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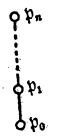
Fibre Products of Noetherian Rings

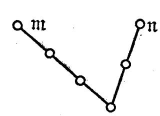
Tetsushi Ogoma

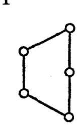
This talk is about fibre product of noetherian rings. The first half is a commentary of the results already published elsewhere, and the second half is new, hence we will give the proof.

Now let me begin with the chain problem of prime ideals. Recall that a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n$ is saturated if there is no proper prime ideal between $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ for any i $(0 \leq i \leq n-1)$ and we draw a picture of it as the following 

. The length of the chain is n. We say that a noetherian ring A is catenary if for any pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals in A, the lengths of all saturated chains between $\mathfrak{p}$ and $\mathfrak{q}$ are the same. A noetherian ring A is universally catenary if any A-algebra of finite type is catenary.

The theorem that a geometric ring (an algebra of finite type over a field) is (universally) catenary is a classical result and it was a problem whether every noetherian ring is (universally) catenary. Nagata constructed counter examples to the problem for the first time, and his construction is suggestive and useful to our topic of today. So let us review:

Nagata's example ([9, Example 2]). Let K be a field and take a regular semilocal K-algebra domain $(R, \mathfrak{m}, \mathfrak{n})$ such that $\dim R_\mathfrak{m} > \dim R_\mathfrak{n}$ and that $K \simeq R/\mathfrak{m} \simeq R/\mathfrak{n}$. Of course, we need an idea to construct such an example and really he had, this is the first point. The second point is to take the subring $A = K + (\mathfrak{m} \cap \mathfrak{n})$ of R. What happens in the process of taking A from R? A picture of typical maximal chain in R looks like 

and that in A looks like , identifying $\mathfrak{m}$ with $\mathfrak{n}$.

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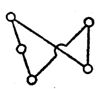
This example raised other problems on chain conditions ([15], [10]), which I do not mention here. Anyway, in spite of keen study on the universally catenary property, it seems to me that much attention was not paid to a problem as the following:

For a given picture, does there exist a noetherian ring whose spectrum contains saturated chains corresponding to the picture? (cf. [2], [3] and [8])

The following are two of such types.

Question 1 ([11, Remark 5.7]). *If a ring A is universally catenary, then does A have a codimension function?*

Recall that φ: Spec $A \to \boldsymbol{Z}$ is a codimension function if $\varphi(\mathfrak{q}) - \varphi(\mathfrak{p}) = \operatorname{ht} \mathfrak{q}/\mathfrak{p}$ for any pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$. It is well known that if a ring A has a dualizing complex, then A is universally catenary and has a codimension function [7, Chapter 5, § 7]. But considering sufficient conditions for a ring of having a dualizing complex, we come to a problem whether the above two are independent each other or not. Really, if A has a codimension function then it is easy to see that A is catenary. But A is not necessarily universally catenary by Nagata's example (take R as $\dim R_n = 1$). And Question 1 asks whether the converse holds. In fact, if we get a ring with spectrum which contains saturated chains corresponding to the picture

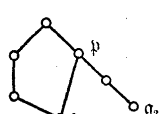


then Question 1 is negative.

Question 2 (R.Y. Sharp [18, Problem 3.11]). *Let M be a balanced big Cohen Macaulay A-module and let $\mathfrak{p}$ be an associated prime of A-module $M/(a_1, \cdots, a_r)M$ for some M-sequence $a_1, a_2, \cdots a_r$. Then is it true that the localization $M_\mathfrak{p}$ of M at $\mathfrak{p}$ is a balanced big Cohen Macaulay $A_\mathfrak{p}$-module?*

Recall that M is a balanced big Cohen Macaulay A-module for a local ring A if M is a big Cohen Macaulay A-module for any system of parameters of A. For a big Cohen Macaulay module, even the change of order of a regular sequence could violate the regularity of new sequence, which makes it difficult to deal with such modules. So Sharp defined this notion and proved several properties similar to finite Cohen Macaulay module ([19], [20]). But as for the localization, it was left open, which is Question 2.

If we get a ring containing



as saturated chains with $\mathfrak{q}_1$, $\mathfrak{q}_2$ minimal, then Question 2 is negative. The point is that for an element a in A to be a part of system of parameters of A, a must not be contained in $\mathfrak{q}_1 \cup \mathfrak{q}_2$, but for b in $A_\mathfrak{p}$ to be a part of system of parameters of $A_\mathfrak{p}$, b can be contained in $\mathfrak{q}_1 A_\mathfrak{p}$.

Now we come up to the fibre product. We recall that a diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & A_0 \\ \uparrow & & \uparrow{\scriptstyle \varphi_2} \\ A & \longrightarrow & A_2 \end{array}$$

is a fibre product of $A_1 \xrightarrow{\varphi_1} A_0 \xleftarrow{\varphi_2} A_2$ in a category $\mathscr{C}$ when, for any commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & A_0 \\ \uparrow & & {\scriptstyle \varphi_2}\uparrow \\ X & \longrightarrow & A_2 \end{array}$$

in $\mathscr{C}$, there exists a unique morphism $X \rightarrow A$ which makes the diagram commutative. When $\mathscr{C}$ is the category of commutative rings, the subring $\{(a_1, a_2) \in A_1 \times A_2 \mid \varphi_1(a_1) = \varphi_2(a_2)\}$ of the direct product $A_1 \times A_2$ satisfies the above condition and the fibre product always exists in this category.

From now on, let us assume the fibre product

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & A_0 \\ {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle \varphi_2} \\ A & \xrightarrow[p_2]{} & A_2 \end{array}$$

is given and fix the following notation.

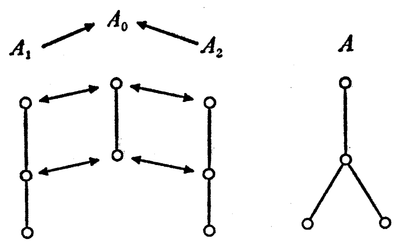
$$\Phi = \varphi_1 \circ p_1 = \varphi_2 \circ p_2, \quad C = \varphi_1(A_1) \cap \varphi_2(A_2), \quad \mathrm{Ker}\,\varphi_i = \mathfrak{a}_i,$$
$$V_0 = \{\Phi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\, C\}, \qquad V_i = \{p_i^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\, A_i,\ \mathfrak{p} \not\supseteq \mathfrak{a}_i\} \quad (i = 1, 2).$$

Then we have,

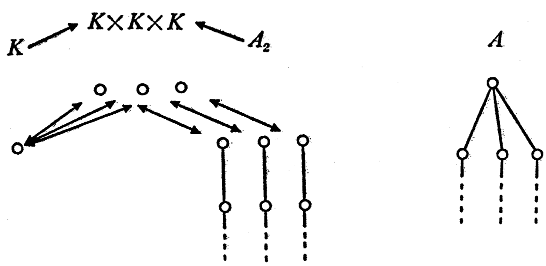
Theorem 1 ([12, Theorem 3.1], see also [4] and [6]).
1) V_0 *is a closed subset of* Spec A *and is isomorphic to* Spec C.
2) Spec $A - V_0$ *is a disjoint union of open subsets* V_1 *and* V_2 *in* Spec A.
3) V_i *is also isomorphic to open set defined by* $\mathfrak{a}_i$ *in* Spec A_i.
4) $V_0 \cup V_i$ *is isomorphic to* Spec $A/\mathrm{Ker}\, p_i$. $i = 1, 2$.

We shall get a general idea of the theorem if we understand the following special cases.

a) Suppose φ_1 and φ_2 are surjective. In this case, $\operatorname{Spec} A_0 = \operatorname{Spec} C$ and it is understood as a closed subset of $\operatorname{Spec} A_1$ and $\operatorname{Spec} A_2$. Then $\operatorname{Spec} A$ is the union of the spaces of $\operatorname{Spec} A_1$ and $\operatorname{Spec} A_2$ identified at this closed subset.

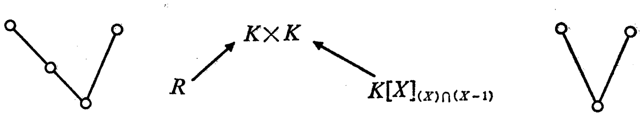


b) Suppose $A_1 = K$ is a field and that A_2 is a semilocal ring such that $A_2/\mathfrak{m} \simeq K \times K \times K$ where $\mathfrak{m}$ is the Jacobson radical. Then we see that $\operatorname{Spec} A$ is the space identifying all the maximal ideals.

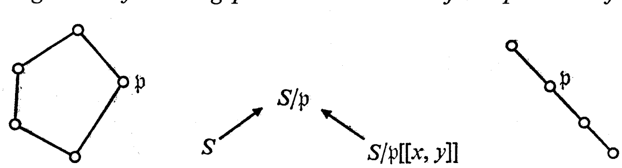


Now we have the example of Nagata as a fibre product of $K \to K \times K = R/(\mathfrak{m} \cap \mathfrak{n}) \leftarrow R$ and we get the counter examples to Question 1 and Question 2 as follows.

Question 1. *With R in Nagata's example, take the fibre product of*



Question 2. *Let S be a ring whose spectrum contains a saturated chains corresponding to the following picture. Take the fibre product of*



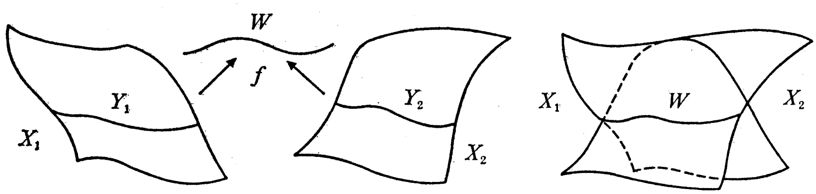
In Question 2, we can really find a required balanced big Cohen Macaulay module which gives a counter-example to Sharp's question [12, Example II].

In these examples, we can directly prove that the rings are noetherian but it is natural to ask when the fibre product of noetherian rings is again noetherian. The answer is the following:

Theorem 2. *Suppose A_1 and A_2 are noetherian. Then the fibre product $A_1 \times_{A_0} A_2$ is noetherian if and only if*
1) *C is noetherian and*
2) *$\mathfrak{a}_1/\mathfrak{a}_1^2$ and $\mathfrak{a}_2/\mathfrak{a}_2^2$ are finite C-modules.*

Proof. See [12, Theorem 2.1].

We can apply these two results to scheme theory. That is, suppose two noetherian schemes X_1 and X_2 with closed subschemes Y_1 and Y_2 respectively are given. For an affine morphism f: $Y_1 \coprod Y_2 \to W$ with some condition, consider the quotient space g: $X_1 \coprod X_2 \to Z$ which is universal in the category of spaces such that $g|_{Y_1 \amalg Y_2} = f$ and that g is isomorphic outside the closed subset $Y_1 \coprod Y_2$. We can state the condition that such a quotient space exists in the category of noetherian schemes. (See [12, § 4] for the exact definition and the proof.)



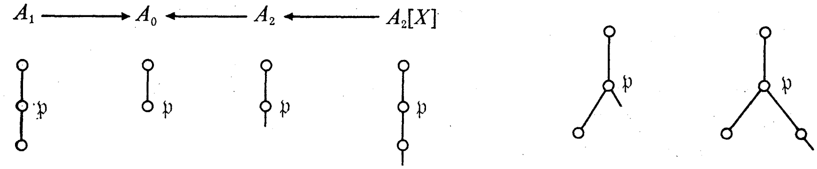
In particular, for a hypersurface Y in a projective n space $\boldsymbol{P}^n$ and for a finite covering f: $Y \to W$, we obtain the quotient space of $\boldsymbol{P}^n$ by f in the category of algebraic varieties [12, Theorem 4.2].

Now we apply fibre product to another topic. For injective modules and dualizing complexes, they decompose as follows.

Proposition 3. *Let $E = E_A(A/\mathfrak{p})$ ($\mathfrak{p} \in \operatorname{Spec} A$) be the injective envelope of the A-module $A/\mathfrak{p}$. Suppose A_i ($i=1, 2$) are finite A-algebras. Then we have $E = E_1 \coprod_{E_0} E_2$ where $E_i = \operatorname{Hom}_A(A_i, E)$ ($i=0, 1, 2$). In particular, if A_i has a fundamental dualizing complex $I_i^\cdot$ ($i=1, 2$) such that $\operatorname{Hom}_A(A_0, I_1^\cdot) \simeq \operatorname{Hom}_A(A_0, I_2^\cdot)$, say $I_0^\cdot$, then $I_1 \coprod_{I_0} I_2$ is a fundamental dualizing complex of A.*

Proof. See [11, § 3]. A similar assertion of the second half is found in [5].

Now as an analogy of chain problems, we consider the situation where φ_i is surjective ($i=1, 2$), $\operatorname{Ker} \varphi_2 = \mathfrak{p}$ is a non-zero prime and 0 in A_2 is primary to $\mathfrak{p}$. Then the picture of the fibre product $A_1 \times_{A_0} A_2$ looks like $\operatorname{Spec} A_1$ with "tail" or something corresponding to the $\operatorname{Ker} \varphi_2$. And if $\dim (A_1)_{\mathfrak{p}} \geq 1$, then $\mathfrak{p}$ is an embedded prime [13, Lemma 2.1]. (By abuse of notation, we write $\mathfrak{p}$ instead of $\varphi_2(\mathfrak{p})$, etc.)



If we change A_2 by $A_2[X]$ in the above fibre product, we see that the prime corresponding to $\mathfrak{p}$ is not an associated prime.

Conversely, if $\mathfrak{p}$ is an embedded prime of a noetherian ring A, take a decomposition $0 = \mathfrak{b}_1 \cap \mathfrak{b}_2$ such that $\mathfrak{b}_2$ is a $\mathfrak{p}$-primary component of some normal primary decomposition of 0 and that $\mathfrak{p} \notin \operatorname{Ass}_A A/\mathfrak{b}_1$. Then we have $A = A/\mathfrak{b}_1 \times_{A/\mathfrak{b}_0} A/\mathfrak{b}_2$, where $\mathfrak{b}_0 = \mathfrak{b}_1 + \mathfrak{b}_2$.

The picture in the above argument is not so accurate and contains some problem, but the process of 'excluding' the embedded primes works like that and we can apply this to another Sharp conjecture in small dimensional cases.

Conjecture (Sharp [17, (4.4)]). *If R has a dualizing complex, then R would be a homomorphic image of a finite dimensional Gorenstein ring.*

This is a generalization of a theorem of Reiten [16, (7) Theorem], which says that if R is a Cohen Macaulay ring with canonical module K, then the principal idealization $R \ltimes K$ is a Gorenstein ring.

By the above argument and by virtue of Proposition 3, we can reduce the conjecture to Cohen Macaulay case if $\dim A \leq 1$. With a little more device of fibre product, we can prove the conjecture in case of $\dim A \leq 2$ [13, Theorem 3.7]. Recently, Aoyama and Goto proved the conjecture in case of $\dim A \leq 4$ with A local [1], but the conjecture is still open in general case.

Another result on fibre product is the following.

Theorem 4. *With surjective φ_1 and φ_2, suppose that A_1 is Cohen Macaulay local ring with grade $\mathfrak{a}_1 \geq 1$ and that A_2 is local, (S_1) and equidimensional. Then A is Gorenstein if and only if one of the following holds,*

1) *trivial, that is, $\mathfrak{a}_2 = 0$ and $A \simeq A_1$ is Gorenstein or*

2) *A_2 is Cohen Macaulay and $\mathfrak{a}_i$ are canonical modules of A_i $(i = 1, 2)$. In case 2) A_0 is also Gorenstein.*

The proof was improved by J. Nishimura, and we will show his proof here.

Proof (Nishimura). We have only to consider non-trivial case. Then $\mathfrak{a}_2 \neq 0$ and we have an irredundant decomposition $0=\mathfrak{a}_1 \cap \mathfrak{a}_2$ in A by the property of fibre product.

Since grade $\mathfrak{a}_1 \geq 1$ and since A_2 is local, (S_1) and equidimensional, we see that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have no common prime divisor if A is (S_1). So if $A_1 = A/\mathfrak{a}_2$ is Cohen Macaulay and A is Gorenstein, then $A_2 = A/\mathfrak{a}_1$ is Cohen Macaulay by [14, Proposition 1.3]. In this case $\mathfrak{a}_1 = \mathrm{Hom}_A(A_1, A)$ and $\mathfrak{a}_2 = \mathrm{Hom}_A(A_2, A)$ are canonical modules of A_1 and A_2 respectively.

On the converse, it is sufficient to prove that A is Gorenstein under the assumption of 2). To do this we may assume that A is a complete local ring. Then we can take a Gorenstein ring B such that $\dim A = \dim B$ and that A is a homomorphic image of B. Now to the exact sequences of A-modules

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{a}_1 & \simeq & \mathfrak{a}_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a}_2 & \longrightarrow & A & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \wr\Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a}_2 & \longrightarrow & A_2 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

gained by fibre product, apply the functor $\mathrm{Hom}_B(\ \ , B)$. We get

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \mathrm{Hom}_B(A_0, B) & \longrightarrow & \mathrm{Hom}_B(A_2, B) & \longrightarrow & \mathrm{Hom}_B(\mathfrak{a}_2, B) & \longrightarrow \mathrm{Ext}_B^1(A_0, B) \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \quad\downarrow \\
0 \to & \mathrm{Hom}_B(A_1, B) & \longrightarrow & \mathrm{Hom}_B(A, B) & \longrightarrow & \mathrm{Hom}_B(\mathfrak{a}_2, B) & \longrightarrow \mathrm{Ext}_B^1(A_1, B) \\
 & \downarrow & & \downarrow & & & \\
 & \mathrm{Hom}_B(\mathfrak{a}_1, B) & \longrightarrow & \mathrm{Hom}_B(\mathfrak{a}_1, B) & & & \\
 & \downarrow & & \downarrow & & & \\
 & \mathrm{Ext}_B^1(A_0, B) & \longrightarrow & \mathrm{Ext}_B^1(A_2, B) & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}
$$

Since A_1 and A_2 are Cohen Macaulay rings of the same dimension equal to $\dim B$ with canonical modules $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively, we have

$$\begin{aligned}
&\mathrm{Ext}_B^1(A_1, B)=\mathrm{Ext}_B^1(A_2, B)=0,\\
&\mathrm{Hom}_B(\mathfrak{a}_1, B)\simeq A_1, \qquad \mathrm{Hom}_B(\mathfrak{a}_2, B)\simeq A_2,\\
&\mathrm{Hom}_B(A_1, B)\simeq \mathfrak{a}_1, \qquad \mathrm{Hom}_B(A_2, B)\simeq \mathfrak{a}_2
\end{aligned}$$

and that

$$\mathrm{Hom}_B(A_0, B)=0.$$

Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{a}_2 & \simeq & \mathfrak{a}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a}_1 & \longrightarrow & \mathrm{Hom}_B(A, B) & \longrightarrow & A_2 & \longrightarrow & 0\\
 & & \wr\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a}_1 & \longrightarrow & A_1 & \longrightarrow & \mathrm{Ext}_B^1(A_0, B) & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

because two homomorphisms from $\mathrm{Hom}_B(A, B)$ to $\mathrm{Ext}_B^1(A_0, B)$ are natural maps defined by taking residue classes modulo the submodule $\mathrm{Hom}_B(A_1, B)$ $\oplus\mathrm{Hom}_B(A_2, B)=\mathfrak{a}_1\oplus\mathfrak{a}_2$. So we have $\mathrm{Hom}_B(A, B)=A_1\times_{A_0}A_2=A$ and A is Gorenstein.

Appendix.

In connection with Theorem 2, Professor Buchsbaum asked me if the fibre product always exists in the category of noetherian rings. Unfortunately, I found that the answer was no, but since his question might lead us to better understanding of fibre product, we show the counter example.

Take an algebraic extension field L over a field K such that $[L:K]=\infty$. Consider two natural homomorphisms $\varphi_1\colon K\to L$ and $\varphi_2\colon L[X]\to L$ defined by $\varphi_2(X)=0$. Then we see that the fibre product A of φ_1 and φ_2 in the category of general commutative rings has the form $A=K+XL[X]$ which is not noetherian.

Suppose there exists the fibre product B of φ_1 and φ_2 in the category of noetherian rings. Then there exists a unique homomorphism $f\colon B\to A$

which makes the diagram commutative. We show that the image $f(B)$ is equal to A, which is a contradiction because the homomorphic image of a noetherian ring is again noetherian.

Take $\alpha \in L$ and consider the ring $C(\alpha)=K+XK(\alpha)[X]$. Since natural homomorphisms $C(\alpha)\to L[X]$ and $C(\alpha)\to K$ make the diagram commutative and since $C(\alpha)$ is noetherian, we see that there exists a homomorphism $C(\alpha)\to B$ which makes the diagram commutative. That is, $C(\alpha)\subseteq f(B)\subseteq A$. Since $\bigcup_{\alpha\in L} C(\alpha)=A$, we have $f(B)=A$.

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Department of Mathematics
Faculty of Science
Kochi University
Kochi, 780 Japan