Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 161–166

On the Injective Envelope of the Residue Field of a Local Ring

Hideyuki Matsumura

0. In this note a ring will mean a commutative noetherian ring. When we say '(A, m, k) is a local ring', we mean that A is a local ring, m is its maximal ideal and k is its residue field.

1. Let A be a ring and M be an A-module. The injective envelope $E=E_A(M)$ of M is defined by the following properties:

(1) E contains M as a submodule, and E is an essential extension of M.

(2) E is an injective A-module.

The first condition is usually easy to check, while the second is sometimes not so easy to verify.

When (A, m, k) is a local ring, $E:=E_A(k)$ has the following remarkable property: if $N\neq 0$ is an A-module (not necessarily finitely generated), then $\mathrm{Hom}_A(N, E)\neq 0$. (To see this, take a non-zero element x of N and set $I:=\mathrm{ann}(x)$. Then there is a non-zero A-linear mapping $Ax\simeq A/I\to A/m=k\to E$; extend it to an A-linear mapping $N\to E$.) Moreover, since E is an essential extension of k, it is easy to see that $\mathrm{Ass}_A(E)=\{m\}$, and consequently every element of E is killed by a suitable power of m, so that E can be viewed as a module over the completion $\hat{A}$ of A. Matlis ([1] Theorem 3.7) showed that $\mathrm{Hom}_A(E, E)=\hat{A}$, in other words every endomorphism of the A-module E is realized by multiplication by exactly one element of $\hat{A}$. In particular, E is a faithful $\hat{A}$-module (i.e. $a\in\hat{A}$, $aE=0\Rightarrow a=0$). Using these facts we obtain the following characterization of $E_A(k)$ when A is complete.

Theorem 1. *Let (A, m, k) be a complete local ring and E be an A-module containing k as submodule. Then E is the injective envelope of k if and only if*

(a) *E is an essential extension of k, and*

(b) *E is a faithful A-module.*

Received January 11, 1986.

Proof. The necessity of these conditions has been already explained. (Here the completeness of A is not needed.) Let us prove the sufficiency. Let D be the injective envelope of E. Then D is also an essential extension of k, so that $D=E_A(k)$. We have to show that $D=E$. Consider the exact sequence $0\to E\to D\to D/E\to 0$, and apply the contravariant exact functor $\mathrm{Hom}_A(\ , D)$ to this sequence. We get the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_A(D/E, D)\longrightarrow \mathrm{Hom}_A(D, D)\longrightarrow \mathrm{Hom}_A(E, D)\longrightarrow 0.$$

Since $\mathrm{Hom}_A(D, D)=A$ by the cited theorem of Matlis, the hypothesis (b) on E implies that the arrow $\mathrm{Hom}_A(D, D)\to \mathrm{Hom}_A(E, D)$ in the last exact sequence is injective. Therefore $\mathrm{Hom}_A(D/E, D)=0$, hence $D/E=0$ as wanted.

Remark 1. The completeness of A is superfluous when A is a one-dimensional analytically irreducible local domain, because in that case any non-zero ideal of $\hat{A}$ contracts to a non-zero ideal of A, so that E is faithful as $\hat{A}$-module if it is so as A-module. On the other hand, if $\hat{A}$ has an ideal $I\neq 0$ such that $A\cap I=0$ and if we set $B=\hat{A}/I$, then $\{x\in E_A(k)\,|\,Ix=0\}=\mathrm{Hom}_{\hat{A}}(B, E_A(k))=E_B(k)$ is an A-module satisfying (a) and (b) but not isomorphic to $E_A(k)$, because it is not a faithful $\hat{A}$-module.

Remark 2. In the case when A is an artinian local ring this characterization was found by Yuji Yoshino several years ago and was used in his Matser's Thesis [2]. In his proof he used the fact that if M is an A-module of finite length then its Matlis dual $\mathrm{Hom}_A(M, E_A(k))$ has the same length as M. I put Yoshino's criterion in my Japanese book [3] as an exercise. It was when I was revising the book for the English edition that I noticed the possibility of generalization as above. In afterthought, the theorem is implicitly contained in [1; Theorem 4.2] or [9; Theorem 5.21 Corollary, p. 154]. But in the present form it does not seem to be widely known, and the purpose of the present paper is to show its usefulness by applications.

One can transform Theorem 1 in a more symmetric form, as follows.

Corollary. *Let (A, m, k) and E be as in the theorem. Then E is the injective envelope of k if and only if*

(a′) *for every non-zero element x of E, there are elements $a\in A$ such that $ax\in k$ and $ax\neq 0$; and*

(b′) *for every non-zero element a of A, there are elements $x\in E$ such that $ax\in k$ and $ax\neq 0$.*

Proof. (a′) is just a restatement of (a). The condition (b′) is stronger than (b), while (a) and (b) together imply (b′).

Now we turn to applications.

2. Let (R, m_0, k) be a complete local ring and set $F := E_R(k)$. Let $A := R[\![X]\!]$ denote the formal power series ring over R in one variable X. Then A is a complete local ring with residue field k. Let $E = F[X^{-1}]$ be the set of the polynomials in X^{-1} with coefficients in F, and make it an A-module by the following convention:

for $a \in R$, $\xi \in F$, and for integers $n, m \geq 0$,

$$aX^n \cdot \xi X^{-m} = \begin{cases} (a\xi)X^{-(m-n)} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Since a polynomial $f(X^{-1}) = \xi_0 + \xi_1 X^{-1} + \cdots + \xi_q X^{-q}$ is killed by X^i for $i > q$, the product of $f(X^{-1})$ and a formal power series $\sum a_i X^i$ has a meaning, and E becomes an A-module. This E is an injective envelope of the A-module $k = A/(m_0, X)$. In fact, if

$$f(X^{-1}) = \xi_0 + \xi_1 X^{-1} + \cdots + \xi_q X^{-q} \in E, \quad \xi_q \neq 0,$$

then there exists $a \in R$ such that $0 \neq a\xi_q \in k$, and then $(aX^q)f(X^{-1}) = a\xi_q \in k$. Therefore E is an essential extension of k. If

$$\psi(X) = a_p X^p + a_{p+1} X^{p+1} + \cdots \in A, \quad a_p \neq 0.$$

then there exists $\xi \in F$ such that $a_p \xi \neq 0$ (since F is a faithful R-module), and so $\psi(X) \cdot (\xi X^{-p}) = a_p \xi \neq 0$. Therefore E is a faithful A-module.

3. Let k be a field and $A = k[\![X_1, \cdots, X_n]\!]$ be the formal power series ring over k. Then $k[X_1^{-1}, \cdots, X_n^{-1}]$ is an A-module by the convention

$$X^\alpha \cdot X^{-\beta} = \begin{cases} X^{-(\beta-\alpha)} & \text{if } \beta_i \geq \alpha_i \text{ for } i = 1, \cdots, n, \\ 0 & \text{otherwise.} \end{cases}$$

By repeated use of the preceding example, or by direct verification of the conditions (a) and (b) of the theorem, we see that $k[X_1^{-1}, \cdots, X_n^{-1}]$ is the injective envelope of the residue field k of A. This was proved by Northcott [4] by a different method. The elements of $k[X_1^{-1}, \cdots, X_n^{-1}]$ are called inverse polynomials. They were known and used by Macaulay [5] in 1916 already.

4. Let A be a discrete valuation ring and let π, k, K be, respectively, a prime element, the residue field and the quotient field of A. It is well known that an A-module M is injective if and only if it is divisible (i.e. $M=\pi M$). Since A has only two prime ideals, there are only two indecomposable injective A-modules (up to isomorphisms), and they are K and K/A. In particular, $K/A=E_A(k)$. This follows also from our theorem, as the conditions (a) and (b) are easily verified for K/A. Note that, since $K=A[\pi^{-1}]$, we can write $E_A(k)=(\pi^{-1}A+\pi^{-2}A+\cdots)/A=\bigcup_{n>0}(\pi^{-n}A/A)$.

5. According to the structure theorem of I.S. Cohen, every complete local ring (A, m, k) is of the form

$$A=C[\![X_1, \cdots, X_n]\!]/I,$$

where C is either a field (in this case we have $C\simeq k$), or a discrete valuation ring with residue field k, and I is an ideal of $C[\![X_1, \cdots, X_n]\!]$. If E' denotes the injective envelope of the $C[\![X_1, \cdots, X_n]\!]$-module k, then $E_A(k)=\mathrm{Hom}_{C[[X]]}(A, E')=\{\xi \in E' \mid I\xi=0\}=(0: I)_{E'}$. One can explicitly describe E' by the results of the preceding three sections, and so one can hope to get an explicit description of $E_A(k)$ when the ideal I has a simple form.

6. Let $N=\{0, 1, 2, \cdots\}$ and let Λ be a finitely generated submonoid of N^n. Let $\{\alpha, \beta, \cdots, \gamma\}\subset\Lambda$ be a system of generators of Λ, and let

$$A:=k[\![X^\alpha, X^\beta, \cdots, X^\gamma]\!]\subset k[\![X_1, \cdots, X_n]\!]$$

where k is a field. Thus,

$$A=\{\sum_{\lambda\in\Lambda} c_\lambda X^\lambda \mid c_\lambda \in k\}.$$

We set $E:=k[X^{-\alpha}, X^{-\beta}, \cdots, X^{-\gamma}]\subset k[X_1^{-1}, \cdots, X_n^{-1}]$. Then E is a vector space over k with basis $\{X^{-\lambda} \mid \lambda \in \Lambda\}$. We make E an A-module by the following formula:

for $\nu, \mu \in \Lambda$,

$$X^\nu\cdot X^{-\mu}=\begin{cases} X^{-(\mu-\nu)} & \text{if } \mu-\nu \in \Lambda, \\ 0 & \text{otherwise.}\end{cases}$$

Note that, if $\nu_i>\mu_i$ for some i, then $\mu-\nu$ has a negative coordinate and consequently $\mu-\nu \notin \Lambda$. The set $k[X_1^{-1}, \cdots, X_n^{-1}]$ is a $k[\![X_1, \cdots, X_n]\!]$-module by the convention in Section 3, hence also an A-module by restriction of scalars, but E is not a submodule of this module; E is rather a factor module of $k[X_1^{-1}, \cdots, X_n^{-1}]$ by the A-submodule $\Sigma_{\alpha\in N^n-\Lambda} kX^{-\alpha}$.

Again it is easy to check (a) and (b) of the theorem, so that E is the injective envelope of the residue field k of A.

7. Let (R, m) and (S, n) be local rings such that $R \subset S$, $n \cap R = m$, and $S/n = R/m =: k$. An R-linear map $\rho: S \to R$ satisfying $\rho|_R = 1_R$ is called a Reynolds operator. Let N be an S-module and M be an R-submodule of N. Suppose S has a Reynolds operator $\rho: S \to R$. Then an R-linear map $\rho^*: N \to M$ is called a Reynolds operator compatible with ρ if

$$\rho^*(s\xi) = \rho(s)\xi \qquad \text{for } s \in S,\ \xi \in M.$$

(Note that we have, in particular, $\rho^*(\xi) = \xi$ for $\xi \in M$.)

The following theorem (in the 0-dimensional case) was the main application of Theorem 1 in Yoshino [2].

Theorem 2 (Y. Yoshino). *Let R, S, k be as above and assume that*

(1) *R is complete,*

(2) *S has a Reynolds operator $\rho: S \to R$, and*

(3) *$E_S(k)$ contains an R-submodule M containing k and there is a Reynolds operator $\rho^*: E_S(k) \to M$ compatible with ρ. Then $M = E_R(k)$.*

Proof. Let $0 \neq \xi \in M$. Then we can find an element $s \in S$ such that $0 \neq s\xi \in k$. Since $s\xi = \rho^*(s\xi) = \rho(s)\xi$ and $\rho(s) \in R$, we see that M is an essential extension of k as R-module. Next, let $0 \neq r \in R$. There is an element η of $E_S(k)$ such that $0 \neq r\eta \in k$. Then $r\eta = \rho^*(r\eta) = r\rho^*(\eta)$, hence $rM \neq 0$. Thus M is a faithful R-module.

8. Graded case

Let $R = \bigoplus_{n \in \boldsymbol{Z}} R_n$ be a Noetherian $\boldsymbol{Z}$-graded ring. In the following, when we say graded module we mean $\boldsymbol{Z}$-graded R-module even when R is $\boldsymbol{N}$-graded. For graded modules M and N, define

$$\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_n \underline{\mathrm{Hom}}_R(M, N)_n,$$

where $\underline{\mathrm{Hom}}_R(M, N)_n$ is the abelian group of the R-linear maps $M \to N$ which increase the degree by n. Denote by $H(R)$ the abelian category whose objects are graded R-modules and in which the group of the morphisms from M to N is given by $\underline{\mathrm{Hom}}_R(M, N)$. This category has enough injectives by [6] (1.10). An object N of $H(R)$ is, by definition, injective if the functor $\underline{\mathrm{Hom}}_R(\ , N)$ is exact. If M is a graded module and N is a graded submodule of M, M is said to be an essential extension of N (in $H(R)$) if every non-zero graded submodule of M has a non-zero intersection with N. The injective envelope of a graded R-module M is denoted by $\underline{E}_R(M)$; this is, by definition, an essential extension of M in $H(R)$ which is injective in $H(R)$.

The category $H(R)$ was systematically discussed by S. Goto and K.

Watanabe in [7]. They mainly considered the case where R is N-graded and R_0 is a field k. Then the field k is denoted by $\underline{k}$ when it is viewed as the graded R-module $R/(R_1+R_2+\cdots)$. They proved that $\underline{E}_R(\underline{k})$ is identical with $E_R(k)$ and $\underline{E}_R(\underline{k})=\underline{\mathrm{Hom}}_k(R, k)$. Ikeda [8] discussed the more general case where R is N-graded and R_0 is a local ring with maximal ideal m and residue field k. Let E denote the injective envelope of the R_0-module k, and let $\underline{E}$ denote the graded R-module such that $\underline{E}_0=E$ and $\underline{E}=0_n$ for $n\neq 0$. Set $M=m+R_1+R_2+\cdots$ and $\underline{k}=R/M$. Then Ikeda showed that $\underline{E}_R(\underline{k})=\underline{\mathrm{Hom}}_{R_0}(R, E)$, and that $\underline{\mathrm{Hom}}_R(\underline{E}_R(\underline{k}), \underline{E}_R(\underline{k}))=R\otimes_{R_0}\hat{R}_0$. Therefore, in this situation, assuming that R_0 is a complete local ring we can prove the graded version of our Theorem 1 by a similar method.

References

[1] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math., **8** (1958), 511–528.

[2] Y. Yoshino, Invariant subrings and Gorenstein rings, Master Thesis, Nagoya University, 1979 (in Japanese and unpublished).

[3] H. Matsumura, Theory of Commutative Rings, Kyoritsu Shuppan, Tokyo, 1980 (in Japanese).

[4] D. G. Northcott, Injective envelopes and inverse polynomials, Quart. J. Math. Oxford (2), **23** (1972), 289–297.

[5] F. S. Macaulay, Algebraic Theory of Modular Systems, Cambridge Tracts no. 19, 1916.

[6] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J., **9** (1957), 119–221.

[7] S. Goto-K. Watanabe, On graded rings, I, J. Math. Soc. Japan, **30** (1978), 179–213.

[8] S. Ikeda, On the Gorensteinness of Rees algebras over local rings, Nagoya Math. J., **102** (1986), 135–154.

[9] D. W. Sharpe and P. Vámos, Injective modules, Cambridge Tracts No. 62, Cambridge University Press, 1972.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, 464 Nagoya
Japan