ to an open set containing $\mathfrak{p}$).

The other notations in (5.2) are as follows. First, $\det(\mathscr{F},\mathrm{sw}_\eta(\mathscr{G}))$ and $\det(\mathscr{G},\mathrm{sw}_\eta(\mathscr{F}))$ are homomorphisms $\mathrm{Gal}(k^{\mathrm{ab}}/k)\to\Lambda^\times$ defined in the following way. If $\mathrm{char}(k)=0$, we define them to be the trivial homomorphism. Assume $\mathrm{char}(k)=p>0$. Then $\det(\mathscr{F},\mathrm{sw}_\eta(\mathscr{G}))$ is the product $\sigma\mapsto\prod_x h_x(\sigma)^{s(x)}$ where x ranges over all closed points of $\mathrm{Spec}(A\otimes_{O_k}k)$ at which $\mathscr{G}$ is not smooth, $s(x)\in\boldsymbol{Z}$ denotes the classical Swan conductor of $\mathscr{G}$ at the unique point of $\mathrm{Spec}(A\otimes_{O_k}k^{1/p^\infty})$ lying over x (we made the perfection of k since the classical Swan conductor works with the perfectness of the residue field), and h_x is the following composite map

$$\mathrm{Gal}(k^{\mathrm{ab}}/k)\longrightarrow\mathrm{Gal}((k')^{\mathrm{ab}}/k')\xleftarrow{\cong}\mathrm{Gal}(\kappa(x)^{\mathrm{ab}}/\kappa(x))\longrightarrow\Lambda^\times.$$

Here k' is the maximum separable subextension of $\kappa(x)/k$, the first arrow is the transfer, and the last map is defined by $\det(\mathscr{F}\,|_x)$ (note $\mathscr{F}$ is smooth at x by the assumption in (5.2)). We define $\det(\mathscr{G}, \mathrm{sw}_\eta(\mathscr{F}))$ in the same way.

Finally, for any homomorphism h: $\mathrm{Gal}(k^{\mathrm{ab}}/k) \to \Lambda^\times$, $h(\pi) \in \Lambda^\times$ denotes the image of π under

$$k^\times \longrightarrow \mathrm{Gal}(k^{\mathrm{ab}}/k) \xrightarrow{h} \Lambda^\times$$

where the first arrow is the reciprocity map of the local class field theory.

Remark (5.4). In (5.2), assume $\mathrm{char}(k)=0$. Then the sw_η-terms vanish and hence the formula (5.2) has the simple form

$$\bar{\varepsilon}(\mathscr{F}\,|_{\mathfrak{p}}, \mathrm{sw}_{s,\pi}(\mathscr{G})) = \psi(\mathscr{F}, \mathscr{G}).$$

Hence, as is seen by changing the prime element π, (5.2) implies that $\psi(\mathscr{F}, \mathscr{G})$ is a tame representation of $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ in the mixed characteristic case.

Remark (5.5). The theorem [20] (3.4) of Laumon is regarded as the following case of (5.2): $k = \boldsymbol{F}_q((\pi))$, $\mathscr{F}$ comes from a Λ-sheaf on $\mathrm{Spec}(F)$ by the base change $F \xrightarrow{\subset} A[T^{-1}]$, and $\mathscr{G}$ is the sheaf of rank one defined by the character $\mathrm{Gal}(M^{\mathrm{sep}}/M) \to \Lambda^\times$ of order p associated to the Artin-Schreier equation $X^p - X = T\pi^{-p}$. In this case, $\det(\mathscr{F}, \mathrm{sw}_\eta(\mathscr{G})) = \det(\mathscr{G}, \mathrm{sw}_\eta(\mathscr{F})) = 1$ and the Swan conductor of $\mathscr{G}$ with respect to K is $[\pi^p] - [dT]$, and so the formula (5.2) becomes

$$\bar{\varepsilon}(\mathscr{F}, dT) = \psi(\mathscr{F}, \mathscr{G})(-\pi).$$

The following (5.6) will be proved in [17] (I hope that in the final version of [17], we can weaken the assumptions in (5.6)).

Theorem (5.6). *The conjecture* (5.2) *is true if $\mathscr{F}$ and $\mathscr{G}$ are of rank one, and $\mathscr{G}$ is defined by a character* $\mathrm{Gal}(M^{\mathrm{sep}}/M) \to \Lambda^\times$ *of order p.*

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