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Study of Four-Dimensional Gorenstein ASL Domains I

(Integral posets arising from triangulations of a 2-sphere)

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To the memory of late Professor Akira Hattori

Introduction

The concept of ASL (algebras with straightening laws) plays an important role in the interaction between commutative algebra, theory of partially ordered sets and simplicial complexes. Among many examples of ASL, the most interesting ones are those which are integral domains. We are interested in the condition for a poset (a finite partially ordered set) to satisfy when there is an ASL domain R on it. This question was answered if dim $R=2$ or dim $R=3$ and R is Gorenstein in [8] and [3]. We call such posets integral posets.

In this article and subsequent works, we will study Gorenstein ASL domains of dimension 4, or equivalently, homogeneous coordinate rings of three-dimensional Fano varieties. As the first step for this study, we will determine the integral posets defined by attaching a minimal element T to a poset H' which is a triangulation of a 2-sphere, where we will say that a poset H is a triangulation of a toplogical space X if the underlying topological space of the simplicial complex $\Delta(H)$ associated to H is homeomorphic to X. Our classification is described in (2.2) and there are 18 such posets up to isomorphisms.

The concept of Cohen-Macaulay posets is defined by Cohen-Macaulay property of discrete ASL on the posets. As is shown in [8], the analogous definition of Gorenstein posets would be too strong a property for the study of Gorenstein ASL and the concept of a weakly Gorenstein poset was introduced in [8]. This property turns out to fit very well with the axiom of ASL via the theory of canonical modules.

This article is divided into 4 sections. In Section 1, we will give a characterization of a poset H' which is a triangulation of a 2-sphere by the aid of weakly Gorenstein property for H'.

In Section 2, we will review a graph-theoretical method to describe a

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poset which is a triangulation of a 2-sphere from [4] and we will give examples of ASL domains for each of 18 integral posets arising from triangulations of a 2-sphere.

As a preliminary work for the proof of our main theorem, we study ASL's on the poset C_n which is defined by attaching a minimal element to a circle with $2n$ vertices in Section 3. In Section 4, we will prove that the posets arising from triangulations of a 2-sphere are not integral except for the 18 ones listed in (2.2).

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§ 1. Preliminaries and a characterization of posets which define triangulations of a 2-sphere

First we recall some fundamental facts about ASL (algebras with straightening laws). Our definition of an ASL is the same as the one in [1].

(1.1) Let k be a fixed field, $R=\bigoplus_{n\geqq 0} R_n$ be a Noetherian graded algebra with $R_0=k$ and H be a poset (a finite partially ordered set) with an embedding into R. In this article, we only treat homogeneous ASL. So, we always assume that $H\subset R_1$. A product of elements of H is called a "monomial" and if $\alpha_1, \alpha_2, \cdots, \alpha_n \in H$, $\alpha_1\leqq\alpha_2\leqq\cdots\leqq\alpha_n$, the product $\alpha_1\alpha_2\cdots\alpha_n$ is called a "standard monomial". We say that R is an ASL on H over k if

(ASL-1) R is a free k-module admitting the set of standard monomials as a free basis over k.

(ASL-2) If α, β are incomparable elements of H and

$$\alpha\beta=\sum_{i=1}^{s} c_i\gamma_i\delta_i \qquad (c_i \in k, i=1, \cdots, s)$$

is the unique expression for $\alpha\beta \in R$ as a linear combination of standard monomials with $\gamma_i\leqq\delta_i$ for every i, then $\gamma_i\leqq\alpha, \beta$ for every i. Note that if there is no element $\gamma \in H$ with $\gamma\leqq\alpha$ and $\gamma\leqq\beta$, then $\alpha\beta=0$.

(1.2) We need some terminology on posets.

A totally ordered subset of H is called a chain of H. We define the length of a chain so that the cardinality of the set is the length plus one. We say that H is of rank r if every maximal chain in H has length r. If H is of rank r and $\alpha \in H$, we define the rank of α, denoted $r(\alpha)$ by the length of a maximal chain descending from α. For example, if α is minimal, $r(\alpha)=0$ and if α is maximal, then $r(\alpha)=r$. We denote by $c_i(H)$ the number of chains of H of length i. For example, $c_0(H)=\sharp(H)$, the cardinality of H. The number of maximal chains of H, $c_r(H)$ is called the degree of H.

$\Delta(H)$ is the simplicial complex consisting of chains of H. We denote by $|\Delta(H)|$ the geometric realization of $\Delta(H)$. We have $\dim|\Delta(H)|=r(H)=\dim R-1$.

A subset $I\subset H$ is a poset ideal of H if $\alpha\in I$ and $\beta\leqq\alpha$ implies $\beta\in I$. The fact that R/IR is an ASL over $H-I$ is very important throughout this paper. As a particular example of a poset ideal, we define $I(\alpha)=\{\beta\in H|\beta\ngeqq\alpha\}$ and $H_\alpha=H-I(\alpha)=\{\beta\in H|\beta\geqq\alpha\}$ for $\alpha\in H$. When there is no fear of confusion, we denote the ideal IR of R by the same letter as the poset ideal I.

If α, β are incomparable elements of H, we write $\alpha\not\sim\beta$.

(1.3) (i) $P(H,\lambda)=P(R,\lambda)=\sum_{n\geqq 0}\dim_k R_n\cdot\lambda^n$ is the Poincaré series of R or of H. In our case $P(H,\lambda)$ is expressed as $P(H,\lambda)=\sum_{i\geqq 0}c_{i-1}(H)\cdot\lambda^i(1-\lambda)^{-i}$ (cf. [7], II, 1.4), where we put $c_{-1}=1$.

(ii) If $P(H,\lambda)=g(\lambda)/f(\lambda)$, where $f(\lambda)$, $g(\lambda)\in \boldsymbol{Z}[\lambda]$, we define

$$a_P(H)=\deg g(\lambda)-\deg f(\lambda).$$

If R is a Cohen-Macaulay ring, then $a_P(H)$ coincides with the invariant $a(R)$ defined in [2], Chapter 2. (cf. [8], (1.2))

(iii) We say that H is numerically Gorenstein if $P(H,\lambda)$ satisfies the functional equation

$$P(H,\lambda^{-1})=(-1)^{r+1}\cdot\lambda^{-a}\cdot P(H,\lambda),$$

where $r=r(H)$ and $a=a_P(H)$. We say that H is weakly Gorenstein if there exists a Gorenstein ASL on H and that H is strongly Gorenstein if the discrete ASL $k[H]=k[X_\alpha|\alpha\in H]/(X_\alpha X_\beta|\alpha\not\sim\beta)$ is Gorenstein. Weakly Gorenstein posets of rank 1 are completely classified in [8], (4.10). Recall that strongly Gorenstein implies weakly Gorenstein and weakly Gorenstein implies numerically Gorenstein ([6]).

If R is a Gorenstein ASL domain on H and $\dim R=4$, then $r(H)=3$ and H has unique minimal element, say, T and R/TR is a Gorenstein ASL on $H'=H-\{T\}$. Thus the study of weakly Gorenstein posets of rank 2 serves as a primary step for the study of Gorenstein ASL domains of dimension 4.

(1.3) Let H be a numerically Gorenstein poset of rank two.

(i) If $a_P(H)=0$, then $c_2(H)=2(c_0(H)-2)$ and $c_1(H)=3(c_0(H)-2)$.

(ii) If $a_P(H)=-1$, then $c_2(H)=c_0(H)-1$ and $c_1(H)=2(c_0(H)-1)$.

(iii) If $a_P(H)=-2$, then $c_0(H)=4$ and $c_2(H)=2$.

(iv) If $a_P(H)=-3$, then $c_0(H)=3$. (H itself is a chain of length 2.) Case (i) (resp. Case (ii)) occurs only if $c_0(H)\geqq 6$ (resp. $c_0(H)\geqq 5$).

We will investigate the relation between the two invariants $a(R)$ and $a_P(H)$ for the case R is not necessarily Cohen-Macaulay.

The invariant $a(R)$ of a graded ring over k with $\dim R=d$ is defined by

$$a(R)=\max\{n|(H_m^d(R))_n\neq 0\}=-\min\{n|(K_R)_n\neq 0\},$$

where m is the unique graded maximal ideal and K_R is the canonical module of R.
From (1.3), (ii), it is obvious that $a_P(R)\leqq 0$ for every ASL R. We can prove the same inequality for the invariant $a(R)$.

Proposition (1.4). *If R is a homogeneous ASL on a ranked poset H with $r(H)=r$, then*
(i) $a(R)\leqq 0$.
(ii) *If H has unique minimal element, then $a(R)<0$ and $a_P(H)<0$.*

To prove this, we need a lemma.

Lemma (1.5). *Let R be a graded ring over k and $x\in R_m$ such that* $\dim R/xR=\dim R-1$. *Then $a(R/xR)\geqq a(R)+m$.*

Proof. If x is a non-zero divisor on R, we have an exact sequence of local cohomology modules,

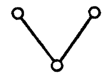
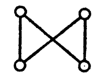
$$H_m^{d-1}(R)\xrightarrow{f}H_m^{d-1}(R/xR)\longrightarrow H_m^d(R)(-m)\xrightarrow{x}H_m^d(R)\longrightarrow H_m^d(R)\longrightarrow 0$$
$$(d=\dim R).$$

We can deduce our inequality from this exact sequence. In this case, we have equality in (1.5) if and only if $\max\{n|(\mathrm{Im}(f))_n\neq 0\}\leqq a(R)+m$.

In general case, put $R'=R/U_R(0)$, where $U_R(0)$ is the intersection of primary components q of (0) with $\dim R/q=\dim R$. Then x is a non-zero divisor on R' and $a(R/xR)\geqq a(R'/xR')\geqq a(R')+m=a(R)+m$.

Proof of (1.4). (i) Put $f_i=\sum_{\alpha\in H,\,\mathrm{rank}(\alpha)=i}\alpha$ $(i=0,\cdots,r)$. Then $(f_0,\cdots,f_r)$ forms a system of parameters of R and by (1.5), it suffices to show that $a(R/(f_0,\cdots,f_r))\leqq r+1$. But by the aid of (ASL-2), it is easy to see that every element of $R/(f_0,\cdots,f_r)$ can be written in a linear combination of monomials of the type $\alpha_1\cdot\alpha_2\cdots\alpha_s$ with $r(\alpha_1)<r(\alpha_2)<\cdots<r(\alpha_s)$. This shows that $a(R/(f_0,\cdots,f_r))\leqq r+1$ and so $a(R)\leqq 0$.

(ii) This is obvious as R/TR is again an ASL on $H-\{T\}$ and we have $0\geqq a(R/TR)\geqq a(R)+1$, $0\geqq a_P(H-\{T\})=a_P(H)+1$.

Remark (1.6). If H is the disjoint union of  and  and if R is an ASL on H, then $a_P(H)=-1$ and $a(R)=0$. So, $a_P(H)=a(R)$ does not hold in general.

We can characterize the posets with $|\Delta(H)|\cong S^2$ by the aid of weakly Gorenstein porperty of H.

Proposition (1.7). *Let H be a poset of* rank 2. *Then H gives a triangulation of a 2-sphere if and only if H satisfies the following conditions;*

(i) *H is weakly Gorenstein with $a(H)=0$.*

(ii) *For every $\beta\in H$, $r(\beta)=1$, $\sharp\{\alpha\in H\,|\,\alpha<\beta\}\geqq 2$.*

(iii) *For every $\gamma\in H$, $r(\gamma)=2$, $\sharp\{\beta\in H\,|\,r(\beta)=1$ and $\beta<\gamma\}\geqq 2$.*

Remark (1.8). It is easy to show from [5] or [7] that H is strongly Gorenstein with $r(H)=2$ and $a_P(H)=0$ if and only if $|\Delta(H)|\cong S^2$. On the other hand, we can construct examples of weakly Gorenstein posets with $a_P(H)=0$ which do not satisfy the condition (ii) or (iii) of (1.7). In fact, if H' is a weakly Gorenstein poset with $a_P(H')=0$, $r(H')=1$ and if we put $H=H'\cup\{\delta,\varepsilon\}$, where δ and ε are "bigger" or "smaller" than any element of H' and $\delta\not\sim\varepsilon$, then H is weakly Gorenstein with $r(H)=2$ and $a_P(H)=0$. As another example, the 4-th Veronese subring R of a polynomial ring of 4 variables can be shown to be an ASL over the following poset H (cf. (3.8)). As R is Gorenstein with $a(R)=-1$, $H'=H-\{T\}$ is weakly Gorenstein with $a_P(H')=0$.

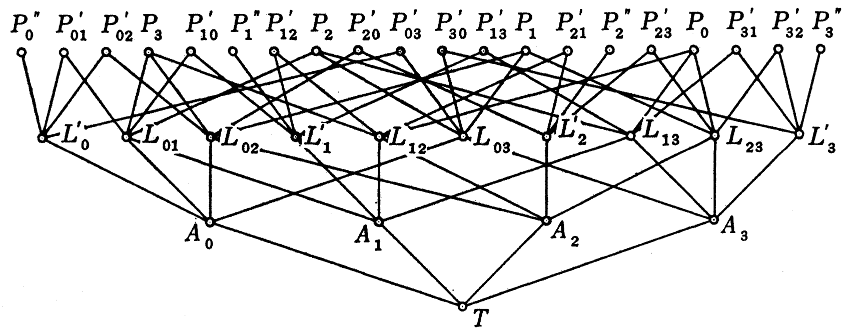


Fig. 1

To prove (1.7), we need some more properties of ASL.

Proposition (1.9). *Let R be an ASL on a poset H with $r(H)\geqq 2$.*

(i) *If H has unique minimal element T and $H'=H-\{T\}$ is not connected, then R does not satisfy the condition (S_2).*

(ii) *If R satisfies (S_2) and if γ is a maximal element of H, then the* *poset* $H^{\gamma}=\{\alpha \in H \mid \alpha<\gamma\}$ *is connected.*

(iii) *If R is Gorenstein, then for every* $\beta \in H$, $r(\beta)=1$, $\#\{\alpha \in H \mid \alpha<\beta\} \leqq 2$. (*Here, the assumption* $r(H)\geqq 2$ *is not necessary.*)

Proof. (i) Assume that $H-\{T\}=H_1 \cup H_2$, where the elements of H_1 and H_2 are not comparable at all. Then as in [3], Proposition A, we may assume $\phi\cdot\psi=0$ for every $\phi \in H_1$ and $\psi \in H_2$ after a suitable fundamental transformation (cf. [3], § 4). Also, if $\phi, \psi \in H_1$ (resp. H_2), $\phi \not\sim \psi$, $[\phi\psi]$ does not contain the monomial T^2. (Where $[\phi\psi]$ denotes the set of standard monomials appearing with non-zero coefficient in the right-hand side of (ASL-2) for $\phi\cdot\psi$.) Thus if we put $I_i=RH_i$ $(i=1, 2)$, then $R/(I_1+I_2)\cong k[T]$ and $I_1\cdot I_2=0$. So, putting $p=I_1+I_2$, $\dim R_p=r\geqq 2$ and depth $R_p=1$.

(ii) Assume that $H^{\gamma}=I \cup J$, where there are no relations between elements of I and J. As γ is a maximal element of H, the poset ideal $I(\gamma)$ generates a prime ideal p of R with ht $(p)=r$. If $\delta \in I(\gamma)$ and if $\delta \not\sim \gamma$, then by (ASL-2) applied to $\delta\cdot\gamma$, $\delta \in H^{\gamma}\cdot R_p$. Thus the maximal ideal of R_p is generated by $I\cdot R_p$ and $J\cdot R_p$ with $(I\cdot R_p)\cdot(J\cdot R_p)=(0)$. So, depth $R_p=1$ and R does not satisfy (S_2).

(iii) The ideal $I(\beta)$ of R is an intersection of prime ideals as $R/I(\beta)$ is reduced being an ASL. If S is the complement of the union of minimal prime ideals of $I(\beta)$, then $\beta+I(\beta)$ is included in S as β is a non-zero divisor of $R/I(\beta)$. By (ASL-2), $I(\beta)\cdot S^{-1}R=(\alpha_1, \cdots, \alpha_s)\cdot S^{-1}R$, where $\{\alpha_1, \cdots, \alpha_s\} =\{\alpha \in H \mid \alpha<\beta\}$. So, if we put $q_j=\{\alpha \in H \mid \alpha<\beta, \alpha\neq\alpha_j\}$ and if p is a minimal prime ideal of $I(\beta)$, $R_p/q_j\cdot R_p$ is a regular local ring of dimension 1 whose maximal ideal is generated by α_j. As $\alpha_i\alpha_j=0$ $(i\neq j)$ being minimal elements of H, $(R_p)^{\wedge}$ is isomorphic to $k(p)[[X_1, \cdots, X_s]]/(X_iX_j \mid i\neq j)$. As R_p is Gorenstein, $s\leqq 2$.

Lemma (1.10). *Let H be a poset with* $r(H)\geqq 2$, $a_P(H)=0$ *and R be a Gorenstein ASL on H. If* α *is a minimal element of H, then*

(i) $\#\{\beta \in H \mid r(\beta)=1$ *and* $\beta>\alpha\}\geqq 2$.

(ii) *If H satisfies the condition* (ii) *of* (1.7) *and if* H_α *is a Cohen-Macaulay poset or* $r(H)=2$, *then* H_α (*resp.* $H'_\alpha=H_\alpha-\{\alpha\}$) *is a weakly Gorenstein poset with* $a_P(H_\alpha)=-1$ (*resp.* $a_P(H'_\alpha)=0$).

Proof. Put $I=I(\alpha)$. By (1.4) (ii), H has at least two minimal elements.

(i) If is β the unique element with $r(\beta)=1$, $\beta>\alpha$, $\{\alpha, \beta\}$ is a regular sequence on R/I and as $R/(I, \alpha, \beta)$ is again an ASL, $a(R/I)\leqq -2$ by (1.4) and (1.5). On the other hand, $K_{(R/I)}\cong[0\colon I]_R$ contains α, which implies $a(R/I)\geqq -1$. Contradiction!

(ii) By our assumption, $K_{(R/I)}\cong[0\colon I]$ is generated by α, which implies $K_{(R/I)}\cong(R/I)(-1)$. As R/I is an ASL on H_α, R/I is Gorenstein if H_α is a Cohen-Macaulay poset. As $R/I \cong K_{(R/I)}(1)$ and a canonical module always satisfies (S_2), $H_\alpha - \{\alpha\}$ is connected by (1.9) (i), which implies that H_α is Cohen-Macaulay if $r(H)=2$.

Proof of (1.7). First, we will prove that for every element $\delta \in H$, the link $Lk_\delta(\Delta)$ of $\Delta = \Delta(H)$ is a cycle. If α is a minimal element of H, then $H'_\alpha = H_\alpha - \{\alpha\}$ is weakly Gorenstein with $a_P(H'_\alpha)=0$ by (1.10) (ii). By [8], (4.10) and by the condition (iii) of (1.7), $H'_\alpha \cong C'_n$ for some $n \geqq 2$, where C'_n is the poset given by the following diagram.

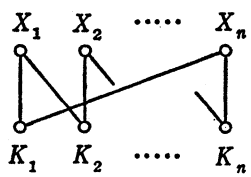


Fig. 2

Thus $Lk_\alpha(\Delta)$ is a circle with $2n$ vertices. If $\beta \in H$, $r(\beta)=1$, by the argument as above and by (1.9) (iii) and (1.10) (i), $\sharp\{\gamma \in H \mid \gamma > \beta\} = \sharp\{\alpha \in H \mid \alpha < \beta\} = 2$. Hence $Lk_\beta(\Delta)$ is a circle with 4 vertices. Now, put $\{\gamma_1, \cdots, \gamma_n\} = \{\gamma \in H \mid r(\gamma)=2\}$ and put $d_i = \sharp\{\beta \in H \mid \beta < \gamma_i, r(\beta)=1\}$ and $e_i = \sharp\{\alpha \in H \mid \alpha < \gamma_i, r(\alpha)=0\}$. Then as each $\beta \in H$, $r(\beta)=1$ dominates exactly two elements of rank 0, $d_i \leqq e_i \leqq 2d_i$ and $d_i = e_i$ if and only if the poset $\{\delta \in H \mid \delta < \gamma_i\}$, which is connected by (1.9) (ii), forms a cycle. Now, as H is weakly Gorenstein with $a(H)=0$, $c_2(H)=2(c_0-2)=4\cdot\sharp\{\beta \in H \mid r(\beta)=1\}$ and $c_1(H) = 3(c_0-2)$. This implies $\sum_{i=1}^n d_i = c_0 - 2 = \sum_{i=1}^n e_i$ and we have $e_i = d_i$ for every i. So, for every maximal element γ of H, $Lk_\gamma(\Delta)$ is again a circle. Thus $|\Delta|$ is a manifold of dimension 2 without boundary with $\chi(|\Delta|)=2$ and so $|\Delta| \cong S^2$. Thus we have proved the "if" part of (1.7) and "only if" part also follows from the argument above as $|\Delta(H)| \cong S^2$ implies that H is strongly Gorenstein with $a(H)=0$.

§ 2. Integral posets arising from the triangulations of S^2

In this section, let H' be a poset with $|\Delta(H')| \cong S^2$ and $H = H' \cup \{T\}$, where T is the unique minimal element of H. We say that H is an integral poset if there exists an ASL on H which is an integral domain. By abuse of language, we say sometimes H' is integral or not integral when H is integral or not integral, although there cannot exist an ASL on H' which is an integral domain as H' has more than one minimal elements.

The purpose of this section is to classify all the integral posets arising in this way. As to describe H' is a big job, we start with a simpler way to describe such H' introduced in [4].

(2.1) For a poset H' with $|\Delta(H')| \cong S^2$, we introduce a graph $\Gamma = \Gamma(H')$ as follows;

(i) The vertex set $v(\Gamma)$ of Γ is the set of minimal elements of H'.

(ii) The set $e(\Gamma)$ of edges of Γ is the set elements of H' of rank 1. As each element β of H with $r(\beta)=1$ dominatesa exactly two minimal elements α, α' of H, the "edge" β connects the vertices α and α'. In this graph, the maximal elements of H correspond to the regions determined by edges of $\Gamma(H')$. As H' defines a triangulation of S^2, $\Gamma(H')$ is a plane graph. Note that there may be two or more edges connecting given two vertices.

Conversely, if Γ is a 2-connected (that is, for every $v \in v(\Gamma)$, $\Gamma - \{v\}$ is connected) plane graph, we can construct a poset $H' = H'(\Gamma)$ putting the elements of H' of rank 0, 1, 2 respectably as vertices, edges, regions determined by the edges of Γ, respectively and defining the order of elements of H' by incidence relations.

Now, we can state our theorem.

Theorem (2.2). *Let H' be a poset of rank two with $|\Delta(H')| \cong S^2$ and $H = H' \cup \{T\}$, where T is the unique minimal element of H. Then there exists an ASL domain on H if and only if $\Gamma(H')$ is isomorphic to one of the following* 18 *graphs.*

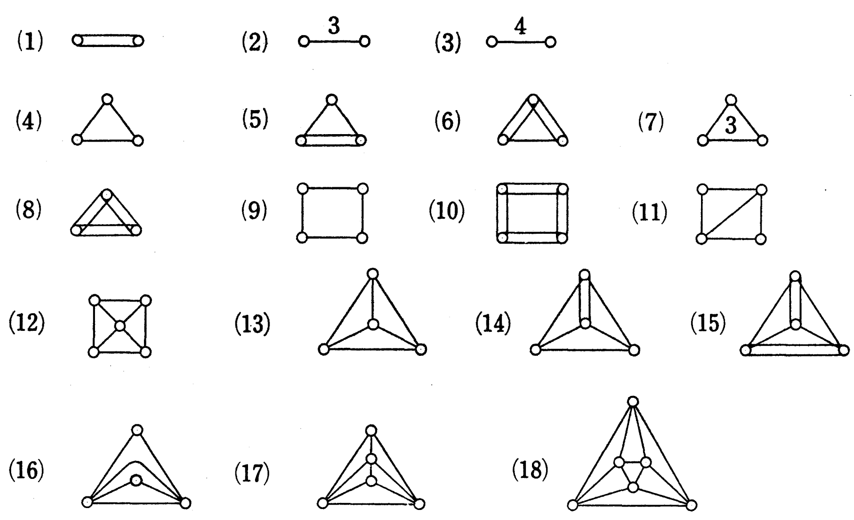


Fig. 3

Where $\alpha \overset{n}{\circ\!-\!\circ} \alpha'$ indicates that there are n edges of Γ connecting the two vertices α and α' or, there exist n elements of rank 1 in $H'=H'(\Gamma)$ which dominate α and α'.

In the rest of this section, we construct examples of ASL domains on the 18 posets defined above and in the following sections we will show that the poset H' with $|\Delta(H')|\cong S^2$ and which are not isomorphic to either one of the 18 posets are not integral.

Notation (2.3). Hereafter, we will denote the minimal elements of H' by $A, B, \cdots, F$, the elements of rank 1 of H' by $J, K, \cdots, Q$ or $J_0, J_1, \cdots$ and maximal elements of H' by $U, V, \cdots, Z$ or $U_0, U_1, \cdots$, and so on. When we indicate the order relations in H', we indicate only the order relations between the elements of rank 0 and rank 1 and rank 1 and rank 2. Always, T is the unique minimal element of the poset.

(2.4) (i) If $H'=H''\cup\{A, B\}$, where A and B are smaller than any element of H'' and $A \not\sim B$ and if there exists an ASL domain R' on the poset $H''\cup\{T\}$, we can construct an ASL domain on H by putting $R=R'[A, B]/(AB-Tf)$, where f is a sutitable linear form. In this manner the posets (1), (2) and (3) of (2.2) are integral (integral posets of rank 2 are completely classified in [3]).

(ii) If $H'=H''\cup\{X, Y\}$, where X and Y are bigger than any element of H'', $X \not\sim Y$ and if $H''\cup\{T\}$ is integral, we can show that H is integral in the similar manner as in (i). In this manner the posets (1), (4), (9) of (2.2) are integral.

(2.5) If R' (resp. R'') is an ASL on a poset H' (resp. H''), then the Segre product $R' \sharp R''$ is an ASL on the poset $H'\times H''$ ([9], § 10). If we define the posets H_2 and C_n ($n\geqq 2$) as follows, then H_2, C_2, C_3 and C_4 are integral posets (cf. [3]) and $H_2\times C_2$ (resp. $H_2\times C_3$, $H_2\times C_4$) is isomorphic to the poset H coming from (16) (resp. (17), (18)) of (2.2). Thus the posets (16), (17), (18) of (2.2) are integral.

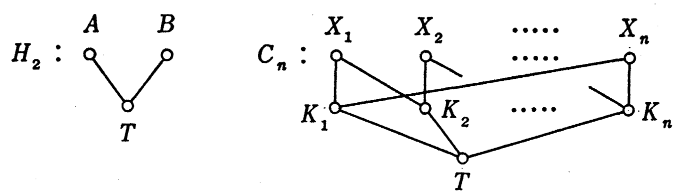


Fig. 4

(2.6) Let R' be an ASL on a poset H_0 with $r(H_0)=r$ and with unique minimal element T, x be a variable over R' and put $u=x^{-1}$. We define a subring R of $R'[x, u]$ by $R=R'[u, \alpha xT | \alpha \in H_0]$. Then R is an ASL on a new poset $H=H_0 \cup H'_0 \cup \{u\}$ with $r(H)=r+1$, where $H'_0=\{\alpha' | \alpha \in H_0\}$ and the order relations in H are defined as

(i) $\alpha'<\beta'$ if and only if $\alpha<\beta$, $\alpha<\alpha'$, $\alpha<\beta'$ if and only if $\alpha<\beta$.

(ii) u is "bigger" than any element of H_0 and not comparable to any element of H'_0.

Taking an ASL domain R' on $H_0=C_2$ (resp. C_3, C_4) indicated above, we can prove that the poset H defined by (5) (resp. (13), (12)) is integral.

(2.7) (Toric ASL domains) We can construct an ASL domain generated by monomials of $k[x, x^{-1}, y, y^{-1}, z, z^{-1}, T]$ by defining a mapping $t: H' \to \mathbf{Z}^3$ which satisfy the following conditions;

(i) If $\{\alpha, \beta, \gamma\}$ is a chain of length 2 of H', then $\{t(\alpha), t(\beta), t(\gamma)\}$ generates $\mathbf{Z}^3$ as $\mathbf{Z}$-module.

(ii) If $\delta, \tau \in H'$, $\delta \not\sim \tau$, one of the following cases occur.

(a) $t(\delta)=-t(\tau)$, (b) $t(\delta)+t(\tau) \in t(H')$, (c) $t(\delta)+t(\tau)=t(\delta')+t(\tau')$ where $\delta'<\delta, \tau$ and $\delta' \leqq \tau'$.

(iii) $\{t(\sigma) | \sigma \in \Delta(H')\}$ defines a triangulation of a polyhedron in $\mathbf{R}^3$ which contains the origin in its interior.

Given such t, we define $i: H \to S=k[x, x^{-1}, y, y^{-1}, z, z^{-1}, T]$ by $i(\alpha)=x^a y^b z^c T$ if $t(\alpha)=(a, b, c)$ and $i(T)=T$. Then the subring R of S generated by $\{i(\alpha) | \alpha \in H\}$ is an ASL domain on H. In fact, if $f=x^a y^b z^c T^n \in R$, then the line connecting the point (a, b, c) and the origin in $\mathbf{R}^3$ intersects unique $t(\sigma)$ ($\sigma \in \Delta(H)$). (Here, $t(\sigma)$ is the open simplex.) By (i), (a, b, c) can be expressed as a sum of elements of $t(\sigma)$ in a unique way, say, $(a, b, c)=t(\alpha_1)+\cdots+t(\alpha_m)$. As $f \in R$, $m \leqq n$ and $f=\alpha_1 \cdots \alpha_m T^{n-m}$. Thus R satisfies (ASL-1) and the condition (ii) implies (ASL-2).

Now, we will give an example of a toric ASL domain for each of the remaining seven posets in (2.2).

(6) $H'=\{A, B, C, J_1, J_2, K, L_1, L_2, X, Y, U, W\}$ (the order relation in H is indicated by the following figure),

$$t(A)=(1, 0, 0), \qquad t(B)=(0, 1, -1),$$

$$t(C)=(0, -1, 1), \qquad t(J_1)=(1, 1, 0),$$

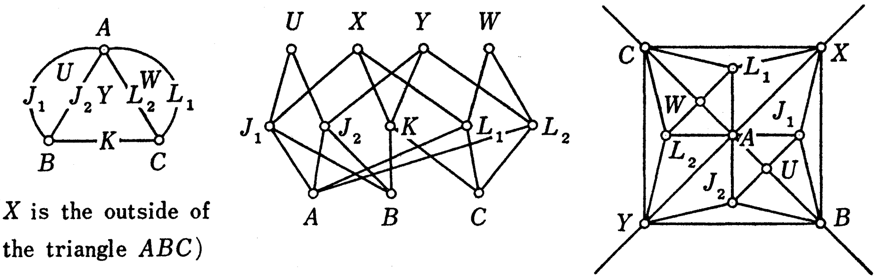
$$t(J_2)=(0, 0, -1), \qquad t(K)=(-1, 0, -1),$$

$$t(L_1)=(1, 0, 1), \qquad t(L_2)=(0, -1, 0),$$

$$t(x)=(1, 1, 1), \qquad t(Y)=(-1, -1, -1),$$

$$t(U)=(1, 1, -1), \qquad t(W)=(1, -1, 1).$$

To illustrate the situation, let us figure the graph, the poset and the triangulation of S^2 for this first example.



(X is the outside of the triangle ABC)

(K is at the infinity)

Fig. 5

(7) $H'=\{A, B, C, J, K_0, K_1, K_2, L, X, Y, U_1, U_2\}$: $J>A, B$; $K_0, K_1, K_2>B, C$; $L>C, A$; $X>J, K_1, L$; $Y>J, K_2, L$; $U_1>K_0, K_1$; $U_2>K_0, K_2$.

$$
\begin{aligned}
&t(A)=(1, 0, 0), && t(B)=(0, 1, 0),\\
&t(C)=(0, -1, 0), && t(J)=(1, 1, 0),\\
&t(K_0)=(-1, 0, 0), && t(K_1)=(0, -1, -1),\\
&t(K_2)=(0, 1, 1), && t(L)=(1, -1, 0),\\
&t(X)=(1, -1, -1), && t(Y)=(1, 1, 1),\\
&t(U_1)=(-1, -1, -1), && t(U_2)=(-1, 1, 1).
\end{aligned}
$$

(8) $H'=\{A, B, C, J_1, J_2, K_1, K_2, L_1, L_2, X, Y, U, V, W\}$; $J_1, J_2>A, B$; $K_1, K_2>B, C$; $L_1, L_2>C, A$; $X>J_1, K_1, L_1$; $Y>J_2, K_2, L_2$; $U>J_1, J_2$; $V>K_1, K_2$; $W>L_1, L_2$.

$$
\begin{aligned}
&t(A)=(1, 0, 0), && t(B)=(-1, 0, 0),\\
&t(C)=(0, 1, 1), && t(J_1)=(0, 0, -1),\\
&t(J_2)=(0, -1, 0), && t(K_1)=(0, 1, 0),\\
&t(K_2)=(0, 0, 1), && t(L_1)=(1, 1, 0),\\
&t(L_2)=(1, 0, 1), && t(X)=(1, 1, -1),\\
&t(Y)=(1, -1, 1), && t(U)=(-1, -1, -1),\\
&t(V)=(-1, 1, 1), && t(W)=(1, 1, 1).
\end{aligned}
$$

(10) $H'=\{A, B, C, D, J_1, J_2, K_1, K_2, L_1, L_2, M_1, M_2, X, Y, U, V, Z, W\}$; $J_1, J_2>A, B$; $K_1, K_2>B, C$; $L_1, L_2>C, D$; $M_1, M_2>D, A$; $X>J_1, K_1, L_1, M_1$; $Y>J_2, K_2, L_2, M_2$; $U>J_1, J_2$; $V>K_1, K_2$; $Z>L_1, L_2$; $W>M_1, M_2$.

$$
\begin{aligned}
&t(A)=(1, 0, 0), && t(B)=(1, 1, -1),\\
&t(C)=(-1, 0, 0), && t(D)=(-1, -1, 1),
\end{aligned}
$$

$t(J_1)=(1, 0, -1)$, $t(J_2)=(1, 1, 0)$,
$t(K_1)=(0, 0, -1)$, $t(K_2)=(0, 1, 0)$,
$t(L_1)=(-1, -1, 0)$, $t(L_2)=(-1, 0, 1)$,
$t(M_1)=(0, -1, 0)$, $t(M_2)=(0, 0, 1)$,
$t(X)=(0, -1, -1)$, $t(Y)=(0, 1, 1)$,
$t(U)=(2, 1, -1)$, $t(V)=(0, 1, -1)$,
$t(Z)=(-2, -1, 1)$, $t(W)=(0, -1, 1)$.

(11) $H'=\{A, B, C, D, J, K, L, M, N, X, Y, Z\}$; $J>A, B$; $K>B, C$; $L>C, D$; $M>D, A$; $N>B, D$; $X>J, K, L, M$; $Y>J, M, N$; $Z>K, L, N$.

$t(A)=(1, 0, 0)$, $t(B)=(0, 1, 0)$,
$t(C)=(-1, 0, 0)$, $t(D)=(0, -1, 0)$,
$t(J)=(1, 1, 0)$, $t(K)=(-1, 1, 0)$,
$t(L)=(-1, -1, 0)$, $t(M)=(1, -1, 0)$,
$t(N)=(0, 0, -1)$, $t(X)=(0, 0, 1)$,
$t(Y)=(1, 0, -1)$, $t(Z)=(-1, 0, -1)$.

(14) $H'=\{A, B, C, D, J, K, L, M_1, M_2, N, P, X, Y, Z, W, U\}$; $J>A, B$; $K>B, C$; $L>C, A$; $M_1, M_2>A, D$; $N>B, D$; $P>C, D$; $X>J, K, L$; $Y>J, M_1, N$; $Z>K, N, P$; $W>L, M_2, P$; $U>M_1, M_2$.

$t(A)=(1, 0, 0)$, $t(B)=(0, 0, -1)$,
$t(C)=(-1, -1, 0)$, $t(D)=(0, 1, 0)$,
$t(J)=(1, 0, -1)$, $t(K)=(-1, -1, -1)$,
$t(L)=(0, -1, 0)$, $t(M_1)=(1, 1, 0)$,
$t(M_2)=(0, 0, 1)$, $t(N)=(0, 1, -1)$,
$t(P)=(-1, 0, 0)$, $t(X)=(0, -1, -1)$,
$t(Y)=(1, 1, -1)$, $t(Z)=(-1, 0, -1)$,
$t(W)=(-1, -1, 1)$ $t(U)=(1, 1, 1)$.

(15) $H'=\{A, B, C, D, J, K_1, K_2, L, M_1, M_2, N, P, X, Y, Z, W, U, V\}$; $J>A, B$; $K_1, K_2>B, C$; $L>C, A$; $M_1, M_2>A, D$; $N>B, D$; $P>C, D$; $X>J, K_1, L$; $Y>J, M_1, N$; $Z>K_2, N, P$; $W>L, M_2, P$; $U>M_1, M_2$; $V>K_1, K_2$.

$t(A)=(1, 0, 0)$, $t(B)=(0, 1, 0)$,
$t(C)=(-1, 0, -1)$, $t(D)=(0, -1, 1)$,

$t(J)=(1, 1, 0)$, $t(K_1)=(0, 1, -1)$,
$t(K_2)=(-1, 0, 0)$, $t(L)=(0, 0, -1)$,
$t(M_1)=(1, 0, 1)$, $t(M_2)=(0, -1, 0)$,
$t(N)=(0, 0, 1)$, $t(P)=(-1, -1, 0)$,
$t(X)=(1, 1, -1)$, $t(Y)=(1, 1, 1)$,
$t(Z)=(-1, -1, 1)$, $t(W)=(-1, -1, -1)$,
$t(U)=(1, -1, 1)$, $t(V)=(-1, 1, -1)$.

Thus we have shown that the 18 posets indicated in (2.2) are integral.

Remark (2.8). (i) It can be shown that an ASL domain over the poset H defined by the graph (18) of (2.2) is isomorphic to the Segre product $(k[x, y])^{(2)}\#(k[u, v])^{(2)}\#(k[s, t])^{(2)}$, or, in another word, the homogeneous coordinate ring of the product of 3 projective lines embedded in $\boldsymbol{P}^{26}$ by an anti-canonical divisor. It will be our further problem to study the family of ASL domains on each of our posets.

(ii) By our construction in (2.7), if R is one of the toric ASL domains defined in (2.7), then the localization of R by T, $R_T=k[x, x^{-1}, y, y^{-1}, z, z^{-1}, T, T^{-1}]$. In another word, the open affine scheme $D_+(T)$ of Proj (R) is isomorphic to $(k^*)^3$.

§ 3. Further investigation of the axiom of ASL

First we recall some facts which appeared in [8] and [3].

(3.1) ([8], (3.2)) If $H=H_1\cup H_2$, where $H_i=H-I_i$ for some poset ideal I_i $(i=1, 2)$ with $I_1\cap I_2=\phi$ and if R is an ASL on H, then R is isomorphic to the fiber product $R_1\times_{R_0}R_2$, where $R_i=R/I_i$ $(i=1, 2)$, $R_0=R/(I_1\cup I_2)R$ and the homomorphism $R_i\rightarrow R_0$ $(i=1, 2)$ is the canonical surjection.

(3.2) (Inductive construction of an ASL). Let H be a poset with $r(H)=r$ and put $H|_i=\{\alpha\in H\,|\,r(\alpha)\geqq i\}$, $I|^i=\{\alpha\in H\,|\,r(\alpha)<i\}$ for $i=0, \cdots, r$. Then any ASL R on H can be constructed as follows;

We start from an ASL on $H|_r$, which is necessarily discrete. Then take an element $\alpha\in H$ with $r(\alpha)=r-1$ and construct an ASL $R(\alpha)$ on H_α. Then $R/(I|^{r-1})R$ which is an ASL on $H|_{r-1}$ is given by the fiber product of $\{R(\alpha)|\alpha\in H, r(\alpha)=r-1\}$. Then take an element β of H with $r(\beta)=r-2$ and consturct an ASL $R(\beta)$ on H_β, with the property that $R(\beta)/\beta R(\beta)$ is the ASL already constructed on $H_\beta\cap H|^{r-1}$, and so on.

(3.3) (Fundamental transformations for an ASL). Let R be an ASL on H. We specify the inclusion i of H in R. A fundamental transformation of R with respect to $\alpha\in H$ is a new embedding j: $H\rightarrow R$ of the form

$j(\alpha)=c_\alpha\cdot i(\alpha)+\sum_{\beta<\alpha} d_\beta i(\beta)$ and $j(\alpha')=i(\alpha')$ for $\alpha'\neq\alpha$, where c_α, $d_\beta\in k$ and $c_\alpha\neq 0$. Then R is again an ASL with respect to the new embedding j of H in R ([3], Proposition D). For simplicity, we call a composition of several fundamental transformations a fundamental transformation, too. Fundamental transformations are used to obtain an isomorphic ASL with simpler (ASL-2) relations. For example, if

$$AB=T(aA+bB+cT), \qquad (a, b, c\in k),$$

then define a new embedding j: $H\to R$ by $j(A)=A-bT$, $j(B)=B-aT$ so that $j(A)j(B)=(c+ab)j(T)$. In this manner, we may assume $AB=dT^2$ $(d\in k)$, conserving the isomorphism class of graded rings. An isomorphism of two ASL's on a poset H is called an isomorphism as an ASL if it is given by a fundamental transformation. In the above example, if $d\neq 0$, define a new embedding j' by $j'(A)=d^{-1}\cdot A$, $j'(B)=B$, $j'(T)=T$ so that $j'(A)\cdot j'(B)=j'(T)^2$. To sum up;

Example (3.4). If R is an ASL on H_2 indicated by (2.5), Fig. 4, we may assume that $R\cong k[A, B, T]/(AB-\varepsilon T^2)$ ($\varepsilon=0$ or 1) as an ASL.

Notation (3.5). If R is an ASL on H, m is a non-standard monomial in R and if

$$m=\sum_{i=1}^{s} c_i\cdot n_i$$

($c_i\in k$, $c_i\neq 0$ and n_i is a standard monomial, $i=1,\cdots,s$),

we put

$$[m]=\{n_1,\cdots,n_s\}.$$

The set $[\alpha\beta]$ of the standard monomials appearing on the right hand side of (ASL-2) is much more restricted than it seems to be. In fact,

Lemma (3.6). *Let H be a poset with unique minimal element T, α, $\beta\in H$, with the property that the unique element of H smaller than α and β is T and R be an ASL on H. If $T\delta\in[\alpha\beta]$, $\delta\neq T,\alpha,\beta$, then one of the following cases occur.*

(i) *$\delta>\alpha,\beta$ and for every $\xi<\delta$, $\xi\not>\alpha$, either*

(a) *$\xi<\beta$, $T\eta\in[\alpha\xi]$ for some $\eta>\xi$ with $\xi\delta\in[\beta\eta]$, or*

(b) *$\alpha'\eta\in[\alpha\xi]$ for some $\alpha'<\alpha,\xi$, $\alpha'\neq T$, $\eta>\xi$. In this case, for some $\zeta>\xi$, $T\zeta\in[\alpha'\beta]$ and $\xi\delta\in[\eta\zeta]$.*

(ii) *$\delta>\alpha$, $\delta\not>\beta$. In this case $\xi\eta\in[\beta\delta]$ for some $\xi<\beta$, $\xi\neq T$, $\eta>\delta$ with $T\zeta\in[\alpha\xi]$, $\zeta>\delta$ and $\delta^2\in[\eta\zeta]$. In particular, this case happens only* *when* $r(H) \geqq 3$.

(ii′) *The same as* (ii) *with* α *and* β *interchanged.*

(iii) $\delta < \beta$. *In this case for some* $\eta > \delta$, $T\eta \in [\alpha\delta]$ *and* $\delta^2 \in [\beta\eta]$.

(iii′) *The same as* (iii) *with* α *and* β *interchanged.*

(iv) $\delta \not< \alpha$, $\delta \not< \beta$. *In this case the following conditions* (a), (a′) *are satisfied.*

(a) *For some* ξ, η, ζ *with* $\xi < \alpha, \delta$ *and* $\eta, \zeta > \delta$, $\xi\eta \in [\alpha\delta]$, $T\zeta \in [\beta\xi]$ *and* $\delta^2 \in [\eta\zeta]$.

(a′) *The same as* (a) *with* α *and* β *interchanged.*

Proof. In case (i) we have $T\xi\delta \in [\alpha\beta\xi]$. What we should do is only to deform the non-standard monomial to the standard one and to write down the conditions in (ASL-2) at each step. And the same procedure for other cases.

Lemma (3.7). *If R is an ASL on the poset* C_n *indicated by* (2.5), *Fig.* 4, *then after suitable fundamental transformation we may assume,*

(i) *if* $n=2$, $K_1K_2 = T(aX_1 + bX_2 + cT)$,

$X_1X_2 = T(a'K_1 + b'K_2 + c'T) + \varepsilon_1 K_1^2 + \varepsilon_2 K_2^2$,

($a, b, c, a', b', c' \in k$, ε_1 *and* ε_2 *are either* 0 *or* 1).

In the following, the index i denotes the equivalence class modulo n. That is, K_i *or* X_i *means* K_{i-n} *or* X_{i-n} *if* $i>n$ *and* K_{i+n} *or* X_{i+n} *if* $n \leqq 0$.

(ii) *if* $n=3$, *the standard monomials that may appear in*

$[K_iK_{i+1}]$ *are* T^2 *and* TX_i,

$[K_iX_{i+1}]$ *are* T^2, TK_{i+1} *and* TK_{i-1},

$[X_iX_{i+1}]$ *are* T^2, TK_{i+1} *and* K_{i+1}^2 *for* $i=1, 2, 3$.

(iii) *If* $n=4$, *the standard monimial(s) that may appear in*

$[K_iK_{i+1}]$ *is* TX_i,

$[K_iK_{i+2}]$ *is* T^2,

$[K_iX_{i+1}]$ *is* TK_{i+1},

$[X_iK_{i+2}]$ *is* TK_{i+1},

$[X_iX_{i+1}]$ *are* K_{i+1}^2 *and* TK_{i+1},

$[X_iX_{i+2}]$ *is* T^2 *for* $i=1, 2, 3, 4$.

(iv) *If* $n \geqq 5$, *then we may assume*

$K_iK_j = 0$ *for* $j \neq i$, $i \pm 1$,

$K_iX_j = 0$ *for* $j \neq i$, $i \pm 1$, $i-2$,

$X_iX_j = 0$ *for* $j \neq i$, $i \pm 1$ *for every* i, $1 \leqq i \leqq n$.

Moreover, we may put

$$K_iK_{i+1}=a_iTX_i,\ X_{i-1}X_i=\varepsilon_iK_i^2+b_iTK_i,\ X_{i-1}K_{i+1}=c_iTK_i,$$

$K_{i-1}X_i=d_iTK_i$ *for some* a_i, b_i, c_i, $d_i\in k$, $\varepsilon_i=0$ *or* 1 *with the relations* $a_ib_i=a_ib_{i+1}=0$, $c_i=\varepsilon_ia_i$ *and* $d_{i+1}=\varepsilon_{i+1}a_i$ *for every* i, $1\leqq i\leqq n$.

Proof. We only sketch the proof of (iv) here. The rest of the proof is in the same line and our method is unique, that is, to express the non-standard monomials of degree 3 in two different ways as the linear combinations of standard monomials and to compare the coefficients of the standard monomials.

Now, by (3.6), the only standard monomials that may appear in $[K_iK_{i+2}]$ (resp. $[X_iX_{i+2}]$) are T^2, TK_i, TK_{i+2} (resp. T^2, TK_i, TX_{i+2}). As in [3], Proposition A, we have the equation of the type

$$(K_i-aT)(X_{i+2}-bT)=0.$$

Now, by a fundamental transformation, we may assume $K_iX_{i+2}=0$ for every i. Then, put $K_iK_{i+2}=T(aT+bK_i+cK_{i+2})$ and $K_iK_{i+3}=T(a'T+b'K_i$ $c'K_{i+3})$. As $(K_iK_{i+2})X_{i+2}=(K_iK_{i+3})X_{i+2}=0$, we have $a=c=a'=c'=0$. That is, the only standard monomial that may appear in $[K_iK_{i+2}]$, $[K_iK_{i+3}]$ is TK_i. If $n=5$, by the same argument with the index i interhcanged by $i-2$, we have $K_iK_{i+2}=K_iK_{i+3}=0$. If $n\geqq 6$, put $K_iX_{i+3}=T(aT+bK_i+$ $cX_{i+3})$. As $[K_iK_{i+3}]\subset\{TK_i\}$, comparing the coefficients of standard monomials appearing in $[K_iK_{i+3}X_{i+3}]$ we have $a=c=0$. Repeating this procedure inductively we have $[K_iK_j]$ (resp. $[K_iX_j]$) $\subset\{TK_i\}$ for $i+2\leqq j\leqq i+n-2$ (resp. $i+2\leqq j\leqq i+n-3$).

On the other hand, put $X_{i-1}X_{i+2}=T(a'T+b'X_{i-1}+c'X_{i+2})$. As $K_iX_{i+2}=0$, we have $a'=b'=0$ and by the same argument as above, we have $[K_jX_{i+2}]$ (resp. $[X_jX_{i+2}]$) $\subset\{TX_{i+2}\}$ for $i\geqq j\geqq i-n+5$ (resp. $i-1\geqq j$ $\geqq i-n+5$) for every i. Now, take $j\neq i$, $i\pm 1$ and we claim $K_iK_j=0$. As we have seen, we can put $K_iK_j=aTK_i$ and $K_iX_j=bTX_j$ (if $j=i-2$, take X_{j-1} instead of X_j). Then we have $(K_iK_j)X_j=aTK_iX_j=abT^2X_j$, while $K_j(K_iK_j)=K_j(bTX_j)=bTK_jX_j$, which shows $b=0$ and by $X_iK_j=0$ (or $X_{i-1}K_j=0$) we have $a=0$.

The other parts of (iv) can be proved similary putting $X_iX_{i+1}=\varepsilon_iK_{i+1}^2$ in R/TR.

Example (3.8). As an example of "inductive construction" of an ASL, we will explain the construction of the poset indicated by (1.5), Fig. 1, on which $R=(k[X_0, X_1, X_2, X_3])^{(4)}$ is an ASL. As the anti-canonical divisor of $\boldsymbol{P}^3$ is the union of 4 planes, take 4 planes H_0, H_1, H_2, H_3 in general position so that $H_i=V_+(A_i)$, or, R/A_iR is isomorphic to the 4-th Veronese subring of a polynomial ring of 3 variables for every i. Then $V_+(I(L_{ij}))$ is the line determined by the intersection of H_i and H_j and $V_+(I(L'_i))$ is another line on A_i in general position. The point $V_+(I(P_i))$ (resp. $V_+(I(P'_{ij}))$) is the intersection of three planes H_j $(j\neq i)$ (resp. the intersection of L'_i and L_{ij} and the point $V_+(I(P''_i))$ is a point on L'_i different from the three points $V_+(I(P'_{ij}))$ $(j\neq i)$. The poset H is anti-isomorphic to the poset of subspaces of $\boldsymbol{P}^3$ given above with inclusion of spaces as order relation.

§ 4. Proof of the main theorem

In this section we will prove that the posets which are triangulations of S^2 are not integral except for the 18 ones given in (2.2). Our method is very elementary; to construct zero-divisors or to show that the multiplication by a certain element is not injective. We always denote by R an ASL on $H=H'\cup\{T\}$, where H' is a poset with $|\Delta(H')|\cong S^2$ and T is the unique minimal element of H.

Definition (4.1). Let γ, δ be elements of H with $\gamma\not\sim\delta$. We say that the product $\gamma\delta$ is banal if $[\gamma\delta]$ is contained in $\{T^2, T\gamma, T\delta\}$.

Lemma (4.2). ([3], Proposition A). *Let α, β, γ be elements of H such that $\alpha\not\sim\beta$, $\alpha\not\sim\gamma$, $\gamma>\beta$ and the products $\alpha\beta$ and $\alpha\gamma$ are banal. Then R is not an integral domain.*

Now we will rewrite (3.6) to our poset H using the terminology of the graph $\Gamma(H')$. We will identify points or edges or regions of S^2 determined by $\Gamma(H')$ to the elements of H and also to the elements of R.

We will always assume that for every minimal element α of H', $R/I(\alpha)$ is normalized as in (3.7).

Lemma (4.3). (i) *Let A, B be distinct points of $\Gamma(H')$. If $T\delta\in[AB]$, $\delta\neq T$, A, B, then either δ is an edge connecting A and B or δ is a two-sided region determined by two edges connecting A and B. In particular, if there exist no edges connecting A and B, then the product AB is banal.*

(ii) *Let A be a point of $\Gamma(H')$, L be an edge of $\Gamma(H')$ with $A\not\sim L$. If $T\delta\in[AL]$, $\delta\neq T$, A, L, then either*

(a) *δ is an end point of L connected with A by some edge and for some $\gamma>A$, δ, $T\gamma\in[A\delta]$ and $\delta^2\in[L\gamma]$.*

(b) *δ is an edge connecting an end point β of L with A. In this case, for some region $\eta>\delta$, L and a two-sided region $\zeta>A$, δ, $\beta\eta\in[\delta L]$, $T\zeta\in[A\beta]$ and $\delta^2\in[\eta\zeta]$.*

(c) *δ is a triangle with $\delta>A$, L. In this case, for either end point*

β of L and an edge μ connecting A and β with $\mu<\delta$, $T\mu\in[A\beta]$ and $\beta\delta\in[\mu L]$.

(iii) *Let A be a point of $\Gamma(H')$, X be a region determined by $\Gamma(H')$ with $X \not> A$. If $T\delta\in[AX]$, $\delta\neq T, A, X$, then $\delta<X$ and either*

(a) *δ is a point with $\deg(\delta)=3$ and there is an edge μ connecting δ and A with the property $T\mu\in[A\delta]$ and $\delta^2\in[\mu X]$.*

(b) *δ is a point with $\deg(\delta)=4$ and there is a two-sided region η with the property $\eta>A$, δ, $T\eta\in[A\delta]$ and $\delta^2\in[\eta X]$.*

(c) *δ is an edge of X and there is a triangle η with $\eta>A$, δ. In this case $T\eta\in[A\delta]$ and $\delta^2\in[\eta X]$.*

Where, for a point B of $\Gamma(H')$, $\deg(B)$ is the number of edges of $\Gamma(H')$ which are incident with B.

Proof. (i) and (ii) are almost direct consequences of (3.6). For (iii), apply (3.6) for our case. The cases (i) and (iv) do not occur in our situation. If $\delta>A$, $\delta \not> X$, in the notation of (3.6) (ii), ξ is a point incident with X and δ is an edge connecting ξ with A and for some region $\eta>\delta$, $\xi\eta\in[\delta X]$. But the last statement does not occur in $R/I(\xi)$ by our normalization (3.7). If $\delta<X$ and δ is a point, our normalization (3.7) of $R/I(\delta)$ implies $\deg(\delta)=3$ (resp. 4) in the case (a) (resp. (b)).

Corollary (4.4). *Assume that there exists a point A and an edge L of $\Gamma(H')$ with $A \not> L$ and either edge point of L is not connected with A by an edge of $\Gamma(H')$. Then H is not an integral poset.*

Proof. If B is an edge point of L, the products AB and AL are banal by (4.3). Then (4.2) shows that R is not an integral domain.

Corollary (4.5). *Any region on S^2 determined by $\Gamma(H')$ is either a two-sided region, a triangle or a quadrangle.*

Now, a vertex A of $\Gamma(H')$ with $\deg(A)\geqq 5$ causes a serious obstruction for integrality of H.

Corollary (4.6). *Let A, B, C, be three points of $\Gamma(H')$, L be an edge of $\Gamma(H')$ connecting A and B. If A, B, C, L satisfy the following conditions, H is not integral.*

(i) *$\deg(A)\geqq 5$, B and C are not connected by edge of $\Gamma(H')$.*

(ii) *For any region X with $X>L$, C is not incident with X.*

Proof. We apply (4.3) (ii) to the product CL. As $\deg(A)\geqq 5$, $A^2\notin[L\xi]$ for any $\xi>A$, $\xi \not> L$ by (3.7) (iv) applied to $R/I(A)$. So case (a) of (4.3) (ii) does not occur in this situation. Also cases (b) and (c) of (4.3) (ii) do not occur by our condition. Thus the product CL is banal and so is the product CB. Then H is not integral by (4.2).

In general, a graph Γ is a simple graph if for any two vertices v, v' of Γ there exists at most one edge of Γ connecting v and v'. We denote by $|\Gamma(H')|$ the underlying simple graph of $\Gamma(H')$ defined in an obvious way.

Corollary (4.7). *If H is integral, for any vertex v of $|\Gamma(H')|$,* $\deg(v) \leqq 4$.

Proof. If there exists a vertex A of $|\Gamma(H')|$ with $\deg(A) \geqq 5$, then we can find $B, C, L \in H'$ which satisfy the conditions of (4.6) together with A.

Corollary (4.8). *Assume that H is integral. Let v be a vertex of $|\Gamma(H')|$ and X be a region of $|\Gamma(H')|$ not incident with v. Then either*

(a) *there exists a vertex v' with* $\deg(v')=3$, *incident with X and connected to v by an edge of $|\Gamma(H')|$, or*

(b) *there is a triangle in $|\Gamma(H')|$ incident with v and which shares an edge with X.*

Proof. If (b) does not occur, then there is a vertex B of $\Gamma(H')$ not connected to A $(=v)$ by any edge of $\Gamma(H')$. As the product AX is not banal, the case (a) or (b) of (4.3) (iii) occurs and this implies our condition (a).

Now, by (4.4), (4.7) and (4.8), it can be shown that the underlying simple graph of $\Gamma(H')$ is either (4), (9), (11), (12), (13), (16), (17), (18) of (2.2) or one of the followings.

Fig. 6

Now we will give a new criterion for non-integrality of H.

Lemma (4.9). *Assume that H is integral and let A, B be two vertices of $\Gamma(H')$ not connected by an edge of $\Gamma(H')$. If C is any other vertex of $\Gamma(H')$, then the number of edges connecting A and C and the number of edges connecting B and C are equal and is at most* 2.

Proof. Let R be an ASL domain on H. As the product AB is banal, we may put $AB=T^2$ by a fundamental transformation. Now let $\{\alpha_1, \cdots, \alpha_s\}$ (resp. $\{\beta_1, \cdots, \beta_t\}$) be the set of edges of $\Gamma(H')$ connecting A and C (resp. B and C) and $\{\xi_1, \cdots, \xi_{s-1}\}$ (resp. $\{\eta_1, \cdots, \eta_{t-1}\}$) be the set of two-sided regions incident with A and C (resp. B and C). Then by (4.3), the product of A with an element the k-vector space V (resp. V') spanned by $\{T, B, C, \beta_1, \cdots, \beta_t\}$ (resp. $\{T, B, \eta_1, \cdots, \eta_{t-1}\}$) is contained in the vector space W (resp. W') spnnaed by $\{T^2, TA, TC, T\alpha_1, \cdots, T\alpha_s\}$ (resp. $T^2, TA, TC\}$. (Note that as $AB=T^2$, $T\beta_i \notin [A\beta_i]$ and $T\eta_i \notin [A\eta_i]$.) As dim $V=t+3$ and dim $W=s+3$ (resp. dim $V'=t+1$ and dim $W'=3$), we have $t\leqq s$ (resp. $t\leqq 2$) as the multiplication by A must be injective. As our situation is symmetric with respect to A and B, we have $s=t$.

Example (4.10). Assume that H is integral. By (4.9) we can show that (9) and (10) of (2.2) are only possibilities for $\Gamma(H')$ if $|\Gamma(H')|$ is (9) of (2.2). By (4.9) and (4.6), we can show that $\Gamma(H')$ itself must be simple if $\Gamma(H')$ is (11), (12), (16) of (2.2) or the second graph in Fig. 6, after (4.8). Also, by (4.6), we can show that $\Gamma(H')$ itself must be simple if $|\Gamma(H')|$ is (17) or (18) of (2.2). After the next example (4.11), we have only to consider the graph whose simple underlying graph is either (4), (13) of (2.2) or ○—○.

Example (4.11). Let H' be the poset given by the graph illustrated in Fig. 6. We name the points of $\Gamma(H')$ by A, B, C, D, E so that $\deg(A)=\deg(B)=3$ and $\deg(C)=\deg(D)=\deg(E)=2$. As the product CD is banal, we may assume $CD=eT^2$ by a fundamental transformation. There are 3 regions determined by $\Gamma(H')$ and let X be the one which is not incident with D. Then X is incident with other 4 points. Now, by (4.3), (iii), the standard monomials contained in $[DX]$ are restricted to T^2, TD, TX, TA and TB. Put $DX=T(tT+aA+bB+dD+xX)$. If we multiply C to this equation, we conclude that $CD=0$ as $[AC]$ nor $[CB]$ contains TX. Thus we have shown that this poset is not integral.

There are several posets except the ones given in (2.2) on the underlying simple graph (4) or (13) of (2.2) which can not be eliminated directly from our criterions given above. But we will omit the proof of non-integrality of these posets as it will be rather boring work. In the rest of this paper, we will prove that the poset H given by the graph $\circ\overset{n}{—}\circ$ is not integral for $n\geqq 5$.

Example (4.12). Let $H=\{T, A, B, K_1, \cdots, K_n, X_1, \cdots, X_n\}$ with the order relation $A, B<K_i, X_i$ and $K_i, K_{i+1}<X_i$ for every i. As in (3.7), the index i is defined modulo n. Now, we will prove that H is not integral if $n\geqq 5$. In this example, we say that the product K_iK_j (resp. K_iX_j) is banal if $[K_iK_j]$ (resp. $[K_iX_j]$) contains at most T^2, TA, TB and TK_i, TK_j (resp. TK_i, TX_j). If we can find a pair of indexes (i, j) for which K_iK_j and K_iX_j or K_iK_j and K_iX_{j-1} are banal, then we can construct a zero-divisor of the form K_i+aT as in [3], Proposition A. In the following, we always assume $n\geqq 5$.

Now, consider the product K_1K_4. If $T\xi \in [K_1K_4]$, as $K_1K_4 \in TR$ by

(3.7) (iv), $T\xi^2 \in [K_1K_4\xi]$ (we always assume that $\xi \neq T, A, B, K_1, K_4$). The only possible procedure for this is; $A\eta \in [K_1\xi]$, $B\xi \in [K_4\eta]$ and $T\xi \in [AB]$ for some η (or the same procedure with K_1 and K_4 interchanged). Examining (3.7) (iv), we see that this procedure is possible only when $n=5$, $\xi=K_5$, $\eta=X_5$ or $\xi=X_4$, $\eta=K_5$. This shows that if $n\geqq 6$, the product K_iK_{i+3} is always banal. The same argument as above also shows that the product K_1X_4 is banal and so H is not integral if $n\geqq 7$.

Before we proceed, one remark about the result of (3.7) (iv) is necessary. If $K_iK_{i+1}=a_iTX_i$ with $a_i\neq 0$, multiplying K_{i-1} (resp. K_{i+2}), we have $K_{i-1}X_i=0$ (resp. $K_{i+2}X_i=0$). In the notations of (3.7) (iv), $a_i\neq 0$ implies $d_i=0$ and $c_{i+1}=0$.

Now, apply (3.7) (iv) to $R/I(A)=R/(T, B)$ and $R/I(B)=R/(T, A)$. We put (*) $K_iK_{i+1}=a_i'AX_i$, $X_{i-1}X_i=\varepsilon_iK_i^2+b_i'AK_i$, $X_{i-1}K_{i+1}=c_i'AK_i$, $K_{i-1}X_i=d_i'AK_i$ in $R/I(A)$ and $K_iK_{i+1}=a_i''BX_i$, $X_{i-1}X_i=\varepsilon_iK_i^2+b_i''BK_i$, $X_{i-1}K_{i+1}=c_i''BK_i$ and $K_{i-1}X_i=d_i''BK_i$ in $R/I(B)$.

Assume $n=6$. We will prove that H is not integral. As we have shown, the product K_1K_4 is banal. So, it suffices to show the product K_1X_3 or K_1X_4 is banal. Assume $T\xi \in [K_1X_4]$, $\xi\neq T, A, B, K_1, X_4$. Then $X_4 \notin TR$ and if $K_1\xi \in TR$, then $T\eta \in [\xi K_1]$ and $\xi^2 \in [\eta X_4]$ for some η. Only possibility in this case is $\xi=K_5$, $TX_5 \in [K_1K_5]$ and $K_5^2 \in [X_4X_5]$. As $\xi=X_5$ or $\xi=K_6$ does not occur also, $TK_5 \in [K_1X_4]$ if R is an integral domain. Now, $TX_5 \in [K_1K_5]$ only if, say, $AK_6 \in [X_5K_1]$ and $BX_5 \in [K_5K_6]$. In the notation of (*) above, $c_6'\neq 0$, $\varepsilon_4=1$, $a_5''\neq 0$ and consequently, $a_6'\neq 0$, $\varepsilon_6=1$, $c_1'=0$, $d_6'=0$ and $c_6''=d_5''=0$. Also, as the product K_2X_5 must not be banal as K_2K_5 is banal, we must have $TK_6 \in [K_2X_5]$, $TX_6 \in [K_2K_6]$. As $c_1'=0$, we must have $BK_1 \in [X_6K_2]$ and $AX_6 \in [K_6K_1]$, having $a_1''=c_1''\neq 0$, $\varepsilon_1=1$, $c_2''=d_1''=0$. Continuing this way, if R is an integral domain, we must have $\varepsilon_i=1$ for every i, $a_i'=c_i'=d_i''=0$, $d_i'\neq 0$, $a_i''=c_i''\neq 0$ for every odd i and $a_i'=c_i'\neq 0$, $d_i''\neq 0$, $d_i'=a_i''=c_i''=0$ for every even i.

Now, after a suitable fundamental transformation, put

$$K_1K_4=T(tT+aA+bB)$$
$$K_1X_4=T(t'T+a'A+b'B+x'X_4+f'K_5)$$
$$K_1X_3=T(t''T+a''A+b''B+x''X_3+f''K_3),$$

where, f' and $f''\neq 0$ if R is an integral domain. Expressing $K_1K_4X_4$ and $K_1K_4X_3$ in two different ways, we have

$$X_4(tT+aA+bB)=K_4(t'T+a'A+b'B+x'X_4+f'K_5),$$
$$X_3(tT+aA+bB)=K_4(t''T+a''A+b''B+x''X_3+fK_3).$$

Comparing the coefficients of AX_4, BX_4, AX_3 and BX_3 in both equations, we have $a=f'a'_4$, $b=f'a''_4$, $a=f''a'_3$, $b=f''a''_3$, inducing a contradiction if $f'\neq 0, f''\neq 0$ as $a'_3=0$ and $a'_4\neq 0$. Thus R cannot be an integral domain and H is not integral.

Next, we will treat the case $n=5$. We record here only the proof of the case $\varepsilon_i=1$ for every i. The proof of the other cases is accomplished in a similar way. In the notation of (*), a'_i, a''_i, c'_i, c''_i, d'_i, d''_i are zero for at least three values of i. So, we can choose an index i, with $a'_i=a''_i=0$. Assume $a'_1=a''_1=c'_1=c''_1=d'_2=d''_2=0$. Then, K_1K_2, K_1X_2, $X_5K_2\in TR$. By a similar argument as above, we can prove that the product K_1K_3 is banal. If $T\xi\in[K_1X_3]$, $\xi\neq T, A, B, K_1, X_3$, only possibility for ξ is $\xi=K_4$ under the condition $TX_4\in[K_1K_4]$. So, if R is an integral domain, we may assume $AK_5\in[K_1X_4]$ and $BX_4\in[K_4K_5]$. Then $a'_5=d'_1\neq 0$, $a'_4=d'_5=0$, $a''_4=d''_5\neq 0$ and $a''_3=a''_5=d''_4=d''_1=0$. Next, let $T\eta\in[K_1X_2]$, $\eta\neq T, A, B, K_1, X_2$. It is easy to show $\eta=X_1$ or K_2. Put

$$K_1K_3=T(tT+aA+bB)$$
$$K_1X_3=T(t'T+a'A+b'B+x'X_3+f'K_4)$$
$$K_1X_2=T(t''T+a''A+b''B+x''X_2+f''K_2+g''X_1).$$

Then we have

$$X_3(tT+aA+bB)=K_3(t'T+a'A+b'B+x'X_3+f'K_4),$$
$$X_2(tT+aA+bB)=K_3(t''T+a''A+b''B+x''X_2+f''K_2+g''X_1).$$

As $a''_3=0$, $b=0$ by the first equation and also $a'=b'=x'=0$, $a=a'_3f'$. As we are always assuming that R should be an integral domain, $f'\neq 0$ and $a\neq 0$. Comparing the coefficients of AX_2 in the second equation, we have $a'_2f''=a\neq 0$. Then by the coefficients of BX_2, we have $a''_2=0$. So, the non-zero a'_i (resp. a''_i) are a'_2, a'_5, a''_4 and all others are 0. Now, by the same procedure as above, we can show that the products K_3K_5 and K_3X_5 are banal contradicting our assertion that R is an integral domain. So, H can not be an integral poset.

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