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Vanishing Cycles and Differentials of Curves over a Discrete Valuation Ring

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In this paper, we study curves over a discrete valuation ring as a sequel to [13]. We study the relation of l-adic vanishing cycles and differentials, both of which represent how far curves are from being smooth. We compare the length of the cohomology of the torsion parts of the sheaves of differentials and the dimension or the "total dimension" of the cohomology of the sheaves of vanishing cycles. For this, we use a special differential on the special fiber called "the relative canonical differential" defined in Section 2. It gives the dimension of the space of vanishing cycles in a special case.

We always use the following notations and terminology. S: the spectrum of a strict local discrete valuation ring A with algebraically closed residue field k of $\mathrm{ch}=p\geqq 0$. s (resp. η): the closed (resp. the generic) point of S. S-curve: flat and separated S-scheme of finite type purely of relative dimension 1 such that the generic fiber is smooth over η.

A relation between vanishing cycles and differentials will be given by the following conjecture. Let $\Lambda:=\mathbf{Q}_l$, where l is a prime number different from p and $R\phi\Lambda$ (resp. $R\psi\Lambda$) be the complex of the sheaves of the vanishing cycles (resp. the nearby cycles) (for the definition, see [4]). Let dimtot denote the total dimension i.e. $\dim_\Lambda+\mathrm{Sw}$, where Sw is the Swan conductor.

Conjecture (0.1). *Suppose X is a regular flat separated S-scheme of finite type and Z is a subscheme of X_s such that Z is proper over s and that $X-Z$ is smooth over S. Then,*

$$\mathrm{dimtot}\, R\Gamma(Z, R\phi\Lambda)=-\mathrm{length}_{\mathcal{O}_S}\, R\Gamma(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Or equivalently,

$$\mathrm{dimtot}\, R\Gamma(Z, R\psi\Lambda)=\dim R\Gamma(Z, \Lambda)-\mathrm{length}_{\mathcal{O}_S}\, R\Gamma(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Here the supports of $\Omega^{\cdot}_{X/S,\mathrm{tors}}$ and $R\phi\Lambda$ are included in Z. This conjecture generalizes that of P. Deligne, Conjecture 1.9 of [5], which treats

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the case that Z consists of isolated singularities. The case where X is proper over S and $Z=X_s$ is conjectured by K. Kato. There are several special cases where the conjecture has been already known.

0) In the case of relative dimension 0, it immediately follows from the definition of the Swan conductor (cf. [5]).

1) The case that $\dim Z=0$ and that S is of equal characteristics is proved by P. Deligne (loc. cit.).

2) The case that X is proper over S and of relative dimension 1, $Z=X_s$ and S is the strict localization at a closed point of a smooth curve over an algebraically closed field is proved by K. Kato (cf. Section 4).

We consider the case that the relative dimension of X over S is 1. We prove Conjecture (0.1) in some special cases. In Section 3, we prove it under the assumption that the wild ramification group P acts trivially on $R\Gamma(Z, R\psi\Lambda)$ using our relative canonical differential. In Section 4, we prove it in the case that S is the strict localization of a smooth curve over an algebraically closed field at a closed point.

We briefly explain what is the relative canonical differential. Let X be a regular S-curve. For simplicity, we assume $C:=X_{s,\mathrm{red}}$ is regular and irreducible. Let ξ be the maximal point of C and put $l_C:=\mathrm{length}_{\mathcal{O}_{X,\xi}}(\Omega^1_{X/S,\mathrm{tors}})_\xi$ and $r_C:=\mathrm{length}_{\mathcal{O}_{X,\xi}}\mathcal{O}_{X_s,\xi}$. It is well known that always $l_C\geqq r_C-1$ and the equality holds if and only if $p\nmid r_C$ (cf. Proposition 13 Chapter III [14]). We assume $p\,|\,r_C$ so that $l_C\geqq r_C$. Then, in Section 2, we define a canonical differential $\omega_C\in\Gamma(C, \Omega^1_{C/s})$. This ω_C is not 0 if and only if $l_C=r_C$ and C is of type II in the sense of Kurihara [9] (cf. Definition (2.3)). It is expected that ω_C knows the behavior of vanishing cycles in the case that $l_C=r_C$ and C is of type II. We show that this is true for the dimension of the space of vanishing cycles and give a conjecture on its conductor.

After the author wrote this paper, he obtained a preprint of S. Bloch [2] in which he proves a formula representing the left hand side of Conjecture (0.1) as a certain intersection number in the case that X is a proper curve over S and $Z=X_s$. It is probable that Bloch's formula is equivalent to that of Conjecture (0.1) in that case though the author has not yet shown it. As mentioned above, Section 2 of this paper contains some pointwise study of vanishing cycles at non-isolated non smooth points, which is not covered by Conjecture (0.1) and raises further problems (cf. Conjecture (2.10)).

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Notations and terminology

For a scheme X and its geometric point x, $\tilde{X}_x$ denotes the strict localization of X at x. $A\{x, y\}$ denotes the strict localization of the polynomial ring $A[x, y]$ at the maximal ideal (π, x, y), where π is a prime element of A.

For a noetherian scheme X, $K_0'(X)$ denotes the Grothendieck group of the category of the coherent $\mathcal{O}_X$-modules and [ ] denotes the class of a coherent $\mathcal{O}_X$-module or a complex of coherent $\mathcal{O}_X$-modules. When $F \to G$ is a morphism of coherent $\mathcal{O}_X$-modules, $[F \to G]$ denotes [Coker]–[Ker].

An extension S' of S means the spectrum of a strictly local discrete valuation ring finite over A such that the fraction field is separable.

A normal crossing divisor (abbreviated n.c.d.) is a closed subscheme of a regular scheme defined etale locally by an ideal $(\prod_i f_i)$ where $(f_i)_i$ forms a part of a regular system of parameters. Note that an n.c.d. is reduced in our terminology.

§1. The length of cohomology of the torsion part of the differential sheaf

In this section, we always assume that X is a regular S-curve and Z is a subscheme of X_s such that Z is proper over s and that $X-Z$ is smooth over S.

If we admit Conjecture (0.1), the following proposition follows from the proper base change theorem. Here, we prove it directly.

Proposition (1.1). *Suppose (X, Z) and (X', Z') satisfy the assumption above and f: $X' \to X$ is a proper S-morphism which induces isomorphisms $Z'_{\mathrm{red}} \xrightarrow{\sim} (Z \times_X X')_{\mathrm{red}}$ and $X'-Z' \xrightarrow{\sim} X-Z$. Then,*

$$(1.1.1) \qquad \chi(Z) - \chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}) = \chi(Z') - \chi(Z', \Omega^{\cdot}_{X'/S,\mathrm{tors}}),$$

where $\chi(Z) := \dim_\Lambda R\Gamma(Z, \Lambda)$ and

$$\chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}) := \mathrm{length}_{\mathcal{O}_S} R\Gamma(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Proof. We note that $\Omega^i_{X/S,\mathrm{tors}} = 0$ if $i \neq 1$, 2 and $\Omega^2_{X/S}$ is of torsion. First we prove

$$(1.1.2) \qquad \chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}) = \chi(Z', Lf^*\Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

In fact for a coherent $\mathcal{O}_X$-module $\mathscr{F}$, we have $Rf_*Lf^*\mathscr{F} = Rf_*\mathcal{O}_{X'} \overset{L}{\otimes}_{\mathcal{O}_X} \mathscr{F}$. Here $f_*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathrm{Supp}\, R^i f_* \mathcal{O}_{X'}$ are finite over s for $i \neq 0$. Since $\Omega^1_{X/S,\mathrm{tors}}$ and $\Omega^2_{X/S}$ are isomorphic in codimension 1, it is sufficient to show the following fact; for any coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ of finite length whose supports are contained in Z, we have $\chi(Z, \mathcal{F}\,{}^{L}\!\otimes_{\mathcal{O}_X}\mathcal{G})=0$. This fact immediately follows from $\chi(Z, \kappa(x_1)^{L}\otimes_{\mathcal{O}_X}\kappa(x_2))=0$ for any closed points x_1 and x_2 of Z.

We may assume that X' is the blowing-up Y of X at a closed point x of Z (cf. [15]). In this case it is clear that $\chi(Z')=\chi(Z)+1$. Hence it is sufficient to show

$$\chi(E, [\Omega^{\cdot}_{Y/S,\mathrm{tors}}]-[Lf^*(\Omega^{\cdot}_{X/S,\mathrm{tors}})])=1, \tag{1.1.3}$$

where E denotes $f^{-1}(x)$. Since X is regular of dimension 2, locally there is an S-immersion i: $X\to T$ which satisfies the following condition:

(1.1.4) $T\to S$ is smooth and purely of relative dimension 2 and $X\to T$ is a regular immersion of codimension 1.

From this, we have following exact sequences

$$0 \longrightarrow N_{X/T} \longrightarrow i^*\Omega^1_{T/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0, \tag{1.1.5}$$

$$0 \longrightarrow N_{X/T}\otimes\Omega^1_{X/S,\mathrm{tors}} \longrightarrow N_{X/T}\otimes\Omega^1_{X/S} \longrightarrow i^*\Omega^2_{T/S} \longrightarrow \Omega^2_{X/S} \longrightarrow 0. \tag{1.1.6}$$

The second induces the exact sequence

$$0 \longrightarrow \Omega^1_{X/S,\mathrm{tors}} \longrightarrow \Omega^1_{X/S} \longrightarrow \omega_{X/S} \longrightarrow N^{\otimes-1}_{X/T}\otimes\Omega^2_{X/S} \longrightarrow 0. \tag{1.1.7}$$

Here $\omega_{X/S}=N^{\otimes-1}_{X/T}\otimes i^*\Omega^2_{T/S}$ is the relative canonical sheaf of X over S (cf. [7]).

Since the question is local at x, there is a commutative diagram

$$\begin{array}{ccccc} E & \longrightarrow & Y & \xrightarrow{j} & U \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & X & \xrightarrow{i} & T \\ & \searrow & & \searrow & \swarrow\!\!\uparrow\sigma \\ & & s & \longrightarrow & S \end{array} \tag{1.1.8}$$

such that i: $X\to T$ satisfies (1.1.4), $x=X\times_{T\to\sigma} S$, U is the blowing-up of T with center σ and that $Y=X\times_T U$. Hence j: $Y\to U$ satisfies the condition (1.1.4).

Since $f^*N_{X/T}=N_{Y/U}$,

$$[\Omega^2_{Y/S}]-[Lf^*\Omega^2_{X/S}]=[N_{Y/U}\otimes\Omega^2_{Y/S}]-[Lf^*(N^{\otimes-1}_{X/T}\otimes\Omega^2_{X/S})]$$

By this and (1.1.7), we have

$$(1.1.9)\qquad \begin{aligned}&[\Omega^{\cdot}_{Y/S,\mathrm{tors}}]-[Lf^*\Omega^{\cdot}_{X/S,\mathrm{tors}}]\\ &=[\Omega^1_{Y/S}\longrightarrow\omega_{Y/S}]-[Lf^*(\Omega^1_{X/S}\longrightarrow\omega_{X/S})]\\ &=[\Omega^1_{Y/S}\longrightarrow\omega_{Y/S}]-[f^*(\Omega^1_{X/S}\longrightarrow\omega_{X/S})]\\ &\quad(\text{since } Lf^*\Omega^1_{X/S}=f^*\Omega^1_{X/S} \text{ and } Lf^*\omega_{X/S}=f^*\omega_{X/S})\\ &=[f^*\omega_{X/S}\longrightarrow\omega_{Y/S}]-[f^*\Omega^1_{X/S}\longrightarrow\Omega^1_{Y/S}].\end{aligned}$$

Since there are exact sequences

$$(1.1.10)\qquad 0\longrightarrow f^*\Omega^1_{X/S}\longrightarrow\Omega^1_{Y/S}\longrightarrow\Omega^1_{E/x}\longrightarrow 0$$

$$(1.1.11)\qquad 0\longrightarrow f^*\omega_{X/S}\longrightarrow\omega_{Y/S}\longrightarrow\Omega^1_{E/x}\otimes N_{E/Y}\longrightarrow 0,$$

we have

$$(1.1.12)\quad [\Omega^{\cdot}_{Y/S,\mathrm{tors}}]-[Lf^*(\Omega^{\cdot}_{X/S,\mathrm{tors}})]=[\Omega^1_{E/x}\otimes N_{E/Y}]-[\Omega^1_{E/x}].$$

Hence we have

$$(1.1.13)\qquad \begin{aligned}&\chi(E,\ [\Omega^{\cdot}_{Y/S,\mathrm{tors}}]-[Lf^*(\Omega^{\cdot}_{X/S,\mathrm{tors}})])\\ &=\chi(E,\ \Omega^1_{E/x}\otimes N_{E/Y})-\chi(E,\ \Omega^1_{E/x})\\ &=\deg N_{E/Y}=1.\end{aligned}$$

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Remark (1.3). As a consequence of Proposition (1.1), we have the following compatibility. Suppose that (X, Z) and (X', Z') satisfy the condition of Conjecture (0.1) and that there exists a birational S-rational map $f\colon X'\to X$ which induces an isomorphism $X'-Z'\to X-Z$. Then, Conjecture (0.1) holds for (X, Z) if and only if so does it for (X', Z').

In the rest of this section, we give a formula which represent $\chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}})$ by intersection theory. It is an analogue of Bloch's version of S. Saito's formula for the Lefschetz number of algebraic surfaces [1] (cf. [12]). First we give some elementary facts on the structure of the $\mathcal{O}_X$-modules $\Omega^1_{X/S,\mathrm{tors}}$ and $\Omega^2_{X/S}$. Let D (resp. D') be the closed subscheme of X corresponding to the ideal $\mathcal{I}_D:=\mathrm{Ann}(\Omega^1_{X/S,\mathrm{tors}})$ (resp. $\mathcal{I}_{D'}:=\mathrm{Ann}(\Omega^2_{X/S})$) of $\mathcal{O}_X$. Then, $\Omega^1_{X/S,\mathrm{tors}}$ (resp. $\Omega^2_{X/S}$) is an invertible $\mathcal{O}_D$-module (resp. invertible $\mathcal{O}_{D'}$-module) and $i_{D'}{}^*\Omega^1_{X/S}$ is a locally free $\mathcal{O}_{D'}$-module of rank 2. $\mathcal{I}_D$ is an invertible $\mathcal{O}_X$-ideal, $\mathcal{I}_D\supset\mathcal{I}_{D'}$ and $\mathcal{I}_D/\mathcal{I}_{D'}$ is of finite length. These facts are easily deduced from the exact sequences (1.1.5) and (1.1.6).

We define a 0-cycle R of Z by

$$(1.4)\qquad R:=\sum_{z\in Z_0}\mathrm{length}_{\mathcal{O}_{X,z}}(\mathcal{I}_D/\mathcal{I}_{D'})\cdot z.$$

Proposition (1.5). *Under the above notations, we have the following formula.*

$$\chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}) = \deg R - (D, D-K). \tag{1.5.1}$$

Proof. By (1.1.7), the canonical morphism $\Omega^1_{X/S} \to \omega_{X/S}$ induces an exact sequence

$$0 \longrightarrow \Omega^1_{X/S,\mathrm{tors}} \longrightarrow \Omega^1_{X/S} \longrightarrow \mathscr{I}_{D'}\omega_{X/S} \longrightarrow 0. \tag{1.5.2}$$

By applying $i_{D'}{}^*$, we obtain the following sequence

$$0 \longrightarrow \Omega^1_{X/S,\mathrm{tors}} \longrightarrow i_{D'}{}^*\Omega^1_{X/S} \longrightarrow i_{D'}{}^*(\mathscr{I}_{D'}\omega_{X/S}) \longrightarrow 0. \tag{1.5.3}$$

Since $\Omega^1_{X/S,\mathrm{tors}}$ is an invertible $\mathcal{O}_D$-module, this is exact. From this we have an equality in $K'_0(D')$

$$[i_{D'}{}^*\Omega^1_{X/S}] = [\Omega^1_{X/S,\mathrm{tors}}] + [i_{D'}{}^*\mathscr{I}_{D'}\omega_{X/S}]. \tag{1.5.4}$$

Since $i_{D'}{}^*\Omega^1_{X/S}$ (resp. $\omega_{X/S}$) is a locally free $\mathcal{O}_{D'}$-module of rank 2 (resp. an invertible $\mathcal{O}_X$-module), we have an equality in $K'_0(D')$

$$[i_{D'}{}^*\Omega^1_{X/S} \to i_D{}^*\Omega^1_{X/S}] = [i_{D'}{}^*\mathscr{I}_{D'}\omega_{X/S} \to i_D{}^*\mathscr{I}_D\omega_{X/S}]. \tag{1.5.5}$$

Since $i_D{}^*\Omega^1_{X/S}$ is a locally free $\mathcal{O}_D$-module of rank 2, we have an equality in $K'_0(D)$

$$[i_D{}^*\Omega^1_{X/S}] = [\mathcal{O}_D] + [i_D{}^*\Omega^2_{X/S}]. \tag{1.5.6}$$

By (1.5.4) $\sim$ (1.5.6), we have

$$[\mathcal{O}_D] + [i_D{}^*\Omega^2_{X/S}] = [\Omega^1_{X/S,\mathrm{tors}}] + [i_D{}^*\mathscr{I}_D\omega_{X/S}] \quad \text{in} \quad K'_0(D). \tag{1.5.7}$$

It is clear that $R + [i_D{}^*\Omega^2_{X/S}] = [\Omega^2_{X/S}]$. Thus we have

$$[\Omega^2_{X/S}] - [\Omega^1_{X/S,\mathrm{tors}}] = R + [i_D{}^*\mathscr{I}_D\omega_{X/S}] - [\mathcal{O}_D]. \tag{1.5.8}$$

By taking the degrees of both sides, we complete the proof since

$$\deg([i_D{}^*\mathscr{I}_D\omega_{X/S}] - [\mathcal{O}_D]) = (D, K-D).$$

§ 2. The relative canonical differential

In this section, we assume that X is a regular S-curve and C is a component of the special fiber X_s with maximal point ξ. We put $l_C := \mathrm{length}_{\mathcal{O}_{X,\xi}}(\Omega^1_{X/S,\mathrm{tors}})_\xi$ and $r_C := \mathrm{length}_{\mathcal{O}_{X,\xi}}\mathcal{O}_{X_s,\xi}$. We have always $l_C \geqq r_C - 1$ and the equality holds if and only if $p \nmid r_C$. For a component C with $p \mid r_C$, we define the relative canonical differential and its variants.

Lemma (2.1). *Let X and C be as above. Suppose C is contained in the regular part $(X_{s,\mathrm{red}})_{\mathrm{reg}}$ of $X_{s,\mathrm{red}}$, and is a principal divisor. Let $r=r_C$ and assume $p|r$. Let π (resp. x) be a prime element of S at s (resp. of X at C). Then*

$$\tilde{\omega}:=\overline{d\log(\pi/x^r)}\in\Gamma(C,\, i_C^*\Omega^1_{X/S}) \tag{2.1.1}$$

does not depend on the choice of π nor x. The differential $\tilde{\omega}$ is not 0 if and only if $l_C=r_C$. If $l_C=r_C$, the image of $\Omega^1_{X/S,\mathrm{tors}}$ in $i_C^*\Omega^1_{X/S}$ by the canonical morphism is generated by $\tilde{\omega}$.

Proof. The independence from the choices of π and x is straightforwards. For the second assertion, since the question is local, we may assume $X=\operatorname{Spec} B$ where $B:=A\{x, y\}/(\pi-ux^r)$ and $u\in A\{x, y\}^\times$. Then $\Omega^1_{X/S}\simeq(Bdx\oplus Bdy)/\left(x^r\left(du+\frac{r}{x}\cdot dx\right)\right)$. Since $x^2|r$, $l=r$ occurs if and only if du is not divisible by x in $Bdx\oplus Bdy$. Since $i_C^*\Omega^1_{X/S}\simeq B/(x)dx\oplus B/(x)dy$, this means that $du\neq 0$ in $i_C^*\Omega^1_{X/S}$. If $l=r$, $\Omega^1_{X/S,\mathrm{tors}}$ is generated by $du+\frac{r}{x}\cdot dx$. The last assertion follows from this fact.

Definition (2.2). Let X and C be as in the beginning of this section and assume that $r=r_C$ is divisible by p. Then $\tilde{\omega}_{X/S,C}\in\Gamma(C\cap(X_{s,\mathrm{red}})_{\mathrm{reg}},\, i_C^*\Omega^1_{X/S})$ is locally defined by (2.1.1).

For further study of $\tilde{\omega}$, we review the definition of the types of extention of discrete valuation rings of M. Kurihara (cf. [9]).

Definition (2.3). Let X be a regular flat separated S-scheme of finite type and ξ be a maximal point of the special fiber X_s and put $C:=\overline{\{\xi\}}$. Then C is of type I (resp. type II) if the canonical morphism $\Omega^1_{X/S,\mathrm{tors}}\to\Omega^1_{X_{s,\mathrm{red}}/s}$ is 0 (resp. non-0) at ξ.

Definition (2.4). Let X, C and r be as in Definition (2.2). We put $C^\circ:=C\cap(X_{s,\mathrm{red}})_{\mathrm{reg}}$. Then the relative canonical differential $\omega_{X/S,C}=\omega_C\in\Gamma(C^\circ,\, \Omega^1_{C/s})$ of X over S at C is the canonical image of $\tilde{\omega}_{X/S,C}$. If C is of type I, $\nu_{X/S,C}=\nu_C\in\Gamma(C^\circ,\, N_{C/X})$ is $\tilde{\omega}_{X/S,C}$, which is defined by the exact sequence

$$0\longrightarrow N_{C^\circ/X}\longrightarrow i_{C^\circ}^*\Omega^1_{X/S}\longrightarrow\Omega^1_{C^\circ/s}\longrightarrow 0.$$

Remark (2.5). Under the same assumption of Definition (2.4), ω_C is not 0 if and only if $l_C=r_C$ and C is of type II. If C is of type I, ν_C is not 0 if and only if $l_C=r_C$.

We give a relation of ω_C or ν_C with the 0-cycle $R=\sum R_c c$ defined by (1.4).

Proposition (2.6). *Let X be a regular S-curve such that $X_{s,\mathrm{red}}$ is an n.c.d. in X.*

(2.6.1) *Let c be a smooth point of $X_{s,\mathrm{red}}$ and C be the component of X_s which contains c. We assume that C satisfies $l_C=r_C$ and is of type* II *(resp.* I*). Then $R_c=0$ if $\omega_C(c)\neq 0$. (resp. Then $R_c=0$ if and only if $v_C(c)\neq 0$, and if $v_C(c)=0$, then we have $R_c\geqq \mathrm{ord}_c\, v_C$.)*

(2.6.2) *Let c be a singular point of $X_{s,\mathrm{red}}$ and, if $X_{s,\mathrm{red}}$ is irreducible (resp. not irreducible) at c, C and C' (resp. C) be the two components (resp. the unique component) of X_s which contain(s) c. We assume $p\,|\,r_C$. In respect case, we put $C'=C$.*

1) *Assume $p\nmid r_{C'}$. Then $l_C=r_C$, C is of type* II, *$R_c=0$ and* $\mathrm{ord}_c\,\omega_C=-1$.

2) *Assume $p\,|\,r_{C'}$. Then $\tilde{\omega}\in\Gamma(C^\circ, i_C^*\Omega^1_{X/S})$ (resp. $\omega\in\Gamma(C^\circ, \Omega^1_{C/s})$) comes from $\Gamma(C^\circ\cup\{c\}, i_C^*\Omega^1_{X/S})$ (resp. $\Gamma(C^\circ\cup\{c\}, \Omega^1_{C/s})$). Further assume C is of type* I *and let $\varphi\colon \bar{C}\to C$ be the normalization. Then $v\in\Gamma(C^\circ, N_{C/X})$ comes from $\Gamma(C^\circ\cup\varphi^{-1}(c), \varphi^*N_{C/X})$.*

3) *Assume $l_C=r_C$, $l_{C'}=r_{C'}$ and C is of type* I. *Then, if $C'\neq C$, we have $R_c\geqq \mathrm{ord}_c\, v_C$. Further assume C' is of type* I. *Then if $C'\neq C$ (resp. $C'=C$ and $\varphi^{-1}(c)=\{c_1, c_2\}$), we have*

$$R_c\geqq \mathrm{ord}_c\, v_C+\mathrm{ord}_c\, v_{C'}-1 \quad \textit{and} \quad \mathrm{ord}_c\, v_C,\ \mathrm{ord}_c\, v_{C'}\geqq 1$$
$$(\textit{resp. } R_c\geqq \mathrm{ord}_{c_1} v_C+\mathrm{ord}_{c_2} v_C+1).$$

Further if $\mathrm{ord}_c\, v_C=\mathrm{ord}_c\, v_{C'}=1$ (*resp.* $\mathrm{ord}_{c_1} v_C=\mathrm{ord}_{c_2} v_C=0$), *we have* $R_c=1$.

Proof of (2.6.1). By assumption, $\tilde{X}_c\simeq \mathrm{Spec}\, B$, where $B=A\{x, y\}/(\pi-ux^r)$ and $u\in A\{x, y\}^\times$. We have $R_c=\mathrm{length}\, B\Big/\left(\dfrac{\partial u}{\partial x}+\dfrac{r}{x}, \dfrac{\partial u}{\partial y}\right)$ since

$$\Omega^2_{B/A}=B\cdot dx\wedge dy/(d(ux^r)\wedge dy,\ d(ux^r)\wedge dx)\simeq B/(x^r)\cdot\left(\frac{\partial u}{\partial x}+\frac{r}{x}, \frac{\partial u}{\partial y}\right).$$

We note that $\dfrac{r}{x}\in(x)$. Then the assertion for type II follows from the fact that $\dfrac{\partial u}{\partial y}(c)\neq 0$ if and only if $\omega_C(c)\neq 0$. The assertion for type I follows from the fact that, if C is of type I, $\dfrac{\partial u}{\partial y}\in(x)$ and $\mathrm{ord}_c\, v_C=\mathrm{ord}_c\left(\dfrac{\partial u}{\partial x} \bmod x\right)$.

Proof of (2.6.2). By assumption, $\tilde{X}_c\simeq \mathrm{Spec}\, B$, where $B=A\{x, y\}/(\pi-$

$ux^r y^{r'})$, $u \in A\{x, y\}^\times$ and $p \mid r = r_C$. If $p \nmid r' = r_{C'}$, we may take $u = 1$. Then by an easy calculation we have $\omega_C = d \log y^{r'} = r' \cdot \frac{dy}{y}$ and $\Omega^2_{B/A} \simeq B/(x^r y^{r'-1})$. The assertion 1) immediately follows from this. We assume $p \mid r' = r_{C'}$. Then we have $\tilde{\omega}_C = \overline{d \log u}$. Except the last assertion in the case that c is a singular point of C, the assertion 2) follows from this. This exceptional case follows from the inequalities $\mathrm{ord}_c\, v_C$, $\mathrm{ord}_C\, v_{C'} \geqq 1$ of the second assertion of 3), since the question is etale local and $\varphi^* N_{C \cup C'/X} = N_{C/X}(-c) \oplus N_{C'/X}(-c)$ etale locally at c. We prove the assertion 3). Assume $l_C = r_C$, $l_{C'} = r_{C'}$ and C is of type I. Then it is shown similarly as above that we have $R_c = \text{length } B \Big/ \left(\frac{1}{u} \cdot \frac{\partial u}{\partial x} + \frac{r}{x}, \frac{1}{u} \cdot \frac{\partial u}{\partial y} + \frac{r'}{y} \right)$ and $\mathrm{ord}_c\, v_C = \mathrm{ord}_c \left(\left(\frac{1}{u} \cdot \frac{\partial u}{\partial x} + \frac{r}{x} \right) \bmod x \right)$. By the assumption that C is of type I, we have $\frac{1}{u} \cdot \frac{\partial u}{\partial y} + \frac{r'}{y} \in (x)$. The first assertion follows from this. Further assume that C' is of type I. Since the question is etale local and $\varphi^* N_{C \cup C'/X} = N_{C/X}(-c) \oplus N_{C'/X}(-c)$ etale locally at c, it is sufficient to show the case $C \neq C'$. Then similarly as above, we have $R_c = \text{length } B / \left(\frac{1}{u} \cdot \frac{\partial u}{\partial x} + \frac{r}{x}, \frac{1}{u} \cdot \frac{\partial u}{\partial y} + \frac{r'}{y} \right)$, $\mathrm{ord}_c\, v_C = \mathrm{ord}_c \left(\left(\frac{1}{u} \cdot \frac{\partial u}{\partial x} + \frac{r}{x} \right) \bmod x \right)$ and $\mathrm{ord}_c\, v_{C'} = \mathrm{ord}_c \left(\left(\frac{1}{u} \cdot \frac{\partial u}{\partial y} + \frac{r}{y} \right) \bmod y \right)$. By assumption, we have $\frac{1}{u} \cdot \frac{\partial u}{\partial y} + \frac{r'}{y} \in (x)$ and $\frac{1}{u} \cdot \frac{\partial u}{\partial x} + \frac{r}{x} \in (y)$. The rest of the assertion 3) follows from this.

The main result of this section is the following theorem which gives the relation between the vanishing cycles and the relative canonical differential.

Theorem (2.7). *Let X be a regular S-curve such that $X_{s,\mathrm{red}}$ is an n.c.d. in X and c be a closed point of X_s. Let P be the wild ramification group of S. We assume any component of X_s which contains c satisfies $l = r$ and is of type* II. *For an integer r, m denotes the prime-to-p part of r.*

i) *Assume that c is a regular point of $X_{s,\mathrm{red}}$ and that C is the component of X_s which contains c. Then*

$$(R^0\psi\Lambda)_c = (R^0\psi\Lambda)_c^P \quad \textit{and} \quad \dim (R^1\psi\Lambda)_c = (r_C - m_C) \cdot \mathrm{ord}_c\, \omega_C.$$

ii) *Assume that c is a singular point of $X_{s,\mathrm{red}}$. We may assume that c is contained in two components of X_s, C_1 and C_2. Then*

$$(R^0\psi\Lambda)_c = (R^0\psi\Lambda)_c^P \quad \textit{and}$$

$$\operatorname{codim}_{(R^1\psi\Lambda)_c}(R^1\psi\Lambda)_c^P=(r_{C_1}-m_{C_1})(\operatorname{ord}_c\omega_{C_1}+1)+(r_{C_2}-m_{C_2})(\operatorname{ord}_c\omega_{C_2}+1).$$

Remark (2.8). The dimension of the P-invariant part $R^*\psi\Lambda^P$ is determined by Theorem 3.3 of [3]. In particular, under the assumption of i), we have $(R^1\psi\Lambda)_c^P=0$.

Proof. i) It is easy to see that we may assume $m=1$ (cf. Lemma (1.3) of [13]). Let S' be an arbitrary extention of S of degree r. Then $S'\simeq\operatorname{Spec}A'$, where $A'=A\{\pi'\}/(\pi-v\pi'^r)$ and $v\in A\{\pi'\}^\times$. By assumption, $\tilde{X}_c\simeq\operatorname{Spec}B$, where $B=A\{x, y\}/(\pi-ux^r)$, $u\in A\{x, y\}^\times$ and $du \bmod x\neq 0$ in $\Omega^1_{k\{y\}/k}$. It is not difficult to see that the normalization $\tilde{X}'$ of $\tilde{X}_c\times_S S'$ is isomorphic to $\operatorname{Spec}B'$, where $B'=A'\{x, y\}[w]/(\pi'-wx,\ w^r-u\cdot v^{-1})$. Hence $\tilde{X}'$ is essentially smooth over S' outside its unique closed point c'. By the definition of vanishing cycles, the dimension of $R^i\psi\Lambda_c$ is equal to that of $R^i\psi\Lambda_{c'}$ with respect to $\tilde{X}'\to S'$. By the dimension formula (Proposition 5.9 [8] or Proposition (4.2) [13]), it is sufficient to calculate the value of δ (cf. (2.9) below) of $B'\otimes_{A'}k$. Thus we are reduced to show the following lemma since $B'\otimes_{A'}k=k\{y\}[w]/(w^r-\bar{u}\cdot\bar{v}^{-1})$.

Lemma (2.9). *Let $A=k\{x\}$, $t\in A$ such that $dt\neq 0$ in $\Omega^1_{A/k}$ and $r=p^n$. We put $A'=A[w]/(w^r-t)$. For the strict henselization B of the local ring of a reduced curve over k at a closed point, we put $\delta(B)=\dim_k(B^{\text{normal}}/B)$. Then*

$$\delta(A')=\frac{1}{2}(r-1)\operatorname{ord}dt.$$

Proof. For $0\leqq i\leqq n$, we put

$$A_i:=A[t_i]/(t_i^{p^i}-t)=A_{i-1}[t_i]/(t_i^p-t_{i-1}),$$

so that we have $A=A_0\subset\cdots\subset A_n=A'$, $\bar{A}_i:=$(the normalization of A_i) and $A_i'=A_i\otimes_{A_{i-1}}\bar{A}_{i-1}$. Then it is clear that $\delta(A_i)=\delta(A_i')+p\delta(A_{i-1})$. Since $\operatorname{ord}dt$ (in $\Omega^1_{A/k}$)$=\operatorname{ord}dt_i$ (in $\Omega^1_{\bar{A}_i/k}$), we are reduced to show the case $n=1$. We put $\operatorname{ord}dt=m-1$. Then it is clear that $p\nmid m$ and there exist a, $b\in A$ such that $t=a+b$, $\operatorname{ord}a=m$ and $b\in A^p$. Thus we may assume $p\nmid m=\operatorname{ord}t$. In this case, the lemma is just the formula

$$\dim_k k[x]/k[x^m, x^p]=\frac{1}{2}(p-1)(m-1).$$

ii) Similarly as above, we may assume that $m_{C_1}=m_{C_2}=1$. We put $r=r_{C_1}\,|\,r'=r_{C_2}$ and $r=r'\cdot r''$. Let S' be an arbitrary extention of S of degree r. Then $S'\simeq\operatorname{Spec}A'$, where $A'=A\{\pi'\}/(\pi-v\pi'^r)$ and $v\in A\{\pi'\}^\times$.

By assumption, $\tilde{X}_c \simeq \operatorname{Spec} B$, where $B = A\{x, y\}/(\pi - ux^r y^{r'})$, $u \in A\{x, y\}^\times$ and $du_1 := du \bmod x \neq 0$ in $\Omega^1_{k\{y\}/k}$ and $du_2 := du \bmod y \neq 0$ in $\Omega^1_{k\{x\}/k}$. Here $C_1 = (x=0)$ and $C_2 = (y=0)$. It is not difficult to see that the normalization $\tilde{X}'$ of $\tilde{X}_c \times_S S'$ is isomorphic to $\operatorname{Spec} B'$, where $B' = A'\{x, y\}[z, w]/(\pi' - xz, z^{r''} - w \cdot y, w^{r'} - u \cdot v^{-1})$. Then by the same argument as above, it is sufficient to show that

$$(2.7.1) \qquad \delta(B' \otimes_{A'} k) = \frac{1}{2}((r-1) \cdot \operatorname{ord} du_1 + r + (r'-1) \cdot \operatorname{ord} du_2 + r').$$

Since $B' \otimes_{A'} k$ is reduced, the following sequence is exact.

$$0 \longrightarrow B' \otimes_{A'} k \longrightarrow B'/(x) \times B'/(z) \longrightarrow B'/(x, z) \longrightarrow 0.$$

From this, we have

$$(2.7.2) \qquad \delta(B' \otimes_{A'} k) = \delta(B'/(x)) + \delta(B'/(z)) + \dim_k B'/(x, z).$$

Here $B'/(x) = (k\{y\}[w]/(w^{r'} - \bar{u} \cdot \bar{v}^{-1}))[z]/(z^{r''} - w \cdot y)$, $B'/(z) = k\{x\}[w]/(w^{r''} - \bar{u} \cdot \bar{v}^{-1})$ and $B'/(x, z) = k[w]/(w^{r'} - \bar{u} \cdot \bar{v}^{-1})$. It immediately follows from this and Lemma (2.9) that

$$\begin{aligned} \delta(B'/(x)) &= \frac{1}{2}((r'-1) \cdot r'' \cdot \operatorname{ord} du_1 + (r''-1)(\operatorname{ord} du_1 + r')) \\ &= \frac{1}{2}((r-1) \cdot \operatorname{ord} du_1 + (r - r')), \end{aligned}$$

$\delta(B'/(z)) = \frac{1}{2}(r'-1) \cdot \operatorname{ord} du_2$ and $\dim_k B'/(x, z) = r'$. From this and (2.7.2), (2.7.1) follows. Q.E.D.

To conclude this section, we propose the following conjecture. The reason of this conjecture is the compatibility with Lemma (3.5).

Conjecture (2.10). *Under the same assumption of Theorem* (2.7);
i) *Assume that* c *is a regular point of* $X_{s,\mathrm{red}}$ *and that* C *is the component of* X_s *which contains* c. *Then*

$$\operatorname{Sw}(R^1\psi\Lambda)_c = R_c + \operatorname{ord}_c \omega_C.$$

ii) *Assume that* c *is a singular point of* $X_{s,\mathrm{red}}$ *and that* c *is contained in two components of* X_s, C_1 *and* C_2. *Then*

$$\operatorname{Sw}(R^1\psi\Lambda)_c = R_c + \operatorname{ord}_c \omega_{C_1} + \operatorname{ord}_c \omega_{C_2} + 2.$$

§ 3. Comparison of dimension and length

Using the results of previous sections, we prove Conjecture (0.1) in a special case.

To study (0.1), clearly there are not any loss of generality if we assume that X and Z are connected. We show that we may further assume that X_η is geometrically connected over η. If $Z \neq X_s$, there is a component of X_s whose multiplicity in X_s is 1. Hence X_η is geometrically connected over η. If $Z = X_s$, X is proper over S and there is an extension S' ($=\operatorname{Spec} \Gamma(X, \mathcal{O}_X)$) of S such that $X \to S$ factors $S' \to S$ and X_η is geometrically connected over $\eta' := S' \underset{S}{\times} \eta$. We have

Lemma (3.1). *Suppose $S' \to S$ is an extension of S and $f\colon X \to S'$ is a regular proper S'-curve. Then Conjecture* (0.1) *holds for $(X, X_{s'})/S'$ if and only if so does it for $(X, X_s)/S$.*

Proof. Since the $\operatorname{Gal}(\bar\eta/\eta)$-module $R\Gamma(X_s, R\psi\Lambda)$ is the induced representation of the $\operatorname{Gal}(\bar\eta/\eta')$-module $R\Gamma(X_{s'}, R\psi\Lambda)$, we have

(3.1.1) $$\begin{aligned}&\operatorname{dimtot}_S R\Gamma(X_s, R\phi\Lambda)\\ &=\operatorname{dimtot}_S R\Gamma(S'_s, R\phi\Lambda)\cdot \dim R\Gamma(X'_{s'}, R\psi\Lambda)\\ &\qquad\qquad + \operatorname{dimtot}_{S'} R\Gamma(X_{s'}, R\phi\Lambda).\end{aligned}$$

Thus it is sufficient to show that

(3.1.2) $$\begin{aligned}&\chi(X_s, \Omega^{\cdot}_{X/S,\mathrm{tors}})\\ &= \chi(S'_s, \Omega^{\cdot}_{S'/S,\mathrm{tors}})\cdot\chi(X_{\bar\eta'}) + \chi(X_{s'}, \Omega^{\cdot}_{X/S',\mathrm{tors}}).\end{aligned}$$

We have the following exact sequences

(3.1.3) $$0 \longrightarrow f^*\Omega^1_{S'/S} \longrightarrow \Omega^1_{X/S,\mathrm{tors}} \longrightarrow \Omega^1_{X/S',\mathrm{tors}} \longrightarrow 0$$

(3.1.4) $$0 \longrightarrow f^*\Omega^1_{S'/S}\otimes\Omega^1_{X/S'}/\mathrm{tors} \longrightarrow \Omega^2_{X/S} \longrightarrow \Omega^2_{X/S'} \longrightarrow 0.$$

Hence we are reduced to show the following equality

(3.1.5) $$\begin{aligned}&\chi(X_{s'}, f^*\Omega^1_{S'/S}) - \chi(X_s, f^*\Omega^1_{S'/S}\otimes\Omega^1_{X/S'}/\mathrm{tors})\\ &= \operatorname{length}_{\mathcal{O}_{S'}}(\Omega^1_{S'/S})\cdot\chi(X_{\bar\eta'}).\end{aligned}$$

This follows from

(3.1.6) $$\begin{aligned}&\chi(X_{s'}, f^*\Omega^1_{S'/S}) - \chi(X_s, f^*\Omega^1_{S'/S}\otimes\Omega^1_{X/S'}/\mathrm{tors})\\ &= \operatorname{length}_{\mathcal{O}_{S'}}(\Omega^1_{S'/S})\cdot(\chi(X_{s'}, \mathcal{O}_{X_{s'}}) - \chi(X_s, \mathcal{O}_{X_{s'}}\otimes\Omega^1_{X/S'}/\mathrm{tors}) \text{ and}\end{aligned}$$

(3.1.7) $$\begin{aligned}\chi(X_{\bar\eta'}) &= \chi(X_\eta, \mathcal{O}_{X_\eta}) - \chi(X_\eta, \Omega^1_{X_\eta/\eta'})\\ &= \chi(X_{s'}, \mathcal{O}_{X_{s'}}) - \chi(X_{s'}, \mathcal{O}_{X_{s'}}\otimes\Omega^1_{X/S'}/\mathrm{tors})\end{aligned}$$

The equality (3.1.7) follows from the invariance of the Euler-Poincaré characteristic since $\mathcal{O}_X$ and $\Omega^1_{X/S'}/\text{tors}$ are f-flat.

The main results of this section are as follows. We note that if X_η is geometrically connected over η, the action of the inertia group I of S on $H^i(Z, R\psi\Lambda)$ is trivial for $i \neq 1$.

Theorem (3.2). *Suppose X_η is geometrically connected and Z is connected. We assume that, if X is proper with genus $X_\eta = 1$ and if $Z = X_s$, then the g.c.d. of the multiplicities of the components of Z is not divisible by p. Then, if the action of the wild ramification group P on $H^1(Z, R\psi\Lambda)$ is trivial, Conjecture* (0.1) *holds for* (X, Z) *i.e.*

(3.2.1) $$\dim R\Gamma(Z, R\phi\Lambda) = -\chi(Z, \Omega^{\cdot}_{X/S,\,\text{tors}}).$$

In [13] Section 3 (resp. Section 4) we obtain a geometric condition that the action of P on $H^1(Z, R\psi\Lambda)$ (resp. $R^1\psi\Lambda_y$) is trivial when $Z = X_s$ (resp. Y is a normal S-curve and y is an isolated non-smooth point of $Y \to S$). If $Z \neq X_s$, there exist a normal S-curve Y, a closed point y of Y_s and a proper S-morphism $f: X \to Y$ such that f induces an isomorphism $X - Z \xrightarrow{\sim} Y - \{y\}$ and that $Z_{\text{red}} = f^{-1}(y)_{\text{red}}$ (cf. [6] Lemmas A and B). Hence the results of [13] Section 4 apply to this case. By Theorem (3.11) of [13] (resp. Theorem (4.12) of [13] and the argument above), we see that if (X, Z) satisfies the assumption of Theorem (3.2) and if $Z = X_s$ (resp. $Z \neq X_s$), there exist (X', Z') which satisfies the condition i) of Theorem (3.3) below and a birational S-rational map $X' \to X$ which induces an isomorphism $X' - Z' \xrightarrow{\sim} X - Z$. Hence by Remark (1.3), Theorem (3.2) follows from the following Theorem (3.3).

Theorem (3.3). *Suppose X_η is geometrically connected and Z is connected and purely of dimension* 1. *We assume*:

1. *$X_{s,\text{red}}$ is a normal crossing divisor in X.*
2. *For $C \in Z_1$, $p \mid r_C$ implies $r_C = l_C$.*
3. *Further if C is of type* I *and C is exceptional of first kind, then C intersects with other components of X_s at more than three points.*

Here Z_1 denotes the set of points of dimension 1 *of Z. Then we have the inequality*

(3.3.1) $$\dim R\Gamma(Z, R\psi\Lambda) \geqq \chi(Z) - \chi(Z, \Omega^{\cdot}_{X/S,\,\text{tors}}).$$

If the equality holds, (X, Z) satisfies either of the following conditions.

i) *If $C \in Z_1$ satisfies $p \mid r_C$, then it is isomorphic to $\mathbf{P}^1_s$, intersects with other components of X_s at exactly two points and the multiplicities of the components which intersect with C are prime to p.*

ii) *X is proper, $Z = X_s$, genus $X_\eta = 1$ and X is of type* I *in the sense of*

Kodaira's symbol.
Conversely if the condition i) *is satisfied, the equality holds.*

Since Sw $R\Gamma(Z, R\psi\Lambda) = -\mathrm{Sw}\, R^1\Gamma(Z, R\psi\Lambda) \leqq 0$, (3.3.1) follows from Conjecture (0.1). Here we prove it without using it. The assumption 3 is not essential since, if there is an exceptional curve of first kind which intersects with other components at less than two points, its contraction preserves other assumptions.

Proof. First we rewrite (3.3.1) by using intersection theory. For $C \in Z_1$, let r_C be the multiplicity of C in X_s. We put $E := \sum_{C \in Z_1} r_C C$, $E_0 := \sum_{C \in Z_1} C$ and $F = (\ , E_0 + K - E)$, where K is the relative canonical divisor of X over S. Then in [13], we have the equality

$$\dim R\Gamma(Z, R\psi\Lambda) = -F(E). \tag{3.3.2}$$

We put $W := (X_{s,\mathrm{red}})_{\mathrm{sing}} \cap Z$ and also let W denote the 0-cycle $\sum_{x \in W} x$. Since $X_{s,\mathrm{red}}$ is an n.c.d. in X, we can easily see that

$$\chi(Z) = \deg W - (E_0, E_0 + K - E) \tag{3.3.3}$$

by using the Riemann-Roch formula $\chi(\mathcal{O}_C) = -(C, C+K)$ for $C \in Z_1$ and the fact that the number $(C, E_0 - C - E)$ is equal to the intersection number of C with other components of Y_s. By (3.3.2), (3.3.3) and Proposition (1.5), (3.3.1) becomes

$$\deg R + (D - (E - E_0), E_0 + K - E) - (D - (E - E_0), D) - \deg W \geqq 0. \tag{3.3.4}$$

We study each term. Since we have $D - (E - E_0) = \sum_{p \mid r_C} C$ by assumption, the second term is;

$$(D - (E - E_0), E_0 + K - E) = \sum_{p \mid r_C} F(C) \tag{3.3.5}$$

and the third term is;

$$-(D - (E - E_0), D) = \sum_{p \mid r_C} -(C, D). \tag{3.3.6}$$

Since $D = E - \sum_{p \nmid r_{C'}} C'$, by assumption,

$$-(C, D) = (C, -E) + \sum_{p \ \ r_{C'}} (C, C'). \tag{3.3.7}$$

Next we compute R in some easy cases.

Lemma (3.4). *If B is an A-algebra, I (resp. I') denotes the annihilator of $\Omega^1_{B/A,\mathrm{tors}}$ (resp. $\Omega^2_{B/A}$).*

1. *Suppose* $B=A\{x, y\}/(\pi-x^m)$ *such that* $(p, m)=1$. *Then*

$$I=I'=(x^{m-1}). \tag{3.4.1}$$

2. *Suppose* $B=A\{x, y\}/(\pi-x^m y^n)$ *such that* $(p, m)=1$. *Then*

$$I=(x^{m-1}y^{n-1}) \quad \textit{and} \quad I'=I\cdot(x, y) \quad \textit{if} \quad (p, n)=1, \tag{3.4.2}$$

$$I=I'=(x^{m-1}y^n) \quad \textit{if} \quad p\,|\,n. \tag{3.4.3}$$

Since this lemma is very easy, we omit the proof.

We decompose the 0-cycle W. For $c\in W$, we put
$B_c:=\{$the two branches of X_s at $c\}:=((X_s)_c^{\sim})_1$.
$W_0:=\{c\in W$; both of r_C for $C\in B_c$ are not divisible by $p\}$
$W_1:=\{c\in W$; one of r_C for $C\in B_c$ is divisible by p and the other is not$\}$
$W_2:=\{c\in W$; both of r_C for $C\in B_c$ are divisible by $p\}$.

Then clearly $W=W_0+W_1+W_2$ and $\sum_{p|r_C} -(C, D)=\deg W_1$ by (3.3.7). By Lemma (3.4), $R\geqq W_0$ as 0-cycles. Thus (3.3.4) becomes:

$$\deg(R-W_0)+\sum_{p|r_C} F(C)-\deg W_2\geqq 0. \tag{3.3.8}$$

Further we decompose the 0-cycle W_2. We put:
$Z_{\mathrm{I}}:=\{C\in Z_1$; r_C is divisible by p and C is of type I$\}$
$Z_{\mathrm{II}}:=\{C\in Z_1$; r_C is divisible by p and C is of type II$\}$
$W_{2,1}:=\{c\in W_2$; both of $C\in B_c$ are of type I$\}$
$W_{2,2}:=\{c\in W_2$; one of $C\in B_c$ is of type I and the other is of type II$\}$
$W_{2,3}:=\{c\in W_2$; both of $C\in B_c$ are of type II$\}$.

We put $W_C:=W\cap C$, $W'_C:=C_{\mathrm{sing}}$ and $W''_C:=W_C-W'_C$ for $C\in Z_1$, and $W'_{2,1}:=\bigcup_{C\in Z_{\mathrm{I}}} W'_C$, $W''_{2,1}:=W_{2,1}-W'_{2,1}$, $W'_{2,3}:=\bigcup_{C\in Z_{\mathrm{II}}} W'_C$, $W''_{2,3}:=W_{2,3}-W'_{2,3}$.

We define positive 0-cycles $N:=\sum_{c\in Z_0} N_c c$ and $\Omega:=\sum_{c\in Z_0}\Omega_c C$ as follows.

$$N_c:=\begin{cases} \operatorname{ord}_c \nu_C & (\text{if } c \text{ is a regular point of } X_{s,\mathrm{red}},\ c\in C \text{ and } C\in Z_{\mathrm{I}}) \\ \operatorname{ord}_{c_1}\nu_C+\operatorname{ord}_{c_2}\nu_C+1 & (\text{if } c\in W'_{2,1},\ \in C \text{ and } \{c_1, c_2\}=\varphi^{-1}(c), \text{ where } \varphi\colon \bar{C}\to C \text{ is the normalization}) \\ \operatorname{ord}_c \nu_C+\operatorname{ord}_c \nu_{C'}-1 & (\text{if } c\in W''_{2,1} \text{ and } B_c=\{C, C'\}) \\ \operatorname{ord}_c \nu_C & (\text{if } c\in W_{2,2},\ c\in C \text{ and } C\in Z_{\mathrm{I}}) \\ 0 & (\text{otherwise}). \end{cases}$$

$$\Omega_c := \begin{cases} \operatorname{ord}_c \omega_C & (\text{if } c \text{ is a regular point of } X_{s,\mathrm{red}}, \\ & \quad c \in C \text{ and } C \in Z_{\mathrm{II}}) \\ \operatorname{ord}_{c_1} \omega_C + \operatorname{ord}_{c_2} \omega_C & (\text{if } c \in W'_{2,3}, \in C \text{ and} \\ \quad \{c_1, c_2\} = \varphi^{-1}(c), \text{ where } \varphi \colon \bar{C} \to C \text{ is the normalization}) & \\ \operatorname{ord}_c \omega_C + \operatorname{ord}_c \omega_{C'} & (\text{if } c \in W''_{2,3} \text{ and } B_c = \{C, C'\}) \\ \operatorname{ord}_c \omega_C & (\text{if } c \in W_{2,2}, c \in C \text{ and } C \in Z_{\mathrm{II}}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then by Proposition (2.6.2.1), we have

$$\text{(3.3.9)} \qquad \deg N = \sum_{C \in Z_{\mathrm{I}}} (-(C, C) + \deg W'_C) - \deg W''_{2,1},$$

since ν_C is a section of the conormal bundle $N_{C/X}$. Also by Proposition (2.6.2.3), $R \geqq W_0 + N$ as 0-cycles. As for $\deg \Omega$, we put $W''_{i,C} := W''_C \cap W_i$ for $i = 1, 2$. Then by Proposition (2.6.2.1),

$$\text{(3.3.10)} \qquad \deg \Omega = \sum_{C \in Z_{\mathrm{II}}} (-2 \cdot \chi(\mathcal{O}_{\bar{C}}) + \deg W''_{1,C}),$$

since ω_C is a rational section of $\Omega^1_{\bar{C}/s}$.

By the definition of $F(C)$,

$$\text{(3.3.11)} \qquad F(C) = (C, C+K) + (C, E_0 - C) + (C, -E).$$

Here $(C, E_0 - C) + (C, -E) = \deg W''_C$. On the other hand, by the Riemann-Roch formula, we have

$$\text{(3.3.12)} \quad (C, C+K) = -2 \cdot \chi(\mathcal{O}_C) = -2 \cdot \chi(\mathcal{O}_{\bar{C}}) + 2 \cdot \deg W'_C.$$

Thus we have

$$\text{(3.3.13)} \qquad \begin{aligned} F(C) &= (C, C) + (C, K) + \deg W''_C \\ &= -2 \cdot \chi(\mathcal{O}_{\bar{C}}) + 2 \cdot \deg W'_C + \deg W''_C. \end{aligned}$$

Now by (3.3.9), (3.3.10) and (3.3.13), the l.h.s. of (3.3.8) becomes

$$\text{(3.3.14)} \quad \begin{aligned} &\deg (R - W_0) + \sum_{p \mid r_C} F(C) - \deg W_2 \\ &= \deg (R - W_0 - N) + \deg \Omega \\ &\quad + \sum_{C \in Z_{\mathrm{II}}} (F(C) - (-2 \cdot \chi(\mathcal{O}_{\bar{C}}) + \deg W''_{1,C})) \\ &\quad + \sum_{C \in Z_{\mathrm{I}}} (F(C) - (C, C) + \deg W'_C) - \deg W''_{2,1} - \deg W_2 \\ &= \deg (R - W_0 - N) + \deg \Omega + \sum_{C \in Z_{\mathrm{II}}} (2 \cdot \deg W'_C + \deg W''_{2,C}) \end{aligned}$$

$$+\sum_{C\in Z_{\mathrm{I}}}((C,K)+\deg W''_C+\deg W'_C)-\deg W''_{2,1}-\deg W_2$$
$$=\deg(R-W_0-N)+\deg\Omega+\deg W_{2,3}$$
$$+\deg W_{2,2}+\sum_{C\in Z_{\mathrm{I}}}(C,K).$$

Here $R-W_0-N$, Ω and $W_{2,3}$ are positive 0-cycles. We put $W_{2,i,C}=W_{2,i}\cap C$, for $C\in Z_{\mathrm{I}}$, so that we have $W_{2,2}=\sum_{C\in Z_{\mathrm{I}}}W_{2,2,C}$. Thus (3.3.8) becomes

(3.3.15) $$\deg(R-W_0-N)+\deg\Omega+\deg W_{2,3}+\sum_{C\in Z_{\mathrm{I}}}(\deg W_{2,2,C}+(C,K))\geqq 0.$$

We show that $\deg W_{2,2,C}+(C,K)\geqq 0$ for $C\in Z_{\mathrm{I}}$. If $(C,K)\geqq 0$, this is clear. We study the case $(C,K)<0$. Always $(C,C+K)\geqq -2$ and is even by the Riemann-Roch formula, and $(C,C)\leqq 0$. It follows from this that $(C,K)\geqq -2$ and that, if $(C,K)=-2$ (resp. $=-1$), we have $(C,C)=0$ (resp. $=-1$). If $C\in Z_1$ satisfies $(C,K)=-2$, by the assumption of connectedness and $(C,C)=0$, X is proper over S, $C=X_{s,\mathrm{red}}$ and $C\simeq \boldsymbol{P}^1_s$. In this case, we have $r_C=1$. In fact, $2\geqq\chi(X_{\bar\eta})=2\chi(\mathcal{O}_{X_\eta})=2\chi(\mathcal{O}_{X_s})=-(r_CC,r_CC+K)=2r_C\geqq 2$. We show that if $(C,K)=-1$ and $C\in Z_{\mathrm{I}}$, then we have $\deg W_{2,2,C}+(C,K)>0$. In this case, by assumption 3, $\deg W''_C\geqq 3$. On the other hand, we have $-(C,C)=\sum_{c\in C}\mathrm{ord}_c\, v_C\geqq\deg W_{2,1,C}$ by Proposition (2.6). From these inequalities, we have $-(C,C)+\deg W_{2,2,C}\geqq 3$, since $W''_C=W_{2,1,C}+W_{2,2,C}$ (cf. Proposition (2.6)). Thus it is sufficient to note that $(C,C+K)\geqq -2$. Thus we have shown that each term of the l.h.s of (3.3.15) is positive and the inequality (3.3.1) is proved.

Now we assume that the equality holds in (3.3.1) i.e. in (3.3.15). This implies

(3.3.16) $$R=W_0-N,\quad \Omega=0,\quad W_{2,3}=0\quad\text{and}$$
$$\deg W_{2,2,C}+(C,K)=0\qquad\text{for}\quad C\in Z_{\mathrm{I}}.$$

Since $\deg W_{2,2,C}+(C,K)=0$, for all $C\in Z_{\mathrm{I}}$, we have $W_{2,2,C}=0$ and $(C,K)=0$. In fact, as we have seen above, $(C,K)=-1$ or -2 does not occur. Thus we have $W_{2,2}=0$. By the connectedness assumption, $W_{2,2}=0$ and Proposition (2.6), we have $Z_1=Z_{\mathrm{I}}$ or $Z_{\mathrm{I}}=\phi$. We assume $Z_1=Z_{\mathrm{I}}$. Then we have $(C,K)=0$ for all C. It is easily checked that this means that the condition ii) is satisfied. Now we assume $Z_1=\phi$. Let $C\in Z_{\mathrm{II}}$. Then since $W_{2,3}=0$ and $\Omega=0$, we have $W'_C=W'_{1,C}$ and $2\cdot\chi(\mathcal{O}_{\bar C})=\deg W'_{1,C}$, by (3.3.10). Since $\chi(\mathcal{O}_{\bar C})\leqq 1$, we have $\chi(\mathcal{O}_{\bar C})=0$ or 1. We assume there exists $C\in Z_{\mathrm{II}}$ such that $\chi(\mathcal{O}_{\bar C})=0$. Then we have $W'_C=W'_{1,C}=0$. By the connectedness assumption, X is proper over S and $C=X_{s,\mathrm{red}}$. It is also easily checked that the condition ii) is satisfied. Now we assume every $C \in Z_{II}$ satisfies $\chi(\mathcal{O}_{\bar{C}})=1$. Then we have $C \simeq \boldsymbol{P}^1_s$, $W'_C = W'_{1,C}$ and $\deg W'_{1,C} = 2$. This is just the condition i).

Conversely, we assume that (X, Z) satisfies the condition i) of (3.3). Then

$$(3.3.17)\quad \deg(R-W_0-N)+\deg\Omega+\deg W_{2,3}+\sum_{C\in Z_I}(\deg W_{2,2,C}+(C, K)) = \deg(R-W_0)+\deg\Omega.$$

Here

$$\deg\Omega=\sum_{C\in Z_{II}}(-2\cdot\chi(\mathcal{O}_{\bar{C}})+\deg W'_{1,C})=\sum_{C\in Z_{II}}(-2+2)=0.$$

Thus it is sufficient to show that $R=W_0$. Since $\Omega=0$, $\omega_C(c)\neq 0$ for $c\in C\cap(X_{s,\mathrm{red}})_{\mathrm{reg}}$ such that $C\in Z_{II}$. It follows from this, Proposition (2.6) and Lemma (3.4) that $R=W_0$. Q.E.D.

The following is shown in the proof of the theorem.

Lemma (3.5). *Under the same assumption of Theorem* (3.3) *except the condition* 3, *if we admit Conjecture* (0.1),

$$(3.5.1)\quad \mathrm{Sw}\, H^1(Z, R\psi\Lambda) = \deg(R-W_0-N)+\deg\Omega+\deg W_{2,3} + \sum_{C\in Z_I}(\deg W_{2,2,C}+(C, K)).$$

§4. The case that S is the strict localization at a closed point of a smooth curve over an algebraically closed field

In this section, we always assume that S is the strict localization at a closed point of a smooth curve over an algebraically closed field k. Let π be a prime element of S so that $S\simeq \mathrm{Spec}\, k\{\pi\}$. We prove Conjecture (0.1) in this case.

Proposition (4.1). *Let S be as above. Suppose X is a regular S-curve and Z is a closed subscheme of X_s such that Z is proper over s and $X-Z$ is smooth over S. Then Conjecture* (0.1) *holds, i.e.*

$$(4.1.1)\quad \mathrm{dimtot}\, R\Gamma(Z, R\phi\Lambda) = -\chi(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Proof. First, we note that this is clear if X_s is a (reduced) n.c.d. of X. We may assume Z is connected. The proof is divided into the following two cases.

X is proper over S and $Z=X_s$ (the global case).

otherwise (the local case).

We first prove the global case. This proof is due to K. Kato. There exist a proper smooth connected curve C over k, a regular proper C-curve Y smooth over the generic point of C and a closed point c of C which satisfies the following condition. S is k-isomorphic to the strict localization of C at c and by this isomorphism X is S-isomorphic to $Y \underset{C}{\times} S$. We need the following global lemma.

Lemma (4.2). *Suppose C is a proper smooth connected curve over an algebraically closed field k and $f\colon Y \to C$ is a regular proper flat generically smooth C-scheme. Then the summation of the both sides of* (4.1.1) *are equal i.e.*

(4.2.1) $$\sum_{c \in C_0} \operatorname{dimtot} R\Gamma(Y_c, R\phi\Lambda) = -\sum_{c \in C_0} \chi(Y_c, \Omega^{\cdot}_{Y/C,\mathrm{tors}}).$$

Proof. Since Y is a proper smooth variety over k, the Euler-Poincaré characteristic $\chi(Y) := \sum_{i \in \mathbf{Z}} (-1)^i \dim H^i(Y, \Lambda)$ of Y is equal to $\chi(Y, \Omega^{\cdot}_{Y/k}) := \sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} (-1)^{i+j} \dim_k H^i(Y, \Omega^j_{Y/k})$. By the Grothendieck-Ogg-Shafarevich formula, we have

$$\chi(Y) = \chi(C) \cdot \chi(Y_{\bar\eta}) - \sum_{c \in C_0} \operatorname{dimtot} R\Gamma(Y_c, R\phi\Lambda).$$

Hence, it is sufficient to show

$$\chi(Y) - \chi(C) \cdot \chi(Y_{\bar\eta}) = \chi(Y, \Omega^{\cdot}_{Y/C,\mathrm{tors}}).$$

Here we have

$$\chi(Y) = \chi(Y, \Omega^{\cdot}_{Y/k}) \quad \text{and} \quad \chi(C) \cdot \chi(Y_{\bar\eta}) = \deg([f^*\Omega^{\cdot}_{C/k}][\Omega^{\cdot}_{Y/C}]).$$

Thus it is sufficient to show that

$$[\Omega^{\cdot}_{Y/k}] - [f^*\Omega^{\cdot}_{C/k}][\Omega^{\cdot}_{Y/C}] = [\Omega^{\cdot}_{Y/C,\mathrm{tors}}].$$

This follows from the exact sequence

$$0 \longrightarrow f^*\Omega^1_{C/k} \otimes \Omega^{i-1}_{Y/C,\mathrm{tors}} \longrightarrow f^*\Omega^1_{C/k} \otimes \Omega^{i-1}_{Y/C} \longrightarrow \Omega^i_{Y/k} \longrightarrow \Omega^i_{Y/C} \longrightarrow 0.$$

We return to the proof of the global case. By the stable reduction theorem, there is a finite covering $\varphi\colon C' \to C$ which completely splits at c and, for c' such that $\varphi(c') \neq c$, Y admits a stable reduction at c'. By the remark at the beginning of the proof and Lemma (4.2), the global case is proved.

We show the local case. As in Section 3, there exist a normal S-curve Y, a closed point y of Y_s and a proper S-morphism $f\colon X \to Y$ such that

$f^{-1}(y)_{\mathrm{red}}=Z_{\mathrm{red}}$ and $f\colon X-Z \xrightarrow{\sim} Y-\{y\}$. Thus it is sufficient to show the following.

Lemma (4.3). *Suppose Y is a normal S-curve and y is a closed point of Y_s such that $Y-\{y\}$ is smooth over S. Then for any proper surjective S-morphism $f\colon X\to Y$ of a regular S-curve X to Y such that $f\colon X-Z\xrightarrow{\sim} Y-\{y\}$, the pair $(X, f^{-1}(y))$ satisfies* (4.1.1) *i.e.*

(4.3.1) $$\operatorname{dimtot} R\Gamma(f^{-1}(y), R\phi\Lambda)=-\chi(f^{-1}(y), \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Proof. By Proposition 6.4.1 of [11], there exists an integer N such that if $g\colon \tilde{Y}_y\to S$ satisfies $f(\pi)\equiv g(\pi) \bmod m_y^N$, there exists an S-isomorphism of $f\colon \tilde{Y}_y\to S$ to $g\colon \tilde{Y}_y\to S$. Using this fact, we prove the lemma by a global argument. There is a projective normal connected surface $T\subset \boldsymbol{P}_k^n$ over k and a closed point t of T such that $T-\{t\}$ is smooth over k and that $\tilde{Y}_y$ is k-isomorphic to $\tilde{T}_t$. We need the following lemma.

Lemma (4.4). *Suppose $T\subset P=\boldsymbol{P}_k^n$ is a projective normal connected surface over k and t_0 is a closed point of T such that $T-\{t_0\}$ is smooth over k. Suppose a positive integer N and a jet $j_0\in J_T(t_0, N):=\mathcal{O}_{T,t_0}/m_{t_0}^N$ such that the image in $\kappa(t_0)=k$ is 0 are given. Then, if we take a sufficiently large integer r, there exist two homogeneous forms of degree r, F_0 and $F_\infty\in\Gamma(P, \mathcal{O}(r))$ such that $(F_0/F_\infty)_{t_0}=j_0$ in $J(t_0, N)$ and that $T\cap H_0-\{t_0\}$, $T\cap H_\infty$ and $T\cap H_0\cap H_\infty$ are smooth over k, where H_* denotes the hypersurface of degree r defined by $F_*=0$.*

Admitting this lemma we continue the proof. We take r, F_0 and F_∞ satisfying the conditions of Lemma (4.4) for the integer N above and the jet $f(\pi)\in J(t, N)$. Replacing the original immersion $i\colon T\subset \boldsymbol{P}^n$ by i followed by the r-uple embedding, we may assume $r=1$. Then the pencil given by (F_0, F_∞) defines the morphism $g\colon \bar{T}\to \boldsymbol{P}_k^1$, where $\bar{T}$ is the blowing-up of T at $T\cap H_0\cap H_\infty$. Since $T\cap H_\infty$ is smooth and $\bar{T}\to\boldsymbol{P}_k^1$ is flat, $\bar{T}\to\boldsymbol{P}_k^1$ is generically smooth. Since $T\cap H_0-\{t\}$ is smooth, t is the unique singularity of the morphism $\bar{T}\times_{\boldsymbol{P}_k^1}(\boldsymbol{P}_k^1)_0^\sim\to(\boldsymbol{P}_k^1)_0^\sim$. Thus by the global case, Lemma (4.3) holds for $\bar{T}\times_{\boldsymbol{P}_k^1}(\boldsymbol{P}_k^1)_0^\sim\to(\boldsymbol{P}_k^1)_0^\sim$ and t. If we identify S and $(\boldsymbol{P}_k^1)_0^\sim$ by $\pi\mapsto F_0/F_\infty$, $\tilde{Y}_y\to S$ and $\tilde{T}_t\to(\mathbf{P}_k^1)_0^\sim$ are S-isomorphic. Thus we have completed the proof of the local case modulo Lemma (4.4).

Although the fact like Lemma (4.4) is used in [10] (4.0, 4.6), the detail of the proof is not given. So here we give it a proof.

Proof of (4.4). It is easy to see the following facts.
If we take a sufficiently large integer r,

1) for any homogeneous r-form F_1 such that $t_0 \notin H_1 := (F_1 = 0)$, and for any closed point $t \neq t_0$ of $T - T \cap H_1$, the following linear map is surjective.

$$\Gamma(P, \mathcal{O}(r)) \longrightarrow J_T(t_0, N) \times J_T(t, 2) \colon F \mapsto ((F/F_1)_{t_0}, (F/F_1)_t).$$

2) for any homogeneous r-forms F_1 and F_2 such that $F_1 \neq F_2$ and $t_0 \notin H_1$, and for any closed point t of $T \cap H_1 - T \cap H_1 \cap H_2$, the following linear map is surjective.

$$\Gamma(P, \mathcal{O}(r)) \longrightarrow J_T(t_0, N) \times J_{T \cap H_1}(t, 2) \colon F \mapsto ((F/F_1)_{t_0}, (F/F_2)_t).$$

We fix such r. By Bertini's theorem there is a homogeneous r-form F_∞ such that $T \cap H_\infty$ is smooth over k and $t_0 \notin T \cap H_\infty$. Let V be the inverse image of j_0 by $\Gamma(P, \mathcal{O}(r)) \to J_T(t_0, N)$ $(F \mapsto (F/F_\infty)_{t_0})$. By the above surjectivity, for any closed point $t \neq t_0$ of $T - H_\infty \cap T$ (resp. $T \cap H_\infty$), the codimension of the subvariety of V consisting of the r-forms F such that $T \cap H$ is not smooth at t (resp. $T \cap H_\infty \cap H$ is not smooth at t), where $H := (F = 0)$, is greater than 3 (resp. 2). Since $\dim T = 2$ and $\dim T \cap H_\infty = 1$, the codimension of the subvariety W of V consisting of the r-forms F such that $H \cap T$ or $H \cap H_\infty \cap T$ is not smooth is greater than 1. Thus if we take an r-form F_0 contained in $V - W$, F_0 and F_∞ satisfy the conditions of Lemma (4.4).

Added in Proof.

The author recently proved that Bloch's formula is equivalent to that of Conjecture (0.1), as mentioned in the introduction. This fact is shown in "Self-intersection 0-cycles and coherent sheaves on arithmetic schemes" (preprint, Univ of Tokyo, 1987).

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