Advanced Studies in Pure Mathematics 12, 1987
Galois Representations and Arithmetic Algebraic Geometry
pp. 249–258

On Algebraic Groups of Mumford-Tate Type

Takashi Ichikawa

Introduction

In [9], Serre introduced certain algebraic groups, which we call *of Mumford-Tate type* in this paper (for its definition, see 1.1), and classified these groups. The definition of these groups is obtained by generalizing that of Mumford-Tate groups of abelian varieties over $\boldsymbol{C}$. The aim of this paper is to determine algebraic groups of Mumford-Tate type which satisfy certain conditions by using Serre's result, and to apply this result to abelian varieties over $\boldsymbol{C}$ and local fields. The statement and the proof of the main result will be given in 1.4 and 3.3 respectively.

As an application of this result, we can determine the Mumford-Tate group M of a simple abelian variety A over $\boldsymbol{C}$ of type 1, 2, or 3 (cf. [6], p. 201) such that $n=\dim A/mr$ is an odd integer. Here m and r are positive integers such that $m^2=[D:E]$ and $r=[E:\boldsymbol{Q}]$, where $D=\mathrm{End}_{\boldsymbol{C}}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ and E is its center. When A is of type 1 or 2, Tankeev and Ribet proved that the semi-simple part S of M coincides with $R_{E/\boldsymbol{Q}}(S_E)$, where S_E is an algebraic group over E which is isomorphic to $\mathrm{Sp}(n)$ ([10], Theorem 5.1, [7], Theorem 1). Moreover, they proved that the Hodge cycles on A^m (m: positive integers) are generated by those of degree 2. When A is of type 3, we can show that $S=R_{E/\boldsymbol{Q}}(S_E)$, where $S_E\cong\mathrm{SO}(2n)$ or $r_b(SL(2b))$. Here b is the positive integer satisfying $\binom{2b}{b}=2n$, and r_b is the b-fold exterior power of the standard representation of $GL(2b)$. In this case, the Hodge cycles on A^m are not generated by those of degree 2.

As proved by Sen ([8], Theorem 1), for an abelian variety A over an ℓ-adic local field with an algebraically closed residue field, the algebraic envelope H of the ℓ-adic Galois group in $GL(T_\ell(A))$ is of Mumford-Tate type, where $T_\ell(A)$ is the ℓ-adic Tate module of A. Hence we can obtain a result for H, which is similar to the above result (see 2.4). As for an abelian variety over a global field, there is a similar result by Serre (Theorem 2.2.8 in "Résumé des cours de 1984–1985").

The author would like to thank Professor Yasutaka Ihara for his advice and encouragement.

Received May 18, 1986.

Notations

Let $\boldsymbol{Z}$ be the ring of integers, and let $\boldsymbol{Q}$ (resp. $\boldsymbol{R}$, resp. $\boldsymbol{C}$) be the field of rational (resp. real, resp. complex) numbers. For a vector space V over a field K, GL_V (resp. SL_V) denote the general (resp. special) linear algebraic groups defined over K. For a non-degenerate alternating K-bilinear form $\psi: V\times V\to K$, $Sp_{V,\psi}$ denotes the symplectic algebraic subgroup of GL_V over K with respect to ψ. For an integer n, $GL(n)$ (resp. $SL(n)$) denote the general (resp. special) linear algebraic groups of degree n, $Sp(n)$ denotes the symplectic algebraic subgroup of $GL(2n)$, and $SO(n)$ denotes the special orthogonal algebraic subgroup of $GL(n)$. For a finite separable extension E of a field K, and an algebraic group G over E, $R_{E/K}(G)$ denotes the scalar restriction of G from E to K. For a module M with an action of G, M^G denotes the submodule of M consisting of its G-invariant elements, and $\mathrm{End}_G(M)$ denotes the ring of G-endomorphisms of M.

§ 1. Main result

In this section, we first recall the definition of algebraic groups of Mumford-Tate type according to [9]. Then we state the main result of this paper, which determines certain algebraic groups of Mumford-Tate type.

1.1. Let K be a field of characteristic 0 contained in an algebraically closed field C, and V be a finite dimensional K-vector space. An algebraic subgroup G of GL_V defined over K is said to be of Mumford-Tate type, if there exists a homomorphism $h: \boldsymbol{G}_m\to G$ ($\boldsymbol{G}_m$: the multiplicative group) defined over C which satisfies the following conditions:

1. Put $V_C=V\otimes_K C$, and for an integer i, put $V_C(i)=\{v\in V_C \mid h(z)v=z^i v \text{ for all } z\in \boldsymbol{G}_m\}$. Then $V_C=V_C(0)\oplus V_C(1)$.
2. There is no proper normal algebraic subgroup N of G defined over K containing the image of h.

1.2. An example of algebraic groups of Mumford-Tate type is the Mumford-Tate group of an abelian variety over $\boldsymbol{C}$ (cf. [5]). Let A be an abelian variety over $\boldsymbol{C}$, and $V=H_1(A, \boldsymbol{Q})$ be the first homology group with coefficients in $\boldsymbol{Q}$. Let M be the Mumford-Tate group of A, which is a connected reductive algebraic subgroup of GL_V defined over $\boldsymbol{Q}$. Then M is of Mumford-Tate type, and the decomposition $V_C=V_C(0)\oplus V_C(1)$ is the Hodge decomposition.

1.3. Another example concerns the image of the representation of the absolute Galois group by the ℓ-adic Tate module of an abelian variety over a local field. Let ℓ be a rational prime, and K be a complete discrete valuation field of characteristic 0 having an algebraically closed residue field of characteristic ℓ. Let $\bar{K}$ be the algebraic closure of K, and let Gal $(\bar{K}/K)$ denote the Galois group of $\bar{K}$ over K. Let A be an abelian variety over K, and put $V_\ell = T_\ell(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, where $T_\ell(A)$ is the ℓ-adic Tate module of A. Then V_ℓ is a vector space of dimension $2\dim(A)$ over the field $\boldsymbol{Q}_\ell$ of ℓ-adic numbers. The natural action of $\mathrm{Gal}(\bar{K}/K)$ on V_ℓ induces a homomorphism ρ: $\mathrm{Gal}(\bar{K}/K) \rightarrow GL(V_\ell)$ whose image $\mathrm{Im}(\rho)$ is a compact ℓ-adic Lie subgroup of $GL(V_\ell)$. Let H be the Zariski closure of $\mathrm{Im}(\rho)$ in GL_{V_ℓ}. Then Sen proved that $\mathrm{Im}(\rho)$ is open in $H(\boldsymbol{Q}_\ell)$ with respect to the ℓ-adic topology, and that the connected component of 1 in H is of Mumford-Tate type. The decomposition $V_C = V_C(0) \oplus V_C(1)$ is the Hodge-Tate decomposition of V, where C is the completion of $\bar{K}$ ([8], Theorem 1).

1.4. We state the main result. Let G be a connected reductive algebraic subgroup of GL_V defined over K which is of Mumford-Tate type, and S be the connected component of 1 in $M \cap SL_V$. Then M is generated by S and the homothety subgroup $\boldsymbol{G}_m$ of GL_V. Assume that V is a simple $G(K)$-module over K. Let D be the division algebra $\mathrm{End}_{G(K)}(V)$ with center E, and let m and r be positive integers such that $m^2 = [D: E]$ and $r = [E: K]$. Let V_E be the E-vector space V.

Theorem 1. *In the above situation, assume the following conditions:*

(1) *There exists a non-degenerate alternating E-bilinear form ψ: $V_E \times V_E \rightarrow E$ which is $S(K)$-invariant.*

(2) *The integer* $\dim(V)/mr$ *is not divisible by* 4.

Then $\dim(V)/mr$ *must be an even integer, and there exists an algebraic subgroup S_E of GL_{V_E} defined over E such that S conicides with the scalar restriction $R_{E/K}(S_E)$. Moreover, only one of the following cases can happen:*

1. $m=1$, *and the pair (of an algebraic group and its representation space) (S_E, V_E) is E-isomorphic to $(Sp_{V_E,\psi}, V_E)$.*

2. $m>1$, $\dim_K((\overset{2}{\wedge} V)^S) = \binom{m+1}{2} r$, *and (S_E, V_E) is C-isomorphic to $(Sp(n), (C^{2n})^{\oplus m})$ for $n = \dim(V)/2mr$. Here $(Sp(n), C^{2n})$ is induced from the natural inclusion $Sp(n) \rightarrow GL(2n)$.*

3. $m>1$, $\dim_K((\overset{2}{\wedge} V)^S) = \binom{m}{2} r$, *and (S_E, V_E) is C-isomorphic to either $(SO(2n), (C^{2n})^{\oplus m})$ or $(r_b(SL(2b), (\overset{2}{\wedge} C^{2b})^{\oplus m})$ for the positive integer b satisfying $\binom{2b}{b} = 2n$. Here $(SO(2n), C^{2n})$ is induced from the natural inclusion $SO(2n) \rightarrow GL(2n)$, and r_b is the b-fold exterior power of the standard representation of $GL(2b)$.*

Remark. Let Ψ be the set of non-degenerate E-bilinear forms ψ: $V_E \times V_E \to E$, and let Φ be the set of non-degenerate K-bilinear forms ϕ: $V \times V \to K$ such that $\phi(ev, w) = \phi(v, ew)$ for any $e \in E$ and $v, w \in V$. Then by [3], 4.7, the map $\Psi \to \Phi$ given by $\psi \to \mathrm{Tr}_{E/K} \circ \phi$ ($\mathrm{Tr}_{E/K}$: the trace from E to K) is a bijection. By the uniqueness, ψ is alternating (resp. $S(K)$-invariant) if and only if $\mathrm{Tr}_{E/K} \circ \psi$ is alternating (resp. $S(K)$-invariant). Hence Condition (1) can be replaced by the following:

(1′) There exists a non-degenerate alternating K-bilinear form ϕ: $V \times V \to K$ which is $S(K)$-invariant and satisfies the equality $\phi(ev, w) = \phi(v, ew)$ for any $e \in E$ and $v, w \in V$.

§ 2. Applications

In this section, we give some corollaries of Theorem 1.

2.1. Let A be an abelian variety over $\boldsymbol{C}$, and put $V = H_1(A, \boldsymbol{Q})$. Let M be the Mumford-Tate group of A, and S be the connected component of 1 in $M \cap SL_V$. Then M is generated by S and $\boldsymbol{G}_m$, and Lefschetz' theorem implies that $\mathrm{End}_{M(K)}(V) = \mathrm{End}_{\boldsymbol{C}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Let A^* be the dual abelian variety of A, and put $V^* = H_1(A^*, \boldsymbol{Q})$. A polarization of A over $\boldsymbol{C}$ induces an identification of V with V^*, which together with the natural pairing $V \times V^* \to \boldsymbol{Q}$ induces a non-degenerate alternating $\boldsymbol{Q}$-bilinear form ϕ_θ on $V \times V$. Then S is contained in Sp_{V, ϕ_θ}.

Put $D = \mathrm{End}_{\boldsymbol{C}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, and let E be its center. Let m and r be positive integers such that $m^2 = [D: E]$ and $r = [E: \boldsymbol{Q}]$. As proved by Albert, E is either a totally real field or a CM-field, and in the former case, $\dim(A)$ is divisible by mr and D belongs to one of the following types ([6], p. 201):

1. $m = 1$, i.e., $D = E$.
2. $m = 2$, and $D \otimes_E \boldsymbol{R} \cong M_2(\boldsymbol{R})^r$.
3. $m = 2$, and $D \otimes_E \boldsymbol{R} \cong \boldsymbol{H}^r$, where $\boldsymbol{H}$ is the quaternion division algebra over $\boldsymbol{R}$.

2.2. As a corollary of Theorem 1, we have:

Corollary 1. *Let A be a simple abelian variety over $\boldsymbol{C}$ which satisfies the following conditions*:

(1) *E is a totally real field.*

(2) *$\dim(A)/mr$ is an odd integer. Put $n = \dim A/mr$.*

Then there exists an algebraic subgroup S_E of GL_{V_E} over E such that $S = R_{E/\boldsymbol{Q}}(S_E)$. If A is of type 1 *or* 2, *then (S_E, V_E) is $\boldsymbol{C}$-isomorphic to $(Sp(n), (\boldsymbol{C}^{2n})^{\oplus m})$. If A is of type* 3, *then (S_E, V_E) is $\boldsymbol{C}$-isomorphic to either $(SO(2n), (\boldsymbol{C}^{2n})^{\oplus m})$ or $(r_b(SL(2b), (\bigwedge^2 \boldsymbol{C}^{2b})^{\oplus m})$ for the positive integer b satisfying $\binom{2b}{b} = 2n$.*

Proof. By Condition (2), Condition (2) of Theorem 1 is satisfied. By Condition (1), $\phi_\theta(ev, w)=\phi_\theta(v, ew)$ for any $e \in E$ and $v, w \in V$. Then by Remark in 1.4, Condition (1) of Theorem 1 is satisfied. Therefore, by Theorem 1, there exists an algebraic subgroup S_E of GL_{V_E} over E satisfying $S=R_{E/Q}(S_E)$, and only one of Cases 1–3 can happen. The integer $\dim_Q((\overset{2}{\wedge} V)^S)$ is equal to the Picard number $P(A)$ of A, and

$$P(A)=\begin{cases} r & \text{if } A \text{ is of type 1 or 3,} \\ 3r & \text{if } A \text{ is of type 2,} \end{cases}$$

([6], p. 202). If A is of type 1, then $m=1$, so Case 1 happens. If A is of type 2, then Case 2 happens. If A is of type 3, then Case 3 happens. This completes the proof.

2.3. Let A be a simple abelian variety over C satisfying Conditions (1) and (2) of Corollary 1. If A is of type 1 or 2, then Tankeev and Ribet proved that the Hodge cycles on A^m (m: positive integers) are generated by those of degree 2, by using the invariant theory of the symplectic groups. Especially, the Hodge conjecture holds for A^m by Lefschetz' theorem ([10], Theorem 5.1, and [7], Theorems 0 and 1). If A is of type 3, one can check that the Hodge cycles on A^m are not generated by those of degree 2 by the invariant theory of the special orthogonal groups ([12]). The author does not know the validity of the Hodge conjecture in this case.

2.4. Let the notation be as in 1.3. Let A^* be the dual abelian variety of A over K, and put $V_\ell^*=T_\ell(A^*)\otimes_Z Q$. A polarization θ of A over K induces an identification of V_ℓ with V_ℓ^*, which, together with the Weil pairing $V_\ell\times V_\ell^*\to Q_\ell$ induces a non-degenerate alternating Q_ℓ-bilinear form ϕ_θ on $V_\ell\times V_\ell$. We may assume that H is connected by replacing K by its certain finite extension in $\bar{K}$. Let S be the connected component of 1 in $H\cap SL_{V_\ell}$. Then H is generated by S and G_m, and S is contained in Sp_{V_ℓ,ϕ_θ}. Assume that V_ℓ is a simple $\mathrm{Gal}(\bar{K}/K)$-module over Q_ℓ. Then one can see that H is reductive. Put $D=\mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(V_\ell)$, and let E be its center. Then there exists a unique involution ι_θ: $E\to E$ such that

$$\phi_\theta(ev, w)=\phi_\theta(v, \iota_\theta(e)w)$$

for any $e \in E$ and $v, w \in V_\ell$. Let m and r be positive integers such that $m^2=[D: E]$ and $r=[E: Q_\ell]$. Then as another corollary of Theorem 1, we have:

Corollary 2. *Let A be an abelian variety over K satisfying the following conditions*:

(1) *H is connected.*
(2) *V_ℓ is a simple* Gal $(\bar{K}/K)$*-module.*
(3) *ι_θ is the identity map on E.*
(4) $2\dim(A)/mr$ *is not divisible by* 4.

Then $2\dim(A)/mr$ *is an even integer. There exists an algebraic subgroup S_E of GL_{V_E} over E such that $S=R_{E/\mathbf{Q}_\ell}(S_E)$, and the pair (S_E, V_E) is C-isomorphic to one of the following:*

$$(Sp(n), (C^{2n})^{\oplus m}),\ (SO(2n), (C^{2n})^{\oplus m}),\ (r_b(SL(2b)), (\overset{2}{\wedge} C^{2b})^{\oplus m}).$$

Here $n=\dim A/mr$ and b is the positive integer satisfying $\binom{2b}{b}=2n$. If $m=1$, then $(S_E, V_E)\cong(Sp(n), E^{2n})$.

Remark. From the stable reduction theorem ([4], IX, Theorem 3.6) and Tate's result ([11], Proposition 4), Conditions (1) and (2) imply that A admits a model over the valuation ring of a certain finite extension of K in $\bar{K}$, whose special fibre has the property that the associated ℓ-divisible group is connected.

§ 3. Proof of Theorem 1

In this section, we give a proof of Theorem 1.

3.1. We recall Serre's result ([9], p. 182):

Theorem A. *Let $G\subset GL_V$ be a connected reductive algebraic group of Mumford-Tate type, and $S=[G, G]$ be its semi-simple part. Let r be the root system of S, and r_i $(i \in I)$ be all the irreducible components of r. Then r_i are all of classical type. If ω is the highest weight of an absolutely irreducible component of the inclusion representation $S\to GL_V$, then for any $i \in I$, the i-component of ω is either* 0 *or a minimal weight of r_i.*

The definition and the description of the minimal weights of irreducible root systems are given in [2]. Here we review the description of the minimal weights for the classical type according to [9], Appendix.

Theorem B. *Let r be an irreducible root system of classical type, and ι the representation associated with a minimal weight of r. Then a pair (r, ι) is necessarily one of the following:*

r is of type A_n $(n\geqq 1)$, and ι is equivalent to the i-fold $(1\leqq i\leqq n)$ exterior power of the standard representation, whose degree is $\binom{n+1}{i}$.

r is type B_n $(n\geqq 2)$, and ι is equivalent to the spin representation of degree 2^n,

r is of type C_n ($n \geqq 4$), and ι is equivalent to the standard representation of degree $2n$,

r is type D_n ($n \geqq 4$), and ι is equivalent to either the standard representation of degree $2n$, or one of the two half-spin representations of degree 2^{n-1}.

3.2. We recall a result of the invariant theory according to [2], Chap. 8, 13.1. Let K be a field of characteristic 0, and V be a K-vector space of dimension n. For a positive integer m, let W be the m-fold exterior power $\overset{m}{\wedge} V$ of V, on which $GL(V)$ acts as

$$g(v_1 \wedge \cdots \wedge v_m) = g(v_1) \wedge \cdots \wedge g(v_m) \ (g \in SL(V),\ v_1 \wedge \cdots \wedge v_m \in W).$$

Lemma C. *Assume that n is even and $m = n/2$. Then*

$$\dim_K ((\overset{2}{\otimes} W))^{SL(V)}) = 1,$$

and

$$\dim_K ((\overset{2}{\wedge} W)^{SL(V)}) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

3.3. Now we shall prove Theorem 1. Let the notation and the assumption be as in its statement. If S were $\{1\}$, then $G = \boldsymbol{G}_m$ and $\dim V = 1$, which contradicts Condition (1). Hence we have $S \neq \{1\}$. By Condition (1), the center of S is contained in $R_{E/K}(\boldsymbol{G}_{m_E}) \cap R_{E/K}(Sp_{E,\psi})$, which is a finite group. Hence S is semi-simple. Let Γ be the set of all K-isomorphisms of E into C. For each $\sigma \in \Gamma$, put $V_\sigma = V_E \otimes_{E,\sigma} C$, and let ϕ_σ: $R_{E/K}(GL_{V_E}) \to GL_{V_\sigma}$ be the natural projection. Then $\prod_{\sigma \in \Gamma} \phi_\sigma$: $R_{E/K}(GL_{V_E}) \to \prod_{\sigma \in \Gamma} \mathrm{GL}_{V_\sigma}$ is an isomorphism. Let S_σ be the image $\phi_\sigma(S)$. Then S_σ is a connected semi-simple algebraic group defined over $\sigma(E)$. Since S is connected, $S(K)$ is Zariski dense in S ([1], 18.3), hence $\phi_\sigma(S(K))$ is Zariski dense in S_σ. For an element x of $S(K) \subset R_{E/K}(GL_{V_E})$ (K), we have $\phi_\tau(x) = \alpha(\tau, \sigma)$ $(\phi_\sigma(x))$, where $\alpha(\tau, \sigma) = \tau \circ \sigma^{-1}$: $\sigma(E) \to \tau(E)$. Therefore, we have

$$\text{(i)} \qquad S_\tau = S_\sigma \otimes_{\sigma(E), \alpha(\tau,\sigma)} \tau(E) \qquad \text{for any } \tau, \sigma \in \Gamma.$$

Since $\prod_{\sigma \in \Gamma} \phi_\sigma$: $S \to \prod_{\sigma \in \Gamma} S_\sigma$ is injective, S_σ is non-trivial for any $\sigma \in \Gamma$ because $S \neq \{1\}$.

Fix an element σ of Γ, and let W_σ be a simple S_σ-module over C such that $W_\sigma^{\oplus m}$ is S_σ-isomorphic to V_σ. Let $\{S_i | i \in I\}$ be the set of all simple normal subgroups of S_σ over C. Then the homomorphism $\prod_{i \in I} S_i \to S_\sigma$ given by $(x_i)_{i \in I} \to \prod_{i \in I} x_i$ is an isogeny. We regard W_σ as a $\prod_{i \in I} S_i$-module via the above isogeny. Then there exists a simple S_i-module W_i

over C for each $i \in I$ such that W_σ is $\prod_{i \in I} S_i$-isomorphic to $\otimes_{i \in I} W_i$. Let $\rho_\sigma: S_\sigma \to GL_{V_\sigma}$ and $\rho_i: S_i \to GL_{W_i}$ be the natural representation. Then we have

$$\text{(ii)} \qquad \rho_\sigma|_{S_i} \cong \rho_i^{\oplus d}, \qquad \text{where } d = \frac{\dim(V_\sigma)}{\dim(W_i)}.$$

We shall show that $\dim(W_i)$ is even for any $i \in I$. Suppose, on the contrary, that there exists an element $i \in I$ such that $\dim W_i$ is odd. Then by Theorems A and B, there exist positive integers a and b ($2 \leqq a$ and $1 \leqq b \leqq a-1$) such that

$$(\rho_i(S_i), W_i) \cong (r_b(SL(a)), \overset{b}{\wedge} C^a)$$

where $r_b: GL(a) \to GL\left(\binom{a}{b}\right)$ is the b-fold exterior power of the standard representation of $GL(a)$. Especially, $\rho_i(S_i)$ contains $\{\zeta^b \cdot I_{W_i} | \zeta \in \mu_a\}$, where I_{W_i} is the unit matrix in GL_{W_i} and $\mu_a = \{\zeta: a\text{-th roots of } 1 \text{ in } C\}$. Then by (ii), S_σ contains $\{\zeta^b \cdot I_{V_\sigma} | \zeta \in \mu_a\}$. On the other hand, by Condition (1), S_σ is contained in $Sp_{V_\sigma, \psi_\sigma}$ ($\psi_\sigma = \psi \otimes_{E,\sigma} C$), whose center is $\{\pm I_{V_\sigma}\}$. Hence a is even and $b = a/2$, so

$$\dim(W_i) = \binom{2b}{b} = 2\binom{2b-1}{b}$$

is even, which is a contradiction. Therefore, $\dim(W_i)$ is even for any $i \in I$.

By Condition (2), $\prod_{i \in I} \dim(W_i) = \dim(W_\sigma) = \dim(V)/mr$ is not divisible by 4, and as proved above, $\dim(W_i)$ is even for any $i \in I$. Therefore, S_σ is absolutely simple and $\dim(V)/mr$ is even. Put $n = \dim(V)/2mr$, and let b be the positive integer satisfying $\binom{2b}{b} = 2n$. Then b must be either 1 or an even integer because for any odd integer $c > 1$,

$$\binom{2c}{c} = 4\binom{2c-3}{c-3}\frac{2c-3}{c-2}$$

is divisible by 4. Therefore, from Theorems A and B and the above argument, only one of the following cases can happen:

[1] $(S_\sigma, W_\sigma) \cong (r_b(SL(2b)), \overset{b}{\wedge} C^{2b})$ for $b > 1$,
[2] $(S_\sigma, W_\sigma) \cong (Sp(n), C^{2n})$,
[3] $(S_\sigma, W_\sigma) \cong (SO(2n), C^{2n})$.

Hence S_σ are defined over the prime field up to isomorphisms. Then by (i), S_σ ($\sigma \in \Gamma$) are isomorphic to each other. If there were two elements

$\sigma, \tau \in \Gamma$ satisfying $D_\sigma = D_\tau$, then $\phi_\sigma|_{D_\sigma}$ is equivalent to $\phi_\sigma|_{D_\tau}$, which contradicts to the equality $\mathrm{End}_S(V_C) = M_m(C)^r = \oplus_{\sigma\in\Gamma} \mathrm{End}_{S_\sigma}(V_\sigma)$. Hence $D_\sigma \neq D_\tau$ for any $\sigma, \tau \in \Gamma$ such that $\sigma \neq \tau$, so D_σ $(\sigma \in \Gamma)$ generate S. Therefore, the homomorphism $\prod_{\sigma\in\Gamma} \phi_\sigma : S \to \prod_{\sigma\in\Gamma} S_\sigma$ is an isomorphism, so there exists an algebraic subgroup S_E of GL_{V_E} defined over E such that $R_{E/K}(S_E) = S$. Then we have $S_\sigma = S_E \otimes_{E,\sigma} C$ for any $\sigma \in \Gamma$.

Assume that $m=1$. Then Condition (1) implies that $(\overset{2}{\wedge} V)^S \neq \{0\}$. Since $\overset{2}{\wedge} V_C \cong \oplus_{\sigma\in\Gamma}(\overset{2}{\wedge} V_\sigma) \oplus \oplus_{\sigma\neq\tau}(V_\sigma \otimes V_\tau)$, we have $(\overset{2}{\wedge} V_C)^S \cong \oplus_{\sigma\in\Gamma}(\overset{2}{\wedge} V_\sigma)^{S_\sigma}$. From Lemma C, $(\overset{2}{\wedge} V_\sigma)^{S_\sigma} = \{0\}$ for Case 1, and evidently $(\overset{2}{\wedge} V_\sigma)^{S_\sigma} = \{0\}$ for Case 3. Therefore, $(S_\sigma, V_\sigma) \cong (Sp(n), C^{2n})$ for any $\sigma \in \Gamma$, hence we have $S_E = Sp_{V_E,\psi}$.

Assume that $m>1$. Then

$$\overset{2}{\wedge} V_C \cong \bigoplus_{\sigma\in\Gamma}\left(m \cdot \overset{2}{\wedge} W_\sigma \oplus \binom{m}{2} \cdot \overset{2}{\oplus} W_\sigma\right) \oplus \bigoplus_{\sigma\neq\tau} m^2 \cdot (W_\sigma \otimes W_\tau),$$

so we have

$$(\overset{2}{\wedge} V_C)^S = \bigoplus_{\sigma\in\Gamma} (m \cdot (\overset{2}{\wedge} W_\sigma)^{S_\sigma} \oplus \binom{m}{2} \cdot (\overset{2}{\otimes} W_\sigma)^{S_\sigma}).$$

If $S_\sigma \cong Sp(n)$, then $\dim_K((\overset{2}{\wedge} V)^S) = \binom{m+1}{2} r$. If $S_\sigma \cong r_b(SL(2b))$ or $SO(2n)$, then $\dim_K((\overset{2}{\wedge} V)^S) = \binom{m}{2} r$ by Lemma C. This completes the proof.

Added in proof. The group D_σ $(\sigma \in \Gamma)$ in 3.3 is the normal subgroup of S such that $\phi_\sigma|_{D_\sigma}$ is an isogeny $D_\sigma \to S_\sigma$.

References

[1] A. Borel, Linear Algebraic Groups, Benjamin, New York (1969).
[2] N. Bourbaki, Groupes et Algèbres de Lie, Eléments de Math., **38**, Hermann, Paris (1975).
[3] P. Deligne, Hodge Cycles on Abelian Varieties, Lecture Notes in Math., **900** (1982), 9–100.
[4] A. Grothendieck et al., Groupes de monodromie en gèometrie algébrique, S.G.A. 71, Lecture Notes in Math., **288** (1972).
[5] D. Mumford, Families of abelian varieties, Proc. Symp. in Pure Math., vol. **9** (Algebraic Groups), A.M.S. (1966), 347–351.
[6] D. Mumford, Abelian Varieties, Oxford University Press (1974).
[7] K. Ribet, Hodge classes on certain abelian varieties, Amer. J. Math., **105** (1983), 523–538.
[8] S. Sen, Lie algebras of Galois groups arising from Hodge-Tate modules, Ann. of Math., **97** (1973), 160–170.
[9] J. P. Serre, Groupes algébriques associés aux modules Hodge-Tate, Journeés

de Géometrie Algébrique, Astérisque, **65**, Soc. Math. de France (1979), 155–187.
[10] S. G. Tankeev, On algebraic cycles on surfaces and abelian varieties, Math. USSR Izv., **18** (1982), 349–380.
[11] J. Tate, p-Divisible groups, Proc. Conf. Local Fields, Springer-Verlag (1967), (1967), 158–183.
[12] H. Weyl, The Classical Groups, Princeton University Press, Princeton (1946).

Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo 113, Japan