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One-Parameter Family of Linear Representations of Artin's Braid Groups

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§ 1. Introduction

The central theme of this note is linear representations of Artin's braid groups. There is a classical series of such representations called the Burau representations, which is defined by means of an embedding of the braid group in the automorphism group of a free group. Recently V. Jones [9] and several authors studied the one-parameter family of linear representations of the braid groups induced from representations of the Hecke algebra of the symmetric group. These representations turn out to be a generalization of the Burau representation. In this note we propose another generalization. Namely we shall consider the integral of the form

$$F(x_1, \cdots, x_n)=\int \prod_{1\leq i<j\leq n+p} (x_i-x_j)^{-\mu_{ij}} dx_{n+1}\wedge \cdots \wedge dx_{n+p}$$

and we study the monodromy of the above multivalued functions. This permits us to define a one-parameter family of linear representations of the braid groups. As a special case $p=1$, it is shown that this representation is equivalent to the Burau representation. The study of this direction is motivated by the work of K. Aomoto [2], in which he computed the system of differential equations satisfied by the above multivalued functions. This note is organized in the following way. Section 2 is concerned with the definition and basic properties of the Artin's braid groups. In Section 3 we shall explain the principle of the vanishing of cohomology of a "generic" local system and by using this formulation, Section 4 focuses our new one-parameter family of linear representations of the braid groups. In Section 5 we discuss the image and the kernel of the Burau representation for special values. Our principal tool is the theorem of Picard type for hypergeometric functions proved by Deligne-Mostow [5] and Terada [16].

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§ 2. Review of basic facts on Artin's braid groups

Let $X_n=\{(z_1, \cdots, z_n) \in C^n; z_i \neq z_j \text{ if } i\neq j\}$. The symmetric group S_n acts on X_n by $(z_1, \cdots, z_n)\cdot g=(z_{g(1)}, \cdots, z_{g(n)})$, $g \in S_n$. We denote by Y_n the quotient space X_n/S_n. The *Artin's braid group* is by definition the fundamental group of Y_n. We shall denote it by B_n. The fundamental group of X_n, which is denoted by P_n, is called the *pure braid group*. We have an exact sequence

(2.1) $$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1.$$

Let us denote by $p: X_n \rightarrow Y_n$ the natural projection. We choose a base point $x_0=(0, 1, \cdots, n-1) \in X_n$. Any element in $\pi_1(Y_n, p(x_0))$ is represented by a path $f: (I, \{0\}) \rightarrow (X_n, x_0)$. Let b_j, $1\leq j\leq n-1$, be the element of $\pi_1(Y_n, p(x_0))$ corresponding to the path in X_n given by

$$f(t)=(0, \cdots, j-2, f_{j-1}(t), f_j(t), j+1, \cdots, n-1)$$

where

$$f_{j-1}(t)=(j+t-1)-\sqrt{-1}\sqrt{t-t^2}, \quad f_j(t)=(j-t)+\sqrt{-1}\sqrt{t-t^2}.$$

Let A_{ij}, $1\leq i<j\leq n$, denote the element of P_n defined by

(2.2) $$A_{ij}=b_{j-1}b_{j-2}\cdots b_{i+1}b_i^2b_{i+1}^{-1}\cdots b_{j-1}^{-1}.$$

Let us recall the following fundamental theorems.

(2.3) **Theorem** (Artin [1]). *The braid group B_n admits a presentation with generators $b_1, \cdots, b_{n-1}$ and defining relations*

$$b_ib_j=b_jb_i \qquad \text{if } |i-j|\geq 2$$
$$b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1}, \qquad 1\leq i\leq n-2.$$

The pure braid group P_n is generated by A_{ij}, $1\leq i<j\leq n$.

(2.4) **Theorem** (Chow, see [3]). *If $n\geq 3$, the center of B_n is the infinite cyclic group generated by*

$$(b_1b_2\cdots b_{n-1})^n=(A_{12})(A_{12}A_{13})\cdots(A_{1n}A_{2n}\cdots A_{n-1,n}).$$

We have a faithful representation of B_n as an automorphism group of a free group $F_n=\langle x_1, \cdots, x_n\rangle$ ([3] Corollary 1.8.3). The representation is induced by a mapping h from B_n to $\mathrm{Aut}(F_n)$ defined by:

(2.5) $$\begin{aligned} h(b_i): x_i &\mapsto x_ix_{i+1}x_i^{-1} \\ x_{i+1} &\mapsto x_i \\ x_j &\mapsto x_j \qquad \text{if } j\neq i, i+1. \end{aligned}$$

The pure braid group P_n is characterized as the subgroup of $\mathrm{Aut}(F_n)$ consisting of the elements $g \in \mathrm{Aut}(F_n)$ satisfying:

$$(2.6) \qquad \begin{aligned} &g(x_i) \sim x_i \qquad \text{(conjugate)}, \ 1 \leq i \leq n \\ &g(x_1 \cdots x_n) = x_1 \cdots x_n. \end{aligned}$$

From this point of view the *profinite braid groups* defined by Y. Ihara [7] may be considered as an arithmetic analogy of the pure braid group.

There is a well-known family of linear representations called the *Burau representations* which are induced from the above representation h. The Burau representation $\beta_n: B_n \to GL_{n-1}(\boldsymbol{Z}[t, t^{-1}])$ is given by

$$(2.7) \qquad b_1 \mapsto \begin{bmatrix} -t & 1 & & & & \\ 0 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

$$b_i \mapsto \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 0 & 0 & & \\ & & t & -t & 1 & & \\ & & 0 & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \leftarrow i\text{-th row } (1 < i < n)$$

(2.8) **Definition.** For $n \in \boldsymbol{N}$, $n \neq 0$ and $q \in \boldsymbol{C}$, we denote by $H_n(q)$ the algebra over $\boldsymbol{C}$ generated by $(t_i)_{i=1,\cdots,n-1}$ with relations:

$$\begin{aligned} &(t_i+1)(t_i-q)=0, \qquad 1 \leq i \leq n-1 \\ &t_i t_j = t_j t_i, \qquad |i-j| \geq 2 \\ &t_{i+1} t_i t_{i+1} = t_i t_{i+1} t_i, \qquad 1 \leq i \leq n-2. \end{aligned}$$

The algebra $H_n(q)$ is called the *Hecke algebra* (or *Iwahori algebra*) of the symmetric group S_n. The original definition for any Coxeter system is due to [8].

The irreducible representations of $H_n(q)$ are parametrized by Young tableau for a generic g. Since we have a natural homomorphism from the group ring $\boldsymbol{C}[B_n]$ to $H_n(q)$, we obtain a family of linear representations of B_n with one-parameter. V. Jones and several authors studied these representations systematically and obtained new polynomial invariants of links (see [4], [9]). These linear representations of B_n contain the Burau representation in the following way.

(2.9) **Theorem** (Jones [9]). *The representation of B_n corresponding to the Young tableau of type* $(n-1, 1)$ *is the tensor product of the Burau representation and the one dimensional parity representation.*

§ 3. Cohomology of rank one local systems

Let us preserve the notations of Section 2. For $n, p>0$, we consider the natural projection $\pi: X_{n+p} \to X_n$, which has a structure of a fibration. We see that the induced homomorphism $\pi_*: \pi_1(X_{n+p}) \to \pi_1(X_n)$ admits a natural section, which we denote by s. Let $\tau: \pi_1(X_{n+p}) \to \boldsymbol{C}^*$ be a homomorphism which is trivial on $s(\pi_1(X_n))$. Let L be the local system over X_{n+p} associated with the representation τ. Let us recall that $\pi_1(X_{n+p}) = P_{n+p}$ is generated by the elements A_{ij}, $1 \leq i < j \leq n+p$ (see (2.3)). We choose $\mu_{ij} \in \boldsymbol{C}$ such that $\exp 2\pi\sqrt{-1}\,\mu_{ij} = \tau(A_{ij})$. Let us put $I_k = \{(i,j);\ 1 \leq i < j \leq k\}$. Since τ is trivial on $s(\pi_1(X_n))$, we have $\mu_{ij} \in \boldsymbol{Z}$ if $(i,j) \in I_n$.

We shall assume the following condition on μ_{ij}:

(3.1) For any subset S of $I_{n+p} - I_n$, $\sum_{(i,j)\in S} \mu_{ij}$ is not an integer.

(3.2) **Proposition.** *Under the hypothesis* (3.1), *we have*

(i) $\boldsymbol{R}^j \pi_* L = 0$ *if* $j \neq p$.

(ii) *The local system* $\boldsymbol{R}^p \pi_* L$ *has rank* $(n+p-2)!/(n-2)!$.

Proof. Let Z denote a fiber of π. The first assertion is a special case of the vanishing of cohomology of a local system discussed in [12]. Let V be a smooth compactification of Z such that $D = V - Z$ is a divisor with normal crossings. Let $i: Z \to V$ be the inclusion map. By means of the hypothesis (3.1), we have

$$i_* L|_Z = i_! L|_Z \tag{3.3}$$

where $i_! L|_Z$ is an extension of $L|_Z$ by zero. We have the following isomorphisms.

$$\begin{aligned} &\boldsymbol{H}^j(V, i_* L|_Z) \cong \boldsymbol{H}^j(Z, L|_Z) \\ &\boldsymbol{H}^j(V, i_! L|_Z) \cong \boldsymbol{H}^j_C(Z, L|_Z). \end{aligned} \tag{3.4}$$

Here the right hand side stands for the cohomology with compact support. By the Poincaré duality we have an isomorphism

$$H^j_C(Z, L|_Z) \cong H_{2p-j}(Z, L|_Z). \tag{3.5}$$

Since Z has a homotopy type of a CW complex of dimension p, we have $H^j(Z, L|_Z) = 0$ if $j > p$. This completes the proof of (i). By an elementary computation we see that the Euler characteristic of Z is

$$(-1)^p(n+p-2)!/(n-2)!.$$

Hence the assertion (ii) follows immediately. For more details and extensive treatments of the vanishing theorem of this type see [12]. (cf. [2], [11]).

The rest of this section is devoted to the study of Hodge structure on $\boldsymbol{PR}^p\pi_*L$ in the case $p=1$. We put $\mu_i=\mu_{i,n+1}$, $1\leq i\leq n$, $\mu_{n+1}=2-\sum_{i=1}^n \mu_i$. The following Lemma is due to Deligne-Mostow [5].

(3.6) **Lemma.** *If* $0<\mu_i<1$, $1\leq i\leq n+1$, *then the projective local system* $\boldsymbol{PR}^1\pi_*L$ *admits a global Hermitian form of signature* $(1, n-2)$. *This determines a linear representation of the pure braid group* P_n *in* $PU(1, n-2)$.

Proof. Let Z be the fiber over $(a_1, \cdots, a_n)\in X_n$. Any section u of $\Omega^1(L|_Z)$ can be written in the form $u=z^{-\mu_i}\cdot e\cdot f\cdot dz$ locally around a_i, where e is a horizontal section of $L|_Z$ and f is a holomorphic function on a punctured neighbourhood of a_i. If f is meromorphic at a_i, we define $v_i(u)$ by $v_{a_i}(f)-\mu$. Let $H^{1,0}(Z, L|_Z)$ be the space of the forms of the first kind, i.e., the meromorphic forms u on $\boldsymbol{P}^1$ satisfying $v_i(u)+\mu_i\geq 0$ for $1\leq i\leq n+1$, where we put $a_{n+1}=\infty$. Let $H^{0,1}(Z, L|_Z)$ be the complex conjugate of $H^{1,0}(Z, \bar{L}|_Z)$, where $\bar{L}|_Z$ denotes the complex conjugate local system of $L|_Z$. We have the Hodge decomposition:

$$H^1(Z, L|_Z)=H^{1,0}\oplus H^{0,1} \tag{3.7}$$

with $\dim H^{1,0}=1$, $\dim H^{0,1}=n-2$. There exists a Hermitian form on $H^1(Z, L|_Z)$ which is positive definite on $H^{1,0}$ and is negative definite on $H^{0,1}$. Such a form is unique up to positive constant. Hence we obtain a horizontal Hermitian form of signature $(1, n-2)$ on $\boldsymbol{PR}^1\pi_*L$, which proves Lemma.

(3.8) **Notations.** By means of the above argument, we obtain a multivalued holomorphic map $w\colon X_n\to D_{n-2}$, where D_{n-2} denotes the $(n-2)$-dimensional complex ball. If $\mu_1=\cdots=\mu_n=\mu$, this map descends to a multivalued holomorphic map from Y_n to D_{n-2}, which we denote by the same letter w. The corresponding linear representation of B_n in $PGL(n-1, \boldsymbol{C})$ is denoted by $\beta_n\langle\mu\rangle$.

§ 4. Examples of one-parameter families of linear representations of braid groups

The local system $\boldsymbol{R}^p\pi_*L$ over X_n defined in Section 3 determines a linear representation of the pure braid group

$$(4.1) \qquad \rho\colon P_n \longrightarrow \mathrm{Aut}\, H^p(Z, L|_Z)$$

where Z denotes a fiber of π. If we suppose moreover that

$$(4.2) \qquad \mu_{1j} = \cdots = \mu_{nj}, \qquad n+1 \leq j \leq n+p,$$

then the local system $R^p\pi_* L$ is invariant under the operation of S_n on X_n. Hence this defines a local system over $Y_n = X_n/S_n$. In this situation the representation ρ gives a representation of the braid group B_n in $\mathrm{Aut}\, H^p(Z, L|_Z)$, which we denote by the same letter ρ.

Let us consider the case $p=1$. We put $\mu_{1,n+1} = \cdots = \mu_{n,n+1} = \mu$, $\alpha = \exp 2\pi\sqrt{-1}(-\mu)$. We observe that the dual representation $\rho^*\colon B_n \to \mathrm{Aut}\, H^1(Z, L|_Z)$ is obtained from the Burau representation

$$\beta_n\colon B_n \longrightarrow GL_{n-1}(\boldsymbol{Z}[t, t^{-1}]) \qquad \text{(see Section 2)}$$

by putting $t = \alpha$.

We consider the case $p > 1$. Let us suppose that $\mu_{ij} = \mu$ for any $(i, j) \in I_{n+p} - I_n$, with some $\mu \in \boldsymbol{C}$ satisfying the condition (3.2). By considering $\exp 2\pi\sqrt{-1}(-\mu)$ as a parameter, we get a one-parameter family of linear representations:

$$(4.3) \qquad \beta_{n,p}\colon B_n \longrightarrow GL_N(\boldsymbol{Z}[t, t^{-1}]), \qquad N = (n+p-2)!/(n-2)!.$$

Since $\beta_n = \beta_{n,1}$, these representations may be considerd as a generalization of the Burau representations.

For $p > 1$, the representation $\beta_{n,p}$ is not irreducible. In fact the following method permits us to decompose the local system $R^p\pi_* L$. By our hypothesis on μ_{ij}, the symmetric group S_p acts naturally on $H^p(Z, L|_Z)$. Let $\Gamma = (d_1, \cdots, d_k)$ be a partition of p, i.e., $d_1 \geq d_2 \geq \cdots \geq d_k \geq 0, \sum_{i=1}^k d_i = p$. We denote by e_Γ an idempotent element of the group ring $\boldsymbol{C}[S_p]$ corresponding to the Young tableau of type Γ (see [18]). Let V_Γ be the left ideal of $\boldsymbol{C}[S_p]$ generated by e_Γ. We have

$$(4.4) \qquad \mathrm{Hom}_{S_p}(V_\Gamma, H^p(Z, L|_Z)) \cong e_\Gamma \cdot H^p(Z, L|_Z).$$

We denote the right hand side by U_Γ. We observe that the action of S_p on $H^p(Z, L|_Z)$ is commutative with the operation of B_n. Hence we obtain the following Proposition by using standard arguments in representation theory.

(4.5) **Proposition.** *We have a direct sum decomposition*

$$H^p(Z, L|_Z) = \bigoplus_{\Gamma\colon \text{partition of } p} [V_\Gamma \otimes U_\Gamma]$$

and for any Γ, U_Γ and $V_\Gamma \otimes U_\Gamma$ are invariant subspaces with respect to the operation of the braid group B_n.

By means of the above Proposition, the representation $\beta_{n,p}$ has a factor corresponding to Γ, which we denote by $\beta_{n,p,\Gamma}$.

(4.5) **Example.** Let us consider the case $n=3$, $p=2$, $\Gamma=(1, 1)$. The representation $\beta_{3,2,\Gamma}$: $B_3 \to GL_3(Z[t, t^{-1}])$ is given by

$$b_1 \mapsto \begin{bmatrix} -t^2 & 0 & 1 \\ 0 & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b_2 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 0 \\ t^2 & 0 & -t^2 \end{bmatrix}.$$

This representation cannot be obtained from the representations of B_3 induced from the natural homomorphism $C[B_3] \to H_3(q)$.

§ 5. Image and kernel of $\beta_n\langle\mu\rangle$

The object of this section is to study the linear representations $\beta_n\langle\mu\rangle$ for special values of μ (see Section 3 for notations). We shall prove the following Theorem.

(5.1) **Theorem.** *Let μ be a real number such that*

(i) $n^{-1} < \mu < 2n^{-1}$

(ii) $(1-2\mu)^{-1} \in Z \cup \{\infty\}$, $((n-1)\mu - 1)^{-1} \in Z \cup \{\infty\}$.

We put $\kappa = (1-2\mu)^{-1}$, $\kappa_\infty = ((n-1)\mu-1)^{-1}$. Then the kernel of the linear representation $\beta_n\langle\mu\rangle$: $B_n \to PU(1, n-2)$, $n \geq 3$, defined in (3.8) is normally generated by the following elements:

$$b_1^{2\kappa}, \quad (b_1 \cdots b_{n-2})^{(n-1)\kappa_\infty}, \quad (b_1 \cdots b_{n-1})^n.$$

The complete list of μ satisfying the hypothesis of Theorem is given in the following table:

(5.2) Table

the case $n=3$

(i) $\mu = 2^{-1}(1-\kappa^{-1})$, $\kappa \geq 3$, $\kappa \in N$

(ii) $\mu = 2^{-1}$, $\kappa = \infty$

(iii) $\mu = 2^{-1}(1+\kappa_\infty^{-1})$, $\kappa_\infty \geq 3$, $\kappa \in N$

the case $n \geq 4$

	$n=4$					$n=5$			$n=6$	$n=7$
μ	1/3	3/8	2/5	5/12	4/9	1/4	1/3	3/8	1/4	1/4
κ	3	4	5	6	9	2	3	4	2	2
κ_∞	∞	8	5	4	3	∞	3	2	4	2

We devide the proof of Theorem (5.1) into several steps. First we review briefly the theory of M. Kato [10] on branched coverings (see also [19]).

Let us start with a pair (G, M), where M is a connected complex manifold and G is a properly discontinuous group of holomorphic transformations of M. We obtain the orbit space X, which is an irreducible normal analytic space. Let $b: X \to \boldsymbol{N}$ be a function defined by $b(x) = \sharp G_z$ for $x \in X$, where $z \in M$, $G.z = x$ and G_z denote the isotropy subgroup of G at z. We write $G \backslash M = (X, b)$. Conversely, given a pair (X, b), where X is an irreducible normal analytic space and $b: X \to \boldsymbol{N}$ is a function, we shall say that (X, b) is *uniformizable* if and only if there exists (G, M) such that $(X, b) = G \backslash M$. The pair (G, M) is called a *uniformization* of (X, b). We call (X, b) an *orbifold* if (X, b) is locally uniformizable. Let (X, b) be an orbifold. We put $\Sigma X = \{x \in X; b(x) \geq 2\}$ and $X_0 = X - \Sigma X$. Let $\{D_j\}_{j \in J}$ be the set of irreducible hypersurfaces is ΣX. The function b attains a constant value b_j on the regular part of D_j. Let μ_j denote a normal loop around D_j. Let N be the smallest normal subgroup of $\pi_1(X_0)$ containing $\{\mu_j^{b_j}\}_{j \in J}$. We shall only state the correspondence between coverings and groups. The following Proposition is a part of Theorem 1 of [10].

(5.3) **Proposition** (M. Kato [10]). *Let (X, b) be a uniformizable orbifold. There is a one-to-one correspondence between the normal subgroups of $\pi_1(X_0)$ containing N and the covering maps of orbifolds*: $(X', b') \to (X, b)$.

The above correspondence may be illustrated in the following diagram:

$$\left\{\begin{matrix} K\text{: normal subgroup of} \\ \pi_1(X_0) \text{ containing } N \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} X_0' \longrightarrow X_0\text{: covering} \\ \text{corresponding to } K \end{matrix}\right\}$$

$$\longrightarrow \left\{\begin{matrix} X'\text{: Fox completion of} \\ X_0' \text{ over } X \text{ ([6])} \end{matrix}\right\}$$

In particular, if we start with $K = N$, we obtain the universal uniformization $(\pi_1(X_0)/N, M)$.

(5.4) **Notation.** We put $M_n = \{(z_1, \cdots, z_{n+1}) \in \boldsymbol{P}^1 \times \cdots \times \boldsymbol{P}^1; z_i \neq z_j \text{ if } i \neq j\}$. Let $PGL(2, \boldsymbol{C})$ act diagonally on M_n and we put $Q_n = M_n / PGL(2, \boldsymbol{C})$.

We see that there is a natural inclusion $X_n \to M_n$. We denote by $PGL(2, \boldsymbol{C})_\infty$ the isotropy subgroup of $PGL(2, \boldsymbol{C})$ at ∞. We have $Q_n = X_n / PGL(2, \boldsymbol{C})_\infty$, where $PGL(2, \boldsymbol{C})_\infty$ acts diagonally on X_n. We have an isomorphism:

(5.5) $$P_n/\mathrm{Cent}\,(B_n)\cong\pi_1(Q_n).$$

Since the natural projection $\pi_1(X_n)\to\pi_1(Q_n)$ has a section, we shall consider $\pi_1(Q_n)$ as a subgroup of P_n.

We have defined in (3.8) a multivalued holomorphic map $w\colon X_n\to D_{n-2}$. This map descends to a multivalued holomorphic map from Q_n to D_{n-2}, which we denote by the same letter w.

Following Deligne and Mostow [5], we consider the following *stable partial compactification* of Q_n (cf. [14]). Let us fix $(\mu_1, \cdots, \mu_{n+1})$ with $0<\mu_i<1$, $1\le i\le n+1$ and $\sum_{i=1}^{n+1}\mu_i=2$. We shall assume moreover the following *integer condition*:

(5.6) (INT) For any $i\neq j$ such that $\mu_i+\mu_j<1$, $(1-\mu_i-\mu_j)^{-1}$ is an integer.

Let S denote the set $\{1, \cdots, n+1\}$ and let $(\boldsymbol{P}^1)^S$ be the set of functions from S to $\boldsymbol{P}^1$. We consider M_n as a subset of $(\boldsymbol{P}^1)^S$ and we define M_n^{st} to be the set of functions $f\colon S\to\boldsymbol{P}^1$ such that for any $x\in\boldsymbol{P}^1$ with $f^{-1}(x)\neq\phi$ we have

(5.7) $$\sum_{f(s)=x,\, s\in S}\mu_s<1.$$

Let $PGL(2, \boldsymbol{C})$ act diagonally on M_n^{st}. We define Q_n^{st} to be the quotient space $M_n^{st}/PGL(2, \boldsymbol{C})$. The partial compactification Q_n^{st} has a natural structure of a complex manifold.

Let $\tilde{Q}_n\to Q_n$ be the covering corresponding to the kernel of the linear representation of P_n in $PU(1, n-2)$ defined in Section 3. Let Q_n^{st} be the Fox completion of $\tilde{Q}_n\to Q_n$ over Q_n^{st}. Let $\tilde{w}\colon\tilde{Q}_n\to D_{n-2}$ denote the lift of w on $\tilde{Q}_n$. The following Picard type theorem was proved by Deligne and Mostow [5], and independently by Terada [16].

(5.8) **Theorem** (Picard [15], Terada [16], Deligne-Mostow [5]). *If* $(\mu_1, \cdots, \mu_{n+1})$ *satisfies the condition* (INT) (*see* (5.6)), *then the corresponding map* $\tilde{w}\colon\tilde{Q}_n\to D_{n-2}$ *extends to a homeomorphism* $\overline{w}\colon Q_n^{st}\to D_{n-2}$ *which is equivariant with the action of* $\pi_1(Q_n)$.

Under the condition (INT), we consider the orbifold (Q_n^{st}, b) defined in the following way. Let K_{ij} be the divisor of Q_n^{st} corresponding to the divisor $z_i=z_j$ in M_n^{st}. We define the value of b at a regular point of K_{ij} by $\kappa_{ij}=(1-\mu_i-\mu_j)^{-1}$. Let c_{ij} denote a normal loop around K_{ij}. By using the notion of orbifolds, the theorem of Picard type (5.8) may be interpretated in the following way.

(5.9) **Theorem.** *Under the condition* (INT), *the universal uniformization of the orbifold* (Q_n^{st}, b) *defined above is isomorphic to* $(\pi_1(Q_n)/N, D_{n-2})$, *where N is the smallest normal subgroup of $\pi_1(Q_n)$ containing $c_{ij}^{\kappa_{ij}}$ for $i\neq j$ with $\mu_i+\mu_j<1$.*

Proof. Let $(\pi_1(Q_n)/N', M)$ be the universal uniformization of the orbifold (Q_n^{st}, b). We observe that the order of $(\beta_n\langle\mu\rangle)(c_{ij})$ is equal to κ_{ij}. Hence by means of Theorem (5.8) and Proposition (5.3), we obtain an unramified covering $M\to D_{n-2}$. Since D_{n-2} is simply connected we have $M=D_{n-2}$, $N=N'$. This completes the proof of Theorem (5.9).

We shall now complete the proof of Theorem (5.1). By the hypothesis on μ, the condition (INT) is satisfied. We have a natural action of S_n on Q_n^{st}. Hence $(\pi_1(Q_n)/N, D_{n-2})$ may also be considered as the universal uniformization of $(Q_n^{st}/S_n, b)$ with certain b. By using the correspondence in (5.3), we conclude that Ker $\beta_n\langle\mu\rangle$ is equal to N. The proof of Theorem (5.1) is reduced to show the following conjugate relations in $B_n/\text{Cent}(B_n)$:

$$(5.10)\qquad \begin{aligned} &c_{ij}\sim b_{ij}^2, && 1\leq i<j\leq n,\\ &c_{i,n+1}\sim(b_1b_2\cdots b_{n-2})^{n-1}, && 1\leq i\leq n.\end{aligned}$$

These relations are checked by using relations of type:

$$(5.11)\quad (b_1b_2\cdots b_{n-1})^n=(b_1b_2\cdots b_{n-1})^{n-1}(b_{n-1}b_{n-2}\cdots b_2b_1^2b_2\cdots b_{n-1}).$$

This completes the proof of Theorem (5.1).

(5.12) **Remarks.** (i) In the case $n=3$, the image of representations of P_3 in $PU(1, 1)$ associated with $(\mu_1, \cdots, \mu_4)$ satisfying the condition (INT) is the Schwarz triangle group. In general, we obtain a series of complex reflection groups operating on the complex ball D_{n-2} as the image of P_n in $PU(1, n-2)$. The image of B_n in $PU(1, n-2)$ in the case listed in the table (5.2) is arithmetic if $n\geq 4$.

(ii) Let us consider the case $n=4$ and $\mu=2/5$. Let Γ denote the image of P_4 in $PU(1, 2)$ by this representation. In this case the commutator subgroup $[\Gamma, \Gamma]$ acts freely on D_2 and the quotient variety $D_2/[\Gamma, \Gamma]$ is one of the Hirzebruch's examples of surfaces of general type with $c_1^2=3c_2$ (see [19]).

(iii) We now consider the case $\mu=1/2$. This representation gives an isomorphism $B_3/\text{Cent}(B_3)\cong PSL(2, Z)$ in the case $n=3$. This fact was used to show that the Burau representation is faithful if $n=3$ (see [3] Theorem 3. 15).

The faithfulness of the Burau representation is an open problem in the case $n\geq 4$. It is known by Varchenko that the image of the representation of B_n in $SL(n-1, Z)$ obtained in this way is equal to $\text{Sp}(n-1, Z)$

(see [17]). This representation may be lifted to a representation of B_n in the Steinberg group $St(n-1, Z)$. The proof of this fact is based on Lemma 9.4 of [13].

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