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On Quasi-Buchsbaum Modules

An Application of Theory of FLC-modules

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Introduction

One of the main purposes of this article is to give a perspective of the theories of the rings and modules called "rings/modules of FLC" in this article or generalized Cohen-Macaulay rings or modules in [S=T=C]. They have been developed for the recent one and a half decade years under the influence of the advance of the theory of Buchsbaum rings/modules.

Most of the objects are closely related to Cohen-Macaulay rings/modules. Our discussion is based mainly on the study of Koszul complex developed in [S_2], [S_3], [S_4], [S_5], [S_6] and [G=S] and our main technique had been crystallized from them. In the study of modules of FLC, we have another powerful notion—u.s.d-sequences. Since the topic related to it has been already presented in [G=Y] in complete form and also in [S_8] in relation with canonical duality, we never mention it so much any more in this article.

Among the modules of FLC, Buchsbaum modules are of most importance. The theory has achieved a remarkable progress in recent years. Not merely as a derived notion from Cohen-Macaulay, but as the independent objects, they revealed to be endowed with an exquisite structure. In Section 2, we allot some space to demonstrate some results related to the Koszul homology of s.o.p. of Buchsbaum modules that are suppressed in the Cohen-Macaulay cases because of the vanishing homology modules.

Another much important aim of us is to develop a theory of a class of modules called quasi-Buchsbaum modules based on and making very best use of the observations on the modules of FLC in the preceding part of this article. Some part of the main theorem had already been announced in [S_6] and a proto-type of the proof had been given in [S_4] which is available only directly from the author. We give a new and rather improved version and proof of it.

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Also some of the recent results related to the research of quasi-Buchsbaum modules and modules of FLC will be presented.

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§ 1. Koszul complexes

In this section we provide some technical lemmas on Koszul complexes which will be used in this article. It should be noted that the way of treating Koszul complexes as below is quite powerful when used in the induction steps.

Throughout this section, the ring A is a general commutative ring with $1\neq 0$, if it is not specified otherwize.

(1.1) **Definition.** Let A be a commutative ring, M any A-module and $a_1, \cdots, a_s$ a sequence of elements in A. We define the Koszul complex $K_\bullet(\boldsymbol{a}; M)$ generated by the sequence $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ over the module M as the tensor product of complexes $K_\bullet(a_1, \cdots, a_{s-1}; M)\otimes_A K(a_s, A)$, inductively. More explicitly, for $s=1$,

$$K_i(a_1, M):=M \qquad \text{for } i=0 \text{ and } 1;$$
$$K_i(a_1, M):=0 \qquad \text{for } i\neq 0, 1,$$

and as to the differential maps d_*;

$$d_1\colon K_1 \longrightarrow K_0 \text{ be the multiplication of } a_1$$

and for $i\neq 1$, let d_i be the zero map.

For $s>1$, let $L_\bullet=K_\bullet(a_1, \cdots, a_{s-1}; M)$ and $e_\bullet$ be the differential map of $L_\bullet$: then $K_\bullet=K_\bullet(a_1, \cdots, a_s; M)=L_\bullet\otimes K_\bullet(a_s; A)$ is defined as follows:

$$K_p:=L_{p-1}\oplus L_p$$

and the p-th differential map d_p is

$$d_p(u, v):=(-e_{p-1}(u), a_s u+e_p(v)).$$

Given a complex $L_\bullet=(L_\bullet, e_\bullet)$ over A (i.e., *a* complex in the category of A-modules and A-linear mappings), let $-L_\bullet$ denote the complex $(L_\bullet, -e_\bullet)$ and $L_\bullet[n]$ the complex shifted by n: namely,

$$(L[n])_p;=L_{p+n} \text{ and the } p\text{-th boundary operator } :=e_{p+n}.$$

With these notations,

(1.2) **Lemma.** *Let $L_{\bullet}$ be a complex in the category of A-modules and A-linear mappings, x an element of the ring A and $K_{\bullet}=L_{\bullet}\otimes_A K_{\bullet}(x; A)$. Then there exists an exact sequence of complexes*:

$$0\longrightarrow L_{\bullet}\xrightarrow{i}K_{\bullet}\xrightarrow{j}-L_{\bullet}[-1]\longrightarrow 0.$$

Where the map i is the natural injection:

$$i_p\colon L_p\longrightarrow K_p=L_{p-1}\oplus L_p;\ v\longmapsto(0, v),$$

and the map j is defined as

$$K_p=L_{p-1}\oplus L_p\longrightarrow -L_{\bullet}[-1]_p=L_{p-1};\ (u, v)\longmapsto xu.$$

Consequently, there induced an exact sequence of homology modules:

$$\cdots\longrightarrow H_{p+1}(-L_{\bullet}[-1])=H_p(L_{\bullet})\xrightarrow{.x}H_p(L_{\bullet})$$
$$\longrightarrow H_p(K_{\bullet})\longrightarrow H_p(-L_{\bullet}[-1])\longrightarrow\cdots.$$

(1.3) **Corollary.** *Let $\boldsymbol{a}$ be a sequence of elements in A, a an element of A and M be an A-module. Then we have the following exact sequence*

$$0\longrightarrow\frac{H_p(\boldsymbol{a}; M)}{aH_p(\boldsymbol{a}; M)}\longrightarrow H_p(\boldsymbol{a}, a; M)\longrightarrow[0\colon a]_{H_{p-1}(\boldsymbol{a}; M)}\longrightarrow 0.$$

The next lemma aims to control the homology $H_{\bullet}=Z_{\bullet}/B_{\bullet}$ as a submodule of $K_{\bullet}/B_{\bullet}$.

(1.4) **Lemma.** *Let $(L_{\bullet}, e_{\bullet})$ be a complex over A, $\boldsymbol{J}$ an ideal, x and y elements in $\boldsymbol{J}$ and p be an integer. Assume the following*:

(1) $[B_{p-1}(L_{\bullet}):_{L_p}x^m]\subset[B_{p-1}(L_{\bullet}):_{L_{p-1}}\boldsymbol{J}]\subset Z_p(L_{\bullet})$, *for all integer* $m>0$;

(2) $[B_p(L_{\bullet}):_{L_p}x^n]=Z_p(L_{\bullet})$ *for all integer* $n>0$.

Let $K_{\bullet}:=L_{\bullet}\otimes K_{\bullet}(y; A)$. *Then we have* $[B_p(K_{\bullet}):_{K_p}x^l]\subset Z_p(K_{\bullet})$ *for all integer* $l>0$.

If we assume furthermore that $\boldsymbol{J}H_p(K_{\bullet})=0$, *then the equality holds in the above conclusion and both ends coincide with* $[B_p(K_{\bullet}):_{K_p}\boldsymbol{J}]$.

Proof. The latter half is trivially true. Let $(m, n)\in K_p=L_{p-1}\oplus L_p$ and assume that

$$(\sharp)\quad x^l(m, n)=(x^l m, x^l n)=(-e_p(u), yu+e_{p+1}(v)),$$

for some $(u, v)\in L_p\oplus L_{p+1}$. Then

$$m\in[B_{p-1}(L_{\bullet})\underset{L_{p-1}}{:}x^l],$$

and by (1)

$$-e_{p-1}(m)=0 \quad \text{and} \quad ym=e_p(w)$$

for some $w \in L_p$. Hence again by (♯),

$$e_p(x^l w+yu)=x^l e_p(w)+ye_p(u)=x^l ym-x^l ym=0$$

namely

$$x^l w+yu \in Z_p(L_\bullet).$$

By (♯),

$$yu=-e_{p+1}(v)+x^l n$$

hence

$$x^l w+yu=x^l(w+n)-e_{p+1}(v),$$

so

$$x^l(w+n) \in Z_p(L_\bullet).$$

By (2)

$$x^l(w+n) \in [B_p(L_\bullet) \underset{L_p}{:} x]$$

hence

$$w+n \in [B_p(L_\bullet) \underset{L_p}{:} x^{l+1}]=Z_p(L_\bullet).$$

Now

$$ym+e_p(n)=e_p(w)+e_p(n)=e_p(w+n)=0.$$

Thus

$$(-e_{p-1}(m), ym+e_p(n))=(0, 0),$$

namely $(m, n) \in Z_p(K_\bullet)$ as required. Q.E.D.

We end this preparatory discussion with introducing a complex obtained as a direct limit of Koszul complexes. That is quite useful when we treat the local cohomology modules.

Let $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a sequence of elements in a Noetherian ring A. Then there defined a direct system of complexes:

$$(K_\bullet(\boldsymbol{a}^n; A))^* \longrightarrow (K_\bullet(\boldsymbol{a}^{n+1}; A))^*,$$

where $(♯)^*$ denotes the A-dual functor $\mathrm{Hom}_A(♯, A)$. Taking account of self duality of Koszul complexes, considering the limit complex of the above system, the following is quite acceptable and indeed well known:

(1.5) **Proposition** ([Gr]). *Let A be a local ring of dimension s and $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a system of parameters of A. Then for any A-module M and for any integer p, we have the following isomorphism*:

$$H^p_{\mathfrak{m}}(M)=\varinjlim H_{s-p}(a_1^m, \cdots, a_s^m; M)$$

§ 2. Modules of FLC—the generalized Cohen-Macaulay modules

Among the various properties of Cohen-Macaulay rings, let us consider the following: a local ring A is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(A)=0$ for all $i \neq \dim A$.

When we are trying to work at rings/modules that are not Cohen-Macaulay, the following is then a question quite naturally expected:

What can we say about the local cohomology modules still remaining?

or

What do these local cohomology modules tell us about the ring?

Henceforth A is a local ring with maximal ideal $\mathfrak{m}$.

As a first attempt, let us impose the restriction on the local cohomology modules as below:

(2.1) **Definition** ([G=Y, S=C=T, S_8]). *A finitely generated A-module M is called a generalized Cohen-Macaulay module* (*or module of Finite Local Cohomology, abbrev. FLC*) *if*

$$l_A(H^i_{\mathfrak{m}}(M))<\infty,$$

for all $i \neq s=\dim M$.

If this is the case, we define a numerical invariant of M by

$$I(M)=\sum_{i=0}^{s-1}\binom{s-1}{i}h^i(M),$$

where $h^i(M)$ denote the length of $H^i_{\mathfrak{m}}(M)$. We call the above invariant the Buchsbaum invariant of M and it plays a quite important role in the theory of module of FLC.

(2.2) **Definition.** Let M be a finitely generated A-module and $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a system of parameters for M. We define a numerical function $I(*; M)$ by;

$$I(\boldsymbol{a}; M):=l_A(M/qM)-e_0(q; M)$$

with $q=(\boldsymbol{a})A$, where $e_0(\sharp)$ denotes the multiplicity symbol.

Note that, since $e_0(q; M)$ coincides with the *Euler characteristic* of the Koszul Complex $K_{*}(q; M)$ [A=B], the function $I(\boldsymbol{a}; M)$ has the following expression;

$$I(\boldsymbol{a}; M)=\sum_{i=1,\cdots,s}(-1)^{i+1}h_i(\boldsymbol{a}; M),$$

where $h_i(\sharp)$ denotes the length of the i-th Koszul homology.

Remark. Let $s=\dim M\geqq 2$ and $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a s.o.p. for M. Assume that $a=a_1$ satisfies the condition:

$$l_A([0 \underset{M}{:} a])<\infty.$$

Then with $\boldsymbol{a}'=\{a_2, \cdots, a_s\}$, we generally have $e_0(\boldsymbol{a}'; M/aM)=e_0(\boldsymbol{a}; M)$ and we have

$$I(\boldsymbol{a}'; M/aM)=I(\boldsymbol{a}; M).$$

We must note here the fact that the difference of the multiplicity from the colength of the ideal generated by a system of parameters was first payed attension by D. Buchsbaum in [B]. The study of Buchsbaum rings/modules has its origin there and we name the invariant given in (2.1) after the fact.

Let us start with the following rather general lemma;

(2.3) **Lemma.** *Let M be a finitely generated A-module of $s=\dim M\geqq 1$ and x be any parameter for M. Assume that $l_A([0:_M x])<\infty$. Then there exists a long exact sequence as below;*

$$H^0_{\mathfrak{m}}(M)\xrightarrow{.x}H^0_{\mathfrak{m}}(M)\longrightarrow H^0_{\mathfrak{m}}(M/xM)\longrightarrow\cdots$$
$$\cdots\longrightarrow H^i_{\mathfrak{m}}(M)\longrightarrow H^i_{\mathfrak{m}}(M)\xrightarrow{.x}H^i_{\mathfrak{m}}(M/xM)\cdots.$$

Proof. Consider the following diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & xM & \longrightarrow & M & \longrightarrow & M/xM & \longrightarrow & 0 \\
 & & \uparrow & \nearrow_{.x} & & & & & \\
 & & M & & & & & & \\
 & & \uparrow & & & & & & \\
 & & [0:_M x] & & & & & & \\
 & & \uparrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

The required sequence will be induced by applying the local cohomology functor to the diagram since by the assumption on x we know that the mapping

$$H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(xM)$$

is a surjection and

$$H^p_{\mathfrak{m}}(M) \longrightarrow H^p_{\mathfrak{m}}(xM)$$

is an isomorphism for each $p \geqq 1$.

As one can easily guess from the above lemma, if M is a module of FLC and the assumption in the lemma is satisfied for a parameter x for M, then M/xM is also a module with FLC. Then for what parameter the requirement is satisfied? Our next lemma provides the perfect answer and is quoted from [S_8], to which the proof should be referred:

(2.4) **Lemma** (cf. [S_3], [S_6], [S_8] and [A=B]). *Let M be an A-module of FLC of dimension s. Then $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$ module of dimension s-$\dim(A/\mathfrak{p})$ for all prime $\mathfrak{p} \neq \mathfrak{m}$ and for a parameter element a for M,*

$$[0 \underset{M}{:} a] \subset H^0_{\mathfrak{m}}(M).$$

Consequently, for any s.o.p. $\boldsymbol{a} = a_1, \cdots, a_s$ for M, we have the following;

(1) *For any r with $0 < r < s$ and for each $p < s - r$,*

$$h^p(M/(a_1, \cdots, a_r)M) \leqq \sum_{i=0}^{r} \binom{r}{i} h^{i+p}(M).$$

(2) *For r as above and for any $p > 0$,*

$$h_p(a_1, \cdots, a_r; M) \leqq \sum_{i=0}^{r-p} \binom{r}{p+i} h^i(M).$$

(3) $I(\boldsymbol{a}; M) = l_A([(a_1, \cdots, a_{s-1})M : a_s]/(a_1, \cdots, a_{s-1})M)$.

By the above lemma, we see the following important fact on the relation between the Buchsbaum invariant and the function $I(\boldsymbol{a}; M)$,

(2.5) **Proposition.** *Let M be a module of FLC. Then*

$$I(M) = \sup\{I(\boldsymbol{a}; M);\ \boldsymbol{a} \textit{ is a system of parameters for } M\}.$$

Proof. It suffices to show that for some s.o.p. for M, the equality $I(\boldsymbol{a}; M) = I(M)$ holds. Now we go by induction on $s = \dim M$. Let $s = 1$

and a be any parameter for M. Then we have the following ascending chain:

$$[0 \underset{M}{:} a] \subset [0 \underset{M}{:} a^2] \subset \cdots \subset H^0_{\mathfrak{m}}(M)$$

which terminates with $[0 :_M a^N]$ for some integer N. Then the required equality holds for the parameter a^N for M.

Let $s \geqq 2$ and a be a parameter for M such that

$$aH^p_{\mathfrak{m}}(M) = 0$$

for $p=0, \cdots, s-1$. Then the exact sequence in (2.3) factors into the following exact sequences:

$$0 \longrightarrow H^p_{\mathfrak{m}}(M) \longrightarrow H^p_{\mathfrak{m}}(M/aM) \longrightarrow H^{p+1}_{\mathfrak{m}}(M) \longrightarrow 0,$$

for $p=0, \cdots, s-2$. Then by the definition, we have

$$I(M/aM) = \sum_{i=0}^{s-2} \binom{s-2}{i} h^i(M/aM) = \sum_{i=0}^{s-2} \binom{s-2}{i} [h^i(M) + h^{i+1}(M)]$$
$$= \sum_{i=0}^{s-1} \binom{s-1}{i} h^i(M) = I(M).$$

On the other hand, by the induction assumption, we find a s.o.p. $\boldsymbol{a}' = \{a_2, \cdots, a_s\}$ for M/aM such that

$$I(M/aM) = l_A(M'/aM') - e_0(\boldsymbol{a}'; M/aM) = l_A(M/(a, \boldsymbol{a}')M) - e_0(a, \boldsymbol{a}'; M).$$

The s.o.p. $\{a, \boldsymbol{a}'\}$ for M is the one which we wanted.

Conversely,

(2.6) **Proposition** ([S=C=T], [S_8] (3.3)). *A finitely generated A-module M is of FLC if*

$$\sup\{I(\boldsymbol{a}, M);\ \boldsymbol{a} \text{ is a system of parameters for } M\} < \infty.$$

Proof. See say (3.3) of [S_8].

The next is rather easy to see but useful.

(2.7) **Proposition.** *Let M be a module of $s = \dim M \geqq 2$. Then the following conditions are equivalent.*

(1) *M is of FLC.*

(2) *There exists an integer N and a parameter x for M in $\mathfrak{m}^{N+1}$ such that*

$$l_A([0 \underset{M}{:} x]) < 0$$

and

$$\mathfrak{m}^N H^p_{\mathfrak{m}}(M/xM)=0$$

for $p=0, \cdots, s-2$.

Proof. (1)$\Rightarrow$(2): Let k be an integer such that $\mathfrak{m}^k H^p_{\mathfrak{m}}(M)=0$ for any $p=0, \cdots, s-1$. Let $N=2k$. Then by the preceding lemma, we see that $\mathfrak{m}^N$ kills each $H^p_{\mathfrak{m}}(M/zM)$ with $p=0, \cdots, s-2$ for *any* parameter z for M hence for any parameter x in $\mathfrak{m}^{N+1}$.

(2)$\Rightarrow$(1): Since $H^0_{\mathfrak{m}}(M)$ is a submodule of M, we need only to prove that $H^p_{\mathfrak{m}}(M)$ is of finite length for $p=1, \cdots, s-1$. By the assumption on x, we can apply Lemma (2.5) to obtain the exact sequences

$$H^{p-1}_{\mathfrak{m}}(M/xM) \longrightarrow H^p_{\mathfrak{m}}(M) \xrightarrow{.x} H^p_{\mathfrak{m}}(M),$$

for $p=1, \cdots, s-1$. Let H denote $H^p_{\mathfrak{m}}(M)$ for such p. Then from the exact sequence above, we see the inclusion

$$[0 \underset{H}{:} \mathfrak{m}^N] \supset [0 \underset{H}{:} x].$$

By the choice of x, we furthermore have

$$[0 \underset{H}{:} \mathfrak{m}^{N+1}] \subset [0 \underset{H}{:} x].$$

Hence we have that

$$[0 \underset{H}{:} \mathfrak{m}^{N+1}] \subset [0 \underset{H}{:} \mathfrak{m}^N].$$

Let z be any element in H and assume that $\mathfrak{m}^L z=0$ for some $L>N$. Then $\mathfrak{m}^{L-(N+1)} z \subset [0 :_H \mathfrak{m}^{N+1}]$ hence

$$\mathfrak{m}^{L-1} z=\mathfrak{m}^N \mathfrak{m}^{L-(N+1)} z=0,$$

which implies that any element of H is killed by the N-th powers of $\mathfrak{m}$.

The above proof says essentially the following:

(2.8) **Corollary.** *Under the same assumption as above, assume* (2) *of* (2.7), *then we have*

$$\mathfrak{m}^N H^p_{\mathfrak{m}}(M)=0$$

for $p=0, \cdots, s-1$.

In the remaining part of this section, we will observe a Koszul homology of s.o.p.'s for Buchsbaum modules. As will be stated below in (2.10), they satisfy a condition for sequences of a ring relative to the module: namely weakly regular sequences:

(2.9) **Definition** ([St=V₁]). Let $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a sequence of elements in the local ring $(A, \mathfrak{m}, \boldsymbol{k})$ and M an A-module. Then $\boldsymbol{a}$ is called a weak M-sequence or a weakly regular sequence on M if the following holds

$$\mathfrak{m}[(a_1, \cdots, a_{i-1})M \underset{M}{:} a_i] \subset (a_1, \cdots, a_{i-1}),$$

for all $i=1, \cdots, s$.

In [St=V_1], the authors discussed the weak M-sequences and gave a characterization of Buchsbaum Modules in terms of weak M-sequence:

(2.10) **Theorem** (Satz 10. [St=V_1]). *Let M be a finitely generated A-module. Then the following conditions are equivalent*:

(1) *M is a Buchsbaum module*: *i.e., there exists a numerical invariant I of M such that for any paramater ideal $q=(\boldsymbol{a})$ for M, the following holds,*

$$l_A(M/qM)-e_0(q; M)=I,$$

where $l_A(\sharp)$ denotes the length of the A-module and $e_0(\sharp, \sharp)$ the multiplicity.

(2) *Any system of parameters for M forms a weak M-sequence.*

Remark. By the definition and (2.6), Buchsbaum modules are of FLC and the invariant $I=I(M)$. Also by the definition, if M is a Buchsbaum module and x is a parameter for M, then M/xM is also a Buchsbaum module and $I(M/xM)=I(M)$.

To demonstrate how the theory of FLC-modules works, we will give a proof of (2.10) making very best use of the preceding preparations in FLC-modules.

Proof. (1)$\Rightarrow$(2): Note that by its definition and by (2.6), Buchsbaum module M is a module of FLC.

To being with we will deal with dimension one case. Given any parameter x for M, we have

$$I(x; M)=l_A([0 \underset{M}{:} x])=I(M)=h^0(M).$$

Hence

$$[0 \underset{M}{:} x]=H^0_{\mathfrak{m}}(M)$$

holds for *any* parameter for M. Since we may choose a system of generators for $\mathfrak{m}$ consisting of parameters for M, we have

$$\mathfrak{m}H^0_{\mathfrak{m}}(M)=0,$$

and then we see that x is weakly M-regular.

Now let $s=\dim M \geqq 2$. Let x be any parameter for M. Then M/xM is also Buchsbaum module (of dimension $s-1$). By the remark just after (2.2), $I(M/xM)=I(M)$. On the other hand we have the following exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(xM) \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(M/xM) \longrightarrow H^1_{\mathfrak{m}}(M).$$

Hence if $H^0_{\mathfrak{m}}(xM) \neq 0$ then we have

$$h^0(M/xM) < h^0(M) + h^1(M).$$

Since

$$h^p(M/xM) \leqq h^p(M) + h^{p+1}(M)$$

for $p=1, \cdots, s-2$, we have

$$I(M/xM)=\sum_{i=0}^{s-2}\binom{s-2}{i}h^i(M/xM)<\sum_{i=0}^{s-1}\binom{s-1}{i}h^i(M)=I(M).$$

This is a contradiction. So we conclude that

$$H^0_{\mathfrak{m}}(xM)=0$$

and

$$[0 \underset{M}{:} x]=H^0_{\mathfrak{m}}(M).$$

So we see that *any* parameter x for M kills $H^0_{\mathfrak{m}}(M)$. Hence $\mathfrak{m}H^0_{\mathfrak{m}}(M)=0$ and x is in fact weak M-sequence. By the induction assumption, any s.o.p. for M/xM forms a weakly regular sequence on M/xM, we are over the proof of this direction.

Before starting converse implication, we deduce the following lemma from the above proof.

Lemma. *Let M be a Buchsbaum module of $s=\dim M$. Then for any parameter x for M,*

$$xH^p_{\mathfrak{m}}(M)=0,$$

for $i=0, \cdots, s-1$.

Consequently, $\mathfrak{m}H^p_{\mathfrak{m}}(M)=0$ *for any $i=0, \cdots, s-1$.*

Proof. In the above proof, we see that $I(M/xM)=I(M)$ and for each $i=0, \cdots, s-2$,

$$h^i(M/xM) \leqq h^i(M) + h^{i+1}(M),$$

hence the equality for each $i=0, \cdots, s-2$. Then in the exact sequence in (2.3), we see that the map

$$H^i_{\mathfrak{m}}(M) \xrightarrow{.x} H^i_{\mathfrak{m}}(M)$$

must be a zero-map for all $i=0, \cdots, s-1$.

We are now ready to give

Proof of (2)$\Rightarrow$(1). Induction on $s=\dim M$. Let $s=1$. Then for any parameter x for M, we have

$$[0 \underset{M}{:} \mathfrak{m}] \subset [0 \underset{M}{:} x] \subset [0 \underset{M}{:} \mathfrak{m}],$$

hence we have

$$I(x; M) = l_A([0 \underset{M}{:} x]) = l_a([0 \underset{M}{:} \mathfrak{m}]).$$

Let $s \geqq 2$ and x be any parameter for M. By the induction assumption, M/xM is a Buchsbaum module, hence by the lemma above, $\mathfrak{m}H^p_{\mathfrak{m}}(M/xM)=0$ for all $p=0, \cdots, s-2$. Since we may choose $x \in \mathfrak{m}^2$, by (2.7), M is of FLC, or furthermore by (2.8) $\mathfrak{m}H^p_{\mathfrak{m}}(M)=0$ for $p=0, \cdots, s-1$. Then just by the same way as the proof of the lemma above, we see that $I(M/xM)=I(M)$, since $h^i(M/xM)=h^i(M)+h^{i+1}(M)$ for each $i=0, \cdots, s-2$. Now let $\boldsymbol{x}=\{x_1, \cdots, x_s\}$ be any s.o.p. for M. Then

$$I(\boldsymbol{x}; M) = I(\boldsymbol{x}'; M/x_1M) = I(M/x_1M) = I(M)$$

with $\boldsymbol{x}'=\{x_2, \cdots, x_s\}$. Q.E.D.

When we consider Buchsbaum modules as a generalization of Cohen-Macaulay modules/rings, the above corresponds to the following characterization of C.-M.-modules: a finitely generated module M is a C.M.-module if and only if any s.o.p. for M is a regular sequence on M. Note also that a C.-M.-module M is a Buchsbaum module with the invariant $I(M)=0$.

Since one of our main purpose in this article is to set up a criterion of a quasi-Buchsbaum module in terms of weakly regular sequence, to clarify the motivation let us present some results related to the Koszul complex generated by a s.o.p. of Buchsbaum modules. The subsequent discussion on the s.o.p. of Buchsbaum modules corresponds to the characterization of regular sequences by Koszul homology.

The main of this part is

(2.11) **Theorem** (cf. [S_4], [S=S=V]). *Let M be a finitely generated module over a local ring $(A, \mathfrak{m}, k)$ and $s=\dim M$. Then the following conditions are equivalent:*

(1) *M is a Buchsbaum module;*

(2) *for any sub-s.o.p. $\{x_1, \cdots, x_r\}$ for M,*

$$\mathfrak{m}H_p(x_1, \cdots, x_r; M)=0,$$

for $p=1$, (resp. for all $p>0$);

(3) *for any s.o.p $\{x_1, \cdots, x_s\}$ for M,*

$$\mathfrak{m}H_p(x_1, \cdots, x_s; M)=0,$$

for $p=1$, (resp. for all $p>0$).

The proof will be achieved in several statements. To begin with

(2.12) **Lemma.** *Let M be a Buchsbaum module of positive dimension and x be a parameter for M. Then*

$$H^0_{\mathfrak{m}}(M)=[0 \underset{M}{:} x]=[0 \underset{M}{:} \mathfrak{m}].$$

Proof. By (2.10), a parameter x is weakly regular on M. Hence the second equality is valid. Now let z be any element in $H^0_{\mathfrak{m}}(M)$. Then $\mathfrak{m}^n z=0$ for some integer n and we have $x^n z=0$, i.e., $H^0_{\mathfrak{m}}(M)\subset[0:_M x^n]=[0:_M \mathfrak{m}]$. Converse inclusion is trivial.

The next leads the implication (1)⇒(2) for $p=1$.

(2.13) **Proposition.** *Let M be a Buchsbaum module of dimension s and $x_1, \cdots, x_r$ be a sub-s.o.p. for M ($r\leqq s$). Then we have*

(1) $\mathfrak{m}H_1(x_1, \cdots, x_r; M)=0$;

(2) $Z_1(x_1, \cdots, x_{r-1}; M)=[B_1(x_1, \cdots, x_{r-1}; M) :_{K_1} x_r]$.

Proof. Induction on r, the length of the sub-s.o.p. Let $r=1$: The assertion (2) is vacuous. (1) requires nothing but the weak regularity of x_1 on M, which is true.

Let $r=2$. Then (2) requires the following

$$[0 \underset{M}{:} x_1]=[0 \underset{M}{:} x_2],$$

which is also valid by (2.12).

Now let $r\geqq 2$. Since $\{x_1, \cdots, x_{r-2}, x_r^m\}$ forms a sub-s.o.p. for M of length $<r$, we have

$$Z_1(x_1, \cdots, x_{r-2}; M)=[B_1(x_1, \cdots, x_{r-2}; M) \underset{K_1}{:} x_r^m].$$

Since $K_0=Z_0$, the following is clear:

$$[B_0(x_1, \cdots, x_{r-2}; M) \underset{K_0}{:} \mathfrak{m}] \subset Z_0.$$

Since x_r is a parameter for a Buchsbaum module $M/(x_1, \cdots, x_{r-2})M$, by (2.12), we have

$$[B_0(x_1, \cdots, x_{r-2}) \underset{K_0}{:} x_r^m] \subset [B_0 \underset{K_0}{:} \mathfrak{m}].$$

Furthermore by induction hypothesis, $\mathfrak{m}H_1(x_1, \cdots, x_{r-1}; M)=0$. We can now apply (1.4) with $L_{\bullet}=K_{\bullet}(x_1, \cdots, x_{r-2}; M)$, $x=x_r$, $y=x_{r-1}$ and $p=1$, to obtain (2).

Let $(m, n) \in K_1(x_1, \cdots, x_r; M)=L_0 \oplus L_1$, with $L_{\bullet}=K_{\bullet}(x_1, \cdots, x_{r-1}; M)$. Let $e_{\bullet}$ denote the differential map of $L_{\bullet}$. Let $a \in \mathfrak{m}$ and assume that $(m, n) \in Z_1(x_1, \cdots, x_r; M)$. We must show that $a(m, n) \in B_1(x_1, \cdots, x_r; M)$.

The cycle condition is

$$(\sharp) \qquad x_r m+e_1(n)=0,$$

this means at first that

$$m \in [B_0(L_{\bullet}) \underset{K0}{:} x_r]=[(x_1, \cdots, x_{r-1})M \underset{M}{:} \mathfrak{m}]$$

and

$$(\sharp 2) \qquad am=e_1(u)$$

for some $u<-L_1$. By the induction hypothesis $\mathfrak{m}H_1(x_1, \cdots, x_{r-1}; M)=0$ and n determines a cycle in $L_1/x_rL_1=K_1(x_1, \cdots, x_{r-1}; M/x_rM)$, we have

$$(\sharp 3) \qquad an=x_rv+e_2(w)$$

for some $(v, w) \in L_1 \oplus L_2$. Operate e_1 to the equality, and we have

$$ae_1(n)=x_re_1(v).$$

On the other hand, by ($\sharp$),

$$e_1(x_ru+an)=a(x_r+e_1(u))=0.$$

So $x_ru=an \in Z_1(L_{\bullet})$. By ($\sharp$3),

$$x_ru+x_rv \in Z_1(L_{\bullet})=[B_1(L_{\bullet}) \underset{L_1}{:} x_r].$$

Hence

$$(u+v) \in [B_1(L_{\bullet}) \underset{L_1}{:} x_r^2]=Z_1(L_{\bullet}), \quad \text{i.e., } e_1(u)+e_1(v)=0.$$

with (♯2), $am=e_1(u)=-e_1(v)$. Thus we have

$$a(m, n)=(-e_1(v), x_r v+e_2(w)) \in B_1(x_1, \cdots, x_r; M):$$

this is what we have to show. Q.E.D.

The following covers the implication (3)⇒(1) with $p=1$.

(2.14) **Proposition.** *Let M be a finitely generated A-module and $x_1, \cdots, x_r$ be elements in $\mathfrak{m}$. Assume that for any system of elements $\boldsymbol{x}=\{x_2^{b_2}, \cdots, x_r^{b_r}\}$ with $b_i \in \{1, 2\}$ for $i=2, \cdots, r$, we have*

$$\mathfrak{m}H_1(x_1, \boldsymbol{x}; M)=0.$$

Then $\{x_1, \cdots, x_r\}$ forms a weak M-sequence.

Proof. Induction on r. Let $r=1$. Then x is in fact a weakly regular on M.

Let $r \geqq 2$. By (1.3), we have

$$\mathfrak{m}H_1(x_1, x_2^{b_2}, \cdots, x_{r-1}^{b_{r-1}}; M) \subset x_r^2 H_1(x_1, x_2^{b_2}, \cdots, x_{r-1}^{b_{r-1}}; M)$$
$$\subset \mathfrak{m}^2 H_1(x_1, x_2^{b_2}, \cdots, x_{r-1}^{b_{r-1}}; M).$$

By Nakayama's lemma,

$$\mathfrak{m}H_1(x_1, x_2^{b_2}, \cdots, x_{r-1}^{b_{r-1}}; M)=0.$$

Then by the induction assumption, we can conclude that $\{x_1, \cdots, x_{r-1}\}$ forms a weak M-sequence. On the other hand, by (1.3) again, we see that

$$[0: x_r]_{H_0(x_1, \cdots, x_{r-1}; M)}$$

is a homomorphic image of $H_1(x_1, \cdots, x_r; M)$. Hence x_r is weakly regular on $M/(x_1, \cdots, x_{r-1})M$. Q.E.D.

We have already finished the proof of (2.11) for $p=1$ in (2) and (3). To complete the proof we give the following:

(2.15) **Proposition.** *Let M be an A-module of finite type. Assume that the following two conditions hold for any sub-s.o.p. $\boldsymbol{x}=\{x_1, \cdots, x_r\}$ for M:*

(1) $\mathfrak{m}H_1(\boldsymbol{x}; M)=0$;

and

(2) $Z_1(\boldsymbol{x}'; M)=[B_1(\boldsymbol{x}'; M) \underset{K_1}{:} x_r^m]=[B_1(\boldsymbol{x}'; M) \underset{K_1}{:} \mathfrak{m}]$,

for any integer $m>0$ with $\boldsymbol{x}'=\{x_1, \cdots, x_{r-1}\}$.

Then for any sub-s.o.p. $\boldsymbol{x}$ *for* M *and for all integer* $p>0$, *the following hold*:

(1′) $\mathfrak{m}H_p(\boldsymbol{x}; M)=0$;

and

(2′) $Z_p(\boldsymbol{x}'; M)=[B_p(\boldsymbol{x}'; M) \underset{K_p}{:} x_r^m]=[B_p(\boldsymbol{x}'; M) \underset{K_p}{:} \mathfrak{m}]$,

for any integer $m>0$.

Proof. By (1), we see that M is a Buchsbaum module. Let $p\geqq 2$. Then for $r=1$, (1′) is clear. For $r=2$,

$$H_2(x_1, x_2; M)=[0 \underset{M}{:} (x_1, x_2)]\subset[0 \underset{M}{:} x_1],$$

hence for this case (1′) is also over. (2′) is trivial, because $K_p(x_1; M)=Z_p(x_1; M)=0$.

Let $r\geqq 3$. By the induction assumption, on length of the sub-s.o.p., all the requirements in applying (1.4) are fulfilled. (2′) is directly concluded by (1.4) with $x=x_r$, $y=x_{r-1}$ and $L_\bullet=K_\bullet(x_1, \cdots, x_{r-2}; M)$.

As to (1′), let $L_\bullet=K_\bullet(x_1, \cdots, x_{r-1}; M)$ and $e_\bullet$ be the differential map. Let $(m, n)\in Z_p(x_1, \cdots, x_r; M)\subset L_{p-1}\oplus L_p$ and a be any element of $\mathfrak{m}$. We claim that $a(m, n)\in B_p(x_1, \cdots, x_r; M)$. We have by the cycle condition,

$$(\sharp) \qquad e_{p-1}(m)=0 \quad \text{and} \quad x_r+e_p(n)=0.$$

Namely $m\in Z_{p-1}(L_\bullet)$, hence by the induction assumption,

$$am=e_p(u)$$

for some $u\in L_p$. Also by the induction assumption, we have

$$\mathfrak{m}H_p(x_1, \cdots, x_{r-1}; M/x_rM)=0.$$

By ($\sharp$) we see that u determines a cycle in $K_p(x_1, \cdots, x_{r-1}; M/x_rM)$, so

$$(\sharp 2) \qquad an=x_r+e_{p+1}(w)$$

for some $(v, w)\in L_p\oplus L_{p+1}$. Hence

$$e_p(x_rv+x_ru)=e_p(an)+x_re_p(u)=a(e_p(n)+x_rm)=0.$$

So $x_r(u+v)\in Z_p((L_\bullet)=[B_p(L_\bullet) :_{L_p} x_r]$, and

$$u+v\in[B_p(L_\bullet) \underset{L_p}{:} x_r^2]=Z_p(L_\bullet).$$

Namely

$$e_p(u)=-e_p(v).$$

We have

$$am=-e_p(v) \quad \text{and} \quad an=x_r(v)+e_{p+1}(w),$$

i.e., $a(m, n) \in B_p(x_1, \cdots, x_r; M)$, as required. Q.E.D.

Now the proof of (2.11) is completed.

§ 3. A basic theorem of quasi-Buchsbaum rings and modules

In this section we present a theorem which characterizes a quasi-Buchsbaum Modules in terms of system of parameters.

Throughout this section, A denotes a local ring with the maximal ideal $\mathfrak{m}$.

Let us start with the following:

(3.0) **Definition.** *Let M be a finitely generated module over a local ring A with its maximal ideal $\mathfrak{m}$ and the residue field $\boldsymbol{k}$. Then M is called a quasi-Buchsbaum module if*

$$\mathfrak{m}H^p_{\mathfrak{m}}(M)=(0)$$

for all $p \neq s := \dim(M)$. A local ring is called a quasi-Buchsbaum ring if it is so as a module over itself.

(3.1) **Notations.** Let M be an A-module and $\boldsymbol{a}=\{a_1, \cdots, a_s\}$ be a system of parameters of A. Let $L_\bullet$ denote the complex defined as the direct limit of the Koszul complexes $K_\bullet(a_1^m, \cdots, a_s^m; A)$, more precisely,

$$L_\bullet := (\varinjlim (K_\bullet(a^m; A))^*)[s].$$

Let $K_\bullet := L_\bullet \otimes_A M$, and $Z_\bullet = Z_\bullet(K_\bullet)$, $B_\bullet = B_\bullet(K_\bullet)$ and $H_\bullet = H(K_\bullet)$ be the cycle, boundary and the homology of the complex $K_\bullet$, respectively.

(3.2) **Remarks.**

(1) Quasi-Buchsbaum modules are modules of FLC.

(2) As already remarked in (1.5) the local cohomology can be calculated as the (co-)homology module of the complex $K_\bullet$, ((2.3) of [G_r]): namely

$$H^p_{\mathfrak{m}}(M)=H_{s-p}(L_\bullet \otimes_A M),$$

for all p.

Our first job is to prepare an easy but important fact on the action of a regular element on a complex:

(3.3) **Lemma.** *Let $(X_{\bullet}, d_{\bullet})$ be a complex in the category of A-modules and A-linear mappings and x be an element of A which is a non-zero-divisor on $X_{\bullet}$.*

Assume that the p-th homology module H_p is killed by x. Then $Z_p \cap xK_p \subset B_p$.

Proof. Let $z = xw$ be any element in $Z_p \cap xK_p$. Then

$$xd_x(w) = d_p(z) = 0.$$

Since x is a non-zero-divisor on K_{p-1}, we have $d_p(w) = 0$, hence w is a p-th cycle. Since x kills the homology, we have

$$z = xw \in B_{s-p}.$$

Remark. As can be easily guessed from the above, once we reduced to the depth positive cases, it became rather easy to work on the homology of the complex of the above type. Indeed in many cases in the calculation of homology modules related to the s.o.p.'s of Buchsbaum or FLC modules, or also d-sequences the gap between the boundary and the intersection $Z_{\bullet} \cap xK_{\bullet}$ is one obstacle hard to clear.

In the case where depth of the module M is zero, the next lemma is of good help when we treat the parameter elements in relation with the Artinian part $H^0_{\mathfrak{m}}(M)$ of M.

(3.4) **Lemma.** *Let M be a module with FLC and x be a parameter for M.*

If $\mathfrak{m}H^0_{\mathfrak{m}}(M) = 0$, then we have

$$[0 \underset{M}{:} x^n] = H^0_{\mathfrak{m}}(M).$$

for all integer $n \geqq 1$,

Consequently,

$$xM \cap H^0_{\mathfrak{m}}(M) = (0).$$

Proof. Since we assume M to be of FLC, by (2.4), we have

$$[0 \underset{M}{:} x] \subset H^0_{\mathfrak{m}}(M).$$

The first assertion then follows from the following sequence of inclusions:

$$[0 \underset{M}{:} \mathfrak{m}] \subset [0 \underset{M}{:} x] \subset [0 \underset{M}{:} x^n] \subset H^0_{\mathfrak{m}}(M) \subset [0 \underset{M}{:} \mathfrak{m}].$$

The second one is then straightforward.

(3.5) **Lemma.** *Let M be a quasi-Buchsbaum module of* $s=\dim M\geqq 2$ *and x be any parameter for M. Then we have the following exact sequence*:

$$0\longrightarrow H^i_{\mathfrak{m}}(M)\longrightarrow H^i_{\mathfrak{m}}(M/xM)\longrightarrow H^{i+1}_{\mathfrak{m}}(M)\longrightarrow 0,$$

for $i=0, \cdots, s-2$.

Proof. This follows from (2.3).

We are now ready to prove our main result in this section:

(3.6) **Theorem.** *Let M be a finitely generated module over a local ring* $(A, \mathfrak{m}, k)$ *of* $\dim M=s\geqq 2$. *Then the following conditions are equivalent*:

(1) *M is a quasi-Buchsbaum module*;

(2) *For any parameter $x\in\mathfrak{m}^2$ for M, M/xM is a quasi-Buchsbaum module and* $l_A([0:_M x])<\infty$.

(3) *For some parameter $x\in\mathfrak{m}^2$ for M, M/xM is a quasi-Buchsbaum module and* $l_A([0:_M x])<\infty$.

Remark. Restriction on $s=\dim M$ above is indispensable: Indeed for *any* A-module of Krull-dimension one, the second assertion of above theorem is always true.

Proof. (1)$\Rightarrow$(2): By the preceding Lemma, we have

$$\mathfrak{m}^2 H^i_{\mathfrak{m}}(M/xM)=0,$$

for $i=0, \cdots, s-2$. Let $M':=M/xM$ and $I:=[0:_A H^0_{\mathfrak{m}}(M')]$.

Suppose that $I\neq\mathfrak{m}$, then there exists $z\in\mathfrak{m}-I$ which is a parameter for both M and M/xM, since we assumed $s=\dim M\geqq 2$.

We however claim at first that $zH^0_{\mathfrak{m}}(M')=0$: contradiction to the choice of z. From the exact sequence

$$0\longrightarrow M/H^0_{\mathfrak{m}}(M)\xrightarrow{.x} M/H^0_{\mathfrak{m}}(M)\longrightarrow M+(xM/H^0_{\mathfrak{m}}(M))\longrightarrow 0,$$

induced an isomorphism

$$H^0_{\mathfrak{m}}(M/(xM+H^0_{\mathfrak{m}}(M)))=H^1_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M))=H^1_{\mathfrak{m}}(M).$$

Let $m'=m \bmod xM$ be any in $H^0_{\mathfrak{m}}(M')$. Then since $\mathfrak{m}^2$ kills $H^0_{\mathfrak{m}}(M')$, $\mathfrak{m}^2 m\subset xM+H^0_{\mathfrak{m}}(M)$ and by the above isomorphsim we see that $\mathfrak{m}m\subset xM+H^0_{\mathfrak{m}}(M)$. So $zm=xn+t$ for some $n\in M$ and $t\in H^0_{\mathfrak{m}}(M)$. Then $zxm=x^2n$ hence $n\in[zM:_M x^2]$.

Since $\{z, x^2\}$ is a part of s.o.p. for M,

$$[0:x^2]_{(M/zM)}\subset H^0_{\mathfrak{m}}(M/zM)$$

and by (3.5), we have

$$H^0_{\mathfrak{m}}(M/zM)\subset[0:\mathfrak{m}^2]_{(M/zM)}.$$

The right hand side of the above inclusion is contained in $[0:x]_{(M/zM)}$. Thus we see that

$$[0:x^2]_{(M/zM)}\subset[0:x]_{(M/zM)},$$

hence $xn=zu$ for some $u\in M$. So we finally have that

$$t=zm-xn=zm-zu\in zM,$$

hence $t\in zM\cap H^0_{\mathfrak{m}}(M)$, namely $zm=xn\in xM$ and we conclude that $zm'=0$: this is the required contradiction. We thus have proved that

$$\mathfrak{m}H^0_{\mathfrak{m}}(M/xM)=0.$$

Let $1\leqq i\leqq s-2$ and assume that

$$I:=[0\underset{A}{:}H^i_{\mathfrak{m}}(M/xM)]\neq\mathfrak{m}.$$

Again we can choose $z\in\mathfrak{m}-I$ so that z is a parameter for both M and $M':=M/xM$.

By Lemma (3.4), we can induce the following exact sequence

$$0\longrightarrow H^0_{\mathfrak{m}}(M)\longrightarrow M'\longrightarrow M/(H^0_{\mathfrak{m}}(M)+xM)\longrightarrow 0.$$

Thus for our case, we have the isomorphism

$$H^i_{\mathfrak{m}}(M/xM)=H^i_{\mathfrak{m}}(M/(H^0_{\mathfrak{m}}(M)+xM)),$$

hence we can assume that $H^0_{\mathfrak{m}}(M)=0$. Then we have the following exact sequence

$$0\longrightarrow M\overset{.x}{\longrightarrow}M\longrightarrow M/xM\longrightarrow 0.$$

Let $L_{\bullet}$ denote the flat complex

$$(\varinjlim(K_{\bullet}(\boldsymbol{a}^m;A))^*)[s]$$

introduced in (3.1), for some system of parameters $\boldsymbol{a}$ of A. Then there exists an exact sequence of complexes

$$0\longrightarrow M\otimes_A L_{\bullet}\overset{.x}{\longrightarrow}M\otimes_A L_{\bullet}\longrightarrow M'\otimes_A L_{\bullet}\longrightarrow 0.$$

Denote by $(K_\bullet, d_\bullet)$ and $(K'_\bullet, d_\bullet)$ the complexes $M\otimes_A L_\bullet$ and $M'\otimes_A L_\bullet$ with the boundary operators $d_\bullet$ and $d'_\bullet$, respectively.

Let $[m']:=m' \bmod B_{s-i}(K'_\bullet)$ be any homology class in $H_{s-i}(K'_\bullet)$ with $m'=m \bmod xK_{s-i} \in Z_{s-i}(K'_\bullet)$. The cycle condition of m' implies that for some $w \in K_{s-i-1}$,

$$d_{s-i}(m)=xw.$$

Then w must also be a cycle: Indeed

$$xd_{s-i-1}(w)=d_{s-i-1}(xw)=0$$

and since x is a non-zero-divisor on K_{s-i-2}, $d_{s-i-1}(w)=0$. Since the homology $H_{s-i-1}(K_\bullet)$ is killed by $\mathfrak{m}$, we have

(♯): $$zw=d_{s-i}(u)$$

for some $u \in K_{s-i}$. Hence we have

(♯♯): $$d_{s-i}(zm)=zd_{s-i}(m)=zxw=d_{s-i}(xu).$$

Consequently, $zm-xu \in Z_{s-i}(K_\bullet)$. Again the homology is killed by $\mathfrak{m}$, we see that

$$x(zm-xu) \in B_{s-i}(K_\bullet),$$

or

$$x^2u \in zK_{s-i}+B_{s-i}(K_\bullet).$$

Let $u':=u \bmod zK_{s-i}$. Then by (♯) u' is a cycle in K'. On the other hand, by the choice of z and by (3.3), we have

$$\mathfrak{m}^2 H^i_{\mathfrak{m}}(M')=0$$

and hence $x^2u'=x^2(u \bmod zK_{s-i}) \in B_{s-i}((M/zM)\otimes L_\bullet)$ or

$$u' \bmod B_{s-i}((M/zM)\otimes L_\bullet) \in [0\colon x^2]_{H^i_{\mathfrak{m}}(M/zM)}.$$

Since $x \in \mathfrak{m}^2$, by Lemma (3.3), we see that

$$[0\colon x^2]_{H^i_{\mathfrak{m}}(M/zM)} \subset [0\colon x]_{H^i_{\mathfrak{m}}(M/zM)},$$

hence

$$xu'=x(u \bmod zK_{s-i}) \in B_{s-i}((M/zM)\otimes L_\bullet).$$

So there exists $v \in K_{s-i}$ such that $xu-zv \in B_{s-i}(K_\bullet)$, which leads together with (♯♯),

$$d_{s-i}(zv)=d_{s-i}(xu)=d_{s-i}(zm)$$

and

$$z(v-m)\in Z_{s-i}(K_{.})\cap zK_{s-i}.$$

By (3.3), we have

$$z(v-m)\in B_{s-i}(K_{.}).$$

We therefore have

$$zm-xu=(zm-zv)+(zv-xu)\in B_{s-i}(K_{.})$$

and we finally see that

$$z(m \bmod xK_{s-i})\in B_{s-i}(K'_{.}),$$

namely z kills $H^i_{\mathfrak{m}}(M/xM)$, against the choice of z.

The implication (2)$\Rightarrow$(3) is trivially true in our case. As to the impilcation: (3)$\Rightarrow$(1): We can apply (2.8) with $N=1$. Q.E.D.

The following was announced in [S=S=V], which contains no proofs at all.

(3.7) **Corollary** ([S=S=V]). *Let M be a finitely generated A-module of dimension s. Then the following conditions are equivalent*:

(1) *M is a quasi-Buchsbaum module*;

(2) *any system of parameters for M contained in $\mathfrak{m}^2$ forms a weak M-sequence*;

(3) *there exists a weak M-sequence of length s in $\mathfrak{m}^2$.*

Proof. We need only to treat the case where $s>0$. (1)$\Rightarrow$(2): Let x be a parameter for M in $\mathfrak{m}^2$. Then by (3.4), $[0:_M x]=H^0_{\mathfrak{m}}(M)$ and x is weakly regular on M. We are done for the case for $s=1$, and also for the first element $x=x_1$ of the s.o.p. $\boldsymbol{x}=\{x_1, \cdots, x_s\}$ in the case where $s\geqq 2$. By (3.6), M/xM is a quasi-Buchsbaum module of dimension $s-1$ hence by the induction assumption the s.o.p. $x_2, \cdots, x_s$ for M/xM form a weakly regular sequence on M/xM. That is what we have to show.

(2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1): Assume that x is weakly regular on M. Then we have the following inclusion

$$[0\underset{M}{:}\mathfrak{m}^2]\subset[0\underset{M}{:}x]\subset[0\underset{M}{:}\mathfrak{m}],$$

which implies that $\mathfrak{m}H^0_{\mathfrak{m}}(M)=0$: indeed if $z\in M$ is killed by $\mathfrak{m}^K$ for some $K>1$, then $\mathfrak{m}^{L-2}z$ is killed by $\mathfrak{m}^2$ hence by $\mathfrak{m}$; namely $\mathfrak{m}^{L-1}z=0$. Thus any element of $H^0_{\mathfrak{m}}(M)$ is killed by $\mathfrak{m}$.

In order to go further, we need to remark the fact that weak M-sequence is a sub-s.o.p. for M. Indeed, if x is a weakly regular and belongs to some prime ideal $\mathfrak{p}$ in the support of M with $\dim(A/\mathfrak{p})=\dim M$, then since $\mathfrak{p}\neq\mathfrak{m}$, $[0:_M x]_{\mathfrak{p}}=0$. But then $M_{\mathfrak{p}}=0$ for $\dim M_{\mathfrak{p}}$ is 0.

Now we see that M/x_1M is of $\dim s-1$, hence by the induction assumption we can conclude that M/x_1M is a quasi-Buchsbaum module. Then again by (3.6) we can conclude that M is also a quasi-Buchsbaum module. Q.E.D.

By (2.10), any s.o.p. for a Buchsbaum module is a weakly regular on the module. Hence

(3.8) **Corollary.** *A Buchsbaum module is a quasi-Buchsbaum module.*

There naturally arises a question: How far quasi-Buchsbaum modules are from Buchsbaum modules? It is rather difficult to find an acceptable answer to it. Though it is still an intermediate one, let us introduce a result by S. Goto.

(3.9) **Proposition** ([G_6]). *Let M be a quasi-Buchsbaum module of dimension $s>0$ and $\boldsymbol{a}=\{a_1, \cdots, a_v\}$ be a system of generators for the maximal ideal $\mathfrak{m}$. Assume that any s-elements subset of $\boldsymbol{a}$ is a s.o.p. for M and M/a_iM is a Buchsbaum module for each $i=1, \cdots, v$, then M is also a Buchsbaum module.*

On the other hand, as to the examples of quasi-Buchsbaum modules which are not Buchsbaum modules, see for example [G_4]. One sees systematic construction of such examples there.

§ 4. Canonical duality and index of reducibility for parameter ideals

This last section is devoted to introduce some results related to modules of FLC.

We firstly discuss the canonical duality theory. Let us recall the definition of the canonical module.

(4.1) **Definition** (cf. [H=K]). Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field $\boldsymbol{k}$ and E a finitely generated A-module. A finitely generated A-module K is called the canonical module of and E denoted by K_E if the completion of K is isomorphic to

$$\operatorname{Hom}_A(H^s_{\mathfrak{m}}(E), E_A(\boldsymbol{k})),$$

where $s=\dim(E)$ and $E_A(\boldsymbol{k})$ denotes the injective envelope of $\boldsymbol{k}$ over A.

The canonical module of a local ring played an important role as a module invariant in the development of theory of Cohen-Macaulay rings/modules and particularly of Gorenstein rings, (e.g. [H=K], [G=S=W], [G=W]). In fact, one can often get several important informations about the target ring through the canonical module. Also in a very wide sense of words, the canonical module is a dual object that complements the half of the properties of the ring. One can realize the fact through the theorem stated below, (4.3).

On the other hand, when we develop some new notion or objects, the canonical module often serves as a testing object or material for constructing examples.

In fact, from the start of research of Buchsbaum rings, we wanted the elementary and acceptable proof of the following: Is the canonical module of a Buchsbaum ring also a Buchsbaum module?

In the lower dimensional cases, we could rather easily prove affirmatively. In the general form Schenzel applied the Dualizing complex characterization of Buchsbaum rings ([Sch]).

In terms of u.s.*d*.-sequences, our ultimate answer is stated as follows:

(4.2) **Theorem.** *Let A be a commutative ring with $1 \neq 0$ and E an A-module. Assume the a sequence $\boldsymbol{a}=a_1, \cdots, a_s$ of elements in A forms a u.s.d-sequence on E. Then for any injective A-module I, the sequence forms a u.s.d-sequence on*

$$\mathrm{Hom}_A(H^s_{\boldsymbol{a}}(E), I),$$

where $H^s_{\boldsymbol{a}}(E)$ stands for the limit of the direct system of Koszul (co)*-homology modules*

$$H^i(a_1^n, \cdots, a_s^n; E)$$

and mappings

$$\phi^{n,n+1}\colon H^i(\boldsymbol{a}^n; E) \longrightarrow H^i(\boldsymbol{a}^{n+1}; E),$$

where $\boldsymbol{a}^m$ denotes the system of elements $a_1^m, \cdots, a_s^m$, for an integer $m>0$.

Before going further into the theory of u.s.*d*-sequences, we present here the FLC-version of canonical duality. In [S_2] the author gave a proof of it applying the theory of the generalized local cohomology studied in [S_1] and the proof was based on the spectral sequence argument. But once we view the FLC-modules as derived ones from Cohen-Macaulay rings/modules, we can give quite an elementary proof by reducing to the Cohen-Macaulay case. Such a proof is given in [S_8].

Our theory establishes a duality for local cohomology modules:

(4.3) **Theorem** (cf. [S_2], [S_7] and [S_8]). *Let $(A, \mathfrak{m}, k)$ be a local ring and M a finitely generated A-module of dimension s. Suppose that M is with F.L.C. Let us define the functor*

$$D^p(\sharp) := \mathrm{Hom}_A(H^p_{\mathfrak{m}}(\sharp), E_A(k)).$$

Then in case $s \geqq 2$, we have the following exact sequence

$$0 \longrightarrow D^0D^0(M) \longrightarrow \hat{M} \longrightarrow D^sD^s(M) \longrightarrow D^0D^1(M) \longrightarrow 0$$

where $\hat{M}$ denotes the completion of M, and the isomorphisms

$$D^1D^s(M) \cong D^0D^s(M) = 0.$$
$$D^iD^s(M) \cong D^0D^{s-i+1}(M)$$

for $i = 2, \cdots, s-1$.

If $s = 1$ then $D^0D^1(M) = 0$.

Now the following is clear to see.

(4.4) **Corollary.** *Let M be a module of FLC (resp. a quasi-Buchsbaum module) of dimension s. Assume that there exists the canonical module K_M of M. Then K_M is also a module of FLC (resp. a quasi-Buchsbaum module).*

The Buchsbaum version is rather difficult than the above FLC or quasi-Buchsbaum cases. The auther however gave an elementary proof by close observation of Koszul homology of s.o.p. in the three dimensional case, which was expected to extend to higher dimensional cases. But to succeed in it, we needed the recognition of s.o.p.'s for Buchsbaum module as (u.s.) d-sequences. The first version of canonical duality for Buchsbaum modules appeared in [S_5] was based on serveral facts that contributed formulations of fundamental theorems in the theory of u.s.d-sequences developed in [G=Y]. While the success in it motivated the construction of the theory itself.

We have now come to

(4.5) **Definition** (cf. [Hu] and [G=Y]). Let A be a commutative ring and E be an A-module. A sequence of elements $\boldsymbol{a} = a_1, \cdots, a_s$ in A is called a d-sequence on E if for each $i = 1, \cdots, s$ and for any j with $i \leqq j \leqq s$ the following holds,

$$[(a_1, \cdots, a_{i-1})E : a_i a_j] = [(a_1, \cdots, a_{i-1})E : a_j].$$

A sequence $\boldsymbol{a}$ is called a *strong* d-sequence on E, if for any integers $n_1, \cdots, n_s > 0$, the sequence

$$a_1^{n_1}, \cdots, a_s^{n_s}$$

forms a d-sequence on E.

If besides each of the properties is stable under any permutation of the sequence, the term *unconditioned* is attached.

By the following, one can see the role of u.s.d-sequences in the theory of modules of FLC.

(4.6) **Proposition** (cf. [Sch$_2$]). *Let M be a finitely generated A-module of FLC. Then an s.o.p. $\boldsymbol{a}=a_1, \cdots, a_s$ for M is a u.s.d-sequence on M if and only if the following holds*:

$$I(\boldsymbol{a}; M)=I(M),$$

with $I(M)$ the Buchsbaum invariant of M and $I(\boldsymbol{a}; M)$ the function introduced in (2.2).

We next present a characterization of FLC-module in the terms of u.s.d-sequences.

(4.7) **Theorem** (cf. [S=C=T], [G=Y] and [S$_8$]). *Let M be a finitely generated A-module of dimension s. Then the following conditions are equivalent.*

(1) *There exists an s.o.p. for M that forms a u.s.d-sequence on M.*

(2) $\sup\{I(q; M);\ q$ *is a parameter ideal for* $M\}<\infty$.

(3) *M is a module of FLC.*

From the preceding two results, One can deduce the following characterization of Buchsbaum modules, which provides an entrance to the theory of Buchsbaum modules from the theory of u.s.d-sequences.

(4.8) **Theorem** ([Hu]). *A finitely generated module M over a local ring A is a Buchsbaum ring if and only if any s.o.p. for M is a u.s.d-sequence on M.*

By this characterization of Buchsbaum modules in terms of u.s. d-sequences, our canonical duality theory (4.2) for u.s.d-sequences leads directly the canonical duality for Buchsbaum modules.

To close this section let us see another interesting fact of s.o.p. of rings of FLC. To provide a quite different point of view in the theory of modules of FLC, we will quote some results from the joint work [G=S] with S. Goto.

(4.9) **Definition** ([G=S]). Given a finitely generated module M over a local ring A and a s.o.p. $\boldsymbol{a}$ of M, define the index of reducibility of $\boldsymbol{a}$ with respect to M as:

$$r_M(\boldsymbol{a}):=\dim_k(\mathrm{Hom}_A(A/\mathfrak{m}, M/(\boldsymbol{a})M)),$$

the dimension as a k-vector space of the socle of $M/(\boldsymbol{a})M$.

It is well known that if M is C.-M., then the index of reducibility is a constant number, i.e, an invariant of M. However the converse is not true. Such was studied in [E=N]. We could give a partial results of characterization of such rings/modules.

Particularly, in the case of FLC-rings, we found an interesting facts. To state our results we need the following

(4.10) **Definition.** Define the type of a module M over a local ring A as;

$$r(M):=\sup\{r_M(\boldsymbol{a})\colon \boldsymbol{a} \text{ is a s.o.p. for } M\}.$$

Then we have

(4.11) **Theorem** ([G=S]). *If M is of FLC, then the type $r(M)$ of M is bounded from above by*

$$\sum_{i=0}^{s-1}\binom{s}{i}l_A(H^i_{\mathfrak{m}}(M))+v_A(K_M)$$

where $v_A(\sharp)$ denotes the number of minimal generators.

If M is quasi-Buchsbaum ring then the type $r(M)$ coincides with the above number.

One can easily deduce the above from rather detailed expression of the above as follows:

(4.11a) **Theorem** ((2.1), (2.3) and (2.5) [G=S]). *Let M be a module of FLC of dimension s. Then we have*

$$\sum_{i=0}^{s}\binom{d}{i}\dim_k \operatorname{Hom}_A(k, H^i_{\mathfrak{m}}(M))\leqq r_A(M)\leqq\sum_{i=0}^{d-1}\binom{d}{i}l_A(H^i_{\mathfrak{m}}(M))+v(K_{\hat{M}}),$$

where $\hat{M}$ denotes the $\mathfrak{m}$*-adic completion of M.*

Note that the proof was achieved by analysis of interaction of Local cohomology and Koszul homology and some of the key technics used in the preceding part of this article had been developed during the preparation of the paper [G=S].

Related to the above, we close with the following which is also quoted from [G=S].

(4.12) **Theorem** ((3.8) and (3.9) [G=S]). *If A is a local ring of* $\dim A\leqq 3$, *then $r(A)<\infty$.*

On the other hand, there exists a local ring A of given dimension $d \geqq 4$ *such that* $r(A) = \infty$.

With this result one may expect the invariant, the type of a ring to provide another clue to the exploration of the outside of the world of C.-M. rings.

References

[A] Aoyama, Y., Some basic results on canonical modules, J. Math. Kyoto Univ., **23** (1983), 85–94.

[A=B] Auslander, M. and Buchsbaum D. A., Codimension and multiplicity, Ann. of Math., **68** (1958), 625–657.

[A=G] Aoyama, Y. and Goto, S., On the endomorphism ring of the canonical module, J. Math. Kyoto Univ., **25** (1985), 21–30.

[B] Buchsbaum, D. A., Complexes in local ring theory, in Some Aspects of Ring Theory, C.I.M.E., Rome, 1965.

[G_1] Goto, S., On Buchsbaum rings, J. Algebra, **67** (1980), 272–279.

[G_2] ——, On the associated graded rings of parameter ideals in Buchsbaum rings, J. Algebra, **85**, No. 2 (1983), 490–534.

[G_3] ——, Blowing-up of Buchsbaum rings, Commutative Algebra: Durham 1981, London Math. Soc. Lect. Note Ser. **72**, 140–162, Cambridge Univ. Press.

[G_4] ——, A note on Quasi-Buchsbaum rings, Proc. Amer. Math. Soc., **90** (1984), 511–516.

[G_5] ——, On the Cohen-Macaulayfication of certain Buchsbaum rings, Nagoya Math. J., **80** (1980), 107–116.

[G_6] ——, Noetherian local rings with Buchsbaum associated graded rings, J. Algebra, **86** (1984), 336–384.

[G=Shi] Goto, S. and Shimoda, Y., On Rees algebras over Buchsbaum rings, J. Math. Kyoto Univ., **20** (1980), 691–708.

[G=S] Goto, S. and Suzuki, N., Index of reducibility of parameter ideals in a local ring, J. Algebra, **87** (1984), 53–88.

[G=S=W] Goto, S., Suzuki, N. and Watanabe, K., On affine semigroup rings, Japan. J. Math., **2**, No. 1 (1976), 1–12.

[G=W] Goto, S. and Watanabe, K., On graded rings, I, J. Math. Soc. Japan, **30** (1978), 179–213.

[G=Y] Goto, S. and Yamagishi, K., The theory of unconditioned strong *d*-sequences with applications to rings and modules possessing finite local cohomology, in preprint (1983).

[Gr] Grothendieck, A., Local cohomology, Lect. Notes in Math., **41** (1967), Springer.

[H=K] Herzog. J. and Kunz, E., Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Math., **238** (1971), Springer.

[Hu] Huneke, C., The theory of *d*-sequences and powers of ideals, Adv. in Math., **46** (1982), 249–279.

[R=S=V] Renschuch, B., Stuckrad, J. and Vogel, W., Weitere Bemerkungen zu einem Problem der Schnittheorie und über ein mas von A. Seidenberg für die Imperfektheit, J. Algebra, **37** (1975, 447–471.

[Sch] Schenzel, P., On Buchsbaum modules and their canonical modules, Teubner Text zur Math. 29, Seminor Eisenbud/Singh/Vogel, Vol. 1, (1980), 65–77.

[Sch_2] Schenzel, P., Standard systems of parameters and their blowing-up rings, J. reine angew. Math., **344** (1983), 201–220.

[S=C=T] Schenzel, P., Cuong, N. T. and Trung, N. V., Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr., **85** (1978), 57–73.

[St] Stückrad, J., Über die kohomologische Characterisierung von Buchsbaum-Moduln, Math. achr., **95** (1980), 256–272.

[S=S=V] Stückrad, J., Schenzel, P. and Vogel, W., Theorie der Buchsbaum moduln, Preprint der Section Mathematik, Martin-Luther Universität nr. 23/24, 1979, (Preliminary version without proofs.)

[$St=V_1$] Stückrad, J. and Vogel, W., Eine Verallgemeinerung der Cohen-Macaulay-Ringe und Anwendungen auf ein Problem der Multiplizitatstheorie, J. Math. Kyoto Univ., **13** (1973), 513–528.

[$St=V_2$] ——, Toward a theory of Buchsbaum singularities, Amer. J. Math., **100** (1978), 727–746.

[S_1] Suzuki, N., On the generalized local cohomology and its duality, J. Math. Kyoto Univ., **18** (1978), 71–85.

[S_2] ——, On applications of generalized local cohomology, Proceedings of the 1^{-st} symposium on commutative algebra, Karuizawa (1978) 88–96, in Japanese.

[S_3] ——, On the Koszul complex generated by a system of parameters for a Buchsbaum module, Bulletin of Department of General Education of Shizuoka College of Pharmacy, **8** (1979), 27–35.

[S_4] ——, On a basic theorem for quasi-Buchsbaum modules, ibid., **11** (1982), 33–40.

[S_5] ——, Canonical duality for Buchsbaum modules-An application of Goto's lemma on Buchsbaum modules, ibid., **13** (1984), 47–60.

[S_6] ——, The Koszul complex of Buchsbaum modules, R.I.M.S. Koukyuroku 446, Proceedings of R.I.M.S. Symposium on Commutative Algebra and Algebraic Geometry. Dec., 1981, pp. 15–25. Research Institute for Mathematical Sciences, Kyoto University.

[S_7] ——, A note on the generalized local cohomology and a type of local rings, Collected papers for 25-th anniversary of Shizuoka College of Pharmacy, Oct. 1978, 237–243.

[S_8] ——, Canonical duality for u.s. d-sequences, to appear in J. Math. Kyoto Univ., **26**–4 (1986), 571–593.

[T] Trung, N. V., Standard system of parameters of generalized Cohen-Macaulay modules, Commutative Algebra, proceedings of the 4-th Symposium on Commutative Algebra. Nov. 1982, 164–180.

[V] Vogel, W., Über eine Vermutung von D.A. Buchsbaum, J. Algebra, **25** (1973), 106–112.

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